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Geodesic equation in κ-Minkowski spacetime

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In this paper, we derive corrections to the geodesic equation due to the κ-deformation of curved spacetime, up to the first order in the deformation parameter a. This is done by generalizing the method from our previous paper [Phys. Rev. D **84**, 085020 (2011)] to include curvature effects. We show that the effect of κ-noncommutativity can be interpreted as an extra drag that acts on the particle while moving in this κ-deformed curved spacetime. We have derived the Newtonian limit of the geodesic equation and using this, we discuss possible bounds on the deformation parameter. We also derive the generalized uncertainty relations valid in the nonrelativistic limit of the κ-spacetime.

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I. INTRODUCTION

Noncommutative space entered physics in 1947 when Snyder proposed a model of noncommutative spacetime, admitting a fundamental length [1] as a solution for high energy cutoff, as envisaged by Heisenberg. Since then, motivations for investigating noncommutativity have changed and now noncommutative geometry provides a possible paradigm to capture spacetime uncertainty. Since such uncertainties are encountered in approaches to the microscopic theory of gravity [2], noncommutative geometry naturally comes in the discussions of quantum gravity. It is known that in the low energy limit, the symmetry algebra of certain quantum gravity models is the κ-Poincare algebra. The corresponding spacetime, known as κ-Minkowski spacetime is an example for a Lie algebraic type noncommutative spacetime [3,4]. κ-Minkowski spacetime has been studied in the context of deformed special relativity also [5–7]. Various aspects of this spacetime have been brought out in [8–10].

In recent times, different field theory models on κ-spacetime have been constructed, using various approaches, and many interesting aspects of these models have been analyzed [11–13]. Particularly, the scalar field theory on κ-Minkowski spacetime is analyzed in [14]. By investigating the effect of the κ-deformation parameter on different physical models and comparing their predictions with well-known experimental and observational results, bounds on the noncommutative parameter have been obtained by various authors [15–18].

One of the motivations of studying noncommutative geometry is that it naturally encodes the quantum structure of the spacetime. Thus it is of intrinsic interest to see how gravity theories can be constructed on noncommutative spacetimes and to analyze how these models differ from the gravity theories in the commutative spacetime. Several authors have studied these issues using different approaches [19–23]. Some of these authors have constructed gravity theories on Moyal spacetime using $*$-product approach. Tetrad formulation of general relativity was generalized to the noncommutative case, leading to complex gravity models. In [21], notions of Hopf algebra was used to construct a noncommutative diffeomorphism invariant theory and different aspects of these models were analyzed. In [22], by demanding the noncommutative parameter $\theta^{\mu\nu}$ to be covariantly constant, a generalized $*$ product on the curved noncommutative spacetime was obtained. Using this, a possible model of noncommutative gravity theory was studied and the corresponding modification to the geodesic equation was obtained. In [23], the behavior of a scalar field near a black hole in κ-spacetime was investigated.

In this paper, we derive the geodesic equation on the κ-spacetime, valid up to first order in the deformation parameter. We use a generalization of Feynman's approach [24–26] in deriving the geodesic equation on κ-spacetime. It was shown that the homogeneous Maxwell's equations can be derived by starting with the Newton's force equation and the commutators between the coordinates and velocities [24], which has been generalized to the relativistic case in [25]. In [25], it was shown that the consistent interactions possible for a relativistic particle are with scalar, vector and gravitational fields. Various aspects of Feynman's approach have been studied in [27,28]. This method has been generalized to Moyal spacetime in [29] and to κ-spacetime in [30,31].

In [31], we have generalized the approach of [26] to κ-spacetime, and derived the κ-deformed Maxwell's equations and Lorentz equation, valid up to first order in the deformation parameter a and its classical limits were obtained. We found that the modified Newton's equation

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depends on velocities and this can be interpreted as an effect of a background electromagnetic field. It was also shown that the deformed Lorentz equation gets κ-dependent corrections that are quadratic in velocities, and it can be interpreted that these modifications are due to the curvature, induced by the κ-deformation of the spacetime. A similar feature was shown in the case of Moyal spacetime in [32]. In [31], we have found that the electrodynamics depend on the mass of the particle as well as on its charge. We have also investigated the trajectory of the charged particle in a constant electric field in κ-spacetime, showing the effect of the induced electromagnetic field. Thus it is natural to ask what happens if we consider the motion of a particle in κ-spacetime with curvature. We take up this issue of constructing the geodesic equation in this paper.

This paper is organized as follows. In the next section, we generalize the method of [25], along the lines taken in [31], to derive the geodesic equation in the commutative spacetime. This approach is suitable for generalization to κ-spacetime with a nontrivial metric. In Sec. III, we generalize this approach to κ-deformed spacetime with curvature. Here first, we adapt the method of Sec. II to the case of κ-Minkowski spacetime. This is discussed in Sec. III A. Then in Sec. III B, the geodesic equation is derived in the κ-spacetime with curvature. We show that the κ-dependent correction to the geodesic equation is cubic in velocities. Then in Sec. III C, we obtain the correction to the Newtonian limit of the geodesic equation. We see that only the radial force equation gets a κ-dependent modification. Using this modification, we discuss possible bounds on the deformations parameter a. In Sec. III D, we derive the nonrelativistic correction to the commutation relations. Using this, we get the generalized uncertainty principle. Our concluding remarks are given in Sec. IV.

We work with $\eta_{\mu\nu} = (+,-,-,-)$.

II. GRAVITY AND FEYNMAN APPROACH

The Feynman approach of deriving the Maxwell's equations and the Lorentz equation is known to be equivalent to the minimal coupling prescription in the commutative spacetime [26,31]. This equivalence was generalized to κ-Minkowski spacetime in [31]. In [25], the Feynman approach was generalized to the relativistic case and the geodesic equation for a particle was obtained. In this section, we provide a brief summary of the minimal coupling approach, leading to the derivation of the geodesic equation. In the later sections, we will generalize this approach to κ-deformed spacetime and obtain the corresponding corrections to the geodesic equations, valid up to the first order in the deformation parameter a.

A relativistic particle of mass m and electric charge e is described by $x_\mu(\tau)$ obeying

$$\begin{aligned}
&[x_\mu(\tau), x_\nu(\tau)] = 0,\\
&[x_\mu(\tau), p_\nu(\tau)] = -i\eta_{\mu\nu},\\
&F_\mu = \partial_\mu \phi + eF_{\mu\nu}\dot{x}^\nu,\\
&F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),
\end{aligned} \tag{1}$$

where τ is a parameter and $p_\mu = m\dot{x}_\mu + eA_\mu$ is the canonical momentum operator, $F_\mu = m\ddot{x}_\mu$ is the force, $F_{\mu\nu}$ is the electromagnetic strength tensor, A_μ is a gauge field and $\phi(x)$ is an arbitrary function of x. Since, in this paper, we are concerned only with effect of κ-deformation on gravity, we set $A_\mu(x) = 0$, i.e., we are dealing with electrically neutral particles. Thus we have

$$\begin{aligned}
&[x_\mu, x_\nu] = 0, \qquad [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = -i\eta_{\mu\nu}\\
&F_\mu = 0, \qquad p_\mu = m\dot{x}_\mu,
\end{aligned} \tag{2}$$

where we have taken $\phi(x) = 0$, because this choice will lead to the correct geodesic equation. In [25] it is argued that the generalization from flat to curved space can be done by replacing the Minkowskian metric $\eta_{\mu\nu}$ with an arbitrary metric $g_{\mu\nu}(X)$ in Eq. (1). With this modification, the geodesic equation was derived using the Feynman approach. Using this approach, we start with the postulate

$$[X_\mu, X_\nu] = 0, \qquad [X_\mu, P_\nu] = -ig_{\mu\nu}(X). \tag{3}$$

But for now we assume that the above "metric" $g_{\mu\nu}(X)$ is a function of operator X alone and that it is a symmetric tensor [which is implied by the first equation in (3)]. All of the shifting of indices is done with $\eta_{\mu\nu}$ (this is different from what is done in [25]). $X_\mu(\tau)$ is a new position operator and $P_\mu(\tau)$ is the corresponding conjugate momenta, $m\dot{X}_\mu = P_\mu$, and we want to solve (3) in terms of the operators given in (2). It is easy to see that we can construct operators X and P as follows:

$$X_\mu \equiv x_\mu, \qquad P_\mu \equiv g_{\mu\alpha}p^\alpha, \tag{4}$$

where x_μ and p_ν satisfy (2). Now, we take the derivative with respect to τ of Eqs. (3) and get

$$\frac{1}{m}[P_\mu, P_\nu] + [X_\mu, \ddot{X}_\nu] = -i\frac{dg_{\mu\nu}}{d\tau}, \tag{5}$$

where we have used $\dot{P}_\mu = m\ddot{X}_\mu$. Using Eqs. (4) and (2) we have[1]

$$[P_\mu, P_\nu] = i\left(g_{\mu\alpha}\frac{\partial g_{\nu\beta}}{\partial x_\alpha} - g_{\nu\alpha}\frac{\partial g_{\mu\beta}}{\partial x_\alpha}\right)p^\beta, \tag{6}$$

where we have used $[p_\mu, f(x,p)] = i\frac{\partial f}{\partial x^\mu}$. Also we have

[1]Notice that in the Feynman's approach or its generalization in [25], the notion of conjugate momentum is not used. Whereas, in [26] and in [31], conjugate momentum is used and it has been shown that the Feynman's approach and this method of using conjugate momentum are equivalent.

$$\dot{g}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial X_\beta}\dot{X}_\beta = \frac{1}{m}\frac{\partial g_{\mu\nu}}{\partial X_\beta}P_\beta = \frac{1}{m}\frac{\partial g_{\mu\nu}}{\partial x_\beta}g_{\beta\alpha}p^\alpha. \quad (7)$$

Equations (5)–(7), give

$$[X_\mu, m\ddot{X}_\nu] = -\frac{i}{m}\left(g_{\mu\alpha}\frac{\partial g_{\nu\beta}}{\partial x_\alpha} - g_{\nu\alpha}\frac{\partial g_{\mu\beta}}{\partial x_\alpha} + g_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)p^\beta. \quad (8)$$

Since $\ddot{X}$ is a function of x and p, the left-hand side (lhs) of Eq. (8) becomes

$$[X_\mu, \ddot{X}_\nu] = [x_\mu, \ddot{X}_\nu] = -i\frac{\partial \ddot{X}_\nu}{\partial p^\mu}. \quad (9)$$

Using this, we integrate Eq. (8) over p^μ to get

$$m\ddot{X}_\nu = G_\nu + \frac{1}{2m}\left(g_{\mu\alpha}\frac{\partial g_{\nu\beta}}{\partial x_\alpha} - g_{\nu\alpha}\frac{\partial g_{\mu\beta}}{\partial x_\alpha} + g_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)p^\beta p^\mu. \quad (10)$$

Defining

$$\tilde{\Gamma}_{\nu\mu\beta} = -\frac{1}{2}\left(g_{\mu\alpha}\frac{\partial g_{\nu\beta}}{\partial x_\alpha} - g_{\nu\alpha}\frac{\partial g_{\mu\beta}}{\partial x_\alpha} + g_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right), \quad (11)$$

we get

$$m\ddot{X}_\nu + \frac{1}{m}\tilde{\Gamma}_{\nu\mu\beta}p^\beta p^\mu = 0, \quad (12)$$

where we have chosen $G_\nu(x) = 0$. In the above, Eq. (11) is similar to the Christoffel symbol and Eq. (12) is similar to the famous geodesic equation. To make the proper correspondence to gravity we have to go to the classical limit, i.e., we have to take the limit $[,] \longrightarrow \frac{1}{i}\{,\}_{\rm PB}$ and all operators reduces to corresponding commuting c-number functions. We assume that metric is invertible and define an inverse of the symmetric tensor $g_{\mu\alpha}$ by the following relation:

$$g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu. \quad (13)$$

From Eqs. (4), (11), and (13), we get

$$\begin{aligned} g^{\beta\sigma}P_\sigma &= p^\beta, \\ g^{\beta\sigma}\frac{\partial g_{\alpha\beta}}{\partial x^\rho} &= -g_{\alpha\beta}\frac{\partial g^{\beta\sigma}}{\partial x^\rho}, \\ \tilde{\Gamma}_{\nu\mu\beta}g^{\beta\sigma}g^{\mu\sigma} &= \frac{1}{2}g_{\nu\alpha}\left(\frac{\partial g^{\alpha\sigma}}{\partial x_\rho} + \frac{\partial g^{\alpha\rho}}{\partial x_\sigma} - \frac{\partial g^{\rho\sigma}}{\partial x_\alpha}\right) \\ &\equiv \Gamma^{\rho\sigma}_\nu, \end{aligned} \quad (14)$$

where $\Gamma^{\rho\sigma}_\nu$ is the Christoffel symbol. Using Eqs. (14), we re-express the geodesic equation in Eq. (12) as

$$\ddot{X}_\nu + \Gamma^{\mu\beta}_\nu \dot{X}_\beta \dot{X}_\mu = 0. \quad (15)$$

Note that all raising and lowering of indices were done using $\eta_{\mu\nu}$, and tensor $g_{\mu\nu}$ is treated only as a symmetric tensor with an inverse defined in Eq. (13). This shows the derivation of the geodesic equation using the Feynman approach.

III. κ-DEFORMATION OF GRAVITY

In this section, we obtain the geodesic equation for a particle moving in the κ-deformed spacetime with the arbitrary metric, which are the main results of this paper. After discussing modifications required to adapt the Feynman's approach to the κ-Minkowski spacetime, we generalize this to the κ-deformed spacetime with the arbitrary metric and derive the κ-deformed geodesic equation. Then we discuss the Newtonian limit of this geodesic equation and obtain the corrections. The effect of these corrections on observational and experimental results are analyzed and the bounds on the deformation parameter suggested by this correction to the Newtonian results is discussed. We also obtain the κ-modified commutation relations and derive generalized uncertainty relations.

A. κ-Minkowski spacetime

κ-Minkowski spacetime is defined by the coordinates obeying

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu\hat{x}_\nu - a_\nu\hat{x}_\mu). \quad (16)$$

Operators $\hat{x}_\mu$ can be realized in terms of operators x and p [31,33] as

$$\hat{x}_\mu = x_\alpha\varphi^\alpha{}_\mu(p), \quad (17)$$

where $\varphi^\alpha{}_\mu(p)$ must satisfy

$$\frac{\partial\varphi^\alpha{}_\mu}{\partial p^\beta}\varphi^\beta{}_\nu - \frac{\partial\varphi^\alpha{}_\nu}{\partial p^\beta}\varphi^\beta{}_\mu = a_\mu\varphi^\alpha{}_\nu - a_\nu\varphi^\alpha{}_\mu. \quad (18)$$

Solving Eq. (18) up to the first order in deformation parameter a, we get

$$\varphi^\alpha{}_\mu = \delta^\alpha_\mu[1 + \alpha(a\cdot p)] + \beta a^\alpha p_\mu + \gamma p^\alpha a_\mu, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad (19)$$

where parameters of the realization α, β and γ have to satisfy

$$\gamma - \alpha = 1, \qquad \beta \in \mathbb{R}. \quad (20)$$

It will be convenient for later to define an operator $\hat{y}$ which commutes with $\hat{x}$ (for more on the properties of $\hat{y}$ see [33]), i.e.,

$$[\hat{y}_\mu, \hat{x}_\nu] = 0 \Leftrightarrow [\hat{y}_\mu, \hat{y}_\nu] = -i(a_\mu\hat{y}_\nu - a_\nu\hat{y}_\mu), \quad (21)$$

and hence, it is easy to see that any function of $\hat{y}$ also commutes with $\hat{x}$, i.e.,

$$[f(\hat{y}), \hat{x}_\mu] = 0. \quad (22)$$

We can express $\hat{y}$ and $f(\hat{y})$ up to the first order in a, as

$$\hat{y}_\mu = x_\mu + \gamma x_\mu (a\cdot p) + (\gamma - 1)(x\cdot p)a_\mu + \beta(x\cdot a)p_\mu,$$
$$f(\hat{y}) = f(x) + \gamma\left(x\cdot\frac{\partial f}{\partial x}\right)(a\cdot p) + (\gamma-1)\left(a\cdot\frac{\partial f}{\partial x}\right)(x\cdot p) + \beta(a\cdot x)\left(\frac{\partial f}{\partial x}\cdot p\right). \tag{23}$$

The canonical momentum operator (in the $e = 0$ case) $\hat{p}_\mu = m\frac{d\hat{x}_\mu}{d\tau}$ is then obtained [31] as

$$\hat{p}_\mu = p_\alpha \varphi^\alpha{}_\mu \to \hat{p}_\mu = p_\mu + (\alpha+\beta)(a\cdot p)p_\mu + \gamma a_\mu p^2 \tag{24}$$

and obeys

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \tag{25}$$

$$[\hat{p}_\mu, \hat{x}_\nu] = i\eta_{\mu\nu}(1 + s(a\cdot p)) + i(s+2)a_\mu p_\nu + i(s+1)a_\nu p_\mu, \quad s = 2\alpha + \beta. \tag{26}$$

We have shown [31] that this construction via Feynman's approach satisfies all Jacobi identities. By taking the derivative of Eq. (16), with respect to τ, we get a condition,

$$[\hat{p}_\mu, \hat{x}_\nu] + [\hat{x}_\mu, \hat{p}_\nu] = i(a_\mu \hat{p}_\nu - a_\nu \hat{p}_\mu). \tag{27}$$

These results are needed for deriving the geodesic equation in the κ-Minkowski spacetime in the next subsection.

B. κ-dependent corrections to the geodesic equation

In the flat commutative spacetime, our derivation of geodesic equation used the conjugate pairs (x, p), and for flat noncommutative spacetime, we used $(\hat{x}, \hat{p})$ (see previous section). We showed that all the operators in the flat noncommutative spacetime could be written in terms of x, p and deformation parameter a. For the noncommutative spacetime with curvature, we will use $(\hat{X}, \hat{P})$ and our approach is to construct them as functions of x, p and deformation parameter a. In the case of neutral particles, conjugate momenta is given by $\hat{P}_\mu = m\frac{d\hat{X}_\mu}{d\tau}$. After obtaining the realization for $\hat{X}$ and $\hat{P}$ in terms of x and p, consistent with the κ generalization of the relations in Eqs. (3) and (4), we derive the corrections to the geodesic equation due to the κ-deformation of spacetime.

We start with the postulate

$$[\hat{X}_\mu, \hat{X}_\nu] = i(a_\mu \hat{X}_\nu - a_\nu \hat{X}_\mu), \tag{28}$$

where

$$\hat{X}_\mu = X_\alpha \varphi^\alpha{}_\mu, \tag{29}$$

and $\varphi^\alpha{}_\mu$ satisfies Eq. (18). The conjugate momentum $\hat{P}_\mu$ should satisfy all the Jacobi identities and also obey

$$[\hat{P}_\mu, \hat{X}_\nu] + [\hat{X}_\mu, \hat{P}_\nu] = i(a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu). \tag{30}$$

In the commutative limit, $a \to 0$, $\hat{X}_\mu$ and $\hat{P}_\nu$ should satisfy

$$\hat{X}_\mu \to X_\mu = x_\mu, \qquad \hat{P}_\mu \to P_\mu = g_{\mu\alpha}p^\alpha, \qquad [\hat{X}_\mu, \hat{P}_\nu] \to g_{\mu\nu}(x), \tag{31}$$

and in the limit $g_{\mu\nu}(x) \to \eta_{\mu\nu}$, they should reduce to the corresponding flat spacetime results, i.e.,

$$\hat{X}_\mu \to \hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu, \qquad \hat{P}_\mu \to \hat{p}_\mu = p_\alpha \varphi^\alpha{}_\mu, \qquad [\hat{X}_\mu, \hat{P}_\nu] \to [\hat{x}_\mu, \hat{p}_\nu]. \tag{32}$$

We explicitly construct $\hat{P}_\mu$, satisfying the above requirements. This leads to the realization of $\hat{P}_\mu$ as

$$\hat{P}_\mu \equiv g_{\alpha\beta}(\hat{y})p^\beta \varphi^\alpha{}_\mu. \tag{33}$$

Using Eq. (18), it is straightforward to see that this construction satisfies all Jacobi identities as well as Eq. (30) to all orders in a. For more details on the construction of $\hat{P}_\mu$ see Appendix A.

Thus to summarize, the coordinates and conjugate momenta for the κ-deformed spacetime with an arbitrary metric is given as

$$\hat{X} = x_\alpha \varphi^\alpha{}_\mu, \qquad \hat{P}_\mu = g_{\alpha\beta}(\hat{y})p^\beta \varphi^\alpha{}_\mu, \tag{34}$$

satisfying

$$[\hat{X}_\mu, \hat{X}_\nu] = i(a_\mu \hat{X}_\nu - a_\nu X_\mu), \tag{35}$$

and

$$[\hat{X}_\mu, \hat{P}_\nu] = -ig_{\alpha\beta}(\hat{y})\left(p^\beta \frac{\partial \varphi^\alpha{}_\nu}{\partial p^\sigma}\varphi^\sigma{}_\mu + \varphi^\alpha{}_\nu \varphi^\beta{}_\mu\right). \tag{36}$$

Equations (34)–(36) are valid to all orders in a.

Next we take the derivative of Eq. (36) with respect to τ, and by using $\frac{d\hat{P}_\mu}{d\tau} = m\dot{\hat{X}}_\mu$ and $\frac{dp_\mu}{d\tau} = 0$, we get

$$[\hat{X}_\mu, m\dot{\hat{X}}_\nu] = -\frac{1}{m}[\hat{P}_\mu, \hat{P}_\nu] - i\frac{dg_{\alpha\beta}(\hat{y})}{d\tau}\left(p^\beta \frac{\partial \varphi^\alpha{}_\nu}{\partial p^\sigma}\varphi^\sigma{}_\mu + \varphi^\alpha{}_\nu \varphi^\beta{}_\mu\right). \tag{37}$$

Up to this, all of our results are valid to all orders in a. We now calculate the right-hand side (rhs) of Eq. (37) explicitly, up to the first order in the deformation parameter a, using Eqs. (19), (23), and (34). Since our goal is to get the corrections to $\ddot{\hat{X}}_\mu$, we assume

$$\ddot{\hat{X}}_\mu = \ddot{X}_\mu + \delta\ddot{X}_\mu(a) + O(a^2), \tag{38}$$

where $\delta\ddot{X}(a)$ is linear in a and generally a function of x and p, while $\ddot{X}_\mu$ satisfies Eq. (12). For the lhs of Eq. (37) we have

$$[\hat{X}_\mu, \ddot{\hat{X}}_\nu] = [\hat{X}_\mu, \ddot{X}_\nu] + [X_\mu, \delta\ddot{X}_\nu(a)] + O(a^2). \tag{39}$$

Combining Eq. (37) and (39) we have

$$m[X_\mu, \delta\ddot{X}_\nu(a)] = -[\hat{X}_\mu, m\ddot{X}_\nu] - \frac{1}{m}[\hat{P}_\mu, \hat{P}_\nu] - i\frac{dg_{\alpha\beta}(\hat{y})}{d\tau}\left(p^\beta \frac{\partial \varphi^\alpha{}_\nu}{\partial p^\sigma}\varphi^\sigma{}_\mu + \varphi^\alpha{}_\nu \varphi^\beta{}_\mu\right), \tag{40}$$

where the lhs can be written as

$$[X_\mu, \delta\ddot{X}_\nu(a)] = [x_\mu, \delta\ddot{X}_\nu(a)] = -i\frac{\partial[\delta\ddot{X}_\nu(a)]}{\partial p^\mu}. \tag{41}$$

Next we calculate the rhs of Eq. (40) explicitly up to the first order in a, using Eqs. (12), (19), (23), and (34). Then by using Eq. (41), we perform the integration of Eq. (40) and obtain $\delta\ddot{X}_\nu(a)$, which with Eq. (38) finally gives

$$\ddot{\hat{X}}_\nu + \frac{1}{m^2}\tilde{\Gamma}_{\nu\mu\beta}p^\beta p^\mu = \frac{1}{m^2}\tilde{\Sigma}_{\nu\tau\delta\mu}p^\tau p^\delta p^\mu, \tag{42}$$

where $\tilde{\Gamma}_{\nu\mu\beta}$ is given in (11) and $\tilde{\Sigma}_{\nu\tau\delta\mu}$ in Appendix B. We note here that the important feature of the a-dependent corrections is that it is cubic in p, and that it depend on the realization that is on the parameters α, β and γ.

If we now go to the classical limit [as described in text after Eq. (15)] by using Eqs. (13) and (34) we get

$$\begin{aligned} p^\beta p^\mu &= g^{\beta\alpha}g^{\mu\sigma}\hat{P}_\alpha\hat{P}_\sigma \\ &\quad + P_{\sigma_1}P_{\sigma_2}P_{\sigma_3}\{\ldots a\ldots\}^{\beta\mu\sigma_1\sigma_2\sigma_3} + O(a^2), \\ p^\tau p^\delta p^\mu &= g^{\tau\sigma_1}g^{\delta\sigma_2}g^{\mu\sigma_3}P_{\sigma_1}P_{\sigma_2}P_{\sigma_3} + O(a), \end{aligned} \tag{43}$$

and with Eq. (42) we have

$$\begin{aligned} \tilde{\Gamma}_{\nu\mu\beta}p^\beta p^\mu &= \Gamma_\nu^{\alpha\sigma}\hat{P}_\alpha\hat{P}_\sigma + \tilde{\Gamma}_{\nu\mu\beta}\{\ldots a\ldots\}^{\beta\mu\sigma_1\sigma_2\sigma_3} \\ &\quad \times P_{\sigma_1}P_{\sigma_2}P_{\sigma_3} + O(a^2), \\ \tilde{\Sigma}_{\nu\tau\delta\mu}p^\tau p^\delta p^\mu &= \tilde{\Sigma}_{\nu\tau\delta\mu}g^{\tau\sigma_1}g^{\delta\sigma_2}g^{\mu\sigma_3}P_{\sigma_1}P_{\sigma_2}P_{\sigma_3} + O(a^2). \end{aligned} \tag{44}$$

Using $\hat{P}_\mu = m\dot{\hat{X}}_\mu$ and Eq. (44) in (42) we get

$$\ddot{\hat{X}}_\nu + \Gamma_\nu^{\alpha\sigma}\dot{\hat{X}}_\alpha\dot{\hat{X}}_\sigma = m\Sigma_\nu^{\sigma_1\sigma_2\sigma_3}\dot{\hat{X}}_{\sigma_1}\dot{\hat{X}}_{\sigma_2}\dot{\hat{X}}_{\sigma_3} + O(a^2), \tag{45}$$

where

$$\Sigma_\nu^{\sigma_1\sigma_2\sigma_3} \equiv -\tilde{\Gamma}_{\nu\mu\beta}\{\ldots a\ldots\}^{\beta\mu\sigma_1\sigma_2\sigma_3} + \tilde{\Sigma}_{\nu\tau\delta\mu}g^{\tau\sigma_1}g^{\delta\sigma_2}g^{\mu\sigma_3}. \tag{46}$$

The explicit form of $\tilde{\Sigma}_{\nu\tau\delta\mu}$ is given in Appendix C. Above Eq. (45) shows the corrections to the geodesic equation due to the κ-deformation of the Minkowski spacetime.

C. Newtonian limit

Now we will investigate the "Newtonian limit" of the κ-deformed geodesic equation obtained in Eq. (45). From now on we will consider a special case of κ-Minkowski space were we take $a_\mu = (a, \vec{0})$. We define the Newtonian limit by following three requirements [34]:

(1) Particles are moving slowly, and hence we have

$$\frac{d\hat{X}_i}{d\tau} \ll \frac{d\hat{X}_0}{d\tau}. \tag{47}$$

(2) The gravitational field is weak and can be considered as perturbation about the flat spacetime metric, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{48}$$

(3) The gravitational field is static.

From the definition of the inverse metric, $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$, we find that to the first order in h, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Using the above assumptions and keeping only linear terms in a and h, from Eq. (45), we get

$$\ddot{\hat{X}}_0 = 0, \tag{49}$$

$$\ddot{\hat{X}}_i + \frac{1}{2}\frac{\partial h^{00}}{\partial x^i}(\dot{\hat{X}}_0)^2 = m\Sigma_i^{000}(\dot{X}_0)^3. \tag{50}$$

Using Eq. (49) we can easily change the derivatives with respect to τ to derivatives with respect to $\hat{t}$ in Eq. (50). At first glance it seems that the last term in Eq. (50) is not reparametrization invariant. But since we are keeping only linear terms in a and h (hence Σ_i^{000} is already linear in a and h) we have $\dot{X}_0 = \frac{dt}{d\tau} \approx 1 + \frac{1}{2}h_{00} \approx 1 + O(h)$ reparametrization is intact. Generalizing the definition of Newton's gravitational force, given as $F^i = -G\frac{mM}{r^3}x^i = \frac{1}{2}\frac{\partial h^{00}}{\partial x^i}$ to the κ-deformed spacetime as $\hat{F}^i = m\frac{d^2\hat{X}^i}{d\hat{t}^2}$, we get

$$\hat{F}^i = F^i\left(1 - \frac{am}{3}C\right), \tag{51}$$

where $C = 5\alpha + 5\beta + 12\gamma$. Note that the force equation does get a-dependent modification, but there is only radial force as in the commutative case. But this radial force has an a-dependent correction and this can be compared to the prediction of the Pioneer anomaly. Also notice that the a-dependent correction depends on the mass of the test particle. This shows that the equivalence principle is violated. The corrections also depend on the choice of realization, that is, on the parameters α, β, and γ.

The form of correction to the force equation obtained above in Eq. (51) is exactly in the same form as that was obtained in [18] and hence we will get the same bounds on a as obtained in [18]. Thus the Pioneer anomaly sets a bound $|a| \leq 10^{-53}$ m and the violation of equivalence principle sets $|a| \leq 10^{-55}$ m (for a body of mass 1 kg).

D. Uncertainty relations

It is well-known that any quantum theory of spacetime will imply a minimal bound on the localization of particles [35] and that the noncommutativity of spacetime can in fact account for the modification of the Heisenberg uncertainty relations [36]. In [37] it has been argued that even at the Newtonian level there are modifications of the uncertainty relations due to gravity. In order to see modifications of the Heisenberg uncertainty principle in our approach, we first see the nonrelativistic limit of Eqs. (35) and (36),

$$[x_i, x_j] = 0, \qquad [x_i, p_j] = i\hbar(1 + ams)\delta_{ij}, \\ [x_0, p_0] = -i\hbar(1 + 3am(s + 1)), \tag{52}$$

where $s = 2\alpha + \beta$. Now, using $\Delta A \Delta B \geq \frac{1}{2}|\langle[A, B]\rangle|$, we get the deformed uncertainty relations as

$$\Delta x_i \Delta x_j \geq 0, \qquad \Delta x_i \Delta p_j \geq \frac{\hbar}{2}(1 + ams), \\ \Delta E \Delta t \geq \frac{\hbar}{2}(1 + 3am(s + 1)). \tag{53}$$

IV. CONCLUSION AND OUTLOOK

We have analyzed the effect of κ deformation on the motion of a particle in the curved spacetime. Although, one can find various approaches that handle noncommutative space and gravity (most of them on Moyal space [19–23]), our approach is one of the first attempts that deals with effect of gravity on κ-deformed spacetime. In this paper, we have first generalized the Feynman approach of [31] to reproduce results in [25]. This then enables us to derive the geodesic equation for the κ-Minkowski spacetime up to the first order in the deformation parameter a. The main difference between the commutative and κ-deformed case is that there is an "extra" force that is proportional to $\dot{X}^3$ in the κ-deformed case. This term, which is cubic in velocities, can be interpreted as an extra drag that acts on the particle when moving in a κ-deformed curved spacetime. This approach allows one to treat these effects as a perturbation to the commutative, curved spacetime. The principal characteristic of our approach is that all the corrections depend on the choice of realization (parameters α, β and γ) and on the mass of the test particle. Since the photon has $m_\gamma = 0$, there is no change in the geodesic equation for light, and also no change in uncertainty relations, which makes it more difficult to set experimental bounds on deformation parameter a. Since for certain quantum gravity models the low energy limit is the κ-Poincare algebra and corresponding spacetime is κ-Minkowski, our results can be thought of as the effect of quantum gravity. We have derived the a-dependent correction to the Newtonian limit of the geodesic equation. We see that the Newtonian force/potential remains radial, but depends on the mass of the test particle (as well as a). In the "special relativistic" limit (obtained by taking $g_{\mu\nu} \to \eta_{\mu\nu}$), results obtained here reproduce various deformed special relativity models, since these models differ from ours only in explicit choice of the parameters α, β and γ (that is in the choice of realization). It is clear that in this limit we have effects that violate Lorentz symmetry and that there is a change in the dispersion relation [16]. It is interesting to note that the bounds on the deformation parameter a obtained here are same as that obtained in [18], where a different realization of the coordinates of κ-Minkowski spacetime was used.

We have shown that the κ-deformed commutation relations between phase space variables induce modified uncertainty relations. There have been investigations on the possible modifications of atomic spectra due to the generalized uncertainty relations and bounds on deformation parameters were obtained [38]. Thus it is of interest to study the changes in the spectrum of the Hydrogen atom due to the κ-deformed uncertainty relations we have in Eq. (53). This will be taken up separately.

In the commutative limit $[X_\mu, P_\nu]$ gives rise to the metric $g_{\mu\nu}$ and by analogy, one could interpret $[\hat{X}_\mu, \hat{P}_\nu]$ (or just the symmetric part of it) as giving rise to noncommutative metric $\hat{g}_{\mu\nu}$. Then it would be interesting to construct the noncommutative version of Ricci tensor $\hat{R}_{\mu\nu}$ and Ricci scalar $\hat{R}$ in order to get a Lagrangian that would reproduce Eq. (45) by action principal. The question of the invariant line element is still unsolved. These problems are of immense importance and will be reported elsewhere.

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APPENDIX A

We have a more general construction of the operator $\hat{P}_\mu$ up to the first order in the deformation parameter. By differentiating Eq. (28) with respect to τ, we get

$$[\hat{P}_\mu, \hat{X}_\nu] + [\hat{X}_\mu, \hat{P}_\nu] = i(a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu). \tag{A1}$$

This determines the antisymmetric part of $[\hat{P}_\mu, \hat{X}_\nu]$. We can write

$$[\hat{P}_\mu, \hat{X}_\nu] = \hat{S}_{\mu\nu} + \hat{A}_{\mu\nu}, \tag{A2}$$

where $\hat{S}_{\mu\nu} = \hat{S}_{\nu\mu}$ and $\hat{A}_{\mu\nu} = -\hat{A}_{\nu\mu}$. From Eqs. (A1) and (A2) we get

$$\hat{A}_{\mu\nu} = \frac{i}{2}(a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu). \tag{A3}$$

In the limit $a \to 0$, we must have

$$[\hat{P}_\mu, \hat{X}_\nu] \overset{a\to 0}{\longrightarrow} [P_\mu, X_\nu] = i g_{\mu\nu}. \tag{A4}$$

Using these, up to the first order in the deformation parameter a we have $\hat{S}_{\mu\nu} = i g_{\mu\nu} + \delta S(a)_{\mu\nu}$ or more explicitly

$$\hat{S}_{\mu\nu} = ig_{\mu\nu} + ia_\alpha G_{\mu\nu}{}^{\alpha\beta}(x)p_\beta + O(a^2). \tag{A5}$$

Here $G_{\mu\nu}{}^{\alpha\beta} = G_{\nu\mu}{}^{\alpha\beta}$, and then we get

$$[\hat{P}_\mu, \hat{X}_\nu] = ig_{\mu\nu} + ia_\alpha G_{\mu\nu}{}^{\alpha\beta}(x)p_\beta + \frac{i}{2}(a_\mu \hat{P}_\nu - a_\nu \hat{P}_\mu). \tag{A6}$$

We get constraints on $G_{\mu\nu}{}^{\alpha\beta}(x)$ by requiring that the Jacobi identities must be satisfied up to the first order in a. From

$$[[\hat{X}_\mu, \hat{X}_\nu], \hat{P}_\lambda] + [[\hat{X}_\nu, \hat{P}_\lambda], \hat{X}_\mu] + [[\hat{P}_\lambda, \hat{X}_\mu], \hat{X}_\nu] = 0 \tag{A7}$$

we get

$$\begin{aligned} &a_\alpha(G_{\mu\nu}{}^{\alpha}{}_{\beta} - G_{\mu\beta}{}^{\alpha}{}_{\nu}) \\ &= \alpha a^\alpha\left(x_\beta \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - x_\nu \frac{\partial g_{\mu\beta}}{\partial x^\alpha}\right) + \beta x^\alpha\left(a_\beta \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - a_\nu \frac{\partial g_{\mu\beta}}{\partial x^\alpha}\right) \\ &\quad + \gamma(x\cdot a)\left(\frac{\partial g_{\mu\nu}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\nu}\right) + \frac{3}{2}(a_\nu g_{\mu\beta} - a_\beta g_{\nu\mu}). \end{aligned} \tag{A8}$$

Now we can construct $G_{\mu\nu}{}^{\alpha\beta}(x)$ from $\eta_{\mu\nu}$, $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\alpha}x_\beta$, and symbolically we can write this as

$$G_{\mu\nu\alpha\beta} = \sum_i A_i (g\cdot\eta)_{\mu\nu\alpha\beta} + \sum_i B_i\left(\eta\cdot g\cdot \frac{\partial g}{\partial x}x\right)_{\mu\nu\alpha\beta}, \quad A_i, B_i \in \mathbb{R}. \tag{A9}$$

We get constraints on parameters A_i and B_i from Eq. (A8) and also by requiring that the above equations should go to correct limit as $g_{\mu\nu} \to \eta_{\mu\nu}$. Finally $G_{\mu\nu\alpha\beta}$ is determined by the parameters α and β and four more free parameters. Now it is possible to reconstruct operator $\hat{P}_\mu$ from Eq. (A6). This procedure is more general but valid only up to the first order in the deformation parameter a. The construction where $\hat{P}_\mu = g_{\alpha\beta}(\hat{y})p^\beta \varphi^\beta{}_\mu$ is a special case of this general procedure, but we have chosen this special case because it is analogous to the undeformed case (valid up to all orders in a).

APPENDIX B

$$\tilde{\Sigma}_{\nu\tau\delta\mu} = \mathcal{A}_{\nu\tau\delta\mu} + \mathcal{B}_{\nu\tau\delta\mu}, \tag{B1}$$

where

$$\begin{aligned} \mathcal{A}_{\nu\tau\delta\mu} = \frac{1}{3}\Big\{ &-\tilde{\Gamma}_{(\sigma\delta)\alpha}[2\alpha\delta^\alpha_{[\mu}\delta^\sigma_{\nu]}a_\tau + 2\beta\delta^\alpha_{[\mu}\eta_{\nu]\tau}a^\sigma + 2\gamma\delta^\sigma_\tau\delta^\alpha_{[\mu}a_{\nu]}] + 2\delta^\alpha_{[\mu}\delta^\sigma_{\nu]}\Big[\Big(\alpha\Big(a\cdot\frac{\partial g_{\alpha\beta}}{\partial x}\Big)(x_\tau\bullet) \\ &+ \beta(a\cdot x)\Big(\frac{\partial g_{\alpha\beta}}{\partial x^\tau}\bullet\Big) + \gamma\Big(x\cdot\frac{\partial g_{\alpha\beta}}{\partial x}\Big)(a_\tau\bullet)\Big)\frac{\partial g_{\sigma\delta}}{\partial x_\beta} + \alpha g_{\alpha\beta}\Big(\Big\langle\delta^\beta_\tau\bullet, \Big(a\cdot\frac{\partial g_{\sigma\delta}}{\partial x}\Big)\Big\rangle - \Big(\frac{\partial^2 g_{\sigma\delta}}{\partial x^\beta\partial x}\cdot a\Big)(x_\tau\bullet) + \frac{\partial g_{\sigma\delta}}{\partial x^\beta}(a_\tau\bullet)\Big) \\ &+ \beta g_{\alpha\beta}\Big(a^\beta\Big(\frac{\partial g_{\sigma\delta}}{\partial x^\tau}\bullet\Big) + (a\cdot x)\Big(\frac{\partial^2 g_{\sigma\delta}}{\partial x^\tau\partial x_\beta}\bullet\Big)\Big) + \gamma\Big(g_{\alpha\beta}\frac{\partial^2 g_{\sigma\delta}}{\partial x_\beta\partial x^\lambda}x^\lambda - \tilde{\Gamma}_{(\delta\sigma)\alpha}\Big)(a_\tau\bullet) - (\alpha-\gamma)\Big(a\cdot\frac{\partial g_{\sigma\delta}}{\partial x}\Big)\Big(x\cdot\frac{\partial g_{\alpha\tau}}{\partial x}\Big)\Big] \\ &+ 2\delta^\lambda_{[\mu}\delta^\sigma_{\nu]}\Big[\beta g_{\alpha\beta}a^\alpha\Big(\delta^\beta_\tau\bullet\frac{\partial g_{\sigma\delta}}{\partial x^\lambda} + \frac{\partial g_{\sigma\delta}}{\partial x_\beta}\eta_{\lambda\tau}\bullet\Big) + \gamma a_\lambda\Big(g_{\alpha\tau}\bullet\frac{\partial g_{\sigma\delta}}{\partial x_\alpha} - \tilde{\Gamma}_{(\sigma\delta)\tau}\bullet\Big)\Big]\Big\} \end{aligned} \tag{B2}$$

and

$$\begin{aligned} \mathcal{B}_{\nu\tau\delta\mu} = \frac{1}{3}\Big\{ &\delta^\alpha_\nu\delta^\beta_\mu\Big[\frac{\partial g_{\alpha\beta}}{\partial x_\sigma}\Big(\alpha\Big(a\cdot\frac{\partial g_{\sigma\delta}}{\delta x}\Big)x_\tau + \beta(a\cdot x)\Big(\frac{\partial g_{\sigma\delta}}{\partial x^\tau}\bullet\Big) + \gamma\Big(x\cdot\frac{\partial g_{\sigma\delta}}{\partial x}\Big)a_\tau\Big) + \gamma\Big(x^\rho\frac{\partial^2 g_{\alpha\beta}}{\partial x_\sigma\partial x^\rho}g_{\sigma\delta}a_\tau - \tilde{\Gamma}_{(\alpha\beta)\delta}a_\tau\Big) \\ &+ \alpha\Big(a^\rho x_\delta\frac{\partial^2 g_{\alpha\beta}}{\partial x_\sigma\partial x^\rho}g_{\sigma\tau} + \Big(a\cdot\frac{\partial g_{\alpha\beta}}{\partial x}\Big)g_{\tau\delta}\Big) + \beta\Big(a^\rho g_{\rho\tau}\frac{\partial g_{\alpha\beta}}{\partial x^\delta} + (a\cdot x)\frac{\partial^2 g_{\alpha\beta}}{\partial x_\sigma\partial x^\delta}g_{\sigma\tau}\Big)\Big] \\ &- (\alpha+\gamma)a_\mu\tilde{\Gamma}_{(\nu\delta)\tau} - \beta a^\alpha(\tilde{\Gamma}_{(\alpha\tau)\delta}\eta_{\mu\nu} + \tilde{\Gamma}_{(\alpha\nu)\delta}\eta_{\mu\tau} + \tilde{\Gamma}_{(\alpha\mu)\delta}\eta_{\nu\tau}) - 2\gamma\tilde{\Gamma}_{(\mu\tau)\delta}a_\nu - 2\alpha a_\tau\tilde{\Gamma}_{(\nu\mu)\delta} \\ &- \alpha\Big(x_\mu\Big(a\cdot\frac{\partial\tilde{\Gamma}_{\nu\delta\tau}}{\partial x}\Big) - 2\tilde{\Gamma}_{\nu\mu\delta}a_\tau\Big) - \beta\Big((a\cdot x)\frac{\partial\tilde{\Gamma}_{\nu\delta\tau}}{\partial x^\mu} - 2\tilde{\Gamma}_{\nu\delta\tau}a_\mu\Big) - \gamma a_\mu\Big(\Big(x\cdot\frac{\partial\tilde{\Gamma}_{\nu\delta\tau}}{\partial x}\Big) - 2\tilde{\Gamma}_{\nu\delta\tau}\Big)\Big\}. \end{aligned} \tag{B3}$$

Here operator $\bullet$ stands for the position where operator p^τ is to be placed in Eq. (42).

APPENDIX C

$$\Sigma_\nu^{\sigma_1\sigma_2\sigma_3} \equiv -\tilde{\Gamma}_{\nu\mu\beta}\{\ldots a\ldots\}^{\beta\mu\sigma_1\sigma_2\sigma_3} + \tilde{\Sigma}_{\nu\tau\delta\mu}g^{\tau\sigma_1}g^{\delta\sigma_2}g^{\mu\sigma_3}, \tag{C1}$$

where

$$\begin{aligned} \{\ldots a\ldots\}^{\beta\mu\sigma_1\sigma_2\sigma_3} = -2g^{\epsilon\sigma_1}g^{\rho\sigma_2}g^{\sigma_3(\beta}g^{\mu)\kappa}\Big[&g_{\alpha\epsilon}(\alpha\delta^\alpha_\kappa a_\rho + \beta a^\alpha\eta_{\rho\kappa} + \gamma\delta^\alpha_\rho a_\kappa) + \alpha\Big(a\cdot\frac{\partial g_{\alpha\epsilon}}{\partial x}\Big) \\ &+ \beta(a\cdot x)\frac{\partial g_{\alpha\epsilon}}{\partial x^\rho} + \gamma\Big(x\cdot\frac{\partial g_{\alpha\epsilon}}{\partial x}a_\rho\Big)\Big]. \end{aligned} \tag{C2}$$

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