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Spectra, vacua, and the unitarity of Lovelock gravity in *D*-dimensional AdS spacetimes

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We explicitly confirm the expectation that generic Lovelock gravity in D dimensions has a unitary massless spin-2 excitation around any one of its constant curvature vacua just like the cosmological Einstein gravity. The propagator of the theory reduces to that of Einstein's gravity, but scattering amplitudes must be computed with an effective Newton's constant which we provide. Tree-level unitarity imposes a single constraint on the parameters of the theory yielding a wide range of unitary region. As an example, we explicitly work out the details of the cubic Lovelock theory.

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I. INTRODUCTION

In D spacetime dimensions, Lovelock gravity [1,2] is defined by the Lagrangian density as

$$\mathcal{L}_{\text{Lovelock}}(R^{\mu\nu}_{\rho\sigma}) = \sum_{n=0}^{[D/2]} a_n \mathcal{L}_n, \tag{1}$$

where a_n's are dimensionful constants, $[\frac{D}{2}]$ corresponds to the integer part of $\frac{D}{2}$, and all the indices on the tensors take values in $(0, \ldots, D-1)$. Each term in the full Lagrangian density is given as Euler densities

$$\mathcal{L}_n = \delta^{\mu_1 \ldots \mu_{2n}}_{\nu_1 \ldots \nu_{2n}} \prod_{p=1}^{n} R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{2}$$

where $\delta^{\mu_1 \ldots \mu_{2n}}_{\nu_1 \ldots \nu_{2n}}$ is the generalized Kronecker delta or the determinant tensor defined as usual in terms of the Kronecker deltas

$$\delta^{\mu_1 \ldots \mu_{2n}}_{\nu_1 \ldots \nu_{2n}} \equiv \det \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_{2n}}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_{2n}} & \ldots & \delta^{\mu_{2n}}_{\nu_{2n}} \end{vmatrix}. \tag{3}$$

This theory is presumably the most natural generalization of Einstein's gravity in D dimensions with the well-known property that the field equations are second order in the derivatives of the metric tensor (in fact, this is one of the defining properties of the theory). The lowest order term with $n = 0$ corresponds to the cosmological constant which is followed by the Einstein-Hilbert action with $n = 1$ and the Gauss-Bonnet (GB) combination with $n = 2$. The GB combination appears in low energy string theory dictated by supersymmetry [3,4]. As a result of general covariance of the action, the field equations are also covariantly divergence free. Moreover, Lovelock gravity is the only higher derivative theory that does not suffer the Buchdahl inequality [5] that exists between the metric formulation and the Palatini formulation—which assumes a generic connection *a priori*—of higher derivative gravity theories [6]. This result is quite interesting since it yields a dynamical derivation of equivalence principle for a class of torsion-free theories [6,7]. Another property of (1) is that for even D, the highest-order term does not contribute to the field equations since its variation is a total derivative (i.e., $\mathcal{L}_{D/2}$ is a topological invariant in even D dimensions). There is a nontrivial point here: in the first order formalism with vielbeins and the spin connection, one can explicitly show that $\mathcal{L}_{D/2}$ can be written as a boundary term. But, in the metric formulation there is no natural covariant vector made of the metric tensor $g_{\mu\nu}$ and its derivatives $\partial_\rho g_{\mu\nu}$; therefore, to see that the $\frac{D}{2}$ term does not contribute to the field equations, one proceeds by showing that under arbitrary variations of the metric, that term yields a total divergence. For example, see Ref. [8] for the case of the GB combination. If one does not insist that $\mathcal{L}_{D/2}$ be written as a boundary term in a covariant way, one can find noncovariant expressions. For example, see Ref. [9] where an algorithm of constructing boundary terms is given and as an example the Einstein-Hilbert Lagrangian density in two dimensions is explicitly written as a boundary term in a noncovariant way.

Recently [10], canonical analysis of Lovelock theory, more specifically $D = 5$ Einstein-GB theory, was carried out via the ADM decomposition [11]. Several other properties such as cosmological solutions [12], black hole solutions [13], and thermodynamical properties [14–16] of Lovelock theories have been studied.

In this work, we expand the Lovelock action around one of its (anti)-de Sitter [(A)dS] vacua up to $O(h^2)$ in the metric perturbation where $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ to study the propagator structure, perturbative spectrum, and the unitarity. As expected, we find that Lovelock gravity has only a massless spin-2 excitation in its spectrum just like the (cosmological) Einstein theory. The free propagator of Lovelock theory reduces to that of cosmological Einstein's gravity with an effective Newton's constant

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and an effective cosmological constant. The main result of this work is to provide how these two effective constants can be computed in terms of the constants, a_n, in the Lovelock action, and discuss the conditions on the parameters coming from the tree-level unitarity requirement.

The layout of this paper is follows: in Sec. II, we introduce the Lovelock gravity, find its $[\frac{D}{2}]$ constant curvature vacua, find an equivalent action whose propagator matches the propagator of Lovelock gravity, and discuss the tree-level unitarity of the theory. As the first nontrivial example, we work out the spectrum and the vacua of the cubic Lovelock gravity. Most of the details of the computations are delegated to the Appendix.

II. PROPAGATOR STRUCTURE OF THE LOVELOCK ACTION

To study the perturbative spectrum of Lovelock gravity (1) around one of its constant curvature vacua, naively one should find the field equations and linearize them around their maximally symmetric solutions. But, this route is somewhat complicated because of the complexity of the action. Instead, let us use the equivalent quadratic action technique, which was employed in several works before [17–20]. Since the technique is described in more detail in these works, here let us briefly recapitulate how it works. Suppose one would like to find the constant curvature vacua and excitations around the vacua; namely, the propagator of a generic gravity action that is constructed from the metric and the contractions of the Riemann tensor. The form of the Lagrangian density action can be taken as $\mathcal{L} \equiv \sqrt{-g}F(R^{\mu\nu}_{\rho\sigma})$. The question is to find an equivalent quadratic Lagrangian density $\mathcal{L}_{\text{quad-equal}} \equiv \sqrt{-g} f_{\text{quad-equal}}(R^{\mu\nu}_{\rho\sigma})$ whose $O(h)$, representing the maximally symmetric vacua, and $O(h^2)$, representing the propagator, expansions match that of the original Lagrangian. The equivalent quadratic action can be found from the curvature expansion around yet to be found maximally symmetric vacua with the Riemann tensor $\bar{R}^{\mu\nu}_{\rho\sigma} = \frac{2\Lambda}{(D-1)(D-2)}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)$ as

$$f_{\text{quad-equal}}(R^{\mu\nu}_{\rho\sigma}) \equiv \sum_{i=0}^{2} \left[\frac{\partial^i F}{\partial (R^{\mu\nu}_{\rho\sigma})^i}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})^i. \quad (4)$$

The allowed values for Λ come from the order $O(h)$ expansion of the original action, but more directly they can be obtained from the maximally symmetric vacuum equation of the equivalent quadratic theory.

It is not difficult to see[1] that the equivalent quadratic action for Lovelock gravity should be in the Einstein-GB form

$$\mathcal{L}_{\text{EGB}} = \frac{1}{\kappa}(R - 2\Lambda_0) + \gamma\chi_{\text{GB}},$$
$$\chi_{\text{GB}} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \quad (5)$$

What is of course remarkable is that the free theory, which is determined by $O(h)$ and $O(h^2)$ expansion, of (5) also matches that of cosmological Einstein gravity $\mathcal{L}_E = \frac{1}{\kappa_e} \times (R - 2\Lambda)$ with the following effective parameters [20]:

$$\frac{1}{\kappa_e} = \frac{1}{\kappa} + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma,$$
$$\frac{\Lambda - \Lambda_0}{2\kappa} + \gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}\Lambda^2 = 0. \quad (6)$$

The main problem is then to relate κ_e and Λ to the parameters a_n in the Lovelock action. In order to find these relations, we need to compute (4) or more explicitly the following quantities:

$$\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma}), \qquad \left[\frac{\partial \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}),$$
$$\left[\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\lambda\gamma}_{\alpha\beta}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}). \quad (7)$$

From now on, the subindex "0" means that the corresponding quantity is evaluated in the background. Details of these calculations are given in the Appendix, here we simply write the final expressions. The background value of the Lagrangian is

$$\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma}) = D! \sum_{n=0}^{[D/2]} a_n \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^n \frac{1}{(D-2n)!}, \quad (8)$$

the first order term reads

$$\left[\frac{\partial \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}) = \sum_{n=0}^{[D/2]} a_n n \times \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} \frac{2(D-2)!}{(D-2n)!}\left(R - \frac{2D\Lambda}{D-2}\right), \quad (9)$$

and the second order term boils down to

$$\left[\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\lambda\gamma}_{\alpha\beta}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta})$$
$$= 4\sum_{n=0}^{[D/2]} a_n n(n-1)\left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-2} \times \frac{(D-4)!}{(D-2n)!}\chi\text{GB} - 2\sum_{n=0}^{[D/2]} a_n n(n-1) \times \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} \frac{2(D-2)!}{(D-2n)!}\left(R - \frac{D\Lambda}{D-2}\right). \quad (10)$$

[1]The $i = 2$ term in the curvature expansion involves totally antisymmetric contractions of two Riemann tensors giving the GB combination.

As a result, the equivalent quadratic action that has the same $O(h)$ and $O(h^2)$ expansions[2] with the Lovelock theory (1) can be constructed from the equivalent quadratic Lagrangian density

$$f_{\text{quad-equal}}(R^{\mu\nu}_{\rho\sigma}) = -2(D-2)! \sum_{n=0}^{[D/2]} \tilde{a}_n \Big[R - \frac{(n-1)D\Lambda}{n(D-2)} - \frac{(n-1)}{4\Lambda(n-2)} \frac{(D-1)}{(D-3)} \chi_{\text{GB}} \Big], \tag{11}$$

where $\tilde{a}_n$ is defined as

$$\tilde{a}_n \equiv a_n \frac{n(n-2)}{(D-2n)!} \Big[\frac{4\Lambda}{(D-1)(D-2)} \Big]^{n-1}.$$

Here, $\tilde{a}_n$ vanishes for $n = 2$, but because of the $(n-2)$ term in the denominator the contribution does not vanish. The propagator of (1) matches that of (11) which itself has exactly the same propagator as the cosmological Einstein's theory but the Newton's constant is modified. Therefore,

$$\mathcal{L}_{\text{Lovelock}}(h^2) = -\frac{1}{2\kappa_e} h^{\mu\nu} \mathcal{D}^E_{\mu\nu\alpha\beta} h^{\alpha\beta}, \tag{12}$$

where $\mathcal{D}^E_{\mu\nu\alpha\beta}$ is the propagator of the cosmological Einstein theory which propagates a unitary massless spin-2 particle as long as $\kappa_e > 0$ [21,22].

Note that $n = 0, 1, 2$ terms of (11) give the cosmological constant, the Ricci scalar, and the GB combination as expected. The first nontrivial term comes from the cubic Lovelock term whose explicit form is

$$\begin{aligned} \frac{\mathcal{L}_3}{8} = & -8R^{\mu\nu\rho\sigma} R_{\mu\ \rho}^{\ \tau\ \gamma} R_{\nu\tau\sigma\gamma} + 4R^{\mu\nu\rho\sigma} R_{\mu\nu}^{\ \ \tau\gamma} R_{\rho\sigma\tau\gamma} \\ & - 24R^{\mu\nu} R^{\rho\sigma\tau}{}_{\mu} R_{\rho\sigma\tau\nu} + 3RR^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \\ & + 24R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + 16R^{\mu\nu} R^{\rho}_{\mu} R_{\nu\rho} \\ & - 12RR^{\mu\nu} R_{\mu\nu} + R^3. \end{aligned} \tag{13}$$

Since no homogeneous in curvature Lagrangian density can have a nonzero maximally symmetric vacuum, together with $\mathcal{L}_3$ one needs to consider at least one of the lower order Lovelock terms. Here, for the sake of generality, we consider all the lower order terms together with $\mathcal{L}_3$. Then, the equivalent quadratic action to $\mathcal{L} = a_0 + a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3$ follows as

$$\begin{aligned} f_{\text{quad-equal}}(R^{\mu\nu}_{\rho\sigma}) = & \ a_0 + 2a_1 R + 4a_2 \chi_{\text{GB}} \\ & - 96a_3 \frac{(D-3)(D-4)(D-5)\Lambda^2}{(D-1)^2(D-2)} \\ & \times \Big(R - \frac{2D\Lambda}{3(D-2)} \Big) \\ & + 48a_3 \frac{(D-4)(D-5)\Lambda}{(D-1)(D-2)} \chi_{\text{GB}}. \end{aligned} \tag{14}$$

As expected, for $D = 3$, 4, and 5, the terms coming from $\mathcal{L}_3$ explicitly vanish. For $D = 6$, they do not vanish explicitly, but as we show below they do not contribute to the field equations which is consistent with the fact that $\mathcal{L}_3$ is a topological invariant in $D = 6$. This is also true for $\mathcal{L}_2$: for $D = 4$ a_2 does not contribute to the field equations. (Note that for $D = 3$, $\mathcal{L}_2$ also does not contribute to the field equations because the GB combination is identically zero.) Similarly, a_1 does not contribute to the field equations in $D = 2$.

Let us rewrite the equivalent quadratic Lagrangian (11) of the Lovelock theory as

$$f_{\text{quad-equal}} = \frac{1}{\tilde{\kappa}}(R - 2\tilde{\Lambda}_0) + \tilde{\gamma}\chi_{\text{GB}}, \tag{15}$$

where the parameters are defined as

$$\frac{1}{\tilde{\kappa}} \equiv -2(D-2)! \sum_{n=0}^{[D/2]} a_n \frac{n(n-2)}{(D-2n)!} \times \Big[\frac{4\Lambda}{(D-1)(D-2)} \Big]^{n-1}, \tag{16}$$

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} \equiv -\frac{D!}{4} \sum_{n=0}^{[D/2]} a_n \frac{(n-1)(n-2)}{(D-2n)!} \Big[\frac{4\Lambda}{(D-1)(D-2)} \Big]^n, \tag{17}$$

$$\tilde{\gamma} \equiv 2(D-4)! \sum_{n=0}^{[D/2]} a_n \frac{n(n-1)}{(D-2n)!} \Big[\frac{4\Lambda}{(D-1)(D-2)} \Big]^{n-2}. \tag{18}$$

One can have a further reduction in representing the maximally symmetric vacua and the free part of the Lovelock theory with an equivalent action by using the fact discussed above: the vacua and the propagator of the Einstein-GB theory can be represented with cosmological Einstein's gravity that has the modified parameters given in (6). Then, we can write

$$I_{\text{equal-Lovelock}} = \int d^D x \sqrt{-g} \frac{1}{\kappa_e}(R - 2\Lambda), \tag{19}$$

whereby using (6) and (16)–(18), one can find that Λ satisfies the vacuum equation

$$0 = \sum_{n=0}^{[D/2]} a_n \frac{(D-2n)}{(D-2n)!} \Big[\frac{4}{(D-1)(D-2)} \Big]^n \Lambda^n, \tag{20}$$

and κ_e becomes

[2]In fact, $O(h^0)$ of the equivalent quadratic Lagrangian also gives the same order of the Lovelock theory, but this is an irrelevant constant.

$$\frac{1}{\kappa_e} = 2(D-3)! \sum_{n=0}^{[D/2]} a_n \frac{n(D-2n)}{(D-2n)!} \times \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1}. \tag{21}$$

As we discussed, since $\mathcal{L}_{D/2}$ is a topological invariant for even D, its contribution to the field equations should vanish for that dimension, and this fact can be verified by observing the appearance of the $(D-2n)$ factor in (20) and (21). Positivity of κ_e is the single constraint to have unitary massless spin-2 excitation which simply gives a bound on one of the a_n's in terms of the others. Reality of Λ, in general, also puts a constraint on the parameters. Similar results have been found in the first order formalism of Lovelock gravity in two different ways: first, by expanding the action up to second order in one of the components of $h_{\mu\nu}$, and second by using the spherically symmetric black hole solution in AdS backgrounds [23,24].

III. CONCLUSION

We have found the $O(h^2)$ expansion of generic Lovelock gravity in D dimensions around one of its constant curvature vacua, and explicitly confirmed the expectation that just like Einstein's theory, there is a single massless spin-2 excitation.[3] Qualitatively, Lovelock theory is built to have this property, but actual computation of the effective Newton's constant and the effective cosmological constant was lacking which was remedied above. We have given the cubic curvature Lovelock action as an explicit nontrivial example. Our construction is based on the fact that the propagator and the maximally symmetric vacua for any gravity theory whose action involves the contractions of the Riemann tensor can be obtained from an equivalent quadratic action.

Even though we have concentrated on the perturbative spectrum and the vacua, it is not difficult to see that from our construction conserved gravitational charges such as energy and angular momenta of black holes in asymptotically (anti)-de Sitter spacetimes can be found for Lovelock gravity by following the procedure of Refs. [27–29]. The expression for the charge will be just like the one in the cosmological Einstein-Hilbert theory with the effective Newton's constant and the effective cosmological constant.

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[3]Besides Lovelock gravity there are other higher curvature theories which propagate just a single massless spin-2 excitation around (A)dS backgrounds, see for example the critical gravity [25,26].

APPENDIX: TERMS OF THE EQUIVALENT ACTION

In the calculations involving the generalized Kronecker delta $\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}}$, which is nothing but a determinantal form, we frequently use the following relation for an $n\times n$ matrix A:

$$\det A = \epsilon_{\alpha_1\ldots\alpha_n} A_{\alpha_1 1} A_{\alpha_2 2}\ldots A_{\alpha_n n}, \tag{A1}$$

where the convention for the permutation symbol is $\epsilon_{12\ldots 2n} = +1$. At the risk of being pedantic, let us explicitly obtain the form of $\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}}$ with the help of (A1). First, consider the elements of the following $2n\times 2n$ matrix:

$$L = \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_{2n}}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_{2n}} & \ldots & \delta^{\mu_{2n}}_{\nu_{2n}} \end{pmatrix}, \tag{A2}$$

where the index ν counts the rows, and the index μ counts the columns; i.e., one has $L_{ij} = \delta^{\mu_j}_{\nu_i}$, and for example, $L_{\alpha_1 1} = \delta^{\mu_1}_{\nu_{\alpha_1}}$. Then, one can write $\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}}$ as

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} = \epsilon_{\alpha_1\ldots\alpha_{2n}} \delta^{\mu_1}_{\nu_{\alpha_1}} \delta^{\mu_2}_{\nu_{\alpha_2}} \ldots \delta^{\mu_{2n}}_{\nu_{\alpha_{2n}}}. \tag{A3}$$

Here, note that $2n$ should be smaller than the dimension of the spacetime D, but need not be equal to D.

Now, let us discuss how the term $\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2n}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2n}}$ is related to $\delta^{\mu_1\ldots\mu_{2k}}_{\nu_1\ldots\nu_{2k}}$. Using (A3), one can find how the $n\to n$ case is related to the $n\to n-\frac{1}{2}$ case

$$\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2n}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2n}} = [D-(2n-1)] \times \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2n-1}} \delta^{\mu_1}_{\nu_{\alpha_1}} \delta^{\mu_2}_{\nu_{\alpha_2}} \ldots \delta^{\mu_{2k}}_{\nu_{\alpha_{2k}}} \delta^{\nu_{2k+1}}_{\nu_{\alpha_{2k+1}}} \ldots \delta^{\nu_{2n-1}}_{\nu_{\alpha_{2n-1}}}. \tag{A4}$$

Using this recursive relation, it is possible to obtain the desired result which will be sufficient in the computation of the equivalent quadratic action

$$\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2n}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2n}} = \frac{(D-2k)!}{(D-2n)!} \delta^{\mu_1\ldots\mu_{2k}}_{\nu_1\ldots\nu_{2k}}. \tag{A5}$$

1. Zeroth order

Let us calculate $\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma})$ which has the form

$$\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma}) = \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}. \tag{A6}$$

By using

$$\bar{R}^{\mu\nu}_{\rho\sigma} = \frac{2\Lambda}{(D-1)(D-2)} (\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}), \tag{A7}$$

one has

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} = \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^n \delta^{\nu_1\ldots\nu_{2n}}_{\nu_1\ldots\nu_{2n}} = \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^n \frac{D!}{(D-2n)!}, \tag{A8}$$

where the second equality follows from (A5). Note that the value of this form for $n_{\max} = [\frac{D}{2}]$ is same for both even and odd dimensions. Then, $\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma})$ becomes

$$\mathcal{L}_{\text{Lovelock}}(\bar{R}^{\mu\nu}_{\rho\sigma}) = D! \sum_{n=0}^{[D/2]} a_n \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^n \times \frac{1}{(D-2n)!}. \quad \text{(A9)}$$

2. First order

The first order term in the equivalent quadratic action has the form

$$\left[\frac{\partial \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}) = \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \left(\prod_{\substack{p=1\\(p\neq r)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} - \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} n \left(\prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right), \quad \text{(A10)}$$

where the term in the second line is calculated in (A8); and after use of (A7), the term in the first line becomes

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \left(\prod_{\substack{p=1\\(p\neq r)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} = \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} n \delta^{\mu_1\mu_2\nu_3\ldots\nu_{2n}}_{\nu_1\nu_2\nu_3\ldots\nu_{2n}} R^{\nu_1\nu_2}_{\mu_1\mu_2}. \quad \text{(A11)}$$

Using (A5), one can further reduce this form to

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \left(\prod_{\substack{p=1\\(p\neq r)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} = \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} 2n \frac{(D-2)!}{(D-2n)!} R. \quad \text{(A12)}$$

This result together with (A8) yields the first order term of the equivalent quadratic action as

$$\left[\frac{\partial \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}) = \sum_{n=0}^{[D/2]} a_n n \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} \times \frac{2(D-2)!}{(D-2n)!} \left(R - \frac{2D\Lambda}{D-2}\right). \quad \text{(A13)}$$

3. Second order

The second order term in the equivalent quadratic action has the form

$$\left[\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma} \partial R^{\lambda\gamma}_{\alpha\beta}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}) = \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \sum_{\substack{q=1\\(r\neq q)}}^{n} \left(\prod_{\substack{p=1\\(p\neq r,q)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2q-1}\nu_{2q}}_{\mu_{2q-1}\mu_{2q}} - \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} 2(n-1) \sum_{r=1}^{n} \left(\prod_{\substack{p=1\\(p\neq r)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} \times \sum_{n=0}^{[D/2]} a_n \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} n(n-1) \left(\prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right), \quad \text{(A14)}$$

where the second and the third terms on the right-hand side are calculated in (A12) and (A8), respectively. On the other hand, the first term takes the form

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \sum_{\substack{q=1\\(r\neq q)}}^{n} \left(\prod_{\substack{p=1\\(p\neq r,q)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2q-1}\nu_{2q}}_{\mu_{2q-1}\mu_{2q}} = n(n-1) \delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4} \left(\prod_{p=3}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right), \quad \text{(A15)}$$

after renaming the dummy indices and using the totally antisymmetric nature of $\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}}$. Then, employing the (A)dS background Riemann tensor form (A7) and using (A5), one gets

$$\delta^{\mu_1\ldots\mu_{2n}}_{\nu_1\ldots\nu_{2n}} \sum_{r=1}^{n} \sum_{\substack{q=1\\(r\neq q)}}^{n} \left(\prod_{\substack{p=1\\(p\neq r,q)}}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2q-1}\nu_{2q}}_{\mu_{2q-1}\mu_{2q}} = n(n-1) \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-2} \frac{(D-4)!}{(D-2n)!} 4\chi_{\text{GB}}. \quad \text{(A16)}$$

With this result and (A12) and (A8), the second order term of the equivalent quadratic action becomes

$$\left[\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma} \partial R^{\lambda\gamma}_{\alpha\beta}}\right]_0 (R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma})(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}) = 4 \sum_{n=0}^{[D/2]} a_n n(n-1) \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-2} \times \frac{(D-4)!}{(D-2n)!} \chi_{\text{GB}} - 2 \sum_{n=0}^{[D/2]} a_n n(n-1) \times \left[\frac{4\Lambda}{(D-1)(D-2)}\right]^{n-1} \frac{2(D-2)!}{(D-2n)!} \left(R - \frac{D\Lambda}{D-2}\right). \quad \text{(A17)}$$

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