Caged black hole with Maxwell charge

Ryotaku Suzuki,[1] Shunichiro Kinoshita,[2] and Tetsuya Shiromizu[1]

[1]*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*
[2]*Osaka City University Advanced Mathematical Institute, Osaka 558-8585, Japan*

We construct the perturbative solution of the charged black hole in the Kaluza-Klein spacetime with the matched asymptotic expansion method. The corrections to the thermodynamic variables are calculated up to the post-Newtonian order. We confirmed that the method can work very well in the Einstein-Maxwell theory.

I. INTRODUCTION

In the last decade, the higher-dimensional black holes have been known to have much richer structure than in four-dimensional spacetime [1]. The discovery of the black ring solution [2,3] revealed that the uniqueness and the horizon topology theorem in four dimension are no longer valid in higher dimensions.

The similar breakdown happens in the Kaluza-Klein spacetimes. When the mass scale is smaller than the scale of the extra dimension, black string solutions and the black holes localized in the Kaluza-Klein spacetime (the caged black hole solutions) exist for the same total mass, which have different horizon topologies. Furthermore, there exist the uniform black strings (UBS) and the nonuniform black strings (NUBS). The NUBS phase bifurcates from the UBS phase at the marginally stable mode of the Gregory-Laflamme instability [4,5]. This system admits interesting phenomena such as the topology changing phase transition from the NUBS phase to the caged black hole phase, and the critical dimension for the stability of the NUBS phase [6].

Since it is difficult to construct the exact solutions of the black objects with the nontrivial horizon topology, the numerical or perturbative approach is helpful in the analysis of higher dimensions. In the Kaluza-Klein spacetime, the small, caged black holes have been first studied perturbatively using a single coordinate patch [7,8]. Gorbonos and Kol used two coordinate patches and introduced the systematic procedure to obtain the solutions perturbatively [9]. This perturbative method is called the matched asymptotic expansion (MAE). The MAE can be applied to a spacetime having at least two separate scales. Using the MAE, they constructed the small, caged black holes up to the post-Newtonian order in the asymptotic zone [9,10]. This method is also useful in the ultraspinning limit of the black objects [11,12].

The effective field theory (EFT) method [13] is another useful method for systematically obtaining the thermodynamic properties of such spacetime, which is considered to be equivalent with the MAE method. In the EFT method, the small scales are integrated out to give the effective Lagrangian for the coarse-grained black objects. This is

applied to the caged black holes [14,15]. The caged black holes with charge are also studied by the EFT [16].

In this paper, we will generalize the MAE method to the Einstein-Maxwell theory and construct the perturbative solution of the charged black holes in the Kaluza-Klein spacetime. In the neutral limit, the solutions here reproduce the neutral results [10]. We also calculated the correction to the thermodynamic variables up to the post-Newtonian order which confirms the EFT result [16].

The organization of this paper is as follows. In Sec. II, we explain the setup of the caged black hole and the matching method. In Sec. III, the Newtonian order is solved at the asymptotic zone. In Sec. IV, we solve the near zone perturbation. In Sec. V, we match the near zone results with the results in Sec. III and evaluate some of the thermodynamic variables. In Sec. VI, we solve the post-Newtonian equation at the asymptotic zone. In Sec. VII, we compute the post-Newtonian correction to the global charges and analyze the thermodynamic properties of solutions. We summarize our work in Sec. VIII. In the Appendixes, we will present some useful formulae and the details of the computations.

II. MATCHED ASYMPTOTIC EXPANSION

In this section, we introduce the method of the matched asymptotic expansion in the $(n + 3)$-dimensional Kaluza-Klein spacetime with the Einstein-Maxwell theory. The action is given by

$$S = \int \left(\frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \sqrt{-g} d^{n+3} x, \qquad (1)$$

where $F_{\mu\nu}$ is the field strength of the Maxwell field A_μ defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this paper we choose the unit of $G = 1$, hereafter. The Einstein equation becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad (2)$$

where the energy-momentum tensor is given by

$$T_{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}, \qquad (3)$$

and the field equation for the Maxwell field is

$$F^{\mu\nu}{}_{;\mu} = 0. \qquad (4)$$

We consider a static, localized black hole solution in the Kaluza-Klein spacetime which is called "caged black hole." Then we take the small black hole limit in which the scale of the black hole ρ_0 is much smaller than the compactification scale L. Since the Maxwell charge is bounded above by the extreme limit, it can be assumed to be the same order as the mass $\sim \rho_0^n$. Therefore, we can perform a similar treatment with the neutral case [9,10].

In this small black hole limit we have two asymptotic zones. The spacetime asymptotes to the flat Kaluza-Klein spacetime in the asymptotic zone ($\rho \gg \rho_0$) and the charged spherical black hole in the near zone ($\rho \ll L$). Let us introduce two coordinate systems. In the asymptotic zone, it is better to use the cylindrical coordinates (r, z) as

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\Omega_n^2 \qquad (5)$$

in which the z-direction is periodic as $z \sim z + L$. The periodicity requires that we must consider the gravity from the mirror images of the black hole. This is equivalent to the infinite array of black holes in the higher-dimensional Minkowski spacetime. In the near zone, on the other hand, it is better to use the spherical coordinates (ρ, χ) in order to see the black hole perturbation as

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(d\chi^2 + \sin^2\chi d\Omega_n^2). \qquad (6)$$

In the asymptotic zone, the relation between two coordinate systems is

$$r = \rho \sin\chi, \qquad z = \rho \cos\chi. \qquad (7)$$

We expand the metric and the Maxwell field in both zones. In the near zone, we consider the black hole perturbation with the expansion parameter $1/L$ as

$$g_{\mu\nu}^{(\text{near})} = g_{\mu\nu}^{(\text{BH})} + \sum_{k=1}^{\infty} h_{\mu\nu}^{(\text{near},k)}, \qquad (8)$$

where $h^{(\text{near},k)} \sim \mathcal{O}(1/L^k)$. $g_{\mu\nu}^{(\text{BH})}$ is the metric of the $(n+3)$-dimensional background charged black hole spacetime

$$g_{\mu\nu}^{(\text{BH})} dx^\mu dx^\nu = -f(\rho)dt^2 + \frac{1}{f(\rho)}d\rho^2 + \rho^2 d\Omega_{n+1}^2, \qquad (9)$$

where

$$f(\rho) = 1 - \frac{\rho_0^n}{\rho^n} + \frac{\sigma^2}{4}\frac{\rho_0^{2n}}{\rho^{2n}}, \qquad (10)$$

and σ is a dimensionless charge parameter which becomes $|\sigma| = 1$ in the extreme limit. The total mass of this black hole is

$$M_0 = \frac{(n+1)\omega_{n+1}\rho_0^n}{16\pi}, \qquad (11)$$

where $\omega_k = 2\pi^{(k+1)/2}/\Gamma(\frac{k+1}{2})$ is the area of the k-dimensional unit sphere, S^k. The background Maxwell field is

$$A_\mu dx^\mu = \frac{Q_0}{n\omega_{n+1}\rho^n}dt, \qquad (12)$$

where the total charge of the background black hole is given by

$$Q_0 = \int *F = \sqrt{\frac{n(n+1)}{32\pi}}\omega_{n+1}\rho_0^n\sigma \qquad (13)$$

and the nonvanishing component of the field strength is the electric field

$$F_{t\rho} = \sqrt{\frac{n(n+1)}{32\pi}}\sigma\frac{\rho_0^n}{\rho^{n+1}} \equiv \mathcal{E}. \qquad (14)$$

In the asymptotic zone, we consider the post-Newtonian expansion with the expansion parameter ρ_0 as

$$g_{\mu\nu}^{(\text{asym})} = \eta_{\mu\nu} + \sum_{m=1}^{\infty} h_{\mu\nu}^{(\text{asym},m)}, \qquad (15)$$

where $h^{(\text{asym},m)} \sim \mathcal{O}(\rho_0^m)$ and $\eta_{\mu\nu}$ is the metric of the $(n+3)$-dimensional Kaluza-Klein spacetime (5). The Maxwell field is also expanded in the same way. It is worth noting that the Newtonian approximation is the order of ρ_0^n because the mass M is written as $\rho_0^n \sim GM$.

Now we are ready to consider the matching procedure between the two zones. In the limit $\rho_0 \ll L$, there exists the overlap region ($\rho_0 \ll \rho \ll L$) in which we will match the two expansions order by order. The "matching ladder" can be understood by the dimensional counting. If we take the harmonic gauge, the linear part of the post-Newtonian equation in the order of ρ_0^m becomes

$$\Box h_{\mu\nu}^{(\text{asym},m)} = 0, \qquad (16)$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Since we consider the static spacetime, it reduces to the Laplace equation. The homogeneous solutions with a multipole l behave as ρ^l and ρ^{-l-n}. For the asymptotic zone, the near zone black hole seems to have multipole moments and behaves as ρ^{-l-n}. From the dimensional counting, the terms with ρ^{-l-n} in asymptotic solution of $\mathcal{O}(\rho_0^m)$ appear in the dimensionless form

$$h_{\mu\nu}^{(\text{asym},m)} \ni \frac{\rho_0^m}{L^{m-l-n}\rho^{l+n}}. \qquad (17)$$

Therefore, the near solution of $\mathcal{O}(1/L^k)$ contributes to the asymptotic solution with the multipole l of $\mathcal{O}(\rho_0^{k+n+l})$.

Similarly, the gravity from the mirror images of the black hole affects the near zone geometry in the form of multipole moments at the infinity, which are proportional to ρ^l. The dimensional counting shows

$$h_{\mu\nu}^{(\text{near},k)} \ni \frac{\rho_0^{k-l}\rho^l}{L^k}. \qquad (18)$$

Then, the asymptotic solution of $\mathcal{O}(\rho_0^m)$ determines to the near solution with the multipole l of $\mathcal{O}(1/L^{m+l})$.

III. NEWTONIAN POTENTIAL IN ASYMPTOTIC ZONE

We start the matching of the near zone solutions with the Newtonian order solution in the asymptotic zone. We will omit the script "near" or "asym" for brevity. As explained in the previous section, the near zone background metric of $\mathcal{O}(1/L^0)$ gives the correction to the asymptotic metric in $\mathcal{O}(\rho_0^n)$. At this order, the solution is just the Newtonian potential from the array of the point source as in the neutral case [9]. In the harmonic gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$ where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$, the linearized Einstein equation is

$$-\frac{1}{2} \Box \bar{h}_{\mu\nu}^{(n)} = 8\pi T_{\mu\nu}^{(n)}. \quad (19)$$

Since $T_{\mu\nu}$ is written in terms of the square of $F_{\mu\nu}$ and the leading order of $F_{\mu\nu}$ is the Newtonian order, the Einstein equation is the same as the vacuum case at this order (namely, $T_{\mu\nu}^{(n)} = 0$). The boundary condition in the overlap region is determined by the behavior of the near background metric. In the overlap region, the near zone background metric in the harmonic coordinates is given as $\mathcal{O}(1/L^0)$ terms in Eq. (D8), which becomes

$$g_{tt}^{(\mathrm{BH})} \simeq -1 + \frac{\rho_0^n}{\rho^n}, \quad (20a)$$

$$g_{ij}^{(\mathrm{BH})} \simeq \left(1 + \frac{\rho_0^n}{n\rho^n}\right)\delta_{ij}. \quad (20b)$$

Requiring the periodicity in z-direction and the harmonic condition, the homogeneous solution with a pole at the origin is given by

$$h_{tt}^{(n)} = \Phi \equiv \rho_0^n \sum_{k=-\infty}^{\infty} \frac{1}{(r^2 + (z - kL)^2)^{n/2}}, \quad (21a)$$

$$h_{ij}^{(n)} = \frac{1}{n} \Phi \delta_{ij}. \quad (21b)$$

Since $\Phi \simeq \rho_0^n/\rho^n$ as $\rho \ll L$, the coefficients are determined by the match with Eq. (20). This is equivalent to the Newtonian potential from the infinite array of point masses,

$$T_{tt}^{(n)} = \sum_{k=-\infty}^{\infty} M_0 \delta^{n+1}(\mathbf{x})\delta(z - kL). \quad (22)$$

Imposing the Lorenz gauge $\partial^\mu A_\mu^{(n)} = 0$, the Maxwell equation is

$$\Box A_\mu^{(n)} = 0. \quad (23)$$

In the same way, the solution is given by

$$A_t^{(n)} = \phi \equiv \frac{Q_0}{n\omega_{n+1}} \sum_{k=-\infty}^{\infty} \frac{1}{(r^2 + (z - kL)^2)^{n/2}}. \quad (24)$$

This is also equivalent to the potential obtained from the infinite array of point charges,

$$J_t^{(n)} = \sum_{k=-\infty}^{\infty} Q_0 \delta^{n+1}(\mathbf{x})\delta(z - kL). \quad (25)$$

Now we can see that the gauge potential is proportional to the Newtonian potential,

$$\phi = \frac{Q_0}{n\omega_{n+1}\rho_0^n} \Phi = \sqrt{\frac{n+1}{32\pi n}} \sigma\Phi. \quad (26)$$

Φ is expanded in the overlap region for $r, z \ll L$,

$$\Phi = \frac{\rho_0^n}{\rho^n} + \sum_{k=0}^{\infty} 2\zeta(n + 2k)\frac{\rho_0^n \rho^{2k}}{L^{n+2k}} C_{2k}^{n/2}(\cos\chi), \quad (27)$$

where $C_l^\nu(x)$ is the Gegenbauer polynomial (see Appendix A for the definition).

IV. NEAR ZONE PERTURBATION

We now proceed to the leading order in the near zone. The leading order correction in the near zone comes from the asymptotic solution of $\mathcal{O}(\rho_0^n)$ in the previous section. In this paper, we only consider the monopole ($l = 0$) correction which is $\mathcal{O}(1/L^n)$.

Here, we calculate the linear perturbation around the black hole metric of Eq. (9) as $g_{\mu\nu} = g_{\mu\nu}^{(\mathrm{BH})} + h_{\mu\nu}$. Since we consider the static perturbation, we can set the following ansatz for the metric perturbation

$$ds^2 = -f(1 + A_0)dt^2 + f^{-1}(1 + B_0)d\rho^2 + \rho^2(1 + E_0)d\Omega_{n+1}^2, \quad (28)$$

and for the Maxwell field perturbation

$$A_\mu dx^\mu = \left(\frac{Q_0}{n\omega_{n+1}\rho^n} + a_0\right)dt. \quad (29)$$

We write the linearized Einstein equation as

$$\delta R_{\mu\nu} = 8\pi\bigg[\delta T_{\mu\nu} - \frac{1}{n+1}(g^{(\mathrm{BH})\alpha\beta}\delta T_{\alpha\beta} - T^{(0)\alpha\beta}h_{\alpha\beta})g_{\mu\nu}^{(\mathrm{BH})} - \frac{1}{n+1}T^{(0)}h_{\mu\nu}\bigg]$$

$$\equiv 8\pi\delta S_{\mu\nu}, \quad (30)$$

where $T_{\mu\nu}^{(0)}$ is the energy-momentum tensor of the background. The components of $T_{\mu\nu}^{(0)}$ are given by

$$-T^{(0)t}{}_t = -T^{(0)r}{}_r = T^{(0)\chi}{}_\chi = T^{(0)\theta_i}{}_{\theta_i}$$

$$= \frac{1}{2}\mathcal{E}^2 = \frac{n(n+1)\sigma^2}{64\pi}\frac{\rho_0^{2n}}{\rho^{2n+2}}, \quad (31)$$

where $\{\theta_i\}_{i=1\ldots n}$ are coordinates of S^n. The explicit forms for each components of $\delta R_{\mu\nu}$ and $\delta S_{\mu\nu}$ are given in Eqs. (C9) and (C10). The Maxwell equation becomes

$$\partial_\rho\bigg[\rho^{n+1}a_0' - \frac{Q_0}{2\omega_{n+1}}((n+1)E_0 - A_0 - B_0)\bigg] = 0. \quad (32)$$

This can be easily integrated as

$$a_0' = \frac{\mathcal{E}}{2}((n+1)E_0 - A_0 - B_0 + 2\alpha_0), \quad (33)$$

where α_0 is an integral constant. Using the above and Eq. (C10), $\delta S_{\mu\nu}$ becomes

$$\delta S^t{}_t = \frac{n^2\sigma^2}{4}\frac{\rho_0^{2n}}{\rho^{2n+2}}[(n+1)E_0 - A_0 + 2\alpha_0], \quad (34a)$$

$$\delta S^\rho{}_\rho = \frac{n\sigma^2}{4}\frac{\rho_0^{2n}}{\rho^{2n+2}}[(n+1)E_0 - B_0 + 2\alpha_0], \quad (34b)$$

and

$$\delta S^\chi{}_\chi = -\frac{n\sigma^2}{4}\frac{\rho_0^{2n}}{\rho^{2n+2}}[E_0 - A_0 - B_0 + 2\alpha_0], \quad (34c)$$

where $\delta S^\mu{}_\nu = g^{(\mathrm{BH})\mu\alpha}\delta S_{\alpha\nu}$. Subtracting the $\rho\rho$-component from the tt-component in Eq. (30), we obtain the following equation:

$$\rho E_0'' + 2E_0' - A_0' - B_0' = 0, \quad (35)$$

which is the same with the neutral cases. Another independent equation comes from the $\chi\chi$-component in Eq. (30),

$$\frac{1}{2}\rho^2(fE_0'' + f'E_0') + (n+1)\rho fE_0' + \frac{1}{2}\rho f(A_0' - B_0')$$
$$+ n(E_0 - B_0) = \frac{n\sigma^2}{4}\frac{\rho_0^{2n}}{\rho^{2n}}[(n+1)E_0 - B_0 + 2\alpha_0]. \quad (36)$$

Integrating Eq. (35), we have

$$\rho E_0' + E_0 - A_0 - B_0 = -2C_1, \quad (37)$$

where C_1 is an integral constant. Imposing

$$A_0 + B_0 + (n-1)E_0 = 0 \quad (38)$$

as the residual gauge condition, Eq. (37) can be solved as

$$E_0 = \frac{2C_1}{n} + \frac{C_2\rho_0^n}{\rho^n}, \quad (39)$$

where C_2 is an integral constant.

To solve Eq. (36) in the gauge condition (38), we introduce a new variable defined by

$$\Psi \equiv -\frac{1}{n-1}(A_0 + nB_0) = E_0 - B_0, \quad (40)$$

where A_0 and B_0 are written by Ψ as

$$A_0 = \Psi - nE_0, \qquad B_0 = E_0 - \Psi. \quad (41)$$

Using the above and Eq. (39), Eq. (36) becomes

$$\rho(f\Psi)' + nf\Psi = \frac{n\rho_0^{2n}}{2\rho^{2n}}[\sigma^2(\alpha_0 + C_1) + nC_2] \quad (42)$$

and then the solution is given by

$$f\Psi = \frac{\tilde{C}_3\rho_0^n}{\rho^n} - \frac{\rho_0^{2n}}{2\rho^{2n}}[\sigma^2(\alpha_0 + C_1) + nC_2], \quad (43)$$

where $\tilde{C}_3$ is an integral constant. Even after imposing the gauge condition (38), there is still the remaining gauge degree of freedom $\rho \to \rho + \delta C\rho_0^n/2\rho^{n-1}$, which changes Eqs. (39) and (43) as

$$E_0 \to \frac{2C_1}{n} + \frac{C_2'\rho_0^n}{\rho^n}, \quad (44a)$$

$$f\Psi \to \frac{\tilde{C}_3'\rho_0^n}{\rho^n} - \frac{\rho_0^{2n}}{2\rho^{2n}}[\sigma^2(\alpha_0 + C_1) + nC_2'], \quad (44b)$$

where $C_2' = C_2 + \delta C$ and $\tilde{C}_3' = \tilde{C}_3 + n\delta C$. Therefore C_2 can be the pure gauge, and we define the gauge invariant combination $C_3 \equiv \tilde{C}_3 - nC_2$. Here, C_3 is the ambiguity of the bare mass and α_0 is the ambiguity of the bare charge which we can freely choose.

As the boundary condition at the horizon, we impose the regularity conditions for A_0, B_0, and E_0 which fix the horizon position. We write the position of the outer event horizon as ρ_+, which is the larger root of $f(\rho) = 0$,

$$\rho_+^n = \frac{1}{2}\rho_0^n(1 + \sqrt{1 - \sigma^2}). \quad (45)$$

The regularity of Ψ at $\rho = \rho_+$ is guaranteed; this condition is satisfied if the right-hand side of Eq. (43) vanishes as $\rho \to \rho_+$, which gives

$$\frac{1}{2}\sigma^2(\alpha_0 + C_1) + \frac{1}{2}nC_2 = (C_3 + nC_2)\frac{\rho_+^n}{\rho_0^n}. \quad (46)$$

In general, the perturbed spacetime may not become extremal or neutral when the background spacetime is extremal ($\sigma = 1$) or neutral ($\sigma = 0$). Then, we will further fix the parameter region of σ so that the solution becomes extremal when $\sigma = 1$ and neutral when $\sigma = 0$. The extremal condition is attained by imposing that the right-hand side of Eq. (43) has the double roots, $(f\Psi)'|_{\rho=\rho_+,\sigma=1} = 0$, which becomes

$$(\alpha_0 + C_1 + nC_2)|_{\sigma=1} = 0 = (C_3 + nC_2)|_{\sigma=1}, \quad (47)$$

where we used Eq. (46). The neutral condition is

$$(2C_3 + nC_2)|_{\sigma=0} = 0. \quad (48)$$

Although C_i and α_0 may depend on σ in general, we can set these to satisfy the above conditions for not only $\sigma = 0, 1$ but also arbitrary σ as

$$C_2 = C_3 = 0, \qquad \alpha_0 = -C_1. \quad (49)$$

Therefore, Ψ becomes zero, that is,

$$\Psi = 0. \quad (50)$$

Then, from Eqs. (33), (39), and (41), the perturbation becomes

$$A_0 = -2C_1, \tag{51a}$$

$$B_0 = \frac{2C_1}{n}, \tag{51b}$$

$$E_0 = \frac{2C_1}{n}, \tag{51c}$$

$$a_0 = -\frac{C_1 Q_0}{n\omega_{n+1}\rho^n} + \alpha_1, \tag{51d}$$

where α_1 is an integral constant. As a result, the near solutions depend on two parameters, C_1 and α_1, which are determined by matching with the asymptotic solutions in the overlap region.

V. MATCHING FROM THE NEWTONIAN POTENTIAL

As mentioned in Sec. II, the near zone monopole correction of $\mathcal{O}(1/L^n)$ comes from the monopole parts of the Newtonian order asymptotic solution. The monopole from the asymptotic zone behaves as a constant which has the dimensionless form ρ_0^n/L^n at the Newtonian order.

Since the leading order terms in the near zone perturbation are not affected by the gauge transformation into the harmonic coordinates (D7), the near solution in the overlap region ($\rho_0 \ll \rho$) becomes

$$h_{tt}^{(\text{near},n)} = -fA_0 \simeq 2C_1, \tag{52a}$$

$$h_{\rho\rho}^{(\text{near},n)} = \frac{B_0}{f} \simeq \frac{2C_1}{n}, \tag{52b}$$

$$h_{\chi\chi}^{(\text{near},n)} = \rho^2 E_0 = \frac{2C_1}{n}\rho^2. \tag{52c}$$

On the other hand, the ρ_0^n/L^n term in the expansion of the Newtonian potential (27) is

$$h_{tt}^{(\text{asym},n)} = \Phi = \frac{\rho_0^n}{\rho^n} + 2\zeta(n)\frac{\rho_0^n}{L^n} + \cdots \tag{53}$$

and the matching provides

$$C_1 = \zeta(n)\frac{\rho_0^n}{L^n}. \tag{54}$$

We now define the expansion parameter $\lambda = \zeta(n)\rho_0^n/L^n$. Also, for the Maxwell field the leading order term of the near solution matches the gauge potential ϕ of the asymptotic solution in the overlap region as

$$A_t^{(\text{near},n)} \simeq \alpha_1 = \frac{2Q_0}{n\omega_{n+1}\rho_0^n}\lambda. \tag{55}$$

Then, combining Eq. (51) with the ansatz (28) and (29), the near zone metric up to $\mathcal{O}(1/L^n)$ becomes

$$g_{tt}^{(\text{near})} = -f(1 - 2\lambda) + \mathcal{O}(1/L^{2n}), \tag{56a}$$

$$g_{\rho\rho}^{(\text{near})} = \frac{1}{f}\left(1 + \frac{2\lambda}{n}\right) + \mathcal{O}(1/L^{2n}), \tag{56b}$$

$$g_{\chi\chi}^{(\text{near})} = \rho^2\left(1 + \frac{2\lambda}{n}\right) + \mathcal{O}(1/L^{2n}), \tag{56c}$$

and the gauge potential in the near zone is

$$A_t^{(\text{near})} = \frac{Q_0}{n\omega_{n+1}\rho^n} + \frac{2\lambda Q_0}{n\omega_{n+1}\rho_0^n}\left(1 - \frac{\rho_0^n}{2\rho^n}\right) + \mathcal{O}(1/L^{2n}). \tag{57}$$

From the leading order near solution (56), the correction to the local constants can be computed. The surface gravity κ becomes

$$\begin{aligned} \kappa &= \frac{1}{2}\frac{|\partial_\rho g_{tt}|}{\sqrt{-g_{tt}g_{\rho\rho}}}\bigg|_{\rho=\rho_+} \\ &= \frac{1}{2}|f'|\left(1 + \frac{f}{f'}A_0' + \frac{1}{2}(A_0 - B_0)\right)\bigg|_{\rho=\rho_+} \\ &= \frac{n\rho_0^n}{2\rho_+^{n+1}}\sqrt{1-\sigma^2}\left(1 - \frac{n+1}{n}\lambda\right) \\ &= \kappa_0\left(1 - \frac{n+1}{n}\lambda\right), \end{aligned} \tag{58}$$

where $\kappa_0 = \sqrt{1-\sigma^2}n\rho_0^n/2\rho_+^{n+1}$ is the surface gravity for $L = \infty$. The horizon area $\mathcal{A}$ becomes

$$\begin{aligned} \mathcal{A} &= \omega_{n+1}(g_{\chi\chi})^{(n+1)/2}|_{\rho=\rho_+} \\ &= \omega_{n+1}\rho_+^{n+1}\left(1 + \frac{n+1}{n}\lambda\right) \\ &= \mathcal{A}_0\left(1 + \frac{n+1}{n}\lambda\right), \end{aligned} \tag{59}$$

where $\mathcal{A}_0$ is the horizon area for $L = \infty$.

We can also calculate the electrostatic potential U,

$$\begin{aligned} U &\equiv A_t|_{\rho=\rho_+} - A_t|_{\rho=\infty} \\ &= \frac{Q_0}{n\omega_{n+1}\rho_+^n}(1 + \sqrt{1-\sigma^2}\lambda) \\ &= U_0(1 + \sqrt{1-\sigma^2}\lambda), \end{aligned} \tag{60}$$

where U_0 is the electrostatic potential for $L = \infty$. We note that the boundary condition of the asymptotic solution leads to $A_t|_{\rho=\infty} = 0$.

VI. MONOPOLE MATCHING IN POST-NEWTONIAN ORDER

The leading correction to the mass and tension in the asymptotic zone are computed through the monopole perturbation of the post-Newtonian order $\mathcal{O}(\rho_0^{2n})$. The near zone perturbation of $\mathcal{O}(1/L^n)$ considered in the previous section gives the boundary condition in the overlap region, which behaves as $\rho_0^{2n}/L^n\rho^n$. First, we present the post-Newtonian equation for the gravity and the Maxwell field and then solve them.

A. Post-Newtonian order perturbation equation

In the asymptotic zone, we consider perturbations of the metric and Maxwell fields to the post-Newtonian order,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(n)} + h_{\mu\nu}^{(2n)}, \qquad A_\mu = A_\mu^{(n)} + A_\mu^{(2n)}, \quad (61)$$

where $h_{\mu\nu}^{(2n)}$ and $A_\mu^{(2n)}$ are the post-Newtonian corrections. In the harmonic gauge, the post-Newtonian equation becomes

$$-\frac{1}{2}\Box h_{\mu\nu}^{(2n)} + R_{\mu\nu}^{[2]}[h^{(n)}, h^{(n)}]$$

$$= 8\pi\left[T_{\mu\nu}^{(2n)} - \frac{1}{n+1}\eta^{\alpha\beta}T_{\alpha\beta}^{(2n)}\eta_{\mu\nu}\right], \quad (62)$$

where we used the Einstein equation of the Newtonian order $G_{\mu\nu}^{(n)} = T_{\mu\nu}^{(n)} = 0$. $R_{\mu\nu}^{[2]}$ is the second order perturbation of the Ricci tensor, which gives a source term from the Newtonian order of Eqs. (21) and (24). The Maxwell equation in this order is given by

$$\Box A_\mu^{(2n)} + \partial_\nu\left[\frac{h^{(n)}}{2}F^{(n)\nu}{}_\mu - h^{(n)\nu\alpha}F_{\alpha\mu}^{(n)} - h_{\mu\alpha}^{(n)}F^{(n)\nu\alpha}\right] = 0. \quad (63)$$

Using Eq. (26), then, Eqs. (62) and (63) become

$$\Box\left[h_{tt}^{(2n)} + \frac{1}{2}\left(1 + \frac{\sigma^2}{2}\right)\Phi^2\right] = 0, \quad (64a)$$

$$\Box\left[h_{ij}^{(2n)} - \frac{1}{2n^2}\left(1 - \frac{n\sigma^2}{2}\right)\Phi^2\delta_{ij}\right] = -\frac{n+1}{2n}(1 - \sigma^2)\Phi_{,i}\Phi_{,j} + \frac{n+1}{n}(\Phi\Phi_{,i})_{,j}, \quad (64b)$$

$$\Box\left[A_t^{(2n)} + \frac{1}{2n\omega_{n+1}}\frac{Q_0}{\rho_0^n}\Phi^2\right] = 0. \quad (64c)$$

The general solution is constructed by the inhomogeneous and homogeneous solutions. The coefficients of the homogeneous solution are determined by the boundary condition. We write the general solution as follows:

$$h_{tt}^{(2n)} = -\frac{1}{2}\left(1 + \frac{\sigma^2}{2}\right)\Phi^2 + s_t\frac{\rho_0^n}{L^n}\Phi, \quad (65a)$$

$$h_{ij}^{(2n)} = \frac{1}{2n^2}\left(1 - \frac{n\sigma^2}{2}\right)\Phi^2\delta_{ij} + s_{ij}\frac{\rho_0^n}{L^n}\Phi + \text{Pf}\left(\int_{[-(L/2),(L/2)]\times\mathbf{R}^{n+1}}G(x,x')\mathcal{S}_{ij}(x')d^{n+2}x'\right), \quad (65b)$$

$$A_t^{(2n)} = -\frac{Q_0}{2n\omega_{n+1}\rho_0^n}\Phi^2 + s_A\frac{\rho_0^n}{L^n}\Phi, \quad (65c)$$

where Pf means the finite part of the integration and s_t, s_{ij}, s_A are the dimensionless coefficients of the homogeneous solution. The Green function in the compact space, $[-L/2, L/2]\times\mathbf{R}^{n+1}$, is given by

$$G(x,x') = -\frac{1}{n\omega_{n+1}}$$
$$\times \sum_{m=-\infty}^{\infty}\frac{1}{((\mathbf{x}-\mathbf{x}')^2 + (z - z' - mL)^2)^{(n/2)}}, \quad (66)$$

where $\mathbf{x}$ is the coordinate vector of $\mathbf{R}^{n+1}$, which gives $r = |\mathbf{x}|$. We write the integrand as

$$\mathcal{S}_{ij} = -\frac{n+1}{2n}[(1 - \sigma^2)(\Phi_{,i}\Phi_{,j}) - 2(\Phi\Phi_{,i})_{,j}]. \quad (67)$$

B. Matching from the near solution

Now, we determine the coefficients s_t, s_{ij}, s_A of the homogeneous term by matching with the near solution. The monopole moment of the near solution behaves as $\sim\rho^{-n}$. Then the relevant terms of this order in the overlap region have the dependence of $\rho_0^{2n}/L^n\rho^n$ in the dimensionless form. Transforming Eqs. (56) and (57) into the harmonic coordinates (D7), the near zone solution up to the relevant order becomes

$$g_{tt}^{(\text{near})} = -\left[1 - 2\lambda - (1 - 2\lambda)\frac{\rho_0^n}{\rho^n}\right] + \mathcal{O}(1/L^{2n}, 1/\rho^{2n}), \quad (68a)$$

$$g_{ij}^{(\text{near})} = \left[1 + \frac{2\lambda}{n} + \left(1 + \frac{2\lambda}{n}\right)\frac{\rho_0^n}{n\rho^n}\right]\delta_{ij} + \mathcal{O}(1/L^{2n}, 1/\rho^{2n}), \quad (68b)$$

$$A_t^{(\text{near})} = \frac{2\lambda Q_0}{n\omega_{n+1}\rho_0^n} + \frac{Q_0}{n\omega_{n+1}\rho^n}(1 - \lambda) + \mathcal{O}(1/L^{2n}, 1/\rho^{2n}). \quad (68c)$$

We extract the monopole moment of $\mathcal{O}(1/L^n)$,

$$g_{tt}^{(\text{near})} \sim -2\zeta(n)\frac{\rho_0^{2n}}{L^n\rho^n},$$ (69a)

$$g_{ij}^{(\text{near})} \sim \frac{2\zeta(n)}{n^2}\frac{\rho_0^{2n}}{L^n\rho^n}\,\delta_{ij},$$ (69b)

$$A_t^{(\text{near})} \sim -\frac{\zeta(n)Q_0}{n\omega_{n+1}\rho_0^n}\frac{\rho_0^{2n}}{L^n\rho^n}.$$ (69c)

Meanwhile, the corresponding terms in the post-Newtonian solution (65) are

$$h_{tt}^{(2n)} \sim \left[s_t - 2\zeta(n)\left(1 + \frac{\sigma^2}{2}\right)\right]\frac{\rho_0^{2n}}{L^n\rho^n},$$ (70a)

$$h_{ij}^{(2n)} \sim \left[s_{ij} + \frac{2\zeta(n)}{n^2}\left(1 - \frac{n\sigma^2}{2}\right)\delta_{ij}\right]\frac{\rho_0^{2n}}{L^n\rho^n},$$ (70b)

$$A_t^{(2n)} \sim \left[s_A - \frac{2\zeta(n)Q_0}{n\omega_{n+1}\rho_0^n}\right]\frac{\rho_0^{2n}}{L^n\rho^n}.$$ (70c)

The finite part of the integration term in Eq. (65b) does not contribute to the monopole moment in the

overlap region. The matching between two solutions shows

$$s_t = \sigma^2\zeta(n),$$ (71a)

$$s_{ij} = \frac{\sigma^2}{n}\zeta(n)\delta_{ij} = \frac{s_t}{n}\delta_{ij},$$ (71b)

$$s_A = \frac{\zeta(n)Q_0}{n\omega_{n+1}\rho_0^n}.$$ (71c)

Therefore, the post-Newtonian correction is determined as

$$h_{tt}^{(2n)} = -\frac{1}{2}\left(1 + \frac{\sigma^2}{2}\right)\Phi^2 + \sigma^2\lambda\Phi,$$ (72a)

$$h_{ij}^{(2n)} = \frac{1}{2n^2}\left(1 - \frac{n\sigma^2}{2}\right)\Phi^2\delta_{ij} + \frac{\sigma^2}{n}\lambda\Phi\delta_{ij} + \text{Pf}\left(\int_{[-(L/2),(L/2)]\times\mathbf{R}^{n+1}}G(x,x')\mathcal{S}_{ij}(x')d^{n+2}x'\right),$$ (72b)

$$A_t^{(2n)} = -\frac{Q_0}{2n\omega_{n+1}\rho_0^n}\Phi^2 + \frac{Q_0}{n\omega_{n+1}\rho_0^n}\lambda\Phi.$$ (72c)

VII. GLOBAL CHARGES AND THERMODYNAMICS

In this section, we compute the global charges and confirm that the first law of the thermodynamics holds for the current cases.

A. Global charges

Now, we calculate the post-Newtonian correction to the global charges. The $(n+2)$-dimensional mass M and tension τ of the asymptotically Kaluza-Klein spacetime are determined by the asymptotic behavior [17,18]

$$h_{tt}^{(\text{asym})} \simeq \frac{c_t}{r^{n-1}},$$ (73)

$$h_{zz}^{(\text{asym})} \simeq \frac{c_z}{r^{n-1}}.$$ (74)

Then, we find

$$M = \frac{\omega_n L}{16\pi}(nc_t - c_z),$$ (75)

$$\tau = \frac{\omega_n}{16\pi}(c_t - nc_z).$$ (76)

The total electric charge Q is determined in the same way,

$$A_t^{(\text{asym})} \simeq \frac{Q}{(n-1)\omega_n L}\frac{1}{r^{n-1}}.$$ (77)

To extract the global charges, we take the limit $r \gg L, z$. As in Eq. (B1), Φ becomes

$$\Phi \simeq \frac{n\omega_{n+1}}{(n-1)\omega_n}\frac{\rho_0^n}{Lr^{n-1}}.$$ (78)

Since the Green function in Eq. (66) has a similar behavior as Φ, the integration becomes

$$\int_{[-(L/2),(L/2)]\times\mathbf{R}^{n+1}}G(x,x')\mathcal{S}_{ij}(x')d^{n+2}x'$$

$$\simeq -\frac{1}{(n-1)\omega_n}\frac{1}{Lr^{n-1}}\int_{[-(L/2),(L/2)]\times\mathbf{R}^{n+1}}\mathcal{S}_{ij}(x')d^{n+2}x'.$$ (79)

Then, Eq. (72) gives

$$h_{tt}^{(\text{asym},2n)} \simeq \frac{n\omega_{n+1}\sigma^2\zeta(n)}{(n-1)\omega_n}\frac{\rho_0^{2n}}{L^{n+1}}\frac{1}{r^{n-1}},$$ (80)

$$h_{zz}^{(\text{asym},2n)} \simeq \frac{\omega_{n+1}\sigma^2\zeta(n)}{(n-1)\omega_n}\frac{\rho_0^{2n}}{L^{n+1}}\frac{1}{r^{n-1}}$$
$$+ \frac{(n+1)(1-\sigma^2)}{2n(n-1)\omega_n L r^{n-1}}\text{Pf}\left(\int d^{n+1}\mathbf{x}\int_{-L/2}^{L/2} dz\Phi_{,z}^2\right).$$
$$\tag{81}$$

The second term in $\mathcal{S}_{zz}(x)$ does not contribute to Eq. (81) because it is the total derivative. The finite part of the integration is computed in Ref. [10]

$$\text{Pf}\left(\int d^{n+1}\mathbf{x}\int_{-L/2}^{L/2} dz\Phi_{,z}^2\right) = -\frac{(n-1)n\zeta(n)\omega_{n+1}\rho_0^{2n}}{L^n}.$$
$$\tag{82}$$

As a result, the ADM mass and tension become

$$M = \frac{(n+1)\omega_{n+1}\rho_0^n}{16\pi}\left(1 + \frac{1}{2}(1+\sigma^2)\lambda\right)$$
$$= M_0\left(1 + \frac{1}{2}(1+\sigma^2)\lambda\right), \tag{83}$$

$$\tau L = \frac{n(n+1)\omega_{n+1}\rho_0^n}{32\pi}(1-\sigma^2)\lambda$$
$$= \frac{n}{2}M_0(1-\sigma^2)\lambda. \tag{84}$$

From Eq. (72c), the total charge Q becomes

$$Q = Q_0(1+\lambda). \tag{85}$$

At $\sigma = 0$, these results reproduce the results in the neutral case [10].

Moreover, in the extremal case ($\sigma = 1$) we can confirm that our results correspond to the exact solutions in Ref. [19]. The exact solutions of the $(n+3)$-dimensional extremal caged black hole are given by

$$ds^2 = -H^{-2}dt^2 + H^{2/n}\delta_{ij}dx^i dx^j,$$
$$A_\mu dx^\mu = -\sqrt{\frac{n+1}{8\pi n}}H^{-1}dt, \tag{86}$$

where

$$H(x^i) = 1 + \mu\sum_{k=-\infty}^{\infty}\frac{1}{(r^2+(z-kL)^2)^{(n/2)}}. \tag{87}$$

If we set $\mu = \rho_0^n(1+\lambda)/2$, at $\sigma = 1$ the perturbative solutions constructed agree with the above solutions up to $\mathcal{O}(\lambda)$.

There is the apparent difference between our results and the results by the EFT calculation [16]. This simply comes from the ambiguity of the parametrization. For example, see Eqs. (3.7) and (3.25) in Ref. [16] which are M and τ. Changing the gauge condition in Eq. (49) gives the different length scale $\rho_0 \rightarrow \rho_0(1 + \epsilon_1(\sigma))$ and charge parametrization $\sigma \rightarrow \sigma(1 + \epsilon_2(\lambda,\sigma))$. We note that, to keep

$\sigma = 1$ be the extreme limit, one requires $\epsilon_2(\lambda, 1) = 0$ further. To reproduce the results by the EFT, we should impose the following gauge condition instead of Eq. (49),

$$C_2 = -\frac{2\rho_+^n C_1}{n\rho_0^n}, \qquad C_3 = C_1, \qquad \alpha_0 = 0. \tag{88}$$

B. First law and the Smarr formula

Here we confirm that our solutions satisfy the Smarr formula

$$nM = \frac{n+1}{8\pi}\kappa\mathcal{A} + \tau L + nQU, \tag{89}$$

and the first law

$$dM = \tau dL + \frac{1}{8\pi}\kappa d\mathcal{A} + UdQ. \tag{90}$$

Collecting the thermodynamic variables from Eqs. (58)–(60), (83), and (84), we write

$$M = M_0\left(1 + \frac{1}{2}(1+\sigma^2)\lambda\right), \quad \mathcal{A} = \mathcal{A}_0\left(1 + \frac{n+1}{n}\lambda\right),$$

$$Q = Q_0(1+\lambda), \quad \kappa = \kappa_0\left(1 - \frac{n+1}{n}\lambda\right),$$

$$U = U_0(1 + \sqrt{1-\sigma^2}\lambda), \quad \tau L = \frac{n}{2}M_0(1-\sigma^2)\lambda, \tag{91}$$

where

$$M_0 = \frac{(n+1)\omega_{n+1}\rho_0^n}{16\pi},$$

$$\mathcal{A}_0 = \omega_{n+1}\rho_0^{n+1}\left(\frac{1+\sqrt{1-\sigma^2}}{2}\right)^{(n+1)/n},$$

$$Q_0 = \sqrt{\frac{n(n+1)}{32\pi}}\omega_{n+1}\rho_0^n\sigma\kappa_0 \quad = \frac{n\rho_0^n}{2\rho_+^{n+1}}\sqrt{1-\sigma^2}, \tag{92}$$

$$U_0 = \frac{Q_0}{n\omega_{n+1}\rho_+^n}.$$

From the direct computation, the above expressions are easily confirmed to follow Eq. (89) up to $\mathcal{O}(\lambda)$.

To confirm the first law for the current case, we take the variation of M with L, $\mathcal{A}$ and Q. Changing the variables from $(\rho_0, \lambda, \sigma)$ to $(L, \mathcal{A}, Q)$, the variation becomes

$$\begin{bmatrix} (\partial_L M)_{\mathcal{A},Q} \\ (\partial_{\mathcal{A}} M)_{L,Q} \\ (\partial_Q M)_{L,\mathcal{A}} \end{bmatrix} = \frac{\partial(\rho_0, \lambda, \sigma)}{\partial(L, \mathcal{A}, Q)}\begin{bmatrix} (\partial_{\rho_0} M)_{\lambda,\sigma} \\ (\partial_\lambda M)_{\rho_0,\sigma} \\ (\partial_\sigma M)_{\rho_0,\lambda} \end{bmatrix}$$
$$= \frac{\partial(\rho_0, \lambda, \sigma)}{\partial(L, \mathcal{A}, Q)}\begin{bmatrix} nM/\rho_0 \\ M_0(1+\sigma^2)/2 \\ M_0\sigma\lambda \end{bmatrix}. \tag{93}$$

Using $L = \zeta(n)^{(1/n)}\rho_0\lambda^{-(1/n)}$, the Jacobian $\partial(\rho_0, \lambda, \sigma)/\partial(L, \mathcal{A}, Q)$ is computed as

$$\frac{\partial(\rho_0, \lambda, \sigma)}{\partial(L, \mathcal{A}, Q)} = \left(\frac{\partial(L, \mathcal{A}, Q)}{\partial(\rho_0, \lambda, \sigma)}\right)^{-1}$$

$$= \begin{bmatrix} \rho_0 \lambda/L & -n\lambda/L & 0 \\ \rho_0 \kappa_0 (1 - (2n+1)\lambda/n)/8\pi n M_0 & \lambda\kappa_0/8\pi M_0 & -\sigma\kappa_0/8\pi M_0 \\ \rho_0 U_0 (1 - 2\lambda)/n M_0 & \lambda U_0/M_0 & U_0(1 + \sqrt{1-\sigma^2})\sqrt{1-\sigma^2}/\sigma M_0 \end{bmatrix}. \tag{94}$$

Using this, we can show that the variation of M satisfies the first law (90) up to $\mathcal{O}(\lambda)$,

$$\begin{bmatrix} (\partial_L M)_{\mathcal{A}, Q} \\ (\partial_\mathcal{A} M)_{L, Q} \\ (\partial_Q M)_{L, \mathcal{A}} \end{bmatrix} = \begin{bmatrix} \tau \\ \kappa/8\pi \\ U \end{bmatrix}. \tag{95}$$

Note that this is also achieved through the Harrison transformation from the neutral seed solutions [20]. On the other hand, our argument gives us a direct confirmation using the direct construction of the perturbative solutions.

VIII. SUMMARY AND DISCUSSION

In this paper, we constructed the perturbative solution of the small black holes with the Maxwell charge in the caged spacetime using the matched asymptotic expansion. The expansion of the Maxwell field can be performed in the same way as the metric. Although our results seem different with the EFT calculation in Ref. [16], this is just the difference in the parametrization. We also confirmed the first law and the Smarr formula, which were shown for a sequence of charged black objects in Kaluza-Klein spacetime in Ref. [20].

Another way to construct the charged solution is to use the Harrison transformation [20,21], which produces the charged dilatonic solution from the neutral seed solution. However, the charged stationary solution like the rotating black ring cannot be produced by the Harrison transformation. In such case, one should rely on some perturbative methods with the Maxwell field, as we used in this paper. Since the charged black ring is studied only by the black-fold approach [22], it is also interesting to study the charged black ring beyond the blackfold approach.

We have not included the finite size effect which comes from the deformation of the horizon. To see this effect, we must consider the multipole perturbation which will be studied in future work.

APPENDIX A: GEGENBAUER POLYNOMIALS

The Gegenbauer polynomials $C_l^\nu(x)$ are defined as the coefficients of the following generating function:

$$\frac{1}{(1 - 2xt + t^2)^\nu} = \sum_{l=0}^\infty C_l^\nu(x) t^l, \tag{A1}$$

where $C_l^\nu(x)$ is written as

$$C_l^\nu(x) = \frac{(-1)^l}{l! 2^l} \frac{\Gamma(\nu + \frac{1}{2})\Gamma(l + 2\nu)}{\Gamma(2\nu)\Gamma(l + \nu + \frac{1}{2})} (1 - x^2)^{((1-2\nu)/2)}$$

$$\times \frac{d^l}{dx^l}[(1 - x^2)^{((2l+2\nu-1)/2)}]. \tag{A2}$$

It follows the equation

$$(1 - x^2)y'' - (2\nu + 1)y' + l(l + 2\nu)y = 0. \tag{A3}$$

APPENDIX B: ASYMPTOTIC LIMIT OF THE NEWTONIAN POTENTIAL

In the limit, $r \gg z, L$, the Newtonian potential Φ in Eq. (21a) behaves as

$$\Phi = \sum_{m=1}^\infty \frac{\rho_0^n}{(r^2 + m^2 L^2)^{(n/2)}} \left[\left(1 - \frac{2mLz}{r^2 + m^2 L^2} + \frac{z^2}{r^2 + m^2 L^2}\right)^{-(n/2)} + \left(1 + \frac{2mLz}{r^2 + m^2 L^2} + \frac{z^2}{r^2 + m^2 L^2}\right)^{-(n/2)} \right] + \frac{\rho_0^n}{(r^2 + z^2)^{(n/2)}}$$

$$= \frac{2\rho_0^n}{L r^{n-1}} \sum_{m=1}^\infty \frac{L/r}{(1 + (mL/r)^2)^{(n/2)}} + \mathcal{O}(1/r^n) = \frac{2\rho_0^n}{L r^{n-1}} \int_0^\infty \frac{dt}{(1 + t^2)^{(n/2)}} + \mathcal{O}(1/r^n) = \frac{n\omega_{n+1}}{(n-1)\omega_n} \frac{\rho_0^n}{L r^{n-1}} + \mathcal{O}(1/r^n). \tag{B1}$$

APPENDIX C: MULTIPOLE PERTURBATION IN THE NEAR ZONE

In this appendix, we give the equation for the multipole perturbation around the charged black hole.

1. Ansatz

The static spacetime can be written in this form

$$ds^2 = g_{tt}(x)dt^2 + g_{ij}(x)dx^i dx^j. \qquad (C1)$$

In this ansatz, we consider the linear perturbations around the black hole as $g_{\mu\nu} = g_{\mu\nu}^{(BH)} + h_{\mu\nu}$, where the background metric is given by

$$g_{\mu\nu}^{(BH)}dx^\mu dx^\nu = -f(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 d\Omega_{n+1}^2. \qquad (C2)$$

Since the background metric has $SO(n+2)$ symmetry, we can expand the perturbations in terms of the spherical harmonics on the $(n+1)$-dimensional sphere. Here, we consider only scalar perturbations. We explicitly write all nonzero components of the perturbations,

$$h_{tt} = -f \sum_{l=0}^\infty A_l(\rho)\mathsf{Y}_l, \qquad (C3a)$$

$$h_{\rho\rho} = f^{-1} \sum_{l=0}^\infty B_l(\rho)\mathsf{Y}_l, \qquad (C3b)$$

$$h_{\rho a} = f^{-1} \sum_{l=1}^\infty C_l(\rho)\mathcal{D}_a\mathsf{Y}_l, \qquad (C3c)$$

$$h_{ab} = \rho^2 \sum_{l=1}^\infty D_l(\rho)\left[\mathcal{D}_a\mathcal{D}_b - \frac{1}{n+1}\gamma_{ab}\mathcal{D}^2\right]\mathsf{Y}_l$$
$$+ \rho^2 \gamma_{ab} \sum_{l=0}^\infty E_l(\rho)\mathsf{Y}_l, \qquad (C3d)$$

where γ_{ab} is the metric of the $(n+1)$-dimensional unit sphere and $\mathcal{D}_a$ denotes the covariant derivative with respect to γ_{ab}. Here, Y_l is the spherical harmonics with l-th multipole moment, which satisfies $\mathcal{D}^2\mathsf{Y}_l = -l(l+n)\mathsf{Y}_l$. The gauge transformation for the metric perturbations is given by $h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}^{(BH)}$ under the infinitesimal coordinate transformations $x^\mu \to x^\mu - \xi^\mu$. Then, the components transform as

$$A_l \to A_l + \frac{f'}{f}\xi_l^\rho, \qquad B_l \to B_l + 2\xi_l^{\rho\prime} - \frac{f'}{f}\xi_l^\rho,$$

$$C_l \to C_l + \xi_l^\rho + f\rho^2\zeta_l', \qquad D_l \to D_l + 2\zeta_l,$$

$$E_l \to E_l + \frac{2}{\rho}\xi_l^\rho - \frac{2l(l+n)}{n+1}\zeta_l, \qquad (C4)$$

where we have expanded ξ^μ with the harmonics Y_l and ζ_l defined as $\xi^a = \zeta_l \mathcal{D}^a\mathsf{Y}_l$. Note that we have not transformed the time coordinate t. Using the above gauge degrees of freedom ξ_l^ρ and ζ_l, we can set $C_l = D_l = 0$ [9]. In the monopole case, C_0 and D_0 are automatically zero and then ξ_0^ρ becomes the residual gauge. For the vector potential, we write the perturbations δA_t, δA_ρ, and δA_a as

$$\delta A_t = \sum_{l=0}^\infty a_l(\rho)\mathsf{Y}_l, \qquad (C5a)$$

$$\delta A_\rho = \sum_{l=0}^\infty b_l(\rho)\mathsf{Y}_l, \qquad (C5b)$$

$$\delta A_a = \sum_{l=1}^\infty c_l(\rho)\mathcal{D}_a\mathsf{Y}_l. \qquad (C5c)$$

The $U(1)$ gauge transformation of $\delta A_\mu \to \delta A_\mu + \partial_\mu\psi$, where ψ is a scalar function, transforms the potential as

$$b_l \to b_l + \psi_l', \qquad c_l \to c_l + \psi_l, \qquad (C6)$$

where $\psi_l(\rho)$ is the expansion coefficient of ψ for the spherical harmonics Y_l. Therefore, we can set $c_l = 0$ ($l \neq 0$) by choosing ψ_l appropriately. In the monopole case, we also set $b_0 = 0$ because c_0 is automatically zero. The field strength becomes

$$\delta F_{t\rho} = -\sum_{l=0}^\infty a_l'\mathsf{Y}_l, \qquad (C7a)$$

$$\delta F_{ta} = -\sum_{l=1}^\infty a_l \mathcal{D}_a\mathsf{Y}_l, \qquad (C7b)$$

$$\delta F_{\rho a} = -\sum_{l=1}^\infty b_l \mathcal{D}_a\mathsf{Y}_l. \qquad (C7c)$$

2. Perturbation equation

The Einstein equation for the linear perturbation is

$$\delta R_{\mu\nu} = 8\pi\left[\delta T_{\mu\nu} - \frac{1}{n+1}(g^{(BH)\alpha\beta}\delta T_{\alpha\beta}\right.$$
$$\left. - T^{(0)\alpha\beta}h_{\alpha\beta})g_{\mu\nu}^{(BH)} - \frac{1}{n+1}T^{(0)}h_{\mu\nu}\right]$$
$$\equiv 8\pi\delta S_{\mu\nu}. \qquad (C8)$$

Under the metric ansatz (C3), the nonvanishing components of $\delta R_{\mu\nu}$ become

$$\delta R_{tt} = -\frac{f}{4\rho^2}\Big[-2\rho^2 f A_l'' - (3\rho^2 f' + 2(n+1)\rho f)A_l' + \rho^2 f' B_l' - (n+1)\rho^2 f' E_l'$$
$$- 2(\rho^2 f'' + (n+1)\rho f')(A_l - B_l) + 2l(l+n)A_l\Big]Y_l, \tag{C9a}$$

$$\delta R_{\rho\rho} = \frac{1}{4\rho^2 f}\Big[-2\rho^2 f A_l'' - 2(n+1)\rho^2 f E_l'' - 3\rho^2 f' A_l' + (\rho^2 f' + 2(n+1)\rho f)B_l'$$
$$- (n+1)(\rho^2 f' + 4\rho f)E_l' + 2l(l+n)B_l\Big]Y_l, \tag{C9b}$$

$$\delta R_{\rho a} = \frac{1}{4\rho f}\Big[-2\rho f A_l' - 2n\rho f E_l' - (\rho f' - 2f)A_l + (\rho f' + 2nf)B_l\Big]\mathcal{D}_a Y_l, \tag{C9c}$$

$$\delta R_{ab} = \frac{1}{2}\Big[-\rho^2 f E_l'' - (\rho^2 f' + 2(n+1)\rho f)E_l' - \rho f(A_l' - B_l') + 2(\rho f' + nf)(B_l - E_l) + l(l+n)E_l\Big]Y_l\gamma_{ab}$$
$$- \frac{1}{2}\Big[A_l + B_l + (n-1)E_l\Big]\mathcal{D}_a\mathcal{D}_b Y_l, \tag{C9d}$$

where we write only the l-th pole components. On the other hand, the nonvanishing components of $\delta S_{\mu\nu}$ become

$$\delta S_{tt} = -f\frac{n}{n+1}(2\mathcal{E}a_l' + \mathcal{E}^2 B_l)Y_l, \tag{C10a}$$

$$\delta S_{\rho\rho} = \frac{1}{f}\frac{n}{n+1}(2\mathcal{E}a_l' + \mathcal{E}^2 A_l)Y_l, \tag{C10b}$$

$$\delta S_{ab} = \rho^2\gamma_{ab}\frac{1}{n+1}\Big[-2\mathcal{E}a_l'$$
$$+ \mathcal{E}^2(E_l - A_l - B_l)\Big]Y_l, \tag{C10c}$$

$$\delta S_{\rho a} = \frac{1}{f}\mathcal{E}a_l\mathcal{D}_a Y_l, \tag{C10d}$$

$$\delta S_{ta} = f\mathcal{E}b_l\mathcal{D}_a Y_l, \tag{C10e}$$

where $\mathcal{E}$ is defined by Eq. (14). From $\delta R_{ta} = 0$ and Eq. (C10e), we see $b_l = 0$ for $l \neq 0$. Moreover, $a_l(l \neq 0)$ is determined by using the Einstein equation from Eqs. (C9c) and (C10d). Therefore, we do not need to solve the Maxwell equation directly to determine $a_l(l \neq 0)$.

3. Master equations

From the traceless part of Eqs. (C9d) and (C10c), we obtain the following algebraic relation:

$$A_l + B_l + (n-1)E_l = 0 \tag{C11}$$

because $[(n+1)\mathcal{D}_a\mathcal{D}_b - \gamma_{ab}\mathcal{D}^2]Y_l \neq 0$ when $l > 1$. Note that for $l = 0, 1$ we do not have any constraint from the perturbation equations but we can require this equation by using the residual gauge [see Eq. (38)].

Now, we introduce the following master variables:

$$A_l \equiv \Psi - n\Pi, \qquad B_l \equiv \Pi - \Psi, \qquad E_l \equiv \Pi, \tag{C12}$$

which satisfy Eq. (C11). Using the master variables, we obtain the following master equations from the perturbation equations:

$$(n+1)\rho^2 f\Pi'' + (n+1)\rho f\Pi' - (n+1)l(l+n)\Pi = -2l(l+n)\Psi, \tag{C13a}$$

$$\rho^2 f\Psi'' + (2\rho^2 f' + (3n+17)\rho f)\Psi' + (2n^2 - l(l+n))\Psi = 0. \tag{C13b}$$

Using the new dimensionless variable $x \equiv (\rho/\rho_0)^n$, Eq. (C13) becomes

$$fx^2\frac{d^2\Pi}{dx^2} + 2fx\frac{d\Pi}{dx} - p_l(p_l + 1)\Pi$$
$$= -\frac{2}{n+1}p_l(p_l + 1)\Psi \tag{C14a}$$

and

$$fx^2\frac{d^2\Psi}{dx^2} + \Big(2x\frac{df}{dx} + 4f\Big)x\frac{d\Psi}{dx} + (2 - p_l(p_l + 1))\Psi = 0, \tag{C14b}$$

where $p_l \equiv l/n$. Using $f(x) = (x - x_+)(x - x_-)/x^2$, Eq. (C14b) becomes

$$(x - x_+)(x - x_-)\frac{d^2\Psi}{dx^2} + (4x - 2x_+ - 2x_-)\frac{d\Psi}{dx}$$
$$+ (2 - p_l(p_l + 1))\Psi = 0. \tag{C15}$$

This equation has the three singular points : $x = x_-, x_+$, and ∞. Therefore, the equation for Ψ becomes the hypergeometric differential equation. Using $\xi = (x - x_-)/(x_+ - x_-)$, it becomes standard form

$$\xi(1-\xi)\frac{d^2\Psi}{d\xi^2} + (2 - 4\xi)\frac{d\Psi}{d\xi} + (p_l + 2)(p_l - 1)\Psi = 0. \tag{C16}$$

This is the same with one for the neutral case [9]. Similarly, defining $\tilde{\Pi} \equiv x\Pi/(x_+ - x_-)$, Eq. (C14a)

becomes the hypergeometric differential equation with a source term

$$\xi(1-\xi)\frac{d^2\tilde{\Pi}}{d\xi^2} + p_l(p_l+1)\tilde{\Pi} = \frac{2p_l(p_l+1)}{n+1}(\xi+e)\Psi,$$

(C17)

where we defined $e \equiv x_-/(x_+ - x_-)$. So, our multipole solution, in general, can be written by four hypergeometric functions and the characteristic solution for Eq. (C17). In the neutral case, we do not need to solve the equation for Π because Π is obtained directly from Ψ using Eq. (C9c) and $\delta R_{\rho a} = 0$.

APPENDIX D: GAUGE TRANSFORMATION INTO THE HARMONIC COORDINATES

To match the near solution with the asymptotic solution in the overlap region, we need the near solution written in the harmonic gauge. If we write the harmonic coordinates as $x_h^\mu(x^\mu) = (t, \rho_h(\rho), \chi, \ldots)$, the harmonic condition ($\nabla^2 x^\mu = 0$) becomes

$$\nabla^2(\rho_h \cos\chi) = 0.$$

(D1)

In the above, ∇^2 is the Laplacian for the full near metric $g_{\mu\nu} = g_{\mu\nu}^{(\text{BH})} + h_{\mu\nu}$. Since the monopole perturbation does not depend on χ, Eq. (D1) up to $\mathcal{O}(1/L^n)$ becomes

$$\left(1 - \frac{1}{2}h^{(n)}\right)\partial_\rho\left[\left(1 + \frac{1}{2}h^{(n)}\right)\rho^{n+1}f(1-B_0)\partial_\rho\rho_h\right]$$
$$= (n+1)(1+E_0)\rho^{n-1}\rho_h,$$

(D2)

where $h^{(n)} = A_0 + B_0 + (n+1)E_0$ is the trace of $h_{\mu\nu}^{(n)}$.

We expand ρ_h as $\rho_h = \rho_h^{(0)} + \rho_h^{(n)}$, where $\rho_h^{(n)}$ is of $\mathcal{O}(1/L^n)$. Then, the $\mathcal{O}(1/L^0)$ equation becomes

$$\partial_\rho(\rho^{n+1}f\partial_\rho\rho_h^{(0)}) = (n+1)\rho^{n-1}\rho_h^{(0)}.$$

(D3)

Assuming $\rho_h \to \rho$ as $\rho \to \infty$, the solution is given by

$$\rho_h^{(0)} = \rho - \frac{1}{2n}\frac{\rho_0^n}{\rho^{n-1}} + \mathcal{O}(\rho_0^{2n}/\rho^{2n-1}).$$

(D4)

The $\mathcal{O}(1/L^n)$ equation is

$$\partial_\rho(\rho^{n+1}f\partial_\rho\rho_h^{(n)}) - (n+1)\rho^{n-1}\rho_h^{(n)}$$
$$= \partial_\rho(\rho^{n+1}fB_0\partial_\rho\rho_h^{(0)}) - \frac{1}{2}\rho^{n+1}\partial_\rho h^{(n)}f\partial_\rho\rho_h^{(0)}$$
$$- (n+1)\rho^{n-1}E_0\rho_h^{(0)}.$$

(D5)

Note that the $\mathcal{O}(1/L^n)$ equation is trivial in the gauge choice (49). Here we consider cases for general C_2, C_3, α_0. Assuming $\rho_h^{(n)} \to 0$ as $\rho \to \infty$, the solution becomes

$$\rho_h^{(n)} = \frac{C_3 + nC_2}{2n}\frac{\rho_0^n}{\rho^{n-1}} + \mathcal{O}(\rho_0^{2n}/\rho^{2n-1}).$$

(D6)

As a result, we have

$$\rho_h(\rho) = \rho - \frac{\rho_0^n}{2n\rho^{n-1}}(1 - nC_2 - C_3) + \mathcal{O}(\rho_0^{2n}/\rho^{2n-1}).$$

(D7)

This equation gives the coordinate transformation from $(t, \rho, \chi, \ldots)$ into $(t, \rho_h(\rho), \chi, \ldots)$. In the harmonic coordinates, the near zone metric and the gauge field become

$$g_{tt}^{(\text{near})} = -f(\rho)(1 + A_0(\rho)) = -\left(1 - 2C_1 - (1 - 2C_1 - C_3)\frac{\rho_0^n}{\rho_h^n}\right) + \mathcal{O}(1/L^{2n}, 1/\rho_h^{2n}),$$

(D8a)

$$g_{\rho_h\rho_h}^{(\text{near})} = \left(\frac{\partial\rho}{\partial\rho_h}\right)^2 f^{-1}(\rho)(1 + B_0(\rho)) = 1 + \frac{2C_1}{n} + \frac{\rho_0^n}{n\rho_h^n}\left(1 + \frac{2C_1}{n} - C_3\right) + \mathcal{O}(1/L^{2n}, 1/\rho_h^{2n}),$$

(D8b)

$$g_{\chi\chi}^{(\text{near})} = \rho^2(1 + E_0(\rho)) = \rho_h^2\left(1 + \frac{2C_1}{n} + \frac{\rho_0^n}{n\rho_h^n}\left(1 + \frac{2C_1}{n} - C_3\right)\right) + \mathcal{O}(1/L^{2n}, 1/\rho_h^{2n}),$$

(D8c)

$$F_{t\rho_h}^{(\text{near})} = \frac{Q_0}{\omega_{n+1}\rho_h^{n+1}}(1 - \alpha_0 - 2C_1 + \mathcal{O}(1/L^{2n}, 1/\rho_h^n)).$$

(D8d)

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