PHYSICAL REVIEW D **86,** 044011 (2012)

AdS and Lifshitz black holes in conformal and Einstein-Weyl gravities

H. Lü,[1,2] Yi Pang,[3] C. N. Pope,[3,4] and J. F. Vázquez-Poritz[5,6]

[1]*China Economics and Management Academy, Central University of Finance and Economics, Beijing 100081, China*
[2]*Institute for Advanced Study, Shenzhen University, Nanhai Avenue 3688, Shenzhen 518060, China*
[3]*George P. and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*
[4]*DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, United Kingdom*
[5]*New York City College of Technology, The City University of New York, 300 Jay Street, Brooklyn, New York 11201, USA*
[6]*The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, New York 10016, USA*
(Received 23 May 2012; published 7 August 2012)

We study black hole solutions in extended gravities with higher-order curvature terms, including conformal and Einstein-Weyl gravities. In addition to the usual anti-de Sitter (AdS) vacuum, the theories admit Lifshitz and Schrödinger vacua. The AdS black hole in conformal gravity contains an additional parameter over and above the mass, which may be interpreted as a massive spin-2 hair. By considering the first law of thermodynamics, we find that it is necessary to introduce an associated additional intensive/extensive pair of thermodynamic quantities. We also obtain new Lifshitz black holes in conformal gravity and study their thermodynamics. We use a numerical approach to demonstrate that AdS black holes beyond the Schwarzschild-AdS solution exist in Einstein-Weyl gravity. We also demonstrate the existence of asymptotically Lifshitz black holes in Einstein-Weyl gravity. The Lifshitz black holes arise at the boundary of the parameter ranges for the AdS black holes. Outside the range, the solutions develop naked singularities. The asymptotically AdS and Lifshitz black holes provide an interesting phase transition, in the corresponding boundary field theory, from a relativistic Lorentzian system to a nonrelativistic Lifshitz system.

DOI: 10.1103/PhysRevD.86.044011 PACS numbers: 04.50.Kd

I. INTRODUCTION

Theories of gravity extended by the addition of higher-order curvature terms are of interest for a number of reasons. One motivation is to investigate whether suitably extended four-dimensional gravity can be quantized in its own right. It has been shown that Einstein gravity extended by the inclusion of quadratic curvature terms is perturbatively renormalizable [1]. However, the price to be paid for achieving renormalizability is that the theory then contains massive spin-2 modes as well as the massless graviton and, furthermore, that the massive modes are ghostlike (i.e., their kinetic term has the wrong sign). Three-dimensional models of gravity, for which the UV divergence problems are inherently less severe, have also been studied extensively. While Einstein gravity itself is essentially trivial in three dimensions, extensions to include higher-order derivative terms lead to interesting toy models with dynamical content and the possibility of well-controlled UV behavior. Such extensions in three dimensions include topologically massive gravity [2], and more recently, "new massive gravity" [3]. The theory can be rendered ghost free, and equivalent to a theory with a standard Einstein-Hilbert action, after truncating out modes with logarithmic falloff by imposing an appropriate boundary condition of AdS_3. (See, for example, Ref. [4].) Supersymmetric extensions were considered in Refs. [5–9].

The situation is rather more subtle in four dimensions. An analogous "critical gravity" in four dimensions was considered in Ref. [10]. The Lagrangian consists of the Einstein-Hilbert term with a cosmological constant Λ and an additional higher-order term proportional to the square of the Weyl tensor, with a coupling constant α. It was shown that there is a critical relation between α and Λ for which the generically present massive spin-2 modes disappear, and are instead replaced by modes with a logarithmic falloff [11] (see also Refs. [12,13]). These log modes are ghostlike in nature [14,15], but since they fall off more slowly than do the massless spin-2 modes, they can be truncated out by imposing an appropriate anti-de Sitter (AdS) boundary condition. The resulting theory is then somewhat trivial, in the sense that the remaining massless graviton has zero on shell energy. Furthermore, the mass and the entropy of standard Schwarzschild-AdS black holes are both zero in the critical theory. Analogous critical theories exist also in higher-dimensional extended gravities with curvature-squared terms [16].

In fact, it was observed in Ref. [17] that one can generalize critical gravity to a wider class of Weyl-squared extensions to cosmological Einstein gravity, where α does not take the critical value. For a certain range of values for α, the mass squared of the massive spin-2 mode in the AdS_4 background is negative, but not sufficiently negative to imply tachyonic behavior. However, this mode, which is again ghostlike, has a slower falloff than the massless graviton and so it can be truncated by imposing appropriate AdS boundary conditions. This extended class of models has been investigated further in Refs. [18–34].

One possible approach would be to begin with the conformally invariant theory described by a pure Weyl-squared action. Being the local gauge theory of the conformal group, this theory of "conformal gravity" has the virtue of yielding a convergent Euclidean functional integral, and also of renormalizability and asymptotic freedom [35]. One might then argue [36,37] that quantum fluctuations would break the scale invariance, and thereby generate an Einstein-Hilbert term in the low-energy effective action. Thus, the Einstein-Weyl extensions of critical gravity described above effectively describe the emergence [38] of Einstein gravity from conformal gravity. It then becomes of interest to investigate the classical solutions of conformal gravity and Einstein-Weyl gravity. Any solution of Einstein gravity with a cosmological constant is also a solution of Einstein-Weyl gravity. However, the Weyl-squared term gives rise to fourth-order equations of motion, which are highly nonlinear, and so it is in general rather difficult to find the further new solutions that exist over and above the pure Einstein solutions. One of the main purposes of the present paper is to search for such new solutions under the simplifying assumption of spherical symmetry (and certain generalizations of this).

The investigation of solutions in higher-derivative extensions of Einstein gravity is also of interest from the AdS/CFT viewpoint, not least because it is known that such higher-order terms arise in string theory. Furthermore, although originally the AdS/CFT correspondence was conceived as a duality between a conformal field theory and a string theory, the idea of holography has been generalized to broader classes of gauge/gravity duality outside the string theoretical context.

Recently, holographic techniques have been used to study nonrelativistic systems, such as atomic gases at ultra-low temperature. This entails two types of gravitational backgrounds: those which correspond to Lifshitz-like fixed points [39] and Schrödinger-like fixed points [40,41]. In the context of condensed matter theory, various systems exhibit a dynamical scaling near fixed points:

$$t \longrightarrow \lambda^z t, \qquad x_i \longrightarrow \lambda x_i, \qquad z \neq 1. \tag{1.1}$$

In other words, rather than obeying the conformal scale invariance $t \rightarrow \lambda t$, $x_i \rightarrow \lambda x_i$, the temporal and the spatial coordinates scale anisotropically.

Requiring also time and space translation invariance, spatial rotational symmetry, spatial parity and time reversal invariance, the authors of Ref. [39] were led to consider D-dimensional geometries of the form

$$ds^2 = \ell^2\left(-r^{2z}dt^2 + r^2 dx_i dx^i + \frac{dr^2}{r^2}\right). \tag{1.2}$$

This metric obeys the scaling relation (1.1) if one also scales $r \rightarrow \lambda^{-1} r$. If $z = 1$, the metric reduces to the usual AdS metric in Poincaré coordinates with AdS radius ℓ. Metrics of the form (1.2) can be obtained as solutions in general relativity with a negative cosmological constant if appropriate matter is included. For example, solutions were found by introducing 1-form and 2-form gauge fields [39]; a massive vector field [42]; in an Abelian-Higgs model [43]; and with a charged perfect fluid [44]. A class of Lifshitz black hole solutions with nonplanar horizons was found in Refs. [45,46]. String theory and supergravity embeddings have been found in Refs. [47–55].

In a similar vein, D-dimensional geometries which exhibit Schrödinger symmetry are described by a metric of the form [40,41]

$$ds^2 = \ell^2\left(-r^{2z}dt^2 + r^2(-dtd\xi + dx_i dx^i) + \frac{dr^2}{r^2}\right), \tag{1.3}$$

which is conformally related to a pp-wave spacetime. This metric obeys the scaling relation

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow \lambda r, \quad \xi \rightarrow \lambda^{2-z}\xi, \quad z \neq 1. \tag{1.4}$$

If momentum along the ξ direction is interpreted as rest mass, then this metric describes a system which exhibits time and space translation invariance, spatial rotational symmetry, and invariance under the combined operations of time reversal and charge conjugation. These geometries have been embedded in string theory [56,57].

The organization of this paper is as follows: Sec. II contains a brief description of four-dimensional Einstein-Weyl gravity, including the equations of motion. In Sec. III, we summarize some salient features of the AdS_4 solution of Einstein-Weyl gravity and the nature of the linearized fluctuations around the AdS_4 background. We also discuss the Lifshitz solutions of Einstein-Weyl gravity, deriving the relation between the coupling strength α of the Weyl-squared term and the value of the Lifshitz anisotropy parameter z. We also find Schrödinger-type solutions. In Sec. IV, we consider a black hole type ansatz for spherically symmetric solutions of Einstein-Weyl gravity, and generalizations where the spatial sections are flat or hyperbolic instead of spherical. We show that the fourth-order equations of motion can be reduced to second-order equations for the metric functions. In the special case of flat spatial sections, we also derive a conserved Noether charge, which for the standard black hole solution is related to the mass.

In Sec. V, we consider spherically symmetric black holes in the specific case of pure conformal gravity. They can have either AdS or Lifshitz asymptotic behavior. The AdS black holes in conformal gravity have an additional parameter over and above the mass, and this leads to interesting consequences when one considers their thermodynamics. We discuss how one may generalize the first law of thermodynamics to include the additional parameter. We also consider the asymptotically Lifshitz black holes in conformal gravity, which can have either $z = 4$ or $z = 0$. In Sec. VI, we extend this discussion to Einstein-Weyl gravities. Now, it appears that the equations governing the metric functions for the spherically symmetric ansatz are too complicated to be solvable in general, and so we have to resort to numerical methods in order to go beyond the known Schwarzschild-AdS metrics. To do this, we first give a discussion of the forms of the solutions in the near-horizon and the asymptotic regions. Then upon performing numerical integrations outwards from the near-horizon region, we find indications that more general black hole solutions do indeed exist, at least when the α coupling parameter for the Weyl-squared term lies in an appropriate range. Section VII contains our conclusions. We present further solutions of conformal gravity and general extended gravities with quadratic curvature-squared terms in Appendices A and B, respectively. We summarize some results on the calculation of the mass for black holes in the critical theory and in conformal gravity in Appendices C and D, respectively.

II. EXTENDED AND CRITICAL GRAVITY

We begin by considering the action

$$I = \frac{1}{2\kappa^2}\int \sqrt{-g}d^4x(R - 2\Lambda + \alpha R^{\mu\nu}R_{\mu\nu} + \beta R^2 + \gamma E_{\rm GB}), \tag{2.1}$$

where $\kappa^2 = 8\pi G$ and $E_{\rm GB}$ is the Gauss–Bonnet term

$$E_{\rm GB} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}. \tag{2.2}$$

Although this term does not contribute to the equations of motion in four dimensions, it can have nontrivial consequences for thermodynamics in the higher-derivative theory, and so we shall include it in our discussion.

The equations of motion that follow from the action (2.1) are

$$\mathcal{G}_{\mu\nu} + E_{\mu\nu} = 0, \tag{2.3}$$

where

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{2.4}$$

$$\begin{aligned} E_{\mu\nu} = {} & 2\alpha\Big(R_{\mu\rho}R_\nu{}^\rho - \frac{1}{4}R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}\Big) + 2\beta R\Big(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\Big) \\ & + \alpha(\Box R_{\mu\nu} + \nabla_\rho\nabla_\sigma R^{\rho\sigma}g_{\mu\nu} - 2\nabla_\rho\nabla_{(\mu}R_{(\nu)}{}^\rho) \\ & + 2\beta(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R). \end{aligned} \tag{2.5}$$

When $\beta = -\frac{1}{3}\alpha$ and $\gamma = \frac{1}{2}\alpha$, the theory describes what we shall call Einstein-Weyl gravity, with the action

$$I = \frac{1}{2\kappa^2}\int \sqrt{-g}d^4x\Big(R - 2\Lambda + \frac{1}{2}\alpha|{\rm Weyl}|^2\Big), \tag{2.6}$$

where

$$|{\rm Weyl}|^2 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2. \tag{2.7}$$

Note that the equations of motion following from this action can be written as[1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha(2\nabla^\rho\nabla^\sigma + R^{\rho\sigma})C_{\mu\rho\sigma\nu} = 0. \tag{2.8}$$

We shall also sometimes consider the limit of conformal gravity, where only the Weyl-squared term in the action is retained. This can be described as the $\alpha \to \infty$ limit of the Einstein-Weyl action (2.6). The action of conformal gravity is conformally invariant, which implies that the equations of motion determine the metric only up to an arbitrary conformal factor.

A special feature of four-dimensional Einstein gravity with curvature-squared terms is that any solution of the pure Einstein theory continues to be a solution of the theory with the quadratic modifications. Thus, in particular, the Schwarzschild-AdS black hole solution

$$\begin{aligned} ds^2 = {} & -\Big(k - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)dt^2 \\ & + \Big(k - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\Big)^{-1}dr^2 + r^2 d\Omega_{2,k}^2 \end{aligned} \tag{2.9}$$

of Einstein gravity is also a solution in the higher-derivative theory. Here $d\Omega_{2,k}^2$ denotes the metric on a unit 2-sphere ($k = 1$), unit hyperbolic plane ($k = -1$) or 2-torus ($k = 0$), and may be written as

$$d\Omega_{2,k}^2 = \frac{dx^2}{1 - kx^2} + (1 - kx^2)dy^2. \tag{2.10}$$

III. ADS, LIFSHITZ AND SCHRÖDINGER VACUA

Unlike the $D = 3$ or $D \geq 5$ cases, for $D = 4$ the cosmological constant of the (A)dS vacuum is not modified by the quadratic curvature terms, and hence we have only one such vacuum with cosmological constant Λ. In this paper we shall consider only negative Λ, and furthermore, without loss of generality, we shall from now on set $\Lambda = -3$.

The linearized fluctuations around the AdS_4 vacuum in Einstein-Weyl gravity were analyzed in Ref. [10]. It turns out that the scalar trace mode decouples from the spectrum, which then contains just massless and massive spin-2 modes, satisfying

$$\alpha(\Box + 2)(\Box + 2 - m^2)h_{\mu\nu} = 0, \tag{3.1}$$

where $h_{\mu\nu}$ is transverse traceless and

$$m^2 = -2 - \frac{1}{\alpha}. \tag{3.2}$$

The characteristics of the linearized theory depend upon the value of the parameter α, and are summarized in Table I.

Owing to the fact that the background is AdS rather than Minkowski spacetime, there is an allowed range of negative mass-squared values for the massive mode, $-\frac{9}{4} \leq m^2 < 0$, for which it is still nontachyonic. In this range, the massive mode actually falls off less rapidly at infinity

[1]In deriving this result it is helpful to note that, in four dimensions, the Weyl tensor satisfies the identity $C_{\mu\rho\sigma\lambda}C_\nu{}^{\rho\sigma\lambda} = \frac{1}{4}C_{\rho\sigma\lambda\tau}C^{\rho\sigma\lambda\tau}g_{\mu\nu}$. This can be seen easily by employing 2-component spinor notation.

TABLE I. The characteristics of the massive and massless spin-2 modes in Einstein-Weyl gravity for finite values of the parameter α. When not indicated to the contrary, the modes are nonghostlike.

	$-\infty < \alpha < -\frac{1}{2}$	$\alpha = -\frac{1}{2}$	$-\frac{1}{2} < \alpha < 0$	$\alpha = 0$	$0 < \alpha < 4$	$4 \le \alpha < \infty$
	$-\frac{9}{4} \le m^2 < 0$	$m^2 = 0$	$m^2 > 0$	⋯	$m^2 < -\frac{9}{4}$	$-\frac{9}{4} \le m^2 < 0$
Massive		Ghost	Ghost	⋯	Ghost	Ghost
			Log	⋯	Tachyon	
	Truncated	Truncated	Nontruncated	⋯		Truncated
Massless	Ghost	Null				

than the massless mode, and so it can be truncated from the theory by imposing a suitable boundary condition at infinity. In the critical theory, which occurs when $\alpha = -\frac{1}{2}$, the massive mode becomes massless and in fact a new type of mode with logarithmic falloff arises. The usual massless mode has zero norm in the critical theory, and the logarithmic mode is ghostlike. The logarithmic mode could be truncated by imposing appropriate boundary conditions, but this would leave only the zero-norm massless graviton [14,15]. The case $\alpha = 0$ corresponds to ordinary Einstein gravity, in which case there is of course no massive mode. Not depicted in the table is the case $\alpha = \pm\infty$, which corresponds to the pure Weyl-squared conformal theory. In the conformal theory the massive mode has $m^2 = -2$, and so although negative, it is not tachyonic.

In addition to the AdS vacuum, the theory (2.1) also admits Lifshitz solutions, for which the metric is given by

$$ds^2 = \frac{dr^2}{\sigma r^2} - r^{2z} dt^2 + r^2(dx^2 + dy^2), \tag{3.3}$$

where

$$(z^2 + 2)\alpha + 2(z^2 + 2z + 3)\beta = \frac{1}{12}(z^2 + 2z + 3), \qquad \sigma = \frac{6}{z^2 + 2z + 3}. \tag{3.4}$$

For the special case of Einstein-Weyl gravity, where $\beta = -\alpha/3$, we have

$$\alpha = \frac{z^2 + 2z + 3}{4z(z - 4)}. \tag{3.5}$$

For conformal gravity, corresponding to $\alpha = \infty$, Eq. (3.5) implies that the Lifshitz scaling parameter z can take the values $z = 4$ or $z = 0$. At the critical point, on the other hand, where $\alpha = -\frac{1}{2}$, both roots of (3.5) give $z = 1$. For general values of α we have

$$z = \frac{8\alpha + 1 \pm \sqrt{2(1 + 2\alpha)(16\alpha - 1)}}{4\alpha - 1}. \tag{3.6}$$

Thus, the reality of z requires that $\alpha \ge \frac{1}{16}$ or $\alpha \le \frac{1}{2}$.

For conformal gravity, we find that there are also Lifshitz-like solutions with S^2 or H^2 spatial topologies as well as T^2. The metrics for all three cases can be written as

$$ds^2 = -r^{2z}\left(1 + \frac{k}{r^2}\right)dt^2 + \frac{4dr^2}{r^2(1 + \frac{k}{r^2})} + r^2 d\Omega_{2,k}^2, \tag{3.7}$$

with $z = 0$ and 4.

Finally, we consider Schrödinger vacua, whose metric takes the form

$$ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2(-2dtdx + dy^2). \tag{3.8}$$

For $z = 1$ and $z = -\frac{1}{2}$, the metrics are Einstein with $\Lambda = -3$, and hence they are solutions for all α and β. In particular, the $z = 1$ case is simply the AdS metric, while if $z = -\frac{1}{2}$ it is the Kaigorodov metric describing a pp-wave propagating in AdS [58,59]. In general, z satisfies Ref. [12]

$$1 - 24\beta + \alpha(4z^2 - 2z - 8) = 0. \tag{3.9}$$

In Einstein-Weyl gravity, we have

$$\alpha = \frac{1}{2z(1 - 2z)}. \tag{3.10}$$

In conformal gravity, z can take values 1, $\frac{1}{2}$, 0 or $-\frac{1}{2}$. Some asymptotic Schrödinger solutions are presented in Appendix A 1.

IV. BLACK HOLE ANSATZ AND EQUATIONS OF MOTION

A. General equations for $k = 1, 0$ or -1

In this paper, we focus on the construction of static, spherically symmetric (or H^2 or T^2 symmetric) black hole solutions that are asymptotic to either the AdS or the Lifshitz vacua discussed in the previous section. We may therefore, without loss of generality, consider the ansatz

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k}^2. \tag{4.1}$$

The equations of motion for a and f may be derived from the Lagrangian obtained by substituting this ansatz into the action (2.1). Since we are interested specifically in the case of Einstein-Weyl gravity, where $\beta = -\frac{1}{3}\alpha$, and since the equations of motion are greatly simplified in this case,[2] we shall present the results under this specialization.

We then find that the equations can be reduced to the second-order system

$$a'' = \frac{r^2 f a'^2 + 4a^2(k + 6r^2 - f - rf') - raa'(4f + rf')}{2r^2 af}, \tag{4.2}$$

and

$$\begin{aligned} f'' = &\frac{1}{2r^2a^2f(ra' - 2a)}\Big(\frac{4r^2a^2}{\alpha}(a(k + 3r^2 - f) - rfa') \\ &+ r^3f^2a'^3 + 2r^2a^2fa'(8r - f') - r^2afa'^2(3f + rf') \\ &- a^3(48r^4 - 16r^2f + 8f^2 - 24r^3f' \\ &+ 4rff' + 3r^2f'^2) - 4ka^3(4r^2 - 2f - rf')\Big). \end{aligned} \tag{4.3}$$

B. Conserved Noether charge for $k = 0$ case

In the case of a toroidal spatial geometry (i.e., $k = 0$), the system of equations has an additional global symmetry, and hence there is an associated conserved Noether charge. In order to discuss this, it is helpful temporarily to choose a different parameterization of the metric ansatz (4.1), using a new radial coordinate ρ such that $r^2 = b(\rho)$ and $dr^2/f = ab^2hd\rho^2$, so that the metric is now written as

$$ds^2 = -a(\rho)dt^2 + a(\rho)b(\rho)^2h(\rho)d\rho^2 + b(\rho)dx^idx^i. \tag{4.4}$$

Since the additional global symmetry arises regardless of whether or not we choose the Weyl-squared combination $\beta = -\frac{1}{3}\alpha$, we shall keep these two parameters arbitrary in the following discussion. Substituting into the action (2.1) yields an effective Lagrangian for a, b and h. The function $h(\rho)$ can be viewed as parameterizing general coordinate transformations of the radial variable, and its equation of motion yields the Hamiltonian constraint $H = 0$. Having obtained this equation, we can impose $h(\rho) = 1$ as a coordinate gauge condition. In this case H, which must vanish, is given by

$$\begin{aligned} H = &\frac{a'b'}{ab} + \frac{b'^2}{2b^2} - 6ab^2 - 2kab - \frac{\alpha}{4a^5b^6}(10a^4a'^4 + 20ab^3a'^3b' + 22a^2b^2a'^2b'^2 + 36a^3ba'b'^3 + 47a^4b'^4) \\ &+ \frac{\alpha}{a^4b^5}(b(ba' + ab')(3ba' + 2ab)a'' + a(ba' + 3ab')(ba' + 4ab')b'') + \frac{\alpha}{a^3b^4}(b^2a''^2 + 2aba''b'' + 3a^2b''^2) \\ &- \frac{2\alpha k(kab^3 + b'^2)}{b^3} \frac{\beta}{4a^5b^6}(20b^4a'^4 + 64ab^3a'^3b' + 72a^2b^2a'^2b'^2 + 124a^3ba'b'^3 + 125a^4b'^4) \\ &+ \frac{2\beta}{a^4b^5}(3b(b^2a'^2 + 3aba'b' + a^2b'^2)a'' + a(b^2a'^2 + 13aba'b' + 16a^2b'^2)b'') + \frac{\beta(ba'' + 2ab'')^2}{a^3b^4} \\ &- \frac{2\beta(ba' + 2ab')(ba''' + 2ab''')}{a^3b^4} - \frac{2\beta k(2kab^3 + 3b'^2)}{b^3}, \end{aligned} \tag{4.5}$$

where a prime here denotes a derivative with respect to ρ. We may then also set $h = 1$ in the effective Lagrangian, so that the remaining equations, for a and b, can be obtained from

$$\begin{aligned} \mathcal{L} = &\frac{a'b'}{ab} + \frac{b'^2}{2b^2} + 6ab^2 + 2kab + \frac{\alpha}{4a^5b^6}(2b^4a'^4 + 5ab^3a'^3b' + 10a^2b^2a'^2b'^2 + 12ab^3a'b'^3 \\ &+ 11a^4b'4 - 2ab(2b^3a'^2a'' + 2ab^2a'b'a'' + 3a^2bb'^2a'' + 2ab^2a'^2b'' + 4a^2ba'b'b'' + 8a^3b'^2b'') \\ &+ 2a^2b^2(b^2a''^2 + 2aba''b'' + 3a2b''^2)) + \frac{2k\alpha}{b^3}(kab^3 + b'^2 - bb'') \\ &+ \frac{\beta}{4a^5b^6}(4ka^3b^3 + 2b^2a'^2 + 2aba'b' + 5a^2b'^2 - 2ab(ba'' + 2ab''))^2. \end{aligned} \tag{4.6}$$

For the $k = 0$ case, corresponding to a black brane solution, the Lagrangian (4.6) and Hamiltonian (4.5) have a global scaling symmetry with

$$a \to \xi^2 a, \qquad b \to \xi^{-1} b. \tag{4.7}$$

This enables us to derive a conserved Noether charge, λ. Having done this, it is more convenient now to revert to the original radial coordinate r and the metric functions a and f in (4.1). The Noether charge is then given by

[2]The reason for the simplification is that the trace of the Weyl-squared contribution to the equations of motion vanishes [see Eq. (2.8)], and so this projection is identical in Einstein gravity and Einstein-Weyl gravity.

$$\lambda = \frac{\sqrt{f}}{ra^{5/2}}[2r^2a^2(ra' - 2a) - \alpha(8a^3f - 2ra^2fa' - 4r^2afa'^2 + 3r^3fa'^3 + 6ra^3f' - 3r^3aa'^2f' + 3r^2a(2af - 2rfa' + raf')a'' + r^2a^2(ra' - 2a) + 2r^3a^2ff''') + 2\beta(ra' - 2a)(4a^2f + 4rafa' - r^2fa'^2 + 4ra^2f' + r^2aa'f' + 2r^2afa'')]. \tag{4.8}$$

It should be emphasized that this quantity is conserved only for the case $k = 0$. The analogous Noether charge was studied in Ref. [60] for Lifshitz black holes (with T^2 horizon topology) in Einstein gravity coupled to a massive vector field. It was shown [61] that it is related to the energy of the black branes:

$$E = -\frac{\lambda\omega_2}{16\pi(z+2)} = \frac{2}{(z+2)}TS, \tag{4.9}$$

where T and S are the temperature and the entropy of the black brane, and ω_2 is its area.

We can test this formula with the $k = 0$ Schwarzschild-AdS black holes, corresponding to $a = f = r^2 - r_+^3/r$ in the metric ansatz (4.1). This solution exists for all values of the α and β parameters. The temperature and the entropy are given by

$$T = \frac{3r_+}{4\pi}, \qquad S = \frac{1}{4}\omega_2 r_+^2[1 - 6(\alpha + 4\beta)]. \tag{4.10}$$

Note that the Gauss–Bonnet term does not contribute to the entropy in this case. The energy is given by

$$E = \frac{1}{8\pi}[1 - 6(\alpha + 4\beta)]r_+^3 = \frac{2}{3}TS. \tag{4.11}$$

The Noether charge λ in this case is given by

$$\lambda = -6[1 - 6(\alpha + 4\beta)]r_+^3. \tag{4.12}$$

Thus, we find that the relation (4.9) holds for this $z = 1$ case. In general we find that the second equality in (4.9) always holds, while the definition of energy in terms of the Noether charge does not apply for solutions of higher-derivative gravity when massive spin-2 modes are excited.

For the case of Einstein-Weyl gravity, i.e., when $\beta = -\frac{1}{3}\alpha$, tthe Noether charge (4.8) simplifies considerably, and becomes

$$\lambda = \frac{1}{\sqrt{a^3f}(ra' - 2a)}(2ra(18ra^2 - 10a^2f - 2rafa' - r^2fa'^2) - \alpha(4ra - fa' - af')(36r^2a^2 - 8a^2f - rafa' - 2r^2fa'^2 - 9ra^2f')). \tag{4.13}$$

V. ADS AND LIFSHITZ BLACK HOLES IN CONFORMAL GRAVITY

In this section, we focus on the special case of conformal gravity, i.e., the limiting case of Einstein-Weyl gravity when α goes to infinity. The equations of motion are given by

$$(2\nabla^\rho\nabla^\sigma + R^{\rho\sigma})C_{\mu\rho\sigma\nu} = 0. \tag{5.1}$$

Note that for the metric ansatz (4.1), the α-independent trace equation (4.2) does not apply in conformal gravity. Thus, the equation of motion is not simply (4.3) with α sent to ∞.

A. AdS black holes

The most general spherically symmetric solution in conformal gravity was found in Refs. [62–64]. (See also, Refs. [31,65].) The solution can easily be generalized to the other horizon topologies T^2 and H^2. The solution for all three cases is given by

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2 d\Omega_{2,k}^2, \qquad f = br^2 + \frac{c^2 - k^2}{3d}r + c + \frac{d}{r}, \tag{5.2}$$

where b, c and d are arbitrary constants. The coefficients of r^2 and $1/r$ are related to the excitations of the massless graviton, while the coefficient of r and the constant c are related to the massive spin-2 mode. If c is chosen so that $c = k$, the solution reduces to the usual AdS black hole for each of the cases $k = 1$, $k = -1$ and $k = 0$. Of course, since the equations of motion for conformal gravity leave an overall conformal factor undetermined, it follows that $d\tilde{s}^2 = \Omega^2 ds^2$ is also a spherically symmetric static solution, where ds^2 is given by (5.2) and Ω is an arbitrary function of r.

In fact, the solution (5.2) can easily be derived by starting from the Schwarzschild-AdS solution

$$d\tilde{s}^2 = -\left(k - \frac{1}{3}\Lambda\rho^2 - \frac{2M}{\rho}\right)dt^2 + \left(k - \frac{1}{3}\Lambda\rho^2 - \frac{2M}{\rho}\right)^{-1}d\rho^2 + \rho^2 d\Omega_{2,k}^2, \tag{5.3}$$

noting that not only this, but also $ds^2 = \Omega(\rho)^{-2}d\tilde{s}^2$, is therefore a solution of conformal gravity, and then defining a new radial coordinate via $r = \rho\Omega(\rho)^{-1}$. Requiring that the resulting metric have the form $ds^2 = -hdt^2 + h^{-1}dr^2 + r^2 d\Omega_{2,k}^2$ implies that $\Omega = 1 + q\rho$ where q is an arbitrary constant, and hence $r = \rho/(1 + q\rho)$. The function h is therefore given by

$$h = \left(2Mq^3 + kq^2 - \frac{1}{3}\Lambda\right)r^2 - 2q(k + 3Mq)r + (k + 6Mq) - \frac{2M}{r}, \tag{5.4}$$

which precisely reproduces the function f in (5.2) with

$$b = 2Mq^3 + kq^2 - \frac{1}{3}\Lambda, \qquad c = k + 6Mq, \qquad d = -2M. \tag{5.5}$$

The fact that the solution (5.2) is related to the usual Schwarzschild-AdS black hole does not imply that these solutions are completely equivalent. The scaling of the Schwarzschild-AdS metric leaves the thermodynamic properties of the black hole unchanged only if the conformal factor is finite and nonvanishing in the regions between the horizon and asymptotic infinity. However, the conformal factor $\Omega = 1 + q\rho$ that relates (5.2) to the usual Schwarzschild-AdS black hole metric is in fact singular at $\rho = \infty$, and so it alters the global structure. In turn, this affects the thermodynamic properties, as we shall discuss below.

1. Thermodynamics of AdS black holes in conformal gravity

We begin by reviewing the thermodynamic properties of the standard Schwarzschild-AdS black hole (5.3) in the context of conformal gravity. Letting ρ_+ denote the radius of the outer horizon, we can solve for M to get

$$M = \frac{1}{6}\rho_+(3k - \Lambda\rho_+^2). \tag{5.6}$$

The Hawking temperature can be obtained from a calculation of the surface gravity in the standard way. The entropy can be derived from the Wald formula [66], giving

$$S = -\frac{\alpha}{8}\int C^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}d\Sigma. \tag{5.7}$$

Thus we have

$$T = \frac{k - \Lambda\rho_+^2}{4\pi\rho_+}, \qquad S = \frac{1}{6}\alpha(3k - \Lambda\rho_+^2)\omega_2, \tag{5.8}$$

where ω_2 denotes the volume of $d\Omega_{2,k}^2$.

It is worth remarking that at first sight the entropy of the black hole in conformal gravity is not simply proportional to the area of the horizon, but now it is given by

$$S = \frac{1}{2}\alpha\left(k\omega_2 + \frac{1}{3}(-\Lambda)A\right), \tag{5.9}$$

where $A = \rho_+^2\omega_2^2$ is the area of the horizon. However the first term in the above is a pure constant, independent of the parameters in the solution, and can be removed by introducing a Gauss–Bonnet term in the Lagrangian. In fact if we use the action (2.1) with $\beta = -\frac{1}{3}\alpha$ and $\gamma = 0$, the first term in (5.9) vanishes and hence the entropy is then proportional to the area of the horizon.

The free energy F can be obtained from the Euclidean action I_E of conformal gravity, using the relation $F = I_E T$. The action converges for the black hole solution, leading to

$$F = -\frac{\alpha\omega_2}{32\pi}\int_{r_+}^{\infty} r^2 dr|\mathrm{Weyl}|^2 = -\frac{\alpha(3k - \Lambda\rho_+^2)^2\omega_2}{72\pi\rho_+}. \tag{5.10}$$

The energy can be determined by integrating the first law, $dE = TdS$, assuming that Λ is held fixed, giving

$$E = \frac{\alpha\Lambda\rho_+(-3k + \Lambda\rho_+^2)\omega_2}{36\pi} = \frac{\alpha(-\Lambda)\omega_2}{6\pi}M. \tag{5.11}$$

This expression for the energy can also be confirmed independently by using either the Deser-Tekin [67] or the Ashtekar-Magnon-Das (AMD) method [68–71].

In conformal gravity the cosmological constant Λ is a parameter of the solution, rather than of the theory, and hence we may treat Λ as a further thermodynamic variable, leading to the more general thermodynamic relations

$$dE = TdS + \Theta d\Lambda, \qquad F = E - TS, \qquad \Theta = -\frac{\alpha\rho_+(3k - \Lambda\rho_+^2)\omega_2}{72\pi}. \tag{5.12}$$

Thus, treating the cosmological constant as a thermodynamic variable does not affect the relationship between F and E. We can simply start by assuming that Λ is constant and obtain the first law of thermodynamics. The first law can then be straightforwardly modified by treating Λ as a variable, thus determining the corresponding conjugate variable Θ, while the other thermodynamic quantities remain unchanged. Treating the cosmological constant as a thermodynamic variable has been considered previously. See, for example, Refs. [72–74]. In Einstein gravity, where the entropy is simply one quarter of the horizon area, without explicit dependence on the cosmological constant Λ, the quantity $\Theta \sim \rho_+^3$ is proportional to the volume conjugate to the cosmological constant, which can then be interpreted as a pressure [74]. In conformal gravity, on the other hand, the entropy has a manifest dependence on Λ, and hence the quantity Θ given in (5.12) is not simply proportional to the volume, but has a linear ρ_+ dependence as well, for nonvanishing k. It is also worth remarking that the Smarr formula $E = 2TS - 2\Theta\Lambda$ in Einstein gravity [74] now becomes $E = 2\Theta\Lambda$ in conformal gravity. We shall discuss this further in Sec. VA 5.

We are now in a position to discuss the more general AdS black holes in conformal gravity. We shall begin by taking the cosmological constant to be fixed,[3] by setting $b = 1$ in (5.2), corresponding to setting the cosmological constant of the asymptotic AdS space to be $\Lambda = -3$. Letting r_+ be the radius of the outer horizon and writing $d = -r_+\tilde{d}$, we find that

[3]When we refer to the "cosmological constant" in the context of conformal gravity, where of course there is no cosmological term in the action; we always mean the cosmological constant of the asymptotic AdS space.

$$r_+^2 = -c + \frac{c^2 - k^2}{3\tilde{d}} + \tilde{d} > 0. \tag{5.13}$$

The temperature and entropy can then be straightforwardly calculated; they are given by

$$T = \frac{(3\tilde{d} - c)^2 - k^2}{12\pi r_+ \tilde{d}}, \qquad S = \frac{1}{6}\alpha(k + 3\tilde{d} - c)\omega_2. \tag{5.14}$$

The free energy F can also be obtained from the Euclidean action, giving

$$F = -\frac{\alpha\omega_2((c-k)^2 - 3(c-k)\tilde{d} + 3\tilde{d}^2)}{24\pi r_+}. \tag{5.15}$$

When $c = k$, the system reduces to the previous Schwarzschild-AdS black hole with cosmological constant fixed at $\Lambda = -3$. Thus, the general solution with $c \neq k$ contains an additional independent parameter.

As we have remarked in the previous subsection, the general solution can be obtained by performing a conformal transformation of the Schwarzschild-AdS black hole, whose cosmological constant Λ can be promoted to being a parameter of the solution. In the new solution, we have chosen to set $b = 1$. Thus as a local solution, our new variables (c, d) are related to the (M, Λ) variables in the Schwarzschild-AdS solution (5.3). It is natural to ask whether the thermodynamics of the new solution are simply the same as (5.12), but expressed in terms of new variables. In order to address this issue, we note that the transformation described in Sec. VA amounts to

$$q = \frac{c-k}{6M}, \qquad M = \frac{1}{6}\rho_+(3k - \Lambda\rho_+^2), \tag{5.16}$$

with

$$\rho_+ = \frac{3r_+\tilde{d}}{k - c + 3\tilde{d}}, \qquad \Lambda = \frac{(2k+c)r_+ - 3k\rho_+}{(r_+ - \rho_+)\rho_+^2} = \frac{(c - k - 3\tilde{d})^2(c + 2k - 3\tilde{d})}{9r_+^2\tilde{d}^2}. \tag{5.17}$$

It is easy to see that when $c = k$, we have $\rho_+ = r_+$ and $\Lambda = -3$, as we should have expected.

It is straightforward to verify that the temperature and entropy in (5.8) are indeed mapped into those in (5.14). However, the free energy in (5.10) becomes

$$F \to \tilde{F} = -\frac{\alpha(3\tilde{d} - c + k)^3\omega_2}{216\pi r_+\tilde{d}}, \tag{5.18}$$

which is different from the free energy given in (5.15). The reason for this can be easily understood as follows. The r and ρ coordinates are related to each other by

$$r = \frac{\rho}{1 + q\rho}, \qquad \rho = \frac{r}{1 - qr}. \tag{5.19}$$

The temperature and entropy are, in a sense, "local" properties, evaluated on the horizon $r = r_+$ or $\rho = \rho_+$, and related by the above equation. Since the theory is conformal, the temperature and entropy are not modified by the conformal transformation. On the other hand, the free energy, and hence the energy, are evaluated by an integration over the regions $[r_+, \infty)$ of the general black hole or $[\rho_+, \infty)$ of the Schwarzschild-AdS black hole. From the relationship (5.19), we find

$$r_+ \le r < \infty \to \left(-\infty < \rho \le -\frac{1}{q}\right) \cup (\rho_+ \le \rho < \infty), \qquad \rho_+ \le \rho < \infty \to r_+ \le r < \frac{1}{q}. \tag{5.20}$$

Thus, we see that the outer region $\rho \ge \rho_+$ of the Schwarzschild-AdS black hole covers only part of the outer region $r \ge r_+$ of the general black hole (5.15). The exterior of the general black hole maps into disconnected regions of the Schwarzschild-AdS solution. Thus, we see that although the conformal transformation does not affect the location of the horizon or the expression for $\sqrt{-g}|\mathrm{Weyl}|^2$, the structure of the asymptotic region is altered by the transformation. Therefore, the Euclidean actions are different for the two solutions. Analogously, the energy of the two solutions, which are typically evaluated at asymptotic infinity, are also different. The upshot is that the two solutions cannot be viewed as equivalent.

Having established that the new solutions are globally inequivalent to the Schwarzschild-AdS black hole with (M, Λ) parameters, we shall now proceed to investigate the thermodynamics of the general black holes in conformal gravity. We should not expect the usual first law $dE = TdS$ still to be satisfied, since the general solutions are now described by two independent parameters, c and d, rather than just one. As we shall see, it is necessary now to introduce an additional pair of intensive and extensive thermodynamic variables, which we shall call Ψ and Ξ, and the first law will be modified to $dE = TdS + \Psi d\Xi$. Once the additional parameter of the AdS black holes in conformal gravity is turned on, by taking $c \neq k$, we find that neither the Deser-Tekin nor the AMD methods gives a finite result for the mass. In Appendix D, we describe a new procedure for calculating the mass in conformal gravity.

It is instructive first to look at the solution where the parameter d is set to zero. In the parameterization in (5.2), this can be done by first writing $c^2 - k^2 = 3\Xi d$, and then sending d to zero, giving

$$f = r^2 + \Xi r + k. \tag{5.21}$$

This solution has a curvature singularity at $r = 0$, which can be shielded by an horizon at $r = r_0$ provided that Ξ is chosen so that $\Xi^2 \ge 4k$. The temperature is given by

$$T_0 = \frac{r_0^2 - k}{4\pi r_0}. \tag{5.22}$$

However, we find that the entropy and free energy both vanish, suggesting that the energy should vanish also. Thus the solution can be viewed as a "thermalized vacuum." (This is analogous to the Schwarzschild black hole in critical gravity, where all thermodynamic quantities except for temperature vanish [10]). In a Deser-Tekin or AMD calculation, this thermalized vacuum will generate a divergence in the evaluation of the mass, and it should be subtracted.

In fact it is easy to verify that this thermalized vacuum is locally conformal to a de Sitter background. To see this, we define $d\hat{s}^2 = \Omega^2 ds^2$, with

$$\Omega = \frac{1}{\Xi r + 2k}, \tag{5.23}$$

and introduce the new radial coordinate $\rho = r\Omega$. We then have

$$d\hat{s}^2 = -\frac{\hat{f}}{4k}dt^2 + \frac{d\rho^2}{k\hat{f}} + \rho^2 d\Omega_{2,k}^2, \tag{5.24}$$

where

$$\hat{f} = 1 - \frac{1}{3}\Lambda\rho^2, \qquad \Lambda = 3(\Xi^2 - 4k). \tag{5.25}$$

The condition for the solution (5.21) to have real roots defining the horizons is $\Xi^2 - 4k \geq 0$, and so this means the conformally related metric $d\hat{s}^2$ in (5.24) is de Sitter spacetime, with positive cosmological constant. The horizon in the metric with f given by (5.21) maps into the de Sitter horizon in (5.24).

From Appendix D, if we take the conserved quantity Q to furnish a definition of energy, we have

$$E = \frac{\alpha\omega_2}{4\pi}(-d + m), \tag{5.26}$$

where

$$m \equiv \frac{(c-k)(c^2-k^2)}{18d}. \tag{5.27}$$

Note that when $c = k$, it reproduces the energy for the Schwarzschild-AdS black hole. When $d = 0$, it is necessary that $c \to k$ with $\Xi = (c^2 - k^2)/(3d)$ held fixed. In this limit, the quantity m vanishes, and hence we see that the thermalized vacuum indeed has zero energy.

It turns out that with this definition of energy for the general AdS black holes in conformal gravity we have

$$F = E - TS. \tag{5.28}$$

We find that, as mentioned earlier, the standard first law $dE = TdS$ is not satisfied for the general AdS black holes, since the solutions are characterized by two independent parameters. If we first consider the situation where the quantity $\Xi = (c^2 - k^2)/(3d)$ is held fixed, then the first law $dE = TdS$ does hold. This corresponds to allowing only variations that keep the thermalized vacuum fixed. If instead we allow general variations of the two parameters in the solution, by allowing Ξ to vary also, then we find that we should add another term in the first law, which now becomes

$$dE = TdS + \Psi d\Xi, \qquad \Psi = \frac{\alpha\omega_2(c-k)}{24\pi}. \tag{5.29}$$

The quantity Ψ here is a new thermodynamic variable conjugate to Ξ, which is determined from the requirement of integrability of the generalized first law.

2. The Noether charge of the $k = 0$ solution

As discussed in Sec. IV B, for $k = 0$, the system has an additional conserved Noether charge. For conformal gravity, the Noether charge for the ansatz (4.1) is given by

$$\begin{aligned}\lambda = {}& \frac{\alpha}{12ra^2\sqrt{af}(ra' - 2a)}(4a^2f - 10rafa' + 7r^2fa'^2 \\ & + 6ra^2f' - 3r^2aa'f' - 6r^2afa'')(4a^2f - 2afra' \\ & - fr^2a'^2 - 2a^2rf' + ar^2a'f' + 2afr^2a'').\end{aligned} \tag{5.30}$$

Thus for the black hole (5.2) with $k = 0$, we have

$$\lambda = \frac{4\alpha(27d^2 - c^3)}{9d}. \tag{5.31}$$

For the Schwarzschild-AdS black hole (5.3), we have

$$\lambda = 8\Lambda\alpha M. \tag{5.32}$$

Note that in both cases we have $\lambda\omega_2 = -32\pi TS$. In other words, the second equality of (4.9) always holds. Indeed, these two Noether charges for the general and the Schwarzschild-AdS solutions can map to each other by the conformal transformation discussed in the previous subsection. Let us define $\tilde{E}$ as

$$\tilde{E} = -\frac{\lambda\omega_2}{48\pi}. \tag{5.33}$$

For the Schwarzschild-AdS black brane, $\tilde{E}$ is precisely the mass of the solution. It follows from the argument presented in the previous subsection that $\tilde{E}$ cannot be the energy of the more general solution that has an additional parameter c. Thus now we have two conserved quantities; one is the true energy E given in (5.26) and the other is $\tilde{E}$. The difference is

$$E - \tilde{E} = \frac{c^3}{216\pi d} = \frac{m}{12\pi}, \tag{5.34}$$

where m is given in (5.27).

3. Conformal boundary term

It is possible to write a conformally invariant boundary term in four dimensions. Thus for completeness, this boundary term should be included in conformal gravity. The conformal boundary term is given by

$$I_c = \eta\alpha \int d^3x\sqrt{-\tilde{g}}C^{\mu\nu\rho\sigma}n_\mu n_\rho \nabla_\nu n_\sigma, \tag{5.35}$$

where n_μ is the unit outward normal to the boundary, and η is an arbitrary pure numerical constant. This boundary term does not contribute to the equations of motion, and so it has no effect on the local solutions, but it can contribute to the thermodynamics. For example, it yields a nontrivial contribution to the Euclidean action, implying that the free energy is now modified, and is given by

$$F = -\frac{\alpha\omega_2[(c-k)^2 - 3(c-k)\tilde{d} + 3\tilde{d}^2]}{24\pi r_+} - \frac{\eta\alpha\omega_2 m}{16\pi}, \tag{5.36}$$

where m is given by (5.27). It is of interest to note that the contribution of the conformal boundary term to the free energy is of the same form as the m term appearing in the expression (5.26) for the energy in conformal gravity without the boundary term.

4. Extremal limit

Since the general AdS black hole (5.2) has the parameter $c-k$ in addition to the usual d parameter of the Schwarzschild-AdS black hole, it is possible to find an extremal limit for which the temperature vanishes and the near-horizon geometry has an AdS_2 factor. For both $k=\pm 1$, the extremal solution takes the same form, given by

$$f = \frac{(r-r_+)^2(rr_+ - r_+^2 + 1)}{rr_+}. \tag{5.37}$$

For $k=1$, the near-horizon geometry is $AdS_2 \times S^2$, with vanishing temperature and entropy. The energy, free energy and Ψ are given by

$$E = F = -\frac{\alpha(r_+^2 - 1)^2\omega_2}{8\pi r_+}, \qquad \Psi = \frac{\alpha\omega_2(r_+^2-1)}{8\pi}, \tag{5.38}$$

which all vanish for $r_+ = 1$. Thus the $r_+ = 1$ solution may also be stable vacuum of the theory. For $k=0$, it turns out that there is no extremal limit, since $f(r)$ has either a single root or a triple root.

5. Thermodynamics with varying Λ

Finally we consider the thermodynamics of the general AdS black holes in conformal gravity when Λ, the cosmological constant of the asymptotically AdS region, is treated as a thermodynamic variable also. This is natural in conformal gravity since the cosmological constant arises as a parameter of the solution rather than as a fixed parameter of the theory. The solution is given by

$$f = -\frac{1}{3}\Lambda r^2 + \Xi r + c + \frac{d}{r}, \quad \text{with} \quad 3\Xi d = c^2 - k^2. \tag{5.39}$$

Letting r_+ be the radius of the outer horizon, and defining $d = -r_+\tilde{d}$, we have

$$\begin{aligned}
T &= \frac{(3\tilde{d} - c)^2 - k^2}{12\pi r_+ \tilde{d}}, \\
S &= \frac{1}{6}\alpha\omega_2(k + 3\tilde{d} - c), \\
\Psi &= \frac{\alpha\omega_2(c-k)}{24\pi}, \\
\Theta &= \frac{\alpha\omega_2 d}{24\pi}, \\
F &= -\frac{\alpha\omega_2((c-k)^2 - 3(c-k)\tilde{d} + 3\tilde{d}^2)}{24\pi r_+}, \\
E &= 2\Theta\Lambda + \Psi\Xi.
\end{aligned} \tag{5.40}$$

These thermodynamic quantities satisfy the relations

$$dE = TdS + \Theta d\Lambda + \Psi d\Xi, \qquad F = E - TS. \tag{5.41}$$

Note that the last equation in (5.40) is the Smarr formula for the general black holes in conformal gravity. Its rather unusual form can be understood by considering the following scaling argument. Since the parameter α has dimensions of length-squared, L^2, and it is treated as a fixed parameter of the theory (which may be set, without loss of generality, to $\alpha = 1$), it follows that the effective scaling dimensions for the thermodynamic quantities are given by

$$\begin{aligned}
&[E] = \frac{1}{L}, \quad [T] = \frac{1}{L}, \quad [S] = 1, \quad [\Theta] = L, \\
&[\Lambda] = \frac{1}{L^2}, \quad [\Psi] = 1, \quad [\Xi] = \frac{1}{L}.
\end{aligned} \tag{5.42}$$

Thus if E is viewed as a function of S, Λ and Ξ, namely $E = h(S, \Lambda, \Xi)$, then under scaling we shall have $E = \mu h(S, \mu^{-2}\Lambda, \mu^{-1}\Xi)$. Differentiating with respect to μ, setting $\mu = 1$, and using the first law in (5.41) then gives the Smarr relation $E = 2\Theta\Lambda + \Psi\Xi$ we found in (5.40).[4]

The entropy of the general black hole can be decomposed as

$$S = \frac{1}{2}\alpha w_2 k + \frac{1}{6}\alpha(-\Lambda)A + 8\pi\Psi + \frac{1}{2}\alpha\omega_2\Xi r_+, \tag{5.43}$$

where $A = r_+^2\omega_2$ is the area of the horizon, and the first pure numerical term is the contribution from the Gauss–Bonnet term in the action (2.1) with $\beta = -\frac{1}{3}\alpha$ and $\gamma = \frac{1}{2}\alpha$.

[4] A Smarr formula with more conventional coefficients would arise if we were to view the coupling constant α as another thermodynamic variable, so that the thermodynamic quantities would all have their standard "engineering" scaling dimensions. We would then have a generalized first law $dE = TdS + \Theta d\Lambda + \Psi d\Xi + \sigma d\alpha$, where σ is a new thermodynamic variable conjugate to α. The Smarr formula would then be $E = 2TS - 2\Theta\Lambda - \Psi\Xi + 2\sigma\alpha$. However, since α is an overall parameter in conformal gravity, including it as an additional variable represents an overparametrization of the system. This is reflected in the fact that there is then a 1-parameter family of possible Smarr relations, with $E = \lambda TS + 2(1-\lambda)\Theta\Lambda + (1-\lambda)\Psi\Xi + \lambda\sigma\alpha$, where λ is an arbitrary constant.

We see from the constraint (5.39) on the parameters that in the limit $c \longrightarrow k$, we can either set $d = 0$ with Ξ fixed, or set $\Xi = 0$ with d fixed. The former leads to the thermalized vacuum (5.21) and the latter leads to the Schwarzschild-AdS black hole.

B. $z = 4$ Lifshitz black holes

We find that conformal gravity admits static asymptotically Lifshitz black hole solutions also, both for $z = 4$ and $z = 0$. We shall first discuss the case with $z = 4$. The solution is given by

$$ds^2 = -r^8 f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega_{2,k}^2, \qquad f = 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4} + \frac{d}{r^6}. \tag{5.44}$$

This solution for Lifshitz black holes is locally equivalent to the AdS black hole solution (5.2) up to an overall conformal factor. Specifically, it can be seen that the metric $d\hat{s}^2 = \Omega^2 ds^2$ with

$$\Omega = \frac{q}{r(c + 3r^2 - k)}, \tag{5.45}$$

becomes, after transforming to the new radial coordinate $\rho = r\Omega$,

$$d\hat{s}^2 = -\frac{1}{9} q^2 \hat{f} dt^2 + \hat{f}^{-1} d\rho^2 + \rho^2 d\Omega_{2,k}^2, \tag{5.46}$$

where

$$\hat{f} = k + \frac{q}{3\rho} - \frac{1}{3}\Lambda\rho^2, \qquad \Lambda = \frac{(c^3 - 27d - 3ck^2 + 2k^3)}{q^2}. \tag{5.47}$$

The conformal factor (5.45) is nonsingular on the horizon $r = r_+$ of the Lifshitz black hole (except, as we shall see below, in the case of the $k = 1$ extremal limit), and the horizon is mapped to that of the conformally related (A) dS black hole (5.46). However, since the conformal factor becomes singular at $r = \infty$, the asymptotic regions, and hence the global structure, are very different for the two metrics.

The equation $f(r) = 0$ determines the locations of the horizons. This yields a cubic equation for r^2, which will have either three real roots or one real root, according to whether the discriminant

$$\Delta = -\frac{1}{27}(c^3 - 27d - 3ck^2 - 2k^3)(c^3 - 27d - 3ck^2 + 2k^3) \tag{5.48}$$

is positive or negative. In particular, in the case that $\Delta > 0$, the cosmological constant of the conformally related metric (5.46) will be positive, and it describes a de Sitter black hole.

Using r_+ as usual to denote the radius of the outer horizon of the Lifshitz black hole, we have

$$d = -\frac{1}{3} r_+^2 (c^2 - k^2 + 3cr_+^2 + 3r_+^4). \tag{5.49}$$

We find that the temperature and the entropy are given by

$$T = \frac{(c + 3r_+^2 - k)(c + 3r_+^2 + k)}{12\pi}, \qquad S = -\frac{1}{6}\alpha\omega_2(c + 3r_+^2 - k). \tag{5.50}$$

The above expressions suggest that the constant c might be spurious, since it always arises in the combination $c + 3r_+^2$. Indeed we can, locally, remove it by first making the coordinate transformation $r^2 = \tilde{r}^2 - c/3$, and then scaling the metric by the factor $\tilde{r}^2/r^2$. However, if c is negative, this transformation can be singular, if $c + 3r_+^2 < 0$, and so we cannot simply use the above transformation to set $c = 0$. Indeed, one can see from (5.50) that if $c = 0$ then T and S cannot both be positive (if $\alpha > 0$). On the other hand, if c is sufficiently negative then we can arrange the parameters so that T and S are both positive.

There is no obvious way to calculate the energy of an asymptotically Lifshitz black hole directly [for example, the conserved charge given by (D8) diverges]. We can, however, integrate the first law, $dE = TdS$, to obtain a thermodynamic definition of the energy, up to an undetermined additive constant. From (5.50) we find

$$E = -\frac{\alpha\omega_2(c + 3r_+^2 - k)^2(c + 3r_+^2 + 2k)}{216\pi}, \tag{5.51}$$

where we have made a choice for the additive constant that is convenient for the cases $k = 1$ or $k = 0$. The energy definition for $k = -1$ will be given presently. The Euclidean action for the $z = 4$ Lifshitz black hole diverges for large r. We can instead define the free energy from the thermodynamic relation $F = E - TS$, yielding

$$F = \frac{\alpha\omega_2(c + 3r_+^2 - k)^2(2c + 6r_+^2 + k)}{216\pi}. \tag{5.52}$$

We now examine the three cases $k = 0$, 1 or -1 in more detail. For $k = 0$, there exists the Noether charge (5.30), giving

$$\lambda = \frac{4}{9}\alpha(c^3 - 27d). \tag{5.53}$$

Therefore, we see that (4.9) holds for this solution. In particular, we have

$$E = \frac{1}{3}TS. \tag{5.54}$$

The $k = 0$ solution has no extremal limit, since then if the function f has a double real root then it necessarily has a triple real root. For $c + 3r_+^2 > 0$ we can set $c = 0$ without loss of generality, since the conformal factor $\tilde{r}^2/r^2$

is nonsingular, as discussed earlier. In this case, the positivity of both the entropy and energy would require that $\alpha < 0$. On the other hand, when we have $c + 3r_+^2 < 0$, the constant c cannot be set to zero, since now the conformal factor $\tilde{r}^2/r^2$ runs from a negative value to 1 when r goes from the horizon to infinity. The positivity of both the entropy and energy now requires that $\alpha > 0$. When $c + 3r_+^2 = 0$, which would be the extremal limit for the $k = 0$ black holes, the solution instead has a naked singularity at $r = r_+$. Thus for a given α, only one of the two branches ($c + 3r_+^2 > 0$ or $c + 3r_+^2 < 0$) is well-defined, since the entropy of one branch is positive at the price that in the other branch it is negative.

For $k = 1$, then again if $c + 3r_+^2 > 0$ we can set the parameter $c = 0$ without loss of generality. In this case, the solution has an extremal limit with $r_+^2 = 1/3$, for which both the entropy and energy vanish. For this branch of solutions, $r_+^2 \geq 1/3$ and the non-negativeness of the entropy and the energy defined by (5.51) is guaranteed as long as α is negative. If $c + 3r_+^2 < 0$, then the parameter c becomes nontrivial. The range where $-2 < c + 3r_+^2 < 0$ in fact cannot arise, since then the function f actually has a third positive root that is larger than the putative largest root r_+, and so $r = r_+$ is not the outer horizon. If $c + 3r_+^2 < -2$, the entropy and energy are non-negative provided that $\alpha > 0$. There is an extremal limit at $c + 3r_+^2 = 1$, but, since $c + 3r_+^2 > 0$ we can reduce this to the $c = 0$, $r_+^2 = 1/3$ extremal case discussed previously. Although the function f also has a double root, at $r = r_0$ if $c + 3r_0^2 = -1$, there is a larger positive root at $r = \sqrt{r_0^2 + 1}$, so this does not describe an extremal black hole. Note that for a given sign of α, only one of the above two branches of solutions is well-defined, and only the branch with $c + 3r_+^2 > 0$ has an extremal limit. The near-horizon geometry of the extremal limit is $AdS_2 \times S^2$.

The behavior of the metric functions is the same for the $k = -1$ solution as for the $k = 1$ solution. Thus for the $c + 3r_+^2 \geq 0$ branch we can again set $c = 0$, and extremality occurs at $r_+^2 = 1/3$. The near-horizon limit of the extremal black hole is $AdS_2 \times H^2$. For solutions to have positive energy, we shift the previous energy (5.51) by a different additive constant, and define

$$\tilde{E} = -\frac{\alpha\omega_2(c + 3r_+^2 - 1)^2(c + 3r_+^2 + 2)}{216\pi} = E - \frac{\alpha\omega_2}{54\pi}. \tag{5.55}$$

The solution has non-negative energy and entropy provided that $\alpha < 0$. For the $c + 3r_+^2 < -2$ branch, the energy and entropy are non-negative provided that $\alpha > 0$. The solution is extremal at $c + 3r_+^2 = 1$, but this reduces to the $c = 0$, $r_+^2 = 1/3$ extremal case discussed above. For a given α, only one of the two branches of solutions can be well-defined.

C. $z = 0$ Lifshitz black holes

We now turn our attention to the $z = 0$ Lifshitz black hole, for which the solution is given by

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega_{2,k}^2, \\ f &= 1 + \frac{c}{r^2} + \frac{c^2 - k^2}{3r^4}. \end{aligned} \tag{5.56}$$

The solution has a power-law curvature singularity at $r = 0$. For $k = 0$, the singularity is naked. The Noether charge is given by $\lambda = -4\alpha c/3$. Since the $k = 0$ solution is not a black hole, we cannot use this example to test the validity of (4.9).

For $k = \pm 1$, there is an horizon at the largest root of f, given by

$$r_+^2 = \frac{1}{6}\left(\sqrt{3(4 - c^2)} - 3c\right). \tag{5.57}$$

The requirement that $r_+^2 > 0$ implies that $-2 \leq c < 1$. It follows that the temperature and entropy are given by

$$\begin{aligned} T &= \frac{1}{\pi\left(2 - \frac{2\sqrt{3}c}{\sqrt{4-c^2}}\right)}, \\ S &= \frac{1}{12}\alpha\omega_2\left(c + 2k + \sqrt{3(4 - c^2)}\right). \end{aligned} \tag{5.58}$$

As in the case of the $z = 4$ Lifshitz black hole, here too we can define the energy, up to an undetermined additive constant, by integrating the first law $dE = TdS$. Here, we find

$$E = \frac{1}{24\pi}\alpha\omega_2(c + 2), \tag{5.59}$$

where we have chosen the (parameter-independent) additive constant so that the energy is positive for $c > -2$. Note that when $c = -2$, the solution becomes extremal, with f given by

$$f = \frac{(r^2 - 1)^2}{r^4}, \tag{5.60}$$

and the energy defined in (5.59) vanishes. The solution has a double root at $r = 1$, with the near-horizon geometry being $AdS_2 \times S^2$ or $AdS_2 \times H^2$. For $k = 1$, the entropy vanishes in the extremal limit.

The metric (5.56) is conformal to (A)dS. Defining $d\hat{s}^2 = \Omega^2 ds^2$ with

$$\Omega = \frac{qr}{c + k + r^2}, \tag{5.61}$$

we find after defining a new radial coordinate $\rho = r\Omega$ that

$$d\hat{s}^2 = -\frac{q^2}{(c + k)^2}\hat{f}dt^2 + \hat{f}^{-1}d\rho^2 + \rho^2 d\Omega_{2,k}^2, \tag{5.62}$$

where

$$\hat{f} = k + \frac{(c-k)q}{3\rho} - \frac{1}{3}\Lambda\rho^2, \qquad \Lambda = \frac{c+2k}{q^2}. \tag{5.63}$$

Thus the conformally related metric describes an AdS black hole if $c+2k<0$ and a dS black hole if $c+2k>0$. The condition for having real roots for r^2 in the $z=0$ Lifshitz black hole is that $4k^2 - c^2 \geq 0$. In particular, if $k = +1$ then the conformally related metric will describe a de Sitter black hole.

As with the $z = 4$ Lifshitz black hole discussed previously, here too the conformal factor is nonsingular on the horizon (except in the extremal limit), and so the horizon of the nonextremal $z = 0$ Lifshitz black hole maps to the horizon of the (A)dS black hole. Once again, however, the conformal factor becomes singular at infinity, and the asymptotic regions of the two conformally related metrics are very different.

VI. ADS AND LIFSHITZ BLACK HOLES IN EINSTEIN-WEYL GRAVITY

The existence of asymptotically AdS black holes in conformal gravity over and above the standard Schwarzschild-AdS black holes suggests that analogous more general solutions should exist also in Einstein-Weyl gravity, possibly including at the critical point. Furthermore, the existence of Lifshitz vacua in these theories and their generalizations to Lifshitz black holes in conformal gravity suggests that such Lifshitz black holes may also exist in Einstein-Weyl gravity. However, no exact solutions with either type of asymptotic behavior have been found, beyond the usual Schwarzschild-AdS black hole.[5] In this section, we establish their existence by using a numerical approach.

For $k = 0$ AdS and Lifshitz black holes, by studying the horizon expansion, we find the following general relation between the Noether charge and the temperature and entropy:

$$\lambda\omega_2 = -32\pi TS. \tag{6.1}$$

By examining the asymptotic behavior at infinity, we find examples for which the energy is given by $E = -\lambda\omega_2/(16\pi(z+2))$, and hence the relation (4.9) appears to hold in these cases. However, as we have seen in conformal gravity discussed in the previous section, the first equality of (4.9) obtained in Ref. [60] does not hold in general in higher-derivative gravity, when massive spin-2 hair is involved.

[5] There exists a degenerate case with $\alpha = 0$ and $8\beta\Lambda + 1 = 0$, in which the Lagrangian is simply $\sqrt{-g}(R - R_0)^2$. This degenerate case allows any metric with constant scalar curvature R_0 to be a solution, including some Lifshitz black holes [75,76]. Since the equations of motion in this case are reduced to a scalar rather than a tensor equation, the system has no linear massive spin-2 excitations.

A. Horizon expansion

The equations of motion for Einstein-Weyl gravity which follow from (2.8) or from (2.5) with $\beta = -\alpha/3$, appear not to be explicitly solvable for the most general static, spherically symmetric solutions. We shall again consider the ansatz (5.2), and so the equations of motion for the metric functions $a(r)$ and $f(r)$ are again given by (4.2) and (4.3). As remarked previously, the Schwarzschild-AdS metrics (2.9) are solutions of these equations, but now we shall have to resort to numerical methods in order to investigate the most general static, spherically symmetric solutions.

In order to do this, we first construct Taylor expansions for the metric functions $a(r)$ and $f(r)$ in the vicinity of a black hole horizon. These will then be used to set the initial conditions for a, a', f and f' just outside the horizon, so that Eqs. (4.2) and (4.3) for a'' and f'' can be numerically integrated out to large distances. It is instructive first to look at the near-horizon expansions of a and f for the Schwarzschild-AdS black hole (2.9). If we set $\Lambda = -3$ as usual, and define the horizon radius r_0 by $k - 2m/r_0 + r_0^2 = 0$, then we have

$$a = f = r^2 + k - \frac{r_0(k + r_0^2)}{r}, \tag{6.2}$$

and so the expansions are of the form

$$\begin{aligned} a(r) = f(r) &= \left(3r_0 + \frac{k}{r_0}\right)(r - r_0) - \frac{k}{r_0^2}(r - r_0)^2 \\ &\quad + \left(\frac{k}{r_0^3} + \frac{1}{r_0}\right)(r - r_0)^3 + \cdots. \end{aligned} \tag{6.3}$$

Since an overall constant factor in $a(r)$ can be absorbed into a rescaling of the time coordinate, for the general solutions we can consider a near-horizon expansion of the form

$$\begin{aligned} a(r) &= (r - r_0) + a_2(r - r_0)^2 + a_3(r - r_0)^3 \\ &\quad + a_4(r - r_0)^4 + \cdots, \end{aligned} \tag{6.4}$$

$$\begin{aligned} f(r) &= f_1(r - r_0) + f_2(r - r_0)^2 + f_3(r - r_0)^3 \\ &\quad + f_4(r - r_0)^4 + \cdots. \end{aligned} \tag{6.5}$$

(Note that these expansions are for nonextremal black holes. The discussion for extremal black holes will be given presently). Substituting these expansions into (4.2) and (4.3), we may then solve order by order in powers of $(r - r_0)$, thus obtaining expressions for a_n and f_n with $n \geq 2$ in terms of f_1, r_0, k and α. For example, we find

$$a_2 = \frac{3r_0^3 + 5f_1 r_0^2 - 2f_1^2 r_0 + kr_0 + kf_1}{f_1^2 r_0^2} - \frac{(3r_0^2 - f_1 r_0 + k)}{4\alpha f_1^2 r_0},$$
$$f_2 = \frac{(f_1 - 3r_0)(3r_0^2 - 2f_1 r_0 + k)}{f_1 r_0^2} + \frac{3(3r_0^2 - f_1 r_0 + k)}{4\alpha f_1 r_0}. \tag{6.6}$$

The expressions for the coefficients with higher n become rapidly quite complicated, and we shall not present them here. They are easily found, up to any desired order, using algebraic computing methods.

Since we have fixed the cosmological constant, by setting $\Lambda = -3$, we see that r_0 and f_1 are nontrivial parameters characterizing the solutions for each choice of $k = 0$, 1 or -1. The case $f_1 = 3r_0 + k/r_0$ corresponds to the Schwarzschild-AdS solution, for which the series expansions can be found from (6.3).

The temperature and the entropy are given by

$$T = \frac{\sqrt{a_1 f_1}}{4\pi}, \qquad S = \frac{1}{4}\omega_2 r_0 (r_0 + 2\alpha(f_1 - 2r_0)). \tag{6.7}$$

The entropy is calculated with respect to the action of Einstein-Weyl gravity. When $k = 0$, the Noether charge is given by

$$\lambda = -2\sqrt{a_1 f_1} r_0 (r_0 - 2\alpha(2r_0 - f_0)). \tag{6.8}$$

It follows that the relation (6.1) indeed holds in general. In the above entropy calculation, we used the action (2.1) with the Gauss–Bonnet term set to zero ($\gamma = 0$). The Gauss–Bonnet term contributes $S_{\rm GB} = \gamma k$, which is a purely numerical constant, independent of the metric modulus parameters.

In the above consideration, the functions a and f have the same single root $r = r_0$, giving rise to nonextremal black holes. In the extremal limit, these functions have a double root, so that the near-horizon geometry has an AdS_2 factor. The Taylor expansion is given by

$$a(r) = (r - r_0)^2 + a_3 (r - r_0)^3 + a_4 (r - r_0)^4 + \cdots,$$
$$f(r) = f_2 (r - r_0)^2 + f_3 (r - r_0)^3 + f_4 (r - r_0)^4 + \cdots. \tag{6.9}$$

We find that the leading-order expansion of the equations of motion when $r \to r_0$ implies that

$$(4\alpha - 1)(3r_0^2 + k) = 0. \tag{6.10}$$

Thus we see that for $k = 0$, 1, extremal black holes do not exist except for $\alpha = \frac{1}{4}$, on which we shall focus. Taking this α value, we find that

$$a_3 = -\frac{2(4k + 15r_0^2)}{3r_0(k + 6r_0^2)},$$
$$a_4 = \frac{41k^2 + 358kr_0^2 + 759r_0^4}{9r_0^2(k + 6r_0^2)^2},$$
$$f_2 = \frac{k + 6r_0^2}{r_0^2}, \tag{6.11}$$
$$f_3 = -\frac{2(2k + 9r_0^2)}{3r_0^3},$$
$$f_4 = \frac{15k^2 + 148kr_0^2 + 363r_0^4}{9r_0^4(k + 6r_0^2)}.$$

As one would have expected, the Noether charge of the $k = 0$ solution vanishes in the extremal limit. Note that in the extremal limit, there is only one nontrivial parameter $r_0 > 0$. As we shall discuss presently, these near-horizon geometries can extend smoothly to the asymptotic AdS or Lifshitz infinities. When $k = -1$, the constraint (6.10) can be solved with $r_0^2 = 1/3$ for arbitrary α. However, the resulting solution is simply the Schwarzschild-AdS solution whose f can have a double zero for $k = -1$.

B. Asymptotic expansion

1. Asymptotically AdS solutions

For asymptotically AdS solutions, the asymptotic regions behave roughly as follows:

$$a \sim r^2(1 + c_0) + k - \frac{2M}{r} + c_1 r^{n+1} + \frac{c_2}{r^n},$$
$$f \sim r^2(1 + c_0) + k - \frac{2M}{r} - \frac{1}{3}c_1(n - 1)r^{n+1} + \frac{c_2(n + 2)}{3r^n}, \tag{6.12}$$

where we have parameterized α by

$$\alpha = -\frac{1}{n(n + 1)} \Rightarrow n = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{\alpha}}. \tag{6.13}$$

Note that there are a total of four parameters in (6.12), corresponding to four excitations. The coefficients (c_0, M) correspond to the massless spin-2 modes, while the (c_1, c_2) correspond to the massive spin-2 modes. For $0 < \alpha < 4$, the constant n is complex, implying that the excitation takes the form

$$\sqrt{r}\left(c_1 \cos\left(\frac{1}{2}\sqrt{4/\alpha - 1}\log r\right) + c_2 \sin\left(\frac{1}{2}\sqrt{4/\alpha - 1}\log r\right)\right). \tag{6.14}$$

Let us present some explicit examples. The first is $n = -1/2$, corresponding to $\alpha = 4$. The functions a and f at asymptotic infinity are given by

$$a = r^2 + m\sqrt{r} + k - \frac{2M}{r} + \frac{5km}{16r^{3/2}} - \frac{m(m^2+48M)}{96r^{5/2}} + \cdots,$$
$$f = r^2 + \frac{1}{2}m\sqrt{r} + k - \frac{32M+7m^2}{r} - \frac{15km}{32r^{3/2}} - \frac{5m(16M+m^2)}{64r^{5/2}} + \cdots. \quad (6.15)$$

When $k=0$, we have the Noether charge $\lambda = -27(8M+m^2)/2$. In this case, the usual Deser-Tekin and AMD methods of energy calculation lead to divergent results, and hence we do not have an independent method of calculating E to verify whether the first equality of (4.9) holds.

The second example is $n=1/2$, corresponding to $\alpha = -4/3$. We have

$$a = r^2 + k + \frac{m}{r^{1/2}} - \frac{2M}{r} + \frac{11km}{96r^{5/2}} + \frac{m^2}{12r^3} - \frac{Mm}{6r^{7/2}} + \cdots,$$
$$f = r^2 + k + \frac{5m}{6r^{1/2}} - \frac{2M}{r} + \frac{65km}{192r^{5/2}} + \frac{25m^2}{144r^3} - \frac{Mm}{4r^{7/2}} + \cdots. \quad (6.16)$$

For $k=0$, the Noether charge is $\lambda = 20M$, which is independent of m. In principle, the asymptotic behavior could have the $r^{3/2}$ series as well, but it does not appear to be the case.

The third example is $n=2$, corresponding to $\alpha = -1/6$, for which we find

$$a = r^2 + k - \frac{2M}{r} + \frac{m}{r^2} - \frac{5km}{21r^4} + \frac{Mm}{3r^5} + \cdots,$$
$$f = r^2 + k - \frac{2M}{r} + \frac{4m}{3r^2} - \frac{10km}{21r^4} + \frac{5Mm}{9r^5} + \cdots. \quad (6.17)$$

For $k=0$, the Noether charge is $\lambda = -8M$. The energy density can be calculated using the Deser-Tekin or AMD methods, giving $E = M/(6\pi)$. Thus for this case, the first equality in (4.9) holds. However, the numerical results indicate that only the $m=0$ case, i.e., the Schwarzschild-AdS solution, describes a black hole with an horizon.

The final example is the critical point, namely $\alpha = -1/2$, corresponding to $n=1$. We find that

$$a = r^2 + k - \frac{2\tilde{M}}{r} - \frac{7km}{15r^3} + \frac{2m\tilde{M}}{3r^4} + \cdots,$$
$$f = r^2 + k - \frac{2\tilde{M} - 2m/3}{r} - \frac{km}{r^3} + \frac{9(18\tilde{M}+7m)}{18r^4} + \cdots, \quad (6.18)$$

where

$$\tilde{M} = m\log r + M. \quad (6.26)$$

For $k=0$, the Noether charge is $\lambda = -18m$. In Appendix C, we derive the mass formula for this case, and we find the energy density is $E = 3m/(8\pi)$. Thus, in this case, the first equality of (4.9) holds.

2. Asymptotic Lifshitz behavior

In this case, we are primarily concerned with the $k=0$ case. We find that the large r expansion is given by

$$a \sim r^{2z}\Big(1 + \frac{(z^2+2)m}{z(z+2)r^{z+2}} + \tilde{c}_+ r^{-1-(1/2)z+(1/2)\Delta} + \tilde{c}_- r^{-1-(1/2)z-(1/2)\Delta}\Big),$$
$$f \sim \sigma r^2\Big(1 + \frac{m}{r^{z+2}} + c_+ r^{-1-(1/2)z+(1/2)\Delta} + c_- r^{-1-(1/2)z-(1/2)\Delta}\Big),$$
$$\tilde{c}_+ = \frac{4 - 11z + z^2 - 3z^3 + (1 - 3z - z^2)\Delta}{2(z-1)^2(3z-1)} c_+,$$
$$\Delta = \sqrt{3(4 - 4z + 3z^2)},$$
$$\tilde{c}_- = \frac{4 - 11z + z^2 - 3z^3 - (1 - 3z - z^2)\Delta}{2(z-1)^2(3z-1)} c_-. \quad (6.20)$$

The Noether charge is given by

$$\lambda = -\frac{6\sqrt{6}(2 - 3z + z^3)m}{z^2(z-4)\sqrt{z^2+2z+3}}. \quad (6.21)$$

Our numerical results suggest that there are Lifshitz-like black holes with S^2 and H^2 topology that have the same leading-order behavior as the above.

C. Numerical analysis

We have carried out a numerical analysis for a variety of choices for the coefficient α that multiplies the Weyl-squared term in the action. The choice of the horizon radius r_0 is a nontrivial parameter, given that we have fixed the cosmological constant ($\Lambda = -3$). The value of the expansion coefficient f_1 is also a nontrivial parameter in the solutions. The deviation of f_1 from the value $3r_0 + k/r_0$ determines the deviation of the black hole from the usual Schwarzschild-AdS solution.

The Schwarzschild-AdS black hole can be thought of as a solution where only the massless spin-2 modes are excited. Deviating from $f_1 = 3r_0 + k/r_0$ corresponds to setting initial conditions near the horizon that cause the massive spin-2 modes to be excited also. Our numerical investigations suggest that solutions of this type exist, in the sense that the numerical routines give a reasonably stable result with the metric functions showing no sign of runaway behavior, provided that the linearized spin-2 massive mode falls off less rapidly than the spin-2 massless mode. This falloff rate is governed by the mass m of the linearized fluctuation, and in turn, this is related to the value of the parameter α in the Lagrangian. Specifically, the condition of less rapid falloff is achieved if the massive mode has negative mass-squared. Solutions with stable behavior appear to exist regardless of whether the negative m^2 lies in the nontachyonic region $-\frac{9}{4} \le m^2 < 0$ or the tachyonic region $m^2 < -\frac{9}{4}$. Of course in the latter case one would expect the solutions to exhibit time-dependent runaway behavior, but this will not show up with the static metric ansatz that we are considering here.

In terms of the constant α that characterizes the coefficient of Weyl-squared in the action, the condition that the massive linearized mode have $m^2 < 0$ corresponds to $\alpha < -\frac{1}{2}$ or $\alpha > 0$.

We find that if α lies in the region $-\infty < \alpha < -\frac{1}{2}$, then defining

$$f_1 = 3r_0 + k/r_0 + \delta, \qquad (6.22)$$

there is a range for δ, with $\delta_- < \delta < \delta_+$, for which the numerical solutions indicate the occurrence of asymptotically AdS black holes. The lower limit δ_- is negative, while the upper limit δ_+ is positive. If the value of δ is fine-tuned to be *equal* to δ_- or δ_+, then the asymptotic behavior of the black hole changes from AdS to Lifshitz. The value of z in the asymptotically Lifshitz case is given by the larger root in (3.6). If the parameter δ is chosen to lie outside the range $\delta_- \le \delta \le \delta_+$, then the numerical analysis indicates that the solution becomes singular.

As an example, let us consider $\alpha = -\frac{11}{16}$, which from (3.6) implies that there should exist asymptotically Lifshitz solutions with $z = 2$. Taking $k = 0$ and choosing $r_0 = 10$, we find that the limiting values for δ in (6.22) are

$$\delta_- \approx -11.596956988, \qquad \delta_+ \approx 62.826397763. \qquad (6.23)$$

In our numerical routine, we set initial conditions just outside the horizon at $r = r_0 + 0.0001$, and run out to $r = 100000$. For the asymptotically Lifshitz black hole with $\delta = \delta_-$, we obtained plots of $a(r)$, $f(r)$, given in Fig. 1, and $a(r)/r^4$ and $f(r)/r^2$, given in Fig. 2. Note that although we integrated out to $r = 100000$, we only plot the functions out to $r = 100$ in order to be able to generate more illustrative displays. The asymptotic value of the ratio $f(r)/r^2$ reaches about 0.545454545452 as r approaches 100000, which is indeed close to the expected ratio 6/11 [see Eq. (3.4)].

The solution with $\delta = \delta_+$ exhibits very similar Lifshitz behavior. If we choose a value of δ that lies in between the two Lifshitz extremes, we obtain an asymptotically AdS black hole. Figures 3 and 4 illustrate this, again for $\alpha = -\frac{11}{16}$, $k = 0$ and $r_0 = 10$, in the case that $\delta = 20$. Figure 3 shows the functions a and f, while Fig. 4 shows the functions a/r^2 and f/r^2.

The story is very similar for solutions with $k = 1$ or -1. For example, if we consider $k = 1$ solutions, again with $\alpha = -\frac{11}{16}$ and $r_0 = 10$, we find that the upper and lower limits on the range of δ in (6.22) is now

$$\delta_- \approx -11.596956988, \qquad \delta_+ \approx 62.826397763. \qquad (6.24)$$

We find solutions exhibiting asymptotically Lifshitz type of behavior, again with $z = 2$, if δ is taken to be either of the extreme values. If, on the other hand, δ lies in between the limiting values δ_- and δ_+, then we find solutions with

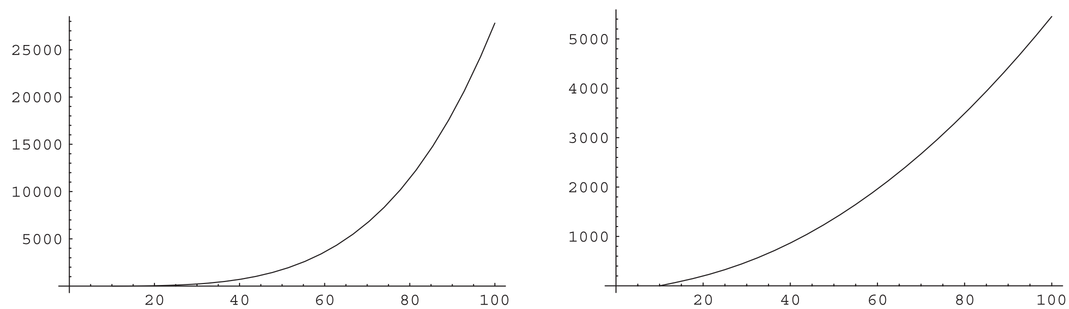

FIG. 1. The metric functions $a(r)$ and $f(r)$ for the asymptotically Lifshitz black hole.

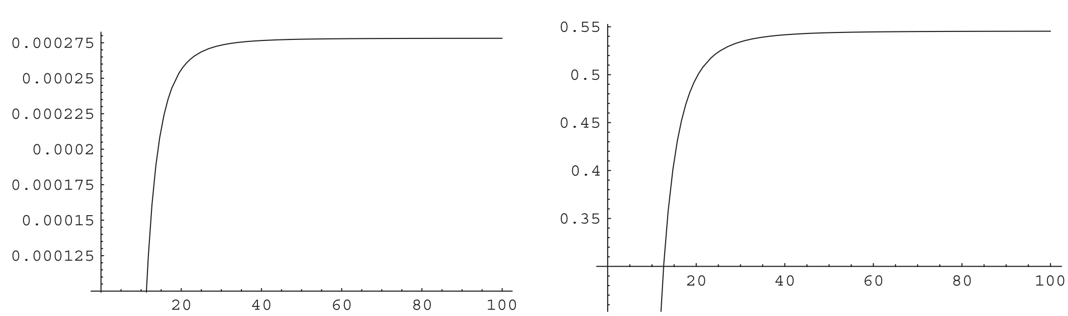

FIG. 2. The asymptotic forms for $a(r)/r^4$ and $f(r)/r^2$, illustrating the $z = 2$ Lifshitz behavior.

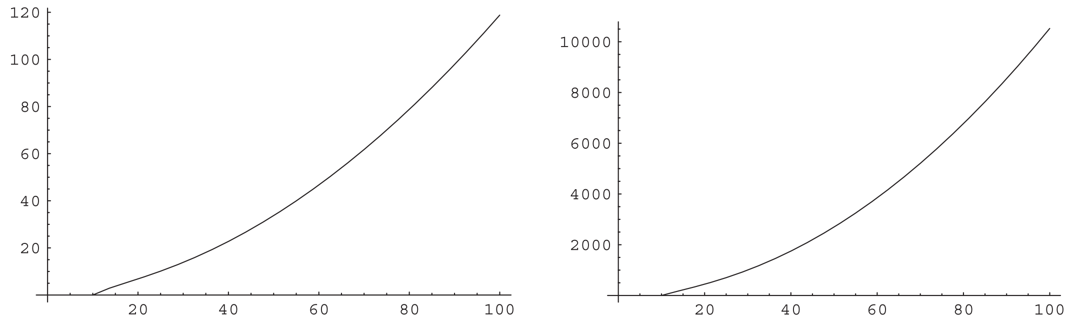

FIG. 3. The metric functions $a(r)$ and $f(r)$ for the asymptotically AdS black hole.

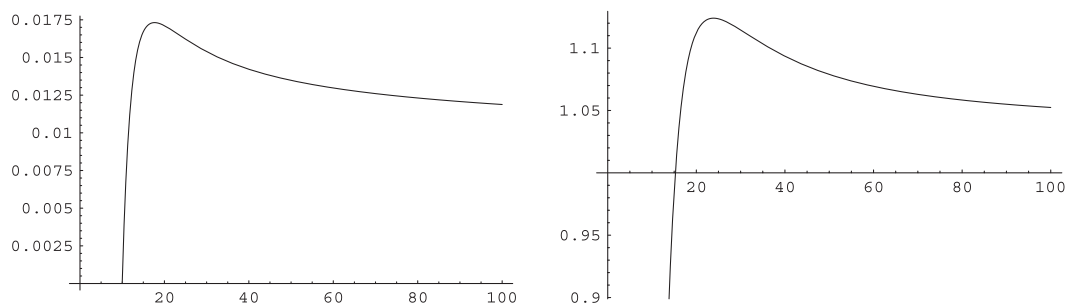

FIG. 4. The asymptotic forms for $a(r)/r^2$ and $f(r)/r^2$, illustrating the AdS behavior.

asymptotically AdS behavior. The forms of the metric functions a and f are qualitatively similar to those illustrated in the $k = 0$ examples above.

When $\alpha = -\frac{1}{2}$, corresponding to the case of critical gravity, numerical analysis indicates that asymptotically AdS black hole solutions again exist, within some range of values for the δ parameter in (6.22). However, $\alpha = -\frac{1}{2}$ is on the borderline for stability of the solutions, with $-\frac{1}{2} < \alpha < 0$ seemingly being unstable, and so it is not easy to extract meaningful quantitative results in the critical case.

We also perform the numerical analysis for the extremal case with the parameter r_0, whose horizon behavior is given by (6.9). For S^2 or T^2 topology, such a solution exists only for $\alpha = 1/4$. For $k = 1$, the numerical results indicate that the horizon can smoothly extend to the asymptotic AdS_4 infinity for all parameters $r_0 > 0.251976578$. When $r_0 = 0.251976578$, the asymptotic behavior becomes Lifshitz-like with exponent $z = -1/2$. For $r_0 < 0.251976578$, the solution becomes singular. For $k = 0$, the horizon can extend smoothly to AdS in the asymptotic region for any $r_0 > 0$. For $k = -1$, we must have $r_0 > 1/\sqrt{6}$. When $r_0 = 1/\sqrt{3}$, the usual Schwarzschild-AdS solution emerges. There is no indication of Lifshitz behavior for $k = -1$.

VII. CONCLUSIONS

In this paper, we have considered four-dimensional Einstein gravity extended by the addition of general quadratic-curvature terms. In addition to the usual AdS vacuum, the theory contains Lifshitz and Schrödinger vacuum solutions. Our primary purpose was to construct black holes obeying asymptotically AdS or Lifshitz boundary conditions, with spherical, 2-torus or hyperbolic H^2 spatial symmetry. We focused on conformal gravity, with a purely Weyl-squared action, as well as Einstein-Weyl gravity, for which the standard Einstein action with cosmological constant is augmented with a Weyl-squared term. The general spherically symmetric local solution in conformal gravity was known previously. It involves two nontrivial parameters, one of which is associated with the mass of the black hole while the other, which we call Ξ, may be thought of as characterizing massive spin-2 hair.

Owing to the presence of the second nontrivial parameter in the general AdS black hole solutions, one can expect that the usual first law of thermodynamics, $dE = TdS$, will need to be augmented by an additional term involving a new pair of intensive and extensive thermodynamic variables. We studied this in detail in the case of AdS black holes in conformal gravity, showing how the first law becomes $dE = TdS + \Psi d\Xi$, where the variable Ψ, conjugate to Ξ, is determined by requiring the integrability of the equation. We also needed to find a satisfactory definition of energy for the black holes in conformal gravity. Its derivation, as a conserved charge evaluated at infinity, is described in Appendix D.

In conformal gravity, the cosmological constant Λ of the AdS black holes is a parameter of the solution rather than a parameter in the action; it characterizes the "AdS radius" of the asymptotically AdS region. It is therefore natural to promote Λ to being another thermodynamic quantity that can be varied in the first law. We showed that this indeed gives a consistent extension of the thermodynamic phase space.

We then constructed Lifshitz black holes in conformal gravity, with a temporal/spatial anisotropic scaling parameter $z = 4$. These solutions involve only a single nontrivial parameter, and hence the thermodynamic quantities can be easily evaluated. We showed that, since the Lifshitz black hole has T^2 spatial sections, there exists a conserved Noether charge λ. Moreover, λ is related to the energy of the black hole, and to the product of temperature and entropy (4.9), in the same way as has previously been observed in Ref. [60] for certain two-derivative theories. However, for the more general AdS black holes (with T^2 spatial sections) involving massive spin-2 hair, characterized by Ξ, the Noether charge no longer seems to provide a natural definition for the energy, although the second equality of (4.9) always holds. By contrast, in the case of Schwarzschild-AdS black holes with T^2 spatial sections, the relation (4.9) always holds. We also obtained Lifshitz-like black holes with S^2 and H^2 spatial sections, with Lifshitz exponent $z = 4$ and 0, and we found that the thermodynamic relations are obeyed in these cases.

The existence of well-defined AdS and Lifshitz black holes in conformal gravity with additional massive spin-2 hair prompted us to seek similar solutions in Einstein-Weyl gravities. It does not appear to be possible to obtain closed-form expressions for such solutions, and so we resorted to numerical integration of the equations of motion. The procedure is to first obtain both the horizon and asymptotic expansions, and then use the horizon expansion as the initial boundary conditions for numerical analysis and compare the resulting solution for large radial values with the asymptotic expansions. We find that the horizon geometry involves an extra parameter over and above that of the usual Schwarzschild-AdS solution which is an Einstein metric. The numerical analysis suggests that asymptotically AdS black holes exist within a continuous range of values for the additional parameter. At the boundary of this parameter region, the asymptotic behavior changes to that of Lifshitz solutions, giving rise to corresponding asymptotically Lifshitz black holes. Beyond these parameter boundaries, the solutions develop naked curvature singularities. For a black brane with $k = 0$, for which there is an additional Noether charge, we find that the second equality in (4.9) always holds, whereas the first equality does not. These solutions provide an interesting phase transition of the corresponding boundary field theory from a relativistic Lorentzian system to a nonrelativistic Lifshitz system. We further examine the existence of extremal solutions whose near-horizon geometry has an AdS_2 factor. It turns out that nontrivial AdS extremal solutions arise only for $\alpha = \frac{1}{4}$. In the case of $k = 1$, there exists an extremal Lifshitz-like black hole with exponent $z = -\frac{1}{2}$.

It would be interesting to explore the possibility of embedding extended gravity within string theory, given that string theory contains higher derivative corrections due to stringy or quantum effects. In fact, other than some special cases for which there is maximal supersymmetry, not much is known regarding the forms of these higher derivative corrections. In light of the vast string landscape, one expects that there are generic corrections, which include the higher-order curvature terms discussed in this paper. One might then invoke holographic techniques, in which case the black hole solutions discussed in this paper could be used to describe three-dimensional field theories or condensed matter systems. For the AdS black holes, the extra parameter of the AdS black hole solution would be mapped to a parameter in the dual field theory associated with finite coupling corrections. The additional global symmetry exhibited by the AdS black brane solutions would then be associated with a particular scaling symmetry in which space and time are rescaled differently, which is present at the conformal fixed point as well as away from it.

ACKNOWLEDGMENTS

H. L. is grateful to the George P. and Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy for hospitality during the course of this work. The research of H. L. is supported in part by NSFC Grant No. 11175269. The research of C. N. P. is supported in part by DOE Grant No. DE-FG03-95ER40917. The research of J. F. V. P. is supported in part by NSF Grant No. PHY-0969482 and a PSC-CUNY Grant. C. N. P. gratefully acknowledges the hospitality of the Mitchell Family Foundation at Cook's Branch Conservancy during the completion of this work.

APPENDIX A: FURTHER SOLUTIONS IN CONFORMAL GRAVITY

1. Asymptotically Schrödinger solutions

Here we construct solutions that are asymptotic to the Schrödinger solutions discussed in Ref. [41]. We consider the metric ansatz

$$ds^2 = -r^{2z} f dt^2 + \frac{dr^2}{r^2 f} + r^2(-2dtdx + dy^2), \tag{A1}$$

for $z = (1, \frac{1}{2}, 0, -\frac{1}{2})$. We find that the equations are reduced to the fourth-order differential equation

$$\begin{aligned} 0 = f'''' &+ \frac{(6f + 18zf + 7rf')f'''}{2rf} + \frac{f''}{2r^2f^2}(4r^2ff'' \\ &+ 2r^2f'^2 + (53z + 5)rff' + 4(16z^2 + 3z - 1)f^2) \\ &+ \frac{f'}{2r^3f^2}(2(3z - 1)r^2f'^2 + (78z^2 - 11z - 9)rff' \\ &+ 4(28z^2 - 10z^2 - 7z + 1)f^2) \\ &+ \frac{4z(z - 1)(2z - 1)(2z + 1)f}{r^4}. \end{aligned} \tag{A2}$$

For $z = 1$, we find a solution $f = 1 - M/r$, giving

$$ds^2 = -r^2 f dt^2 + \frac{dr^2}{r^2 f} + r^2(-2dtdx + dy^2), \qquad f = 1 - \frac{M}{r}. \tag{A3}$$

2. Generalized Plebanski metric

Using the Plebanski metric ansatz [77], we find solutions in conformal gravity given by

$$ds^2 = -\frac{\Delta_x}{x^2 + y^2}(dt + y^2 d\psi)^2 + \frac{\Delta_y}{x^2 + y^2}(dt - x^2 d\psi)^2 + \frac{x^2 + y^2}{\Delta_x} dx^2 + \frac{x^2 + y^2}{\Delta_y} dy^2, \tag{A4}$$

where

$$\Delta_x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4, \qquad \Delta_y = c_0 + d_1 y - c_2 y^2 + \frac{c_1 c_3}{d_1} y^3 + c_4 y^4. \tag{A5}$$

This metric is conformal to the Plebanski-Demianski [78] metric. Namely, the metric $d\hat{s}^2 = \Omega^2 ds^2$ with $\Omega = (1 + c_3 xy/d_1)^{-1}$ is Einstein and satisfies $\hat{R}_{\mu\nu} = \Lambda \hat{g}_{\mu\nu}$ with

$$\Lambda = -\frac{3(c_0 c_3^2 + c_4 d_1^2)}{d_1^2}. \tag{A6}$$

APPENDIX B: FURTHER SOLUTIONS IN EXTENDED GRAVITY

1. Time-dependent metrics

For the general quadratic action (2.1) with arbitrary α and β, we will consider time-dependent and spatially flat isotropic solutions described by the metric

$$ds^2 = -dt^2 + f(t)^2 \sum_{i=1}^{3} dx_i^2. \tag{B1}$$

Applying the trace condition reduces the equations to the single third-order equation

$$3f^2 f'^2 - \Lambda f^4 - 6(\alpha + 3\beta)(3f'^4 - 2ff'^2 f'' + f^2 f''^2 - 2f^2 f' f''') = 0. \tag{B2}$$

For Einstein-Weyl gravity, $\alpha + 3\beta = 0$ and, for a positive cosmological constant, the only solution is de Sitter spacetime.

We will now consider nonisotropic time-dependent solutions for extended gravity with zero cosmological constant, described by the Kasner metric

$$ds^2 = -dt^2 + \sum_{i=1}^{3} t^{2p_i} dx_i^2. \tag{B3}$$

It can be shown that a metric of this form must satisfy the conditions

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1. \tag{B4}$$

In other words, the quadratic terms in the action (2.1) do not modify the Kasner conditions that arise in Einstein gravity. This disallows isotropic expansion and, in particular, one exponent must be negative. However, in conformal gravity the exponents need only satisfy the condition

$$2\sum_{i=1}^{3} p_i + 2\sum_{i=1}^{3} p_i^2 - \left(\sum_{i=1}^{3} p_i\right)^2 = 3. \tag{B5}$$

This solution includes the Kasner metric for which both conditions in (B4) are obeyed.

2. pp-wave metrics

A general class of pp-wave solutions for the general quadratic action (2.1) with arbitrary α and β has the metric

$$ds^2 = H dx^2 + \frac{dr^2}{r^2} + r^2(-2dtdx + dy^2), \tag{B6}$$

where

$$H = f_1 r^2 + \frac{f_2}{r} + f_3 r^{2z_+} + f_4 r^{2z_-} + g_1(1 + y^2 r^2), \tag{B7}$$

the f_i and g_i are functions of x only and $z = z_\pm$ satisfy the equation

$$1 - 24\beta + \alpha(4z_\pm^2 - 2z_\pm - 8) = 0. \tag{B8}$$

For conformal gravity and critical gravity, the H function can have additional terms. Namely, for conformal gravity H has the form

$$H = f_1 r^2 + \frac{f_2}{r} + f_3 r + f_4 + g_1 y^2 r^2 + g_2 y^3 r^2, \tag{B9}$$

while for critical gravity it is given by

$$H = f_1 r^2 + \frac{f_2}{r} + f_3 r^2 \log r + f_4 \frac{\log r}{r} + g_1(1 + y^2 r^2). \tag{B10}$$

For $g_i = 0$ these metrics all reduce to ones presented in Refs. [12,13], for which H is a function only of x and r. Metrics for which the H function involves sinusoidal dependence on the y coordinate are also discussed in Ref. [13]. For $f_1 = f_3 = f_4 = g_i = 0$ all of these metrics reduce to the Kaigorodov [58] metric.

APPENDIX C: ENERGY OF LOGARITHMIC BLACK HOLE IN CRITICAL GRAVITY

In this appendix, we derive the mass of the logarithmic black hole using the Abbott-Deser-Tekin (ADT) and the AMD procedures.

The main idea of the ADT method is to write the asymptotic AdS black hole metric in the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric on AdS, and then interpret the linearized variation of the field equation, given in our case by (2.3), as an effective gravitational energy-momentum tensor $T_{\mu\nu}$ for the black hole field. One then writes the conserved current $J^\mu = T^{\mu\nu}\xi_\mu$, where ξ^μ is a Killing vector that is timelike at infinity, as the divergence of a 2-form $\mathcal{F}_{\mu\nu}$; i.e., $J^\mu = \nabla_\nu \mathcal{F}^{\mu\nu}$. From this, one obtains the ADT mass for the Lagrangian corresponding to (2.1):

$$8\pi G E = (1 + 8\Lambda\beta + 2\Lambda\alpha)\int_{S_\infty} dS_i \mathcal{F}^{0i}_{(0)} + (2\beta + \alpha)\int_{S_\infty} dS_i \mathcal{F}^{0i}_{(1)} + \alpha \int_{S_\infty} dS_{\mu\nu}\mathcal{F}^{0i}_{(2)}, \tag{C1}$$

where dS_i is the area of the sphere at infinity. The definition of $\mathcal{F}^{\mu\nu}$ associated with the various terms in the equations of motion have been calculated in Ref. [67]. One may verify that upon defining

$$\begin{aligned}\mathcal{F}^{\mu\nu}_{(0)} &= \xi_\alpha \nabla^{[\mu} h^{\nu]\alpha} + \xi^{[\mu}\nabla^{\nu]} h + h^{\alpha[\mu}\nabla^{\nu]}\xi_\alpha \\ &\quad - \xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} + \frac{1}{2} h \nabla^\mu \xi^\nu, \\ \mathcal{F}^{\mu\nu}_{(1)} &= 2\xi^{[\mu}\nabla^{\nu]} R^L + R^L \nabla^\mu \xi^\nu, \\ \mathcal{F}^{\mu\nu}_{(2)} &= -2\xi_\alpha \nabla^{[\mu} \mathcal{G}_L^{\nu]\alpha} - 2\mathcal{G}_L^{\alpha[\mu}\nabla^{\nu]}\xi_\alpha,\end{aligned} \tag{C2}$$

it follows that

$$\begin{aligned}\nabla_\nu \mathcal{F}^{\mu\nu}_{(0)} &= \mathcal{G}_L^{\mu\nu}\xi_\nu, \\ \nabla_\nu \mathcal{F}^{\mu\nu}_{(1)} &= [(-\nabla_\mu\nabla_\nu + g^{\mu\nu}\Box + \Lambda g^{\mu\nu})R^L]\xi_\nu, \\ \nabla_\nu \mathcal{F}^{\mu\nu}_{(2)} &= \left[\left(\Box - \frac{2\Lambda}{3}\right)\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{3} R^L g^{\mu\nu}\right]\xi_\nu.\end{aligned} \tag{C3}$$

At the critical point $\Lambda\alpha = -3\Lambda\beta = \frac{3}{2}$, the first term in (C1) vanishes, and the contributions to the mass of the logarithmic black hole from the two remaining terms is

$$E^{\rm ADT}_{\log} = \frac{3m}{8\pi G}. \tag{C4}$$

Since the logarithmic black hole is asymptotically AdS, one can also try to apply the AMD method to this case. The derivation of AMD conserved quantities relies on a detailed analysis of the falloff rate of the curvature near the boundary, which is weighted by a smooth function Ω (the conformal boundary is defined at $\Omega = 0$). For details on the requirement for Ω, the reader is referred to Refs. [68,69]. For n-dimensional asymptotic AdS spacetime, for generic cases in which the leading falloff of the Weyl tensor goes as Ω^{n-5}, the AMD formula for conserved quantities in quadratic curvature theories were explored in Refs. [70,71]. However, in the case of AdS logarithmic black holes, the leading falloff of the Weyl tensor near the boundary is modified to be

$$C_{abcd} \to \Omega^{n-5}K_{abcd} + \Omega^{n-5}\log(\Omega)L_{abcd}. \tag{C5}$$

Here a and b are indices related to a new coordinate system which adopts Ω as the radial coordinate. It is found that, at critical points where a logarithmic term can appear, the falloff behavior of the energy-momentum tensor is still at the order of Ω^{n-3}. Thus, the flux across the boundary is finite. This implies that the AMD conserved quantities for the logarithmic black hole may be well-defined. In Ref. [71], the AMD conserved quantities corresponding to the logarithmic black hole are found to be given by

$$Q_\xi[C] = \frac{\alpha R_0}{8\pi G_{(n)} n(n-3)}\int_C dx^{n-2}\sqrt{\hat{\sigma}}\hat{\mathcal{L}}_{ab}\xi^a \hat{N}^b, \tag{C6}$$

with

$$R_0 = -n(n-1), \tag{C7}$$

where the AdS radius has been set to 1 and $\hat{\mathcal{L}}_{ab} \equiv \ell^2 L_{eafb}\hat{n}^e\hat{n}^f$. Specifically for the four-dimensional AdS-logarithmic black hole solutions with the asymptotic expansion given by (6.18), one finds that

$$E^{\rm AMD}_{\log} = \frac{3m}{8\pi G}. \tag{C8}$$

APPENDIX D: ENERGY OF ADS BLACK HOLES IN CONFORMAL GRAVITY

In this appendix, we present details of the proposal for calculating the mass of AdS black holes in conformal gravity that we discussed in Sec. VA. The Lagrangian for conformal gravity is given by

$$e^{-1}\mathcal{L} = \frac{\alpha}{2} C^{\mu\nu\rho\lambda}C_{\mu\nu\rho\lambda}. \tag{D1}$$

The solutions of conformal gravity discussed in Sec. V can be written as

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{2,k}, \qquad f = r^2 + br + c + \frac{d}{r}, \qquad 3bd - c^2 + k^2 = 0. \tag{D2}$$

To apply the ADT method to this solution, a background subtraction is necessary. It turns out that if we simply choose the static AdS metric as the background, the energy calculated is divergent. On the other hand, the background-independent AMD method will also give a infinite result since the leading falloff of the Weyl tensor of the solution (D2) is slower than that of the usual AdS black hole.

Motivated by finding a proper definition of energy for black hole solutions (D2) in conformal gravity, we adapt the standard Noether method to the Lagrangian of conformal gravity. In the following, we briefly review the Noether procedure for deriving a conserved current associated with symmetry generated by the Killing vector ξ.

The first variation of the Lagrangian 4-form generated by the vector ξ can always be expressed as

$$\mathcal{L}_\xi L = E\mathcal{L}_\xi \phi + d\Theta(\phi, \mathcal{L}_\xi \phi), \tag{D3}$$

where ϕ represents a collection of tensorial fields, E denotes their equations of motion, and $\mathcal{L}_\xi$ denotes the Lie derivative. Using the identity

$$\mathcal{L}_\xi = di_\xi + i_\xi d, \tag{D4}$$

for the Lie derivative of a differential form, we find a conserved current defined by

$$J = \Theta - i_\xi L. \tag{D5}$$

On shell, we have

$$dJ = 0 \Rightarrow J = dQ, \tag{D6}$$

where Q is the conserved charge density associated with the symmetry generated by ξ. Applying this procedure to conformal gravity, we find that

$$Q = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}Q^{\rho\sigma}dx^\mu \wedge dx^\nu, \tag{D7}$$

with

$$Q^{\rho\sigma} = -\frac{\alpha}{8\pi G}(C^{\rho\sigma\mu\nu}\nabla_\mu \xi_\nu - 2\xi_\nu \nabla_\mu C^{\rho\sigma\mu\nu}). \tag{D8}$$

It is well-known that the conserved charge Q derived from the Einstein-Hilbert action only accounts for one half of the true ADM mass. (The other half can be understood as coming from a total derivative term added to the Einstein-Hilbert action [79]). The validity of the proposal to take (D8) as the definition of energy for black holes in conformal gravity can be tested by applying it to the known examples of the Schwarzschild-AdS and Kerr-AdS black holes. We find that the results using (D8) coincide with those obtained from the AMD method and in particular, by setting $\alpha = \frac{1}{2}$, we recover the result presented in Refs. [72,80], thus confirming the tree level equivalence between Einstein gravity and Weyl gravity that was proposed in Ref. [38].

Finally, we calculate the conserved charge for the metric (D2) associated with the timelike Killing vector $\partial/\partial t$. It is given by

$$\int Q = -\frac{\alpha\omega_2}{16\pi G}\left[4d - \frac{2}{3}b(c-k)\right]. \tag{D9}$$

As we discuss in Sec. V, we can use this conserved quantity to provide a definition of energy, which turns out to be consistent with the first law of thermodynamics for the general AdS black holes in conformal gravity.

[1] K. S. Stelle, Phys. Rev. D 16, 953 (1977).
[2] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (Leipzig) 140, 372 (1982); S. Deser, *Quantum Theory of Gravity*, edited by S. M. Christensen (Taylor & Francis, Oxford, 1984), pp. 347–381.
[3] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. 102, 201301 (2009).
[4] W. Li, W. Song, and A. Strominger, J. High Energy Phys. 04 (2008) 082.
[5] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin, and P. K. Townsend, Classical Quantum Gravity 27, 025010 (2010).
[6] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin, and P. K. Townsend, Classical Quantum Gravity 28, 015002 (2011).
[7] E. A. Bergshoeff, O. Hohm, J. Rosseel, and P. K. Townsend, Classical Quantum Gravity 27, 235012 (2010).
[8] H. Lü, C. N. Pope, and E. Sezgin, J. High Energy Phys. 10 (2010) 016.
[9] H. Lü and Y. Pang, J. High Energy Phys. 03 (2011) 050.
[10] H. Lü and C. N. Pope, Phys. Rev. Lett. 106, 181302 (2011).
[11] E. A. Bergshoeff, O. Hohm, J. Rosseel, and P. K. Townsend, Phys. Rev. D 83, 104038 (2011).
[12] M. Alishahiha and R. Fareghbal, Phys. Rev. D 83, 084052 (2011).
[13] I. Gullu, M. Gurses, T. C. Sisman, and B. Tekin, Phys. Rev. D 83, 084015 (2011).
[14] M. Porrati and M. M. Roberts, Phys. Rev. D 84, 024013 (2011).
[15] H. Liu, H. Lü, and M. Luo, Int. J. Mod. Phys. D 21, 1250020 (2012).
[16] S. Deser, H. Liu, H. Lü, C. N. Pope, T. C. Sisman, and B. Tekin, Phys. Rev. D 83, 061502 (2011).
[17] H. Lü, Y. Pang, and C. N. Pope, Phys. Rev. D 84, 064001 (2011).
[18] E. A. Bergshoeff, S. de Haan, W. Merbis, and J. Rosseel, J. High Energy Phys. 09 (2011) 038.
[19] H. Lü, C. N. Pope, E. Sezgin, and L. Wulff, J. High Energy Phys. 10 (2011) 131.
[20] Y. S. Myung, arXiv:1107.3594.
[21] T. Moon and Y. S. Myung, Eur. Phys. J. C 71, 1796 (2011).
[22] Y.-X. Chen, H. Lü, and K.-N. Shao, arXiv:1108.5184.
[23] N. Ohta, Classical Quantum Gravity 29, 015002 (2012).
[24] E. J. Son, M. Eune, and W. Kim, Phys. Lett. B 706, 447 (2012).
[25] H. Lü and K. N. Shao, Phys. Lett. B 706, 106 (2011).
[26] G.-Y. Qi and Y. Guo, arXiv:1110.3448.
[27] G.-Y. Qi, arXiv:1110.3449.
[28] S. J. Hyun, W. J. Jang, J. H. Jeong, and S.-H. Yi, J. High Energy Phys. 01 (2012) 054.

[29] E. A. Bergshoeff, S. de Haan, W. Merbis, M. Porrati, and J. Rosseel, J. High Energy Phys. 04 (2012) 134.
[30] K. Sen, A. Sinha, and N. V. Suryanarayana, Phys. Rev. D **85**, 124017 (2012).
[31] N. Kan, K. Kobayashi, and K. Shiraishi, arXiv:1202.2428.
[32] S. Hyun, W. Jang, J. Jeong, and S. H. Yi, J. High Energy Phys. 04 (2012) 030.
[33] D. Kastor, Classical Quantum Gravity **29**, 155007 (2012).
[34] T. Nutma, Phys. Rev. D **85**, 124040 (2012).
[35] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B201**, 469 (1982).
[36] S. L. Adler, Rev. Mod. Phys. **54**, 729 (1982); **55**, 837 (1983).
[37] A. Zee, Phys. Rev. D **23**, 858 (1981).
[38] J. Maldacena, arXiv:1105.5632.
[39] S. Kachru, X. Liu, and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).
[40] D. T. Son, Phys. Rev. D **78**, 046003 (2008).
[41] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008).
[42] M. Taylor, arXiv:0812.0530.
[43] S. S. Gubser and A. Nellore, Phys. Rev. D **80**, 105007 (2009).
[44] S. A. Hartnoll and A. Tavanfar, Phys. Rev. D **83**, 046003 (2011).
[45] U. H. Danielsson and L. Thorlacius, J. High Energy Phys. 03 (2009) 070.
[46] R. B. Mann, J. High Energy Phys. 06 (2009) 075.
[47] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, J. High Energy Phys. 04 (2010) 120.
[48] K. Balasubramanian and K. Narayan, J. High Energy Phys. 08 (2010) 014.
[49] A. Donos and J. P. Gauntlett, J. High Energy Phys. 12 (2010) 002.
[50] H. Singh, J. High Energy Phys. 12 (2010) 061.
[51] R. Gregory, S. L. Parameswaran, G. Tasinato, and I. Zavala, J. High Energy Phys. 12 (2010) 047.
[52] A. Donos, J. P. Gauntlett, N. Kim, and O. Varela, J. High Energy Phys. 12 (2010) 003.
[53] D. Cassani and A. F. Faedo, J. High Energy Phys. 05 (2011) 013.
[54] N. Halmagyi, M. Petrini, and A. Zaffaroni, J. High Energy Phys. 08 (2011) 041.
[55] H. Singh, arXiv:1202.6533.
[56] C. P. Herzog, M. Rangamani, and S. F. Ross, J. High Energy Phys. 11 (2008) 080.
[57] J. Maldacena, D. Martelli, and Y. Tachikawa, J. High Energy Phys. 10 (2008) 072.
[58] V. R. Kaigorodov, Dokl. Akad. Nauk SSSR **146**, 793 (1962) [Sov. Phys. Dokl. **7**, 893 (1963)].
[59] M. Cvetič, H. Lü, and C. N. Pope, Nucl. Phys. **B545**, 309 (1999).
[60] G. Bertoldi, B. A. Burrington, and A. Peet, Phys. Rev. D **80**, 126003 (2009).
[61] G. Bertoldi, B. A. Burrington, and A. W. Peet, Phys. Rev. D **80**, 126004 (2009).
[62] R. J. Riegert, Phys. Rev. Lett. **53**, 315 (1984).
[63] D. Klemm, Classical Quantum Gravity **15**, 3195 (1998).
[64] G. Cognola, O. Gorbunova, L. Sebastiani, and S. Zerbini, Phys. Rev. D **84**, 023515 (2011).
[65] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635 (1989).
[66] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[67] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
[68] A. Ashtekar and A. Magnon, Classical Quantum Gravity **1**, L39 (1984).
[69] A. Ashtekar and S. Das, Classical Quantum Gravity **17**, L17 (2000).
[70] N. Okuyama and J. I. Koga, Phys. Rev. D **71**, 084009 (2005).
[71] Y. Pang, Phys. Rev. D **83**, 087501 (2011).
[72] M. M. Caldarelli, G. Cognola, and D. Klemm, Classical Quantum Gravity **17**, 399 (2000).
[73] H. Liu, H. Lü, M. Luo, and K. N. Shao, J. High Energy Phys. 12 (2010) 054.
[74] M. Cvetič, G. W. Gibbons, D. Kubiznak, and C. N. Pope, Phys. Rev. D **84**, 024037 (2011).
[75] R. G. Cai, Y. Liu, and Y. W. Sun, J. High Energy Phys. 10 (2009) 080.
[76] E. Ayon-Beato, A. Garbarz, G. Giribet, and M. Hassaine, J. High Energy Phys. 04 (2010) 030.
[77] J. F. Plebanski, Ann. Phys. (Leipzig) **90**, 196 (1975).
[78] J. F. Plebanski and M. Demianski, Ann. Phys. (Leipzig) **98**, 98 (1976).
[79] J. Katz, Classical Quantum Gravity **2**, 423 (1985).
[80] G. W. Gibbons, M. J. Perry, and C. N. Pope, Classical Quantum Gravity **22**, 1503 (2005).