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Super-renormalizable quantum gravity

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In this paper we study perturbatively an extension of the Stelle higher derivative gravity involving an infinite number of derivative terms. We know that the usual quadratic action is renormalizable but suffers from the unitarity problem because of the presence of a ghost (state of negative norm) in the theory. In this paper, we reconsider the theory first introduced by Tomboulis in 1997, but we expand and extensively study it at both the classical and quantum level. This theory is ghost-free, since the introduction of (in general) two entire functions in the model with the property does not introduce new poles in the propagator. The local high derivative theory is recovered expanding the entire functions to the lowest order in the mass scale of the theory. Any truncation of the entire functions gives rise to the unitarity violation, but if we keep all the infinite series, we do not fall into these troubles. The theory is renormalizable at one loop and finite from two loops on. Since only one-loop Feynman diagrams are divergent, then the theory is super-renormalizable. We analyze the fractal properties of the theory at high energy showing a reduction of the spacetime dimension at short scales. Black hole spherical symmetric solutions are also studied omitting the high curvature corrections in the equation of motions. The solutions are regular and the classical singularity is replaced by a "de Sitter-like core" in $r = 0$. Black holes may show a "multihorizon" structure depending on the value of the mass. We conclude the paper with a generalization of the Tomboulis theory to a multidimensional spacetime.

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I. INTRODUCTION TO THE THEORY

One of the biggest problems in theoretical physics is to find a theory that is able to reconcile general relativity and quantum mechanics. There are many reasons to believe that gravity has to be quantum, some of which are the quantum nature of matter in the right-hand side of the Einstein equations, the singularities appearing in classical solutions of general relativity, etc.

The action principle for gravity we are going to introduce in this paper is the result of a synthesis of minimal requirements: (i) classical solutions must be singularity-free, (ii) the Einstein-Hilbert action should be a good approximation of the theory at an energy scale much smaller than the Planck mass, (iii) the spacetime dimension has to decrease with the energy in order to have a complete quantum gravity theory in the ultraviolet regime, (iv) the theory has to be perturbatively renormalizable at quantum level (this hypothesis is strongly related to the previous one), (v) the theory has to be unitary, with no other degree of freedom than the graviton in the propagator, (vi) the spacetime is a single continuum of space and time and, in particular, the Lorentz invariance is not broken. The last requirement is supported by recent observations.

Now let us introduce the theory, step by step, starting from the perturbative non-renormalizable Einstein gravity, through high derivatives gravity theories (the Stelle theory of gravity will be our first example) onto the action which defines a complete quantum gravity theory. The impatient reader can skip to the end of the introduction for the candidate complete quantum gravity bare Lagrangian.

Perturbative quantum gravity is the quantum theory of a spin two particle on a fixed (usually for simplicity is assumed to be flat) background. Starting from the Einstein-Hilbert Lagrangian

$$\mathcal{L} = -\sqrt{-g}\kappa^{-2}R \tag{1}$$

($\kappa^2 = 16\pi G_N$), we introduce a splitting of the metric in a background part plus a fluctuation

$$\sqrt{-g}g^{\mu\nu} = g^{o\mu\nu} + \kappa h^{\mu\nu}, \tag{2}$$

then we expand the action in power of the graviton fluctuation $h^{\mu\nu}$ around the fixed background $g^{\mu\nu}$. Unlikely, the quantum theory is divergent at two loops, producing a counterterm proportional to the Ricci tensor at the third power

$$\sqrt{-g}R^{\alpha\beta}_{\gamma\delta}R^{\gamma\delta}_{\rho\sigma}R^{\rho\sigma}_{\alpha\beta}. \tag{3}$$

In general, in d dimensions the superficial degree of divergence of a Feynman diagram is $D = Ld + 2V - 2I$, where L is the number of loops, V is the number of vertices, and I the number of internal lines in the graph. Using the topological relation between V, I and L, $L = 1 + I - V$, we obtain $D = 2 + (d - 2)L$. In $d = 4$ the superficial degree of divergence $D = 2 + 2L$ increases with the number of loops, and thus we are forced to introduce an infinite number of higher derivative counterterms and then an infinite number of coupling constants, therefore making the theory not predictive. Schematically, we can relate the loop divergences in perturbative quantum gravity to the counterterms we have to introduce to regularize the theory. In short

$$S = -\int d^d x\sqrt{g}\Big[\kappa^{-2}R + \underbrace{\sum_{m,n}^{+\infty}\frac{\alpha_{nm}}{\epsilon}\nabla^n R^m}_{n+2m=2+(d-2)L}\Big], \qquad (4)$$

with "n" and "m" integer numbers, α_{mn} coupling constants, and $1/\epsilon$ the cutoff in dimensional regularization.

A first revolution in quantum gravity was introduced by Stelle [1] with the higher derivative theory

$$S = -\int d^4 x\sqrt{g}[\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \kappa^{-2}R]. \qquad (5)$$

This theory is renormalizable but unfortunately contains a physical ghost (state of negative norm), implying a violation of unitarity in the theory: probability, described by the scattering S matrix, is no more conserved. The classical theory is unstable, since the dynamics can drive the system to become arbitrarily excited, and the Hamiltonian constraint is unbounded from below.

In this paper we generalize the Stelle theory to restore unitarity. This work is inspired by papers about a nonlocal extension of gauge theories introduced by Moffat, Cornish, and their collaborators in the 1990s [2]. The authors further extended the idea to gravity, having in mind the following logic [3]. They considered a modification of the Feynman rules where the coupling constants (g_i for electroweak interactions and G_N for gravity) are no longer constant but a function of the momentum p. They checked the gauge invariance at all orders in gauge theory but only up to the second order in gravity. For particular choices of $g_i(p)$ or $G_N(p)$, the propagators do not show any other pole above the standard particle content of the theory; therefore, the theory is unitary. On the other hand, the theory is also finite if the coupling constants go sufficiently fast to zero in the ultraviolet limit. The problem with gravity is to find a covariant action that self-contains the properties mentioned before: finiteness and/or renormalizability, and unitarity.

In the second and third sections of this work, we expand on the Tomboulis paper [4], with particular emphasis on the unitarity and renormalizability of the theory. Afterward, we will study the fractal properties of the spacetime at short distance, as well as modifications of the Newtonian gravitational potential and spherically symmetric/black hole solutions (Secs. IV, V, and VII). In Sec. VIII we extend the Tomboulis theory to a multidimensional spacetime. In the last Sec. IX we suggest a possible interpretation of the nonlocal nature of gravity.

The theory developed in [4] is very interesting, of great generality, and mainly concentrated on gauge theories. Nevertheless, in [4] Tomboulis also concludes with an extension of the idea to gravity, which we reanalyze through a different perspective.

The action we are going to consider is a generalization of Stelle's theory

$$S = -\int \sqrt{-g}\{R_{\mu\nu}\alpha(\Box_\Lambda)R^{\mu\nu} - R\beta(\Box_\Lambda)R + \gamma\kappa^{-2}R\}, \qquad (6)$$

where $\alpha(\Box_\Lambda)$ and $\beta(\Box_\Lambda)$ are now fixed functionals of the covariant D'Alembertian operator $\Box_\Lambda = \Box/\Lambda^2$ and Λ is a mass scale in the theory.

In the remainder of this section we will summarize the steps and motivations that led us to the generalization (6) and will conclude with an extended version (6).

The covariant action (6) is a collection of terms, but the initial motivation when we started this project was the nonlocal Barvinsky action [5]

$$S = -\frac{1}{\kappa^2}\int d^4 x\sqrt{-g}G^{\mu\nu}\frac{1}{\Box}R_{\mu\nu}, \qquad (7)$$

where $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ is the Einstein tensor. The action (7) looks like a nonlocal action, but indeed reproduces the Einstein gravity at the lower order in the curvature. Barvinsky has shown that the variation of (7), with respect to the metric, gives the following equations of motion (see later in this section for details):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + O(R^2_{\mu\nu}) = 0. \qquad (8)$$

The next step is to modify (7) by introducing an extra operator in the action between the Einstein tensor and the Ricci tensor in order to have a well-defined theory at the quantum level without loss of covariance.

We consider as guideline the value of the spectral dimension in the definition of a finite and/or renormalizable theory of quantum gravity which, of course, is compatible with the Einstein gravity at low energy but also manifests a natural dimensional reduction at high energy. The dimensional reduction is of primary importance in order to have a quantum theory free of divergences in the ultra violet regime. The Stelle theory is characterized by a two-dimensional behavior at high energy, but this is not sufficient to have a well-defined theory at the quantum level because of the presence of ghosts in the propagator. A first easy generalization of (7) is the following action:

$$S = -\frac{1}{\kappa^2}\int d^4 x\sqrt{-g}G^{\mu\nu}F(\Box/\Lambda^2)R_{\mu\nu}, \qquad (9)$$

where $F(\Box/\Lambda^2)$ is a generic function of the covariant d'Alambertian operator which satisfies the already mentioned properties that we are going to summarize below.

(I) Classical limit:

$$\lim_{\Lambda\to+\infty} F(\Box/\Lambda^2) = \frac{1}{\Box}, \qquad (10)$$

if the limit is satisfied, the equations of motion are the Einstein equations plus corrections in $R^2_{\mu\nu}$ (we will explain this in more detail later in the paper).

(II) Finiteness and/or renormalizability of the quantized theory. Our guiding principle is to find a well-defined quantum theory and the dimensional reduction of the spacetime at high energy. The Stelle theory, the Crane-Smolin theory [6], "asimptotically safe quantum gravity" [7], "causal dynamical triangulation" [8], "loop quantum gravity" [9], and "string theory" already manifest this property with a high energy spectral dimension $d_s = 2$ [10–21]. However, such reduction is insufficient if we want a unitary theory free from negative norm states. We can anticipate that, for the model that we are going to introduce in this paper, the spectral dimension is smaller than one in the ultraviolet regime.

The general theory (9) was for the first time derived by Barvinsky [5] in the brane-world scenery and can be written in the following equivalent way:

$$S = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}G^{\mu\nu}\frac{\mathcal{O}^{-1}(\Box_\Lambda)}{\Box}R_{\mu\nu}. \qquad (11)$$

In [5] the author was interested in the infrared modifications to Einstein gravity. In the present paper, however, we are interested to ultraviolet modifications to gravity, as stressed in (10). The form (11) of the action is particularly useful to highlight the classical limit and in the expression of the equations of motion. In analogy to the properties satisfied by the operator $F(\Box/\Lambda^2)$, the operator $\mathcal{O}$ has to satisfy the following limit in order to reproduce the classical theory [see the discussion related to formulas (7) and (8)]:

$$\lim_{\Lambda\to+\infty}\mathcal{O}(\Box/\Lambda^2) = 1. \qquad (12)$$

We now add some details about the Barvinsky derivation of the equation of motion. Taking the variation of (11) with respect to the metric, we find [5]

$$-2\kappa^{-2}\int d^4x\sqrt{-g}G^{\mu\nu}\frac{\mathcal{O}^{-1}(\Box_\Lambda)}{\Box}\delta_g R_{\mu\nu} + O(R^2_{\mu\nu}). \qquad (13)$$

Since the Ricci tensor variation

$$\delta_g R_{\mu\nu} = \frac{1}{2}\Box\delta g_{\mu\nu} - \nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu,$$

[5], integration by parts cancels the $\Box$ operator at the denominator and the contribution of the gauge parameters ϵ_μ vanishes in view of the Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$. All the commutators of covariant derivatives with the $\Box$ operator in $\mathcal{O}^{-1}(\Box_\Lambda)/\Box$ give rise to curvature square operators. Also, the direct variation of the metric gives rise to curvature square terms. Then the equations of motion are very simple, if we omit the squared curvature terms:

$$\mathcal{O}^{-1}(\Box_\Lambda)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + O(R^2_{\mu\nu}) = 8\pi G_N T_{\mu\nu}. \qquad (14)$$

If we truncate the theory to the linear part in the Ricci curvature, the equations of motions simplify to

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N\mathcal{O}(\Box_\Lambda)T_{\mu\nu},\\ &\nabla^\mu(\mathcal{O}(\Box_\Lambda)T_{\mu\nu}) = 0, \end{aligned} \qquad (15)$$

where the second relation is a consequence of the Bianchi identities that we impose on the solution. In particular, such relation implies that the conserved quantity has to be $\mathcal{S}_{\mu\nu} = \mathcal{O}(\Box/\Lambda^2)T_{\mu\nu}$ in the truncation of the theory. These equations of motion are very interesting because black hole solutions, at least for a particular choice of the operator $\mathcal{O}$, are not singular anymore, as recently shown in [22].

At the classical level, Eq. (14) can be derived from the action (9) or (11), but at the quantum level, the Einstein-Hilbert action and the high derivative terms introduced by Stelle are generated. At the quantum as well as classical levels, it is our interest to consider here the consistent theory mainly introduced by Tomboulis in 1997 [4]. Thus, the complete Lagrangian we are going to carefully study is

$$\begin{aligned} \mathcal{L} = -\sqrt{-g}\Big[&\frac{\beta}{\kappa^2}R - \beta_2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \beta_0 R^2\\ &+ \left(R_{\mu\nu}h_2(-\Box_\Lambda)R^{\mu\nu} - \frac{1}{3}Rh_2(-\Box_\Lambda)R\right)\\ &- Rh_0(-\Box_\Lambda)R\Big] - \frac{1}{2\xi}F^\mu\omega(-\Box^\eta_\Lambda)F_\mu\\ &+ \bar{C}^\mu M_{\mu\nu}C^\nu. \end{aligned} \qquad (16)$$

The operator $\Box^\eta_\Lambda$ encapsulates the D'Alembertian of the flat fixed background; whereas, F_μ is the gauge fixing function with the weight functional ω and $\bar{C}^\mu$, C^μ are the ghosts fields ($M_{\mu\nu}$ will be defined in the next section). In general, we introduce two different functions h_2 and h_0. Those functions have not to be polynomial but *entire functions without poles or essential singularities*. While nonlocal kernels can lead to unitary problems, the functions h_2 and h_0 do introduce an effective nonlocality. However, since h_2 and h_0 are transcendental entire functions, their behavior is quite similar to polynomial functions and unitary problems do not occur.

Let us assume for a moment that $h_i(x) = p_n(x)$, where $p_n(x)$ is a polynomial of degree n. In this case, as it will be evident in the next section, the propagator takes the following form:

$$\frac{1}{k^2(1 + p_n(k^2))} = \frac{c_0}{k^2} + \sum_i\frac{c_i}{k^2 - M_i^2}, \qquad (17)$$

where we used the factorization theorem for polynomial and the partial fraction decomposition [4]. When multiplying by k^2 the left and right side of (17) and considering the ultraviolet limit, we find that at least one of the coefficients c_i is negative; therefore. the theory contains a ghost in the spectrum. The conclusion is that h_2 and h_0 cannot be polynomial. In this paper we will reconsider and expand the analysis of the entire functions introduced for the first time by Tomboulis [4], but we will also consider another choice of entire functions inspired by an effective approach to the noncommutative spacetime [23–26].

We can also add to the action the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ but, for spacetimes topologically equivalent to the flat space, we can use the Gauss-Bonnet topological invariant

$$\int d^4x\sqrt{-g}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2] \quad (18)$$

to rephrase the Kretschmann invariant in terms of R^2 and $R_{\mu\nu}R^{\mu\nu}$ already present in the action.

II. GRAVITON PROPAGATOR

We start by considering the quadratic expansion of the Lagrangian (16) in the graviton field fluctuation without specifying the explicit form of the functionals h_2 and h_0 (if not necessary). Following the Stelle paper, we expand around the Minkowski background $\eta^{\mu\nu}$ in power of the graviton field $h^{\mu\nu}$ defined in the following way:

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \quad (19)$$

The form of the propagator depends not only on the gauge choice but also on the definition of the gravitational fluctuation [27]. In the quantum theory the gauge choice is the familiar *harmonic gauge* $\partial_\nu h^{\mu\nu} = 0$ and the Green's functions are defined by the generating functional

$$Z(T_{\mu\nu}) = \mathcal{N}\int\prod_{\mu\leqslant\nu} dh^{\mu\nu}[dC^\sigma][d\bar{C}_\rho][de^\tau]\delta(F^\tau - e^\tau)e^{i(S_g-(1/2\xi)\int d^4x e_\tau\omega(-\Box^\eta_\Lambda)e^\tau + \int d^4x\bar{C}_\tau F^\tau_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha + \kappa\int d^4x T_{\mu\nu}h^{\mu\nu})}, \quad (20)$$

where S_g is the gravitational action defined in (16) subtracted of the gauge and ghost terms and $F^\tau = F^\tau_{\mu\nu}h^{\mu\nu}$ with $F^\tau_{\mu\nu} = \delta^\tau_\mu\partial_\nu$. $D^{\mu\nu}_\alpha$ is the operator which generates the gauge transformations in the graviton fluctuation $h^{\mu\nu}$. Given the infinitesimal coordinates transformation $x^{\mu\prime} = x^\mu + \kappa\xi^\mu$, the graviton field transforms as follows:

$$\begin{aligned}\delta h^{\mu\nu} &= D^{\mu\nu}_\alpha \xi^\alpha \\ &= \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial_\alpha\xi^\alpha + \kappa(\partial_\alpha\xi^\mu h^{\alpha\nu} \\ &\quad + \partial_\alpha\xi^\nu h^{\alpha\mu} - \xi^\alpha\partial_\alpha h^{\mu\nu} - \partial_\alpha\xi^\alpha h^{\mu\nu}).\end{aligned} \quad (21)$$

We have also introduced a weighting gauge functional [to be precise the second term in (21)], which depends on a weight function $\omega(-\Box_\Lambda)$ with the property to fall off at least like the entire functions $h_2(k^2/\Lambda^2)$, $h_0(k^2/\Lambda^2)$ for large momenta [1].

When the gauge symmetry is broken by the addition of the gauge-fixing term, a residual transformation survives for the effective action which involves the gravitational, gauge-fixing, and ghost actions terms. This is the Becchi, Rouet, Stora, and Tyutin (BRST) symmetry defined by the following transformation, which is appropriate for the gauge-fixing term,

$$\begin{aligned}\delta_{\rm BRST}h^{\mu\nu} &= \kappa D^{\mu\nu}_\alpha C^\alpha\delta\lambda, \\ \delta_{\rm BRST}C^\alpha &= -\kappa^2\partial_\beta C^\alpha C^\beta\delta\lambda, \\ \delta_{\rm BRST}\bar{C}^\alpha &= -\kappa\xi^{-1}\omega(-\Box^\eta_\Lambda)F^\tau\delta\lambda,\end{aligned} \quad (22)$$

where $\delta\lambda$ is a constant infinitesimal anticommuting parameter. The first transformation in (22) is nothing but a gauge transformation generated by $\kappa C^\alpha\delta\lambda$, so the functional S_g is BRST invariant since it is a function of $h^{\mu\nu}$ alone. The other two BRST invariant quantities are

$$\begin{aligned}\delta^2_{\rm BRST}C^\alpha &\sim \delta_{\rm BRST}(\partial_\beta C^\sigma C^\beta) = 0, \\ \delta^2_{\rm BRST}h^{\mu\nu} &\sim \delta_{\rm BRST}(D^{\mu\nu}_\alpha C^\alpha) = 0.\end{aligned} \quad (23)$$

The above transformation follows from the anticommuting nature of C^α, $\delta\lambda$ and the following commutation relation of two gauge transformations generated by ξ^μ and η^μ,

$$\frac{\delta D^{\mu\nu}_\alpha}{\delta h^{\rho\sigma}}D^{\rho\sigma}(\xi^\alpha\eta^\beta - \eta^\alpha\xi^\beta) = \kappa D^{\mu\nu}_\lambda(\partial_\alpha\xi^\lambda\eta^\alpha - \partial_\alpha\eta^\lambda\xi^\alpha). \quad (24)$$

Given the second of (23), only the antighost $\bar{C}_\tau$ transforms under the BRST transformation to cancel the variation of the gauge-fixing term. The entire effective action is BRST invariant

$$\delta_{\rm BRST}\bigg(S_g - \frac{1}{2\xi}\int d^4x F_\tau\omega(-\Box^\eta_\Lambda)F^\tau + \int d^4x\bar{C}_\tau F^\tau_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha\bigg) = 0. \quad (25)$$

Let us list the mass dimension of the fields in the gauge-fixed Lagrangian: $[h^{\mu\nu}] = $ mass, $[C^\tau] = $ mass, $[\bar{C}_\tau] = $ mass, $[\kappa] = $ mass^{-1}.

Now we Taylor-expand the gravitational part of the action (11) to the second order in the gravitational perturbation $h^{\mu\nu}(x)$ to obtain the graviton propagator. In the momentum space the Lagrangian, which is purely quadratic in the gravitational field, reads

$$\mathcal{L}^{(2)} = \frac{1}{4}h^{\mu\nu}(-k)K_{\mu\nu\rho\sigma}h^{\rho\sigma}(k) + \mathcal{L}_{\rm GF}, \quad (26)$$

where $\mathcal{L}_{\rm GF}$ is the gauge fixing Lagrangian at the second order in the graviton field

$$\mathcal{L}_{\rm GF} = \frac{1}{4\xi}h^{\mu\nu}(-k)(\omega(k^2/\Lambda^2)k^2P^{(1)}_{\mu\nu\rho\sigma}(k) + 2\omega(k^2/\Lambda^2)k^2\}P^{(0-\omega)}_{\mu\nu\rho\sigma}(k))h^{\rho\sigma}(k). \quad (27)$$

The kinetic operator $K_{\mu\nu\rho\sigma}$ is defined by

$$K_{\mu\nu\rho\sigma} := -\bar{h}_2(z)k^2 P^{(2)}_{\mu\nu\rho\sigma}(k) + \frac{3}{2}k^2\bar{h}_0(z)P^{(0-\omega)}_{\mu\nu\rho\sigma}(k) + \frac{k^2}{2}\bar{h}_0(z)\{P^{(0-s)}_{\mu\nu\rho\sigma}(k) + \sqrt{3}[P^{(0-\omega s)}_{\mu\nu\rho\sigma}(k) + P^{(0-s\omega)}_{\mu\nu\rho\sigma}(k)]\},$$

where the quantities are thus explained

$$\bar{h}_2(z) := \beta - \beta_2\kappa^2\Lambda^2 z + \kappa^2\Lambda^2 z h_2(z), \quad \bar{h}_0(z) := \beta - 6\beta_0\kappa^2\Lambda^2 z + 6\kappa^2\Lambda^2 z h_0(z). \tag{28}$$

In (28) z has to be identified with the D'Alembertian operator in flat spacetime $-\Box^\eta_\Lambda$. We also used the gauge

$$F^\tau = \partial_\mu h^{\mu\tau} \tag{29}$$

and we have introduced the projectors [28]

$$\begin{aligned} P^{(2)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma},\\ P^{(1)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}),\\ P^{(0-s)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \qquad P^{(0-\omega)}_{\mu\nu\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma},\\ P^{(0-s\omega)}_{\mu\nu\rho\sigma} &= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma}, \qquad P^{(0-\omega s)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma}, \end{aligned} \tag{30}$$

where we defined the transverse and longitudinal projectors for vector quantities

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \qquad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \tag{31}$$

Using the orthogonality properties of (30), we can now invert the kinetic matrix in (26) and obtain the graviton propagator. In the following expression the graviton propagator is expressed in the momentum space according to the quadratic action (26),

$$\begin{aligned} D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4}\frac{1}{k^2 + i\epsilon}\Bigg(&\frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{\beta - \beta_2\kappa^2k^2 + \kappa^2k^2h_2(k^2/\Lambda^2)} - \frac{4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\beta - 6\beta_0\kappa^2k^2 + 6\kappa^2k^2h_0(k^2/\Lambda^2)}\\ &- \frac{2\xi P^{(1)}_{\mu\nu\rho\sigma}(k)}{\omega(k^2/\Lambda^2)} - \xi\frac{3P^{(0-s)}_{\mu\nu\rho\sigma}(k) - \sqrt{3}[P^{(0-s\omega)}_{\mu\nu\rho\sigma}(k) + P^{(0-\omega s)}_{\mu\nu\rho\sigma}(k)] + P^{(0-\omega)}_{\mu\nu\rho\sigma}(k)}{\omega(k^2/\Lambda^2)}\Bigg). \end{aligned} \tag{32}$$

Let us consider the graviton propagator in the gauge $\xi = 0$. In this particular gauge, only the first two terms in (32) survive. We will show in the next section that only the physical massless spin-2 pole occurs in the propagator when the theory is renormalized at a certain scale μ_0. The renormalization group invariance preserves unitarity in the dressed physical propagator at any energy scale and no other physical pole emerges at any other scale.

III. RENORMALIZABILITY AND UNITARITY

In this section we want to find an upper bound to the divergences in quantum gravity; before doing this, we closely follow the Tomboulis paper [4] to construct explicitly the entire functions $h_2(z)$ and $h_0(z)$ on which the action (16) depends. Looking at the first two gauge invariant terms in (32), we introduce the following notation:

$$\bar{h}_2(z) = \beta - \beta_2\kappa^2\Lambda^2 z + \kappa^2\Lambda^2 z h_2(z), \quad \bar{h}_0(z) = \beta - 6\beta_0\kappa^2\Lambda^2 z + 6\kappa^2\Lambda^2 z h_0(z), \tag{33}$$

where z will be identified with $-\Box_\Lambda$.

Considering [4], we require the following general properties for the transcendental entire functions h_i ($i = 2, 0$):

(i) $\bar{h}_i(z)$ is real and positive on the real axis, it has no zeroes on the whole complex plane $|z| < +\infty$. This requirement implies that there are no gauge-invariant poles other than the transverse massless physical graviton pole.

(ii) $|h_i(z)|$ has the same asymptotic behavior along the real axis at $\pm\infty$.

(iii) There exists $\Theta > 0$ such that

$$\lim_{|z|\to+\infty} |h_i(z)| \to |z|^\gamma, \qquad \gamma \geqslant 2$$

for the argument of z in the cones

$$C = \{z \mid -\Theta < \arg z < +\Theta, \pi - \Theta < \arg z < \pi + \Theta\}, \quad \text{for } 0 < \Theta < \pi/2.$$

This condition is necessary in order to achieve the (super-)renormalizability of the theory. The necessary asymptotic behavior is imposed not only on the real axis, (ii) but also in conic regions surrounding the real axis. In an Euclidean spacetime, the condition (ii) is not strictly necessary if (iii) applies.

Given the above properties, let us study the ultraviolet behavior of the quantum theory. From the property (iii) in the high energy regime, the propagator in momentum space goes as $1/k^{2\gamma+4}$ [see (32)], but also the n-graviton interaction has the same scaling, since it can be written in the following schematic way:

$$\mathcal{L}^{(n)} \sim h^n \Box_\eta h h_i(-\Box_\Lambda)\Box_\eta h \rightarrow h^n \Box_\eta h(\Box_\eta + h^m \partial h \partial)^\gamma \Box_\eta h, \tag{34}$$

where h is the graviton field and h_i is the entire function defined by the properties (i)–(iii). From (34), the superficial degree of divergence (in four spacetime dimensions) is

$$D = 4L - (2\gamma+4)I + (2\gamma+4)V = 4 - 2\gamma(L-1). \tag{35}$$

In (35) we used again the topological relation between vertexes V, internal lines I, and number of loops L: $I = V + L - 1$. Thus, if $\gamma \geqslant 3$, only 1-loop divergences exist and the theory is super-renormalizable.[1] In this theory the quantities β, β_2, β_0 and eventually the cosmological constant are renormalized,

$$\mathcal{L}_{\rm Ren} = \mathcal{L} - \sqrt{-g}\Big\{\frac{\beta(Z-1)}{\kappa^2}R + \lambda(Z_\lambda - 1) - \beta_2(Z_2-1)\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big) + \beta_0(Z_0-1)R^2\Big\}, \tag{36}$$

where all the coupling must be understood as renormalized at an energy scale μ. On the other hand, the functions h_i are not renormalized. In order to better understand this point we can write the generic entire functions as series, $h_i(z) = \sum_{r=0}^{+\infty} a_r z^r$. For $r \geqslant 1$ there are no counterterms that renormalize a_r because of the superficial degree of divergence (35). Only the coefficient a_0 is renormalized but this is just a normalization convention. The nontrivial dependence of the entire functions h_i on their argument is preserved at quantum level.

Imposing the conditions (i)–(iii) we have the freedom to choose the following form for the entire functions h_i,

$$h_2(z) = \frac{\alpha(e^{H(z)} - 1) + \alpha_2 z}{\kappa^2\Lambda^2 z}, \qquad h_0(z) = \frac{\alpha(e^{H(z)} - 1) + \alpha_0 z}{6\kappa^2\Lambda^2 z}, \tag{37}$$

for three general parameters α, α_2 , and α_0. $H(z)$ is an entire function that exhibits logarithmic asymptotic behavior in the conical region C. Since $H(z)$ is an entire function, $\exp H(z)$ has no zeros in all complex planes for $|z| < +\infty$, according to the property (iii). Furthermore, the nonlocality in the action is actually a "kind" nonlocality, because $\exp H(z)$ is an exponential function and a Taylor expansion of $h_i(z)$ erases the denominator $\Box_\Lambda$ at any energy scale.

The entire function $H(z)$ which is compatible with the property (iii), can be defined as

$$H(z) = \int_0^{p_{\gamma+1}(z)} \frac{1 - \zeta(\omega)}{\omega}\, d\omega, \tag{38}$$

where the following requirements have to be satisfied:

(a) $p_{\gamma+1}(z)$ is a real polynomial of degree $\gamma + 1$ with $p_{\gamma+1}(0) = 0$,

(b) $\zeta(z)$ is an entire and real function on the real axis with $\zeta(0) = 1$,

(c) $|\zeta(z)| \to 0$ for $|z| \to \infty$ in the conical region C defined in (iii).

Let us assume now that the theory is renormalized at some scale μ_0. If we want that the bare propagator to possess no other gauge-invariant pole than the transverse physical graviton pole, we have to set

$$\alpha = \beta(\mu_0), \qquad \frac{\alpha_2}{\kappa^2\Lambda^2} = \beta_2(\mu_0), \qquad \frac{\alpha_0}{6\kappa^2\Lambda^2} = \beta_0(\mu_0). \tag{39}$$

As pointed out in [4], the relations (39) can be used to fix the introduced scale Λ in terms of the Planck scale κ^{-2}. If we fix

$$\beta(\mu_0) = \alpha, \qquad \frac{\beta_2(\mu_0)}{\alpha_2} = \frac{6\beta_0(\mu_0)}{\alpha_0}, \tag{40}$$

the two mass scales are linked by the following relation:

$$\Lambda^2 = \frac{\alpha_2}{\kappa^2\beta_2(\mu_0)}. \tag{41}$$

If the energy scale μ_0 is taken as the renormalization point we get $\bar{h}_2 = \bar{h}_0 = \beta(\mu_0)\exp H(z) := \bar{h}(z)$ and then only the physical massless spin-2 graviton pole occurs in the bare propagator. In the gauge $\xi = 0$ the propagator in (32) simplifies to

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4}\frac{1}{k^2 + i\epsilon}\Bigg(\frac{2P^{(2)}_{\mu\nu\rho\sigma}(k) - 4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\alpha\bar{h}(k^2/\Lambda^2)}\Bigg) = \frac{-i}{(2\pi)^4}\frac{e^{-H(k^2/\Lambda^2)}}{\alpha(k^2 + i\epsilon)}(2P^{(2)}_{\mu\nu\rho\sigma}(k) - 4P^{(0-s)}_{\mu\nu\rho\sigma}(k)). \tag{42}$$

If we choose another renormalization scale μ, then the bare propagator acquires poles; however, these poles cancel in the dressed physical propagator because the shift in the bare part is cancelled with a corresponding shift in the self-energy. This follows easily from the renormalization group invariance. The same procedure is not applicable to the case $h_2(z) = h_0(z) = 0$ [1], because the theory fails to be renormalizable when the unitarity requirement

[1]A *local* super-renormalizable quantum gravity with a large number of metric derivatives was for the first time introduced in [29].

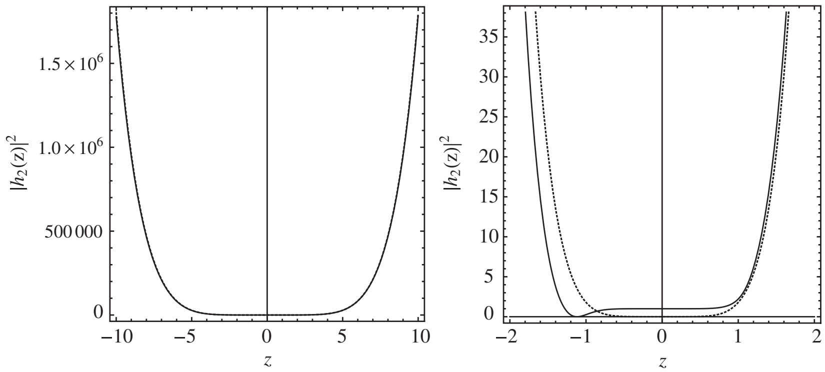

FIG. 1. Plot of $|h_2(z)|^2$ for z real and $\alpha = \alpha_2 = 1$ (solid line). In the first plot $z \in [-10, 10]$ and in the second one $z \in [-2, 2]$. The dashed line represents the asymptotic limit for large real positive and negative values of z. The asymptotic behavior is $|h_2(z)|^2 \approx 1.8z^6$.

$\beta_2 = \beta_0 = 0$ is imposed and an infinite tower of counterterms has to be added to the action.

An explicit example of $\bar{h}(z) = \beta(\mu_0)\exp H(z)$ that satisfies the properties (i)–(iii) can be easily constructed. There are of course many ways to choose $\zeta(z)$, but we focus here on the obvious exponential choice $\zeta(z) = \exp(-z^2)$, which satisfies property (c) in a conical region C with $\Theta = \pi/4$. The entire function $H(z)$ is the result of the integral defined in (38):

$$H(z) = \frac{1}{2}[\gamma_E + \Gamma(0, p^2_{\gamma+1}(z))] + \log[p_{\gamma+1}(z)], \qquad \mathrm{Re}(p^2_{\gamma+1}(z)) > 0, \tag{43}$$

where $\gamma_E = 0.577216$ is the Euler's constant and $\Gamma(a, z) = \int_z^{+\infty} t^{a-1}e^{-t}\,dt$ is the incomplete gamma function. If we choose $p_{\gamma+1}(z) = z^{\gamma+1}$, $H(z)$ simplifies to

$$H(z) = \frac{1}{2}[\gamma_E + \Gamma(0, z^{2\gamma+2})] + \log(z^{\gamma+1}), \qquad \mathrm{Re}(z^{2\gamma+2}) > 0. \tag{44}$$

Another equivalent expression for the entire function $H(z)$ is given by the following series:

$$H(z) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{p_{\gamma+1}(z)^{2n}}{2nn!}, \qquad \mathrm{Re}(p^2_{\gamma+1}(z)) > 0. \tag{45}$$

For $p_{\gamma+1}(z) = z^{\gamma+1}$ the Θ angle, which defines the cone C, is $\Theta = \pi/(4\gamma + 4)$. According to the above expression (45), we find the following behavior near $z = 0$ for the particular choice $p_{\gamma+1}(z) = z^{\gamma+1}$:

$$H(z) = \frac{z^{2\gamma+2}}{2} - \frac{z^{4\gamma+4}}{8} + \frac{z^{6\gamma+6}}{36} + O(z^{6\gamma+7}), \tag{46}$$

where the Taylor expansion is the exact one only for $\arg(z) < \pi/(2\gamma + 2)$, but we already have a stronger constraint on the cone C, $\arg(z) < \Theta = \pi/(4\gamma + 4)$. In particular, $\lim_{z\to 0} H(z) = 0$.[2] A plot of the function $|h_2(z)|^2$ is given in Fig. 1.

IV. SPECTRAL DIMENSION

In this section, we calculate the spectral dimension of the spacetime at short distances, showing that the renormalizability, together with the unitarity of the theory, implies a spectral dimension smaller than one. Let us summarize the definition of spectral dimension in quantum gravity. The definition of spectral dimension is borrowed from the theory of diffusion processes on fractals [50] and easily adapted to the quantum gravity context. Let us study the Brownian motion of a test particle moving on a d-dimensional Riemannian manifold $\mathcal{M}$ with a fixed smooth metric $g_{\mu\nu}(x)$. The probability density for the particle to diffuse from x' to x during the fictitious time T is the heat-kernel $K_g(x, x'; T)$, which satisfies the heat equation

$$\partial_T K_g(x, x'; T) = \Delta_g^{\mathrm{eff}} K_g(x, x'; T), \tag{49}$$

where Δ_g^{eff} denotes the effective covariant Laplacian. It is the usual covariant Laplacian at low energy but it can undergo strong modification in the ultraviolet regime. In particular, we

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[2]We can do a simple choice of the entire function $H(z)$, which gives rise to a condition stronger than (iii). If we take $H(z) = z^2$, then

$$\bar{h}_2(z) = \bar{h}_0(z) = \alpha e^{z^2}, \tag{47}$$

where again $z = -\Box/\Lambda^2$. Another possible choice we wish to analyze is $\bar{h}_2 = \bar{h}_0 = \alpha e^z$, because of its connection with the regular Nicolini-Spallucci black holes [30–49]. For the functions given in (47), the upper bound in (35) can be derived as a truncation of the exponentials and the result does not change: we have divergences at one loop, but the theory is finite for $L > 1$. The exponentials in (47) improve the convergence properties of the theory and the propagator is

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4}\frac{2e^{-k^4/\Lambda^4}}{\alpha(k^2 + i\epsilon)}(P^{(2)}_{\mu\nu\rho\sigma}(k) - 2P^{(0)}_{\mu\nu\rho\sigma}(k)). \tag{48}$$

For $\bar{h}_i(z) = \alpha\exp(z)$, the exponential in the above propagator is replaced with $\exp(-k^2/\Lambda^2)$.

will be interested in the effective Laplacian at high energy and in relation to the flat background. The heat-kernel is a matrix element of the operator $\exp(T\Delta_g)$, acting on the real Hilbert space $L^2(\mathcal{M}, \sqrt{g}\,\mathrm{d}^d x)$, between position eigenstates

$$K_g(x, x'; T) = \langle x'| \exp(T\Delta_g^{\rm eff})|x\rangle. \tag{50}$$

Its trace per unit volume,

$$\begin{aligned} P_g(T) &\equiv V^{-1}\int \mathrm{d}^d x\sqrt{g(x)}K_g(x, x; T) \\ &\equiv V^{-1}\,\mathrm{Tr}\exp(T\Delta_g^{\rm eff}) \end{aligned} \tag{51}$$

has the interpretation of an average return probability. Here $V \equiv \int d^d x\sqrt{g}$ denotes the total volume. It is well-known that $P_g(T)$ possesses an asymptotic expansion for $T \to 0$ of the form $P_g(T) = (4\pi T)^{-d/2}\sum_{n=0}^{\infty} A_n T^n$. The coefficients A_n have a geometric meaning, i.e. A_0 is the volume of the manifold and, if $d = 2$, A_1 is then proportional to the Euler characteristic. From the knowledge of the function $P_g(T)$, one can recover the dimensionality of the manifold as the limit for small T of

$$d_s \equiv -2\frac{\mathrm{d}\ln P_g(T)}{\mathrm{d}\ln T}. \tag{52}$$

If we consider arbitrary fictitious times T, this quantity might depend on the scale we are probing. Formula (52) is the definition of fractal dimension we will use.

From the bare graviton propagator (42), we can easily obtain the heat-kernel and then the spectral dimension. In short, in the momentum space the graviton propagator, omitting the tensorial structure that does not affect the spectral dimension, reads

$$D(k) \propto \frac{1}{k^2\bar{h}(k^2/\Lambda^2)}. \tag{53}$$

We also know that the propagator (in the coordinate space) and the heat-kernel are related by [51]

$$\begin{aligned} G(x, x') &= \int_0^{+\infty} \mathrm{d}T K_g(x, x'; T) \\ &\propto \int \mathrm{d}^4 k e^{ik(x-x')}\int_0^{+\infty} \mathrm{d}T K_g(k; T), \end{aligned} \tag{54}$$

where $G(x, x') \propto \int \mathrm{d}^4 k \exp[ik(x - x')]D(k)$ is the Fourier transform of (53). Given the propagator (53), it is easy to invert (54) with the heat-kernel in the momentum space,

$$K_g(k; T) \propto \exp[-k^2\bar{h}(k^2/\Lambda^2)T], \tag{55}$$

which is the solution of the heat-kernel equation (49) with the effective operator

$$\Delta_g^{\rm eff} = \bar{h}(-\Delta_g/\Lambda^2)\Delta_g, \tag{56}$$

which goes like $(-\Delta_g)^{\gamma+1}\Delta_g$ at high energy. The necessary trace to calculate the average return probability is obtained from the Fourier transform of (55),

$$K_g(x, x'; T) \propto \int \mathrm{d}^4 k e^{-k^2\bar{h}(k^2/\Lambda^2)T}e^{ik(x-x')}. \tag{57}$$

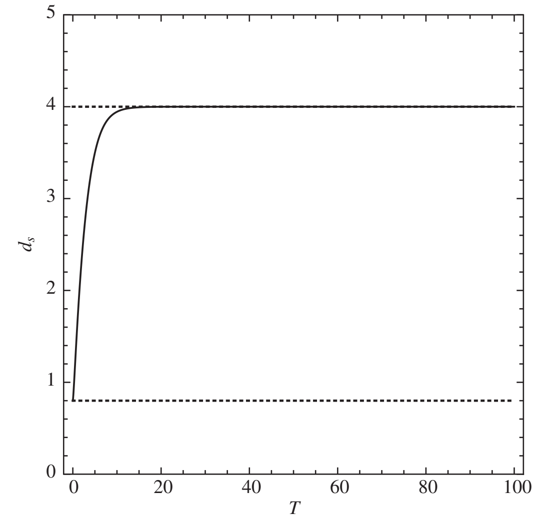

FIG. 2. Plot of the spectral dimension as a function of the fictitious time T for the special case $\gamma = 3$ in (45). The lowest value in the picture is $d_s = 4/5$ at high energy, but at low energy the spectral dimension flows to $d_s = 4$.

Now we are ready to calculate the average return probability defined in (51):

$$P_g(T) \propto \int \mathrm{d}^4 k e^{-k^2\bar{h}(k^2/\Lambda^2)T}. \tag{58}$$

From the requirement (iii) we know that, at high energy, $h(k) \sim k^{2\gamma}$ and then $\bar{h}(k) \sim k^{2\gamma+2}$; therefore, we can calculate the integral (58), and then the spectral dimension defined in (52) for small T will be

$$P_g(T) \propto \frac{1}{T^{2/(2+\gamma)}} \Rightarrow d_s = \frac{4}{\gamma + 2}. \tag{59}$$

The parameter $\gamma \geqslant 3$ implies that the spectral dimension is $d_s < 1$, manifesting a fractal nature of the spacetime at high energy.

We can calculate the spectral dimension at all energy scales as a function of the fictitious time T using the explicit form of the entire function $H(k^2/\Lambda^2)$ given in (45). Integrating numerically (58), we can plot directly the spectral dimension achieving the graphical result in Fig. 2[3] for $\gamma = 3$.

---

[3]For the operators introduced in the previous section $\exp(-\Box/\Lambda^2)^n$ ($n = 1, 2$), the propagator scales as

$$D(k) \propto \frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2}. \tag{60}$$

and the spectral dimension goes to zero at high energy. In particular, for $n = 1$ the heat-kernel can be calculated analytically,

$$K(x, x'; T) = \frac{e^{-((x-x')^2/4(T+1/\Lambda^2))}}{[4\pi(T + 1/\Lambda^2)]^2}, \tag{61}$$

as it is easy to verify by going back to the propagator (60). Now, employing Eq. (52), we find that the spectral dimension is

$$d_s = \frac{4T}{T + 1/\Lambda^2}, \tag{62}$$

which clearly goes to zero for $T \to 0$ and approaches $d_s = 4$ for $T \to +\infty$.

V. ANOTHER CLASS OF THEORIES

In this section we wish to include a more general class of theories following Efimov's study on nonlocal interactions [52]. Let us consider the gauge invariant part of the propagator in the following general form:

$$D(z) = \frac{V(z)}{z\Lambda^2} \tag{63}$$

[the notation is rather compatible with the graviton propagator (42)].

As shown by Efimov [52], the nonlocal field theory is ''unitary'' and ''microcausal'' provided that the following properties are satisfied by $V(z)$:

(I) $V(z)$ is an entire analytic function in the complex z plane and has a finite order of growth $1/2 \leqslant \rho < +\infty$ i.e. $\exists b > 0$, $c > 0$ so that

$$|V(z)| \leqslant c e^{b|z|^\rho}. \tag{64}$$

(II) When $\mathrm{Re}(z) \to +\infty$ $(k^2 \to +\infty)$, $V(z)$ decreases with sufficient rapidity. We can encounter the following cases:
 (a) $V(z) = O(\frac{1}{|z|^a})$ $(a > 1)$,
 (b) $\lim_{\mathrm{Re}(z)\to+\infty} |z|^N |V(z)| = 0$, $\forall N > 0$.

(III) $[V(z)]^* = V(z^*)$ and $V(0) = 1$.

(IV) The function $V(z)$ can be non-negative on the real axis, i.e. $V(x) \geqslant 0$, $x = \mathrm{Re}(z)$.

Here we study the II.b. example of form factor already hinted at in the footnote at the end of Sec. III,

$$V(z) = e^{-z^n} \quad \text{for } n \in \mathbb{N}_+, \qquad \rho = n < +\infty. \tag{65}$$

When omitting the tensorial structure, the high energy propagator in the momentum space reads

$$D(k) = e^{-(k^2/\Lambda^2)^n}/k^2. \tag{66}$$

The ℓ-graviton interaction has the same scaling in the momentum space, as we have similarly highlighted before,

$$\mathcal{L}^{(n)} \sim h^\ell \Box_\eta h \frac{\exp(-\frac{\Box_\eta}{\Lambda^2})^n}{\Box_\eta} \Box_\eta h + \ldots, \tag{67}$$

where "…" indicates other subleading interaction terms coming from the covariant D'Alembertian. Setting an upper bound to the L-loops amplitude, we find

$$\begin{aligned} \mathcal{A}^{(L)} &\leqslant \int (d^4k)^L \Big(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2}\Big)^I (e^{k^{2n}/\Lambda^{2n}} k^2)^V \\ &= \int (dk)^{4L} \Big(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2}\Big)^{I-V} \\ &= \int (dk)^{4L} \Big(\frac{e^{-k^{2n}/\Lambda^{2n}}}{k^2}\Big)^{L-1}, \end{aligned} \tag{68}$$

where in the last step we used again the topological identity $I = V + L - 1$. The L-loops amplitude is UV finite for $L > 1$ and it diverges like "k^4" for $L = 1$. Only 1-loop divergences survive in this theory. The theory is then super-renormalizable, as well as unitary and microcausal [52,53].

Let us conclude by considering the modifications to the gravitational potential due to the form factor $V(z)$. Here we consider a static point particle source of energy tensor $T^\mu_\nu = \mathrm{diag}(-\rho, 0, 0, 0)$ and $\rho = M\delta(\vec{x})$. Given the modified propagator $D(k) = V(k^2/\Lambda^2)/k^2$ (63), the gravitational potential reads

$$\begin{aligned} \Phi(x) &= -\frac{\kappa^2}{8} \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \frac{V(k^2/\Lambda^2)}{k^2} M\delta(\vec{x}') \\ &= -\frac{\kappa^2 M}{8} \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \frac{V(\vec{k}^2/\Lambda^2)}{\vec{k}^2}. \end{aligned} \tag{69}$$

Defining the new variable $p = |\vec{k}|r$, (69) becomes an exclusive function of the radial coordinate and reads

$$\Phi(r) = -\frac{G_N M}{r} \frac{2}{\pi} \int_0^{+\infty} dp J_0(p) V(p^2/r^2\Lambda^2), \tag{70}$$

which can be evaluated for the two classes of form factors here examined.

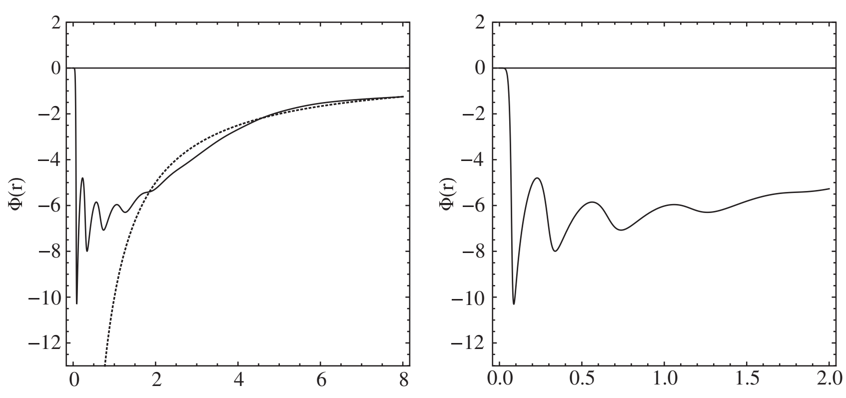

FIG. 3. Plot of the gravitational potential for $\gamma = 3$ and $M = 10$ $(\Lambda = G_N = 1)$. In the first plot the radial coordinate is in the range $r \in [0, 8]$ (in Planck units) and in the second one it is in the range $r \in [0, 2]$.

We first evaluate the integral (70) for the class of theories (45) by which $V(z) = \exp(-H(z))$. For small values of the radial coordinate "r" (large values of "p"), we get $\Phi \approx -2G_N M(\text{const})\Lambda^{2\gamma+2} r^{2\gamma+1}$ (where const $= 3\times 10^7 \pi$ for $\Lambda = 1$ and $G_N = 1$), which is regular for $r \to 0$. A plot of the exact potential for $\gamma = 3$ and $M = 10$ is given in Fig. 3.

For the second class of theories with the propagator given in (66) and $n = 1$, the result of the integral (69) is particularly simple,

$$\Phi(r) = -\frac{G_N M}{r}\,\text{Er}\left(\frac{r\Lambda}{2}\right), \tag{71}$$

which is regular in $r = 0$ and $\Phi(0) = -G_N M\Lambda/\sqrt{\pi}$. For $n > 1$, the potential is again regular in $r = 0$ and $\Phi(0) \propto -G_N M\Lambda$. We can infer that the gravitational potential is regular in the modified renormalizable theories here proposed.

VI. STRUCTURE OF THE INTERACTIONS

We have already shown that the theory is well-defined and power-counting super-renormalizable. However, the calculations are not easy beyond the second order in the graviton expansion. In this section, we give a sketch of how to proceed in the graviton expansion. The reason for the plots in Fig. 4 is to give an operative definition of $\exp H(z)$. One possible approximation in the interactions is the following replacement in the graviton expansion for the minimal renormalizable theory with $\gamma = 3$:

$$e^{H(z)} \approx \begin{cases} z^4 & \text{for } z \gtrsim 1.3, \\ 1 + \frac{z^8}{2} - \frac{z^{24}}{72} + \frac{z^{32}}{288} & \text{for } z \lesssim 1.3. \end{cases} \tag{72}$$

At tree level, or loop amplitudes at high energy, we can just replace $\exp H(z)$ with z^4, at low energy with the second expansion defined in (72) for $z \lesssim 1.3$ and proceed in the calculation, gluing together the results in the two different regimes.

Let us recall again the classical Lagrangian,

$$\mathcal{L} = -\sqrt{-g}\Big\{\frac{\beta}{\kappa^2}R - \left(\beta_2 - \frac{\alpha_2}{\kappa^2\Lambda^2}\right)\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \left(\beta_0 - \frac{\alpha_0}{6\kappa^2\Lambda^2}\right)R^2 - \alpha G^{\mu\nu}\frac{e^{H(-\Box_\Lambda)} - 1}{\kappa^2\Box}R_{\mu\nu}\Big\}, \tag{73}$$

where $\exp H(z)$ is defined in (72). This Lagrangian interpolates between the Einstein-Hilbert Lagrangian at low energy and a high energy theory living in a spacetime of spectral dimension $d_s = 4/(\gamma + 2)$.

A first approximation (but also an operative way to proceed) is to replace the interaction Lagrangian in the UV with the following truncation:

$$\mathcal{L}^{\text{int}}_{\text{UV}} \approx -\frac{\alpha}{\kappa^2\Lambda^2}\sqrt{-g}G^{\mu\nu}\left(\frac{-\Box}{\Lambda^2}\right)^{\gamma}R_{\mu\nu} \quad \text{for } k \gtrsim \Lambda, \tag{74}$$

(k is the energy scale) and the infrared Lagrangian with

$$\mathcal{L}^{\text{int}}_{\text{IR}} = -\sqrt{-g}\Big\{\frac{\beta}{\kappa^2}R - \left(\beta_2 - \frac{\alpha_2}{\kappa^2\Lambda^2}\right)\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \left(\beta_0 - \frac{\alpha_0}{6\kappa^2\Lambda^2}\right)R^2 + \frac{\alpha}{2\kappa^2\Lambda^2}G^{\mu\nu}\left(\frac{-\Box}{\Lambda^2}\right)^{2\gamma+1}R_{\mu\nu}\Big\} \tag{75}$$

for $k \lesssim \Lambda$. In (74) and (75) we used the same expansion of (72) but for a general value of γ. On the other hand, the propagator is the same in both regimes and is given in (32). The three graviton interactions can be obtained performing an $h^{\mu\nu}$ power expansion in (74) and (75). At tree level, n-points functions will be obtain by an interpolation of the amplitude calculated in the two different regimes $k \lesssim \Lambda$ and $k \gtrsim \Lambda$, using, respectively, the $\mathcal{L}^{\text{int}}_{\text{IR}}$ and $\mathcal{L}^{\text{int}}_{\text{UV}}$. In loop amplitudes, we should integrate the interactions terms coming from $\mathcal{L}^{\text{int}}_{\text{IR}}$ up to $k \lesssim \Lambda$ and the interactions coming from $\mathcal{L}^{\text{int}}_{\text{UV}}$ in the range $\Lambda \lesssim k < +\infty$ in the same amplitude.

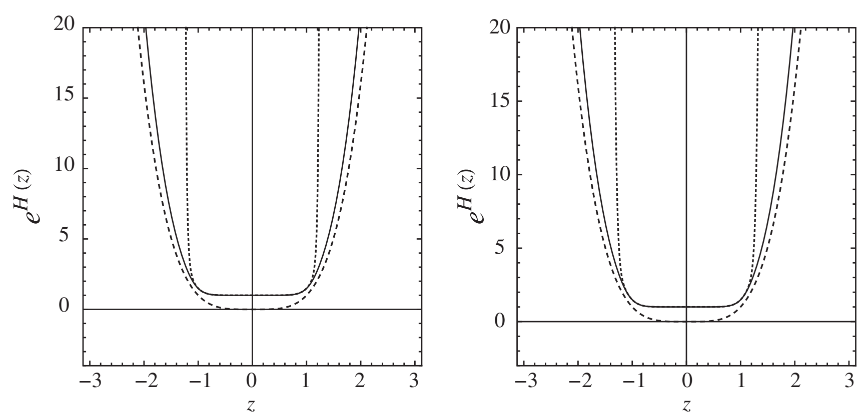

FIG. 4. The first plot is the function $\exp H(z)$ for $p_{\gamma+1} = p_4 = z^4$. The solid line represents the exact function; the large and small dashed lines represent the same function for large and small value of z, respectively: z^4 and $\exp(z^{24}/36 - z^{16}/8 + z^8/2)$. In the second plot, the small dashed line represents a further simplification of the same function: $\exp H(z) \approx 1 + z^8/2 - z^{24}/72 + z^{32}/288$.

VII. BLACK HOLES

In this section, we want to solve the equation of motion coming from the renormalized theory in the case of a spherically symmetric spacetime. Let us start from the classical Lagrangian that we rewrite rearranging the parameters in a different way,

$$\mathcal{L} = -\sqrt{-g}\Big\{\frac{\beta}{\kappa^2}R - \Big(\beta_2 - \frac{\alpha_2}{\kappa^2\Lambda^2}\Big)\Big(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\Big) + \Big(\beta_0 - \frac{\alpha_0}{6\kappa^2\Lambda^2}\Big)R^2 - \alpha G^{\mu\nu}\frac{V^{-1}(-\Box_\Lambda) - 1}{\kappa^2\Box}R_{\mu\nu}\Big\}, \tag{76}$$

where again $V(z) := \exp - H(z)$.[4] From [5], the equations of motion for the above theory up to square curvature terms are

$$G_{\mu\nu} + O(R^2_{\mu\nu}) = 8\pi G_N V(-\Box_\Lambda)T_{\mu\nu}. \tag{77}$$

Since we are going to solve the Einstein equations neglecting curvature square terms, we have to impose the conservation $\nabla^\mu(V(-\Box_\Lambda)T_{\mu\nu}) = 0$ in order for the theory to be compatible with the Bianchi identities. For the exact equations of motion the Bianchi identities are of course satisfied because of the diffeomorphism invariance. The condition $\nabla^\mu(V(-\Box_\Lambda)T_{\mu\nu}) = 0$ compensates the truncation in the modified Einstein equations (77).

Our main purpose is to solve the field equations by assuming a static source, which means that the four-velocity field u^μ has only a non-vanishing timelike component $u^\mu \equiv (u^0, \vec{0})$ $u^0 = (-g^{00})^{-1/2}$. We consider the component $T^0{}_0$ of the energy-momentum tensor for a static source of mass M in polar coordinates to be $T^0{}_0 = -M\delta(r)/4\pi r^2$ [55,56].[5] The metric of our spacetime is assumed to be given by the usual static, spherically symmetric Schwarzschild form

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2\Omega^2, \qquad F(r) = 1 - \frac{2m(r)}{r}. \tag{78}$$

The effective energy density and pressures are defined by

$$V(-\Box_\Lambda)T^\mu{}_\nu = \frac{G^\mu{}_\nu}{8\pi G_N} = \mathrm{Diag}(-\rho^e, P^e_r, P^e_\perp, P^e_\perp). \tag{79}$$

For later convenience, we temporarily adopt free falling Cartesian-like coordinates [56], we calculate the effective energy density assuming $p_{\gamma+1}(z) = z^4$ in $H(z)$,

$$\begin{aligned}\rho^e(\vec{x}) &= -V(-\Box_\Lambda)T^0{}_0 = MV(-\Box_\Lambda)\delta(\vec{x})\\ &= M\int\frac{d^3k}{(2\pi)^3}e^{-H(k^2/\Lambda^2)}e^{i\vec{k}\cdot\vec{x}}\\ &= \frac{2M}{(2\pi)^2r^3}\int_0^{+\infty}e^{-H(p^2/r^2\Lambda^2)}p\sin(p)dp,\end{aligned} \tag{80}$$

where $r = |\vec{x}|$ is the radial coordinate. Here we introduced the Fourier transform for the Dirac delta function and we also introduced a new dimensionless variable in the momentum space, $p = kr$, where k is the physical momentum. The energy density distribution defined in (80) respects spherical symmetry. We evaluated numerically the integral in (80) and the resulting energy density is plotted in Fig. 5. In the low energy limit we can expand $H(z)$ for $z = -\Box/\Lambda^2 \ll 1$ and we can integrate analytically (80)

$$\rho^e(r) = \frac{2M}{(2\pi)^2r^3}\int_0^{+\infty}e^{-p^{16}/(2r^{16}\Lambda^{16})}p\sin(p)dp. \tag{81}$$

The result is really involved and the plot is given in Fig. 5; however, the Taylor expansion near $r \approx 0$ gives a constant leading order

$$\rho^e(r) \approx \frac{M\Lambda^3}{322^{7/16}\Gamma(11/16)\Gamma(7/8)\Gamma(5/4)} + O(r^2). \tag{82}$$

The covariant conservation and the additional condition, $g_{00} = -g^{-1}_{rr}$, completely specify the form of the effective energy tensor $V(-\Box_\Lambda)T^\mu{}_\nu$ and the Einstein's equations read

$$\begin{aligned}\frac{dm(r)}{dr} &= 4\pi\rho^e r^2,\\ \frac{1}{F}\frac{dF}{dr} &= \frac{2(m(r) + 4\pi P^e_r r^3)}{r(r - 2m(r))},\\ \frac{dP^e_r}{dr} &= -\frac{1}{2F}\frac{dF}{dr}(\rho^e + P^e_r) + \frac{2}{r}(P^e_\perp - P^e_r).\end{aligned} \tag{83}$$

Because of the complicated energy density profile, it is not easy to integrate the first Einstein equation in (83),

$$m(r) = 4\pi\int_0^r dr' r'^2\rho^e(r'). \tag{84}$$

However, the energy density goes to zero at infinity, reproducing the asymptotic Schwarzschild spacetime with $m(r) \approx M$ (constant). On the other hand, it is easy to calculate the energy density profile close to $r \approx 0$ since

[4]In general, a differential equation with an infinity number of derivative does not have a well-defined initial value problem and it needs an infinite number of initial conditions. It is shown in [54] that in a general framework each pole of the propagator contributes two initial data to the final solution. This is precisely our case because the only pole in the bare propagator is the massless graviton and the theory has a well-defined Cauchy problem.

[5]Usually, in general relativity textbooks, the Schwarzschild solution is introduced without mentioning the presence of a pointlike source. Once the Einstein equations are solved in the vacuum, the integration constant is determined by matching the solution with the Newtonian field outside a spherically symmetric mass distribution. Definitely, this is not the most straightforward way to expose students to one of the most fundamental solutions of the Einstein equations. Moreover, the presence of a curvature singularity in the origin, where from the very beginning a Þnite mass energy is squeezed into a zero-volume point, is introduced as a shocking, unexpected result. Against this background, we show that, once quantum delocalization of the source is accounted, all these flaws disappear. From this, it follows that for us there is only one physical vacuum solution and this is the Minkowski metric. In other words, the Schwarzschild metric is a vacuum solution with the free integration M equal to zero.

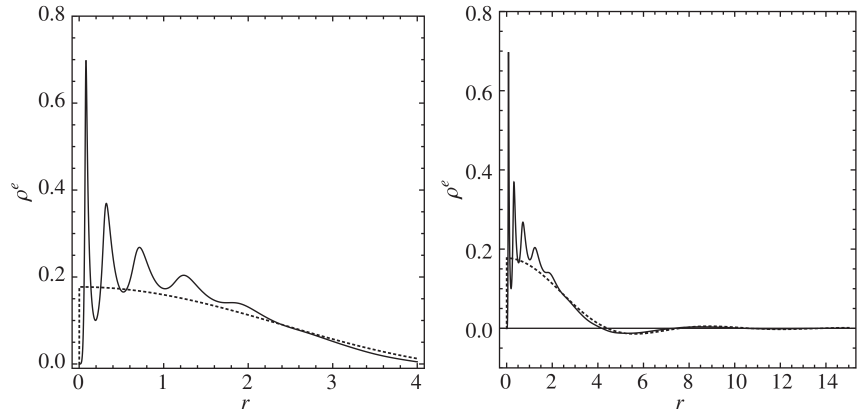

FIG. 5. Plot of the energy density for $M = 10$ in Planck units assuming $\Lambda = m_P$. The solid line is a plot of (80) without any approximation; the dashed line refers to the energy density profile (81) in the limit $-\Box/\Lambda^2 \ll 1$.

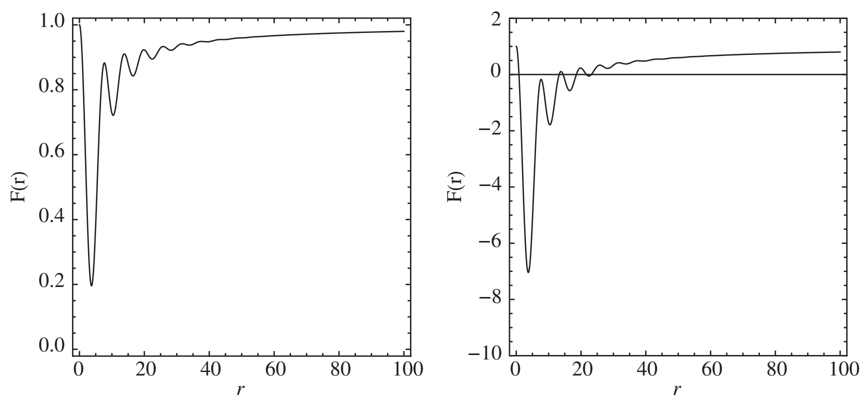

FIG. 6. The two plots show the function $F(r)$, assuming the infrared energy density profile (80). The two plots differ in the value of the ADM mass, which is $M = 1$ in the first plot and $M = 10$ in the second plot (in Planck units, assuming the fundamental scale Λ to be the Planck mass). A crucial property of those black holes is the possibility to have "multihorizon black holes" depending on the mass value. For $M = 10$, for example, we have six horizons according to the second plot.

$H(z) \to \log z^4$ for $z \to +\infty$ [or $r \to 0$ in (80)]. In this regime $m(r) \propto M\Lambda^8 r^8$ and for a more general monomial $p_{\gamma+1}(z) = z^{\gamma+1}$, $m(r) \propto M(\Lambda r)^{2\gamma+2}$. The function $F(r)$ in the metric, close to $r \approx 0$, is

$$F(r) \approx 1 - cM\Lambda^{2\gamma+2} r^{2\gamma+1}, \tag{85}$$

where c is a dimensionless constant.

We show now that the metric has at least two horizons, an event horizon and a Cauchy horizon. The metric interpolates two asymptotic flat regions, one at infinity and the other in $r = 0$, so that we can write the $g_{rr}^{-1} = F$ component in the following way:

$$F(r) = 1 - \frac{2mf(r)}{r}, \tag{86}$$

where $f(r) \to 1$ for $r \to \infty$, $f(r) \propto r^{2\gamma+2}$ for $r \to 0$ and $f(r)$ does not depend on the mass M. The function $F(r)$ goes to "1" in both limits (for $r \to +\infty$ and $r \to 0$) and, since M is a multiplicative constant, we can always choose the mass M for a fixed value of the radial coordinate r, such that $F(r)$ becomes negative. From this, it follows that the function $F(r)$ must change sign at least twice. The second equation in (83) is solved by $P_r^{\rm e} = -\rho^{\rm e}$ and the third one defines the transversal pressure once known the energy density ρ^e. Given the lapse function $F(r)$ in (85), we can calculate the Ricci scalar and the Kretschmann invariant

$$\begin{aligned} R &= cM\Lambda^{2\gamma+2}(2\gamma+2)(2\gamma+3)r^{2\gamma-1}, \\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 4c^2M^2\Lambda^{4\gamma+4}(4\gamma^4+4\gamma^3+5\gamma^2 \\ &\quad + 4\gamma + 2)r^{4\gamma-2}. \end{aligned} \tag{87}$$

By evaluating the above curvature tensors at the origin one finds that they are finite for $\gamma \geqslant 1/2$ and, in particular, for the minimal super-renormalizable theory with $\gamma \geqslant 3$.

The entire function $\exp H(z)$ is able to tame the curvature singularity of the Schwarzschild solution at least for the truncation of the theory here analyzed. We think that the higher order corrections to the Einstein equation will not change the remarkable feature of the solutions found in this section.

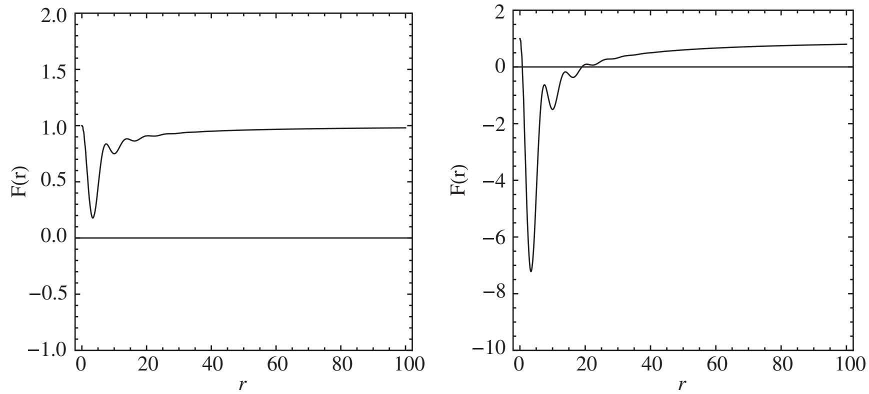

FIG. 7. This plots show the function $F(r)$ for the energy profile (81) and $H(z)$ defined in (44) with the parameter $\gamma = 3$. The ADM mass values are $M = 1$ and $M = 10$ (in Planck units) for the first and the second plot, respectively.

Besides the analysis exposed above, we can integrate numerically the modified Einstein equation of motions (77) for the two energy densities defined, respectively, in (80) and (81). Using the integral form of the mass function (84), we achieve the metric component $F(r)$ defined in (78). The numerical results are plotted in Figs. 6 and 7 for different values of the Arnowitt, Deser, and Misner (ADM) mass M. The metric function $F(r)$ can intersect zero times, twice or more than twice the horizontal axis relative to the value of the ADM mass M. This opens the possibility to have "*multihorizon black holes*" as an exact solution of the equation of motions (15).

VIII. MULTIDIMENSIONAL THEORY

In this section we consider the multidimensional generalization of the action (16). We start from the generalization of the Stelle theory to get to a d-dimensionl renormalizable one.[6] In short, the Lagrangian with at most X derivatives of the metric is

$$\mathcal{L}_{d\text{-Ren}} = a_1 R + a_2 R^2 + b_2 R_{\mu\nu}^2 + \ldots + a_X R^{X/2} + b_X R_{\mu\nu}^{X/2} + c_X \ldots R_{\mu\nu\rho\sigma}^{X/2} + d_X R \Box^{((X/2)-2)} R \ldots . \quad (88)$$

In the second line, the dots on the left imply a finite number of extra terms with fewer derivatives of the metric tensor, and the dots on the right indicate a finite number of operators with the same number of derivatives but higher powers of the curvature ($O(R^2 \Box^{((X/2)-2)} R)$). In (88), the power counting tells us that the maximal superficial degree of divergence of a Feynman graph is

$$D = d - (d - X)(V - I). \quad (89)$$

For $X = d$ the theory is renormalizable since the maximal divergence is $D = d$ and all the infinities can be absorbed in the operators already present in the action (88).

The general action of "derivative order N" can be found combining curvature tensors and covariant derivatives of the curvature tensor. In short the action reads as follows [29]:

$$S = \sum_{n=0}^{N+2} \alpha_{2n} \Lambda^{d-2n} \int d^d x \sqrt{|g|} \mathcal{O}_{2n}(\partial_\rho g_{\mu\nu}) + S_{\rm NL} \quad (90)$$

where Λ is a mass scale in our fundamental theory, $\mathcal{O}_{2n}(\partial_\rho g_{\mu\nu})$ denotes the general covariant scalar term containing "$2n$" derivatives of the metric $g_{\mu\nu}$, while $S_{\rm NL}$ is a nonlocal action term that we are going to set later [4]. The maximal number of derivatives in the local part of the action is $2N + 4$. We can classify the local terms in the following way:

$$\begin{aligned}
&N = 0\colon\ S_0 = \lambda + c_0^{(0)} R + c_1^{(0)} R^2 + c_2^{(0)} R_{\mu\nu} R^{\mu\nu} + c_3^{(0)} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma},\\
&N = 1\colon\ S_1 = S_0 + c_1^{(1)} R^3_{\ldots} + c_2^{(1)} \nabla R_{\ldots} \nabla R_{\ldots},\\
&N = 2\colon\ S_2 = S_0 + S_1 + c_1^{(2)} R^4_{\ldots} + c_2^{(2)} R_{\ldots} \nabla R_{\ldots} \nabla R_{\ldots} + c_3^{(3)} \nabla^2 R_{\ldots} \nabla^2 R_{\ldots},\\
&\ldots\\
&\ldots\\
&N = N\colon\ S_N = \sum_{i=0}^{N-1, N>0} S_i + c_1^{(N)} R^{N+2}_{\ldots} + c_2^{(N)} R^{N-1}_{\ldots} \nabla R_{\ldots} \nabla R_{\ldots} + \ldots + c_M^{(N)} R \ldots \Box^N R \ldots
\end{aligned} \quad (91)$$

[6] Here we use the signature $(+ - \ldots -)$. The curvature tensor is defined by $R^\alpha_{\beta\gamma\delta} = -\partial_\delta \Gamma^\alpha_{\beta\gamma} + \ldots$, the Ricci tensor by $R_{\mu\nu} = R^\alpha_{\mu\nu\alpha}$, and the curvature scalar by $R = g^{\mu\nu} R_{\mu\nu}$ and $g_{\mu\nu}$ is the metric tensor.

In the local theory (88), renormalizability requires $X = d$, so that the relation between the spacetime dimension and the derivative order is $2N + 4 = d$. To avoid fractional powers of the D'Alembertian operator, we take $2N + 4 = d_{\rm odd} + 1$ in odd dimensions and $2N + 4 = d_{\rm even}$ in even dimensions. Given the general structure (90), for $N \geqslant 0$ and $n \geqslant 2$, contributions to the propagator come only from the following operators:

$$R_{\mu\nu}\Box^{n-2}R^{\mu\nu}, \quad R\Box^{n-2}R, \quad R_{\mu\nu\alpha\beta}\Box^{n-2}R^{\mu\nu\alpha\beta}. \tag{92}$$

However, using the Bianchi and Ricci identities one can reduce the terms listed above from three to two (with total $2n$ derivatives),

$$\begin{aligned} &R_{\mu\nu\alpha\beta}\Box^{n-2}R^{\mu\nu\alpha\beta} \\ &= -\nabla_\lambda R_{\mu\nu\alpha\beta}\Box^{n-3}\nabla^\lambda R^{\mu\nu\alpha\beta} + O(R^3) + \nabla_\mu\Omega^\mu \\ &= 4R_{\mu\nu}\Box^{n-2}R^{\mu\nu} - R\Box^{n-2}R + O(R^3) + \nabla_\mu\Omega'^\mu, \end{aligned} \tag{93}$$

where $\nabla_\mu\Omega^\mu$ and $\nabla_\mu\Omega'^\mu$ are total divergence terms. Applying (93) to (92), for $n \geqslant 2$ we discard the third term and we keep the first two.

We now have to define the "non-locl" action term in (90). As we are going to show, both super-renormalizability and unitarity require the following two nonlocal operators:

$$R_{\mu\nu}h_2(-\Box_\Lambda)R^{\mu\nu}, \qquad Rh_0(-\Box_\Lambda)R. \tag{94}$$

The full action, focusing mainly on the nonlocal terms and the quadratic part in the curvature, reads

$$\begin{aligned} S = \int d^dx\sqrt{|g|}\Big[&2\kappa^{-2}R + \bar\lambda + \sum_{n=0}^{N}(a_nR(-\Box_\Lambda)^nR \\ &+ b_nR_{\mu\nu}(-\Box_\Lambda)^nR^{\mu\nu}) + Rh_0(-\Box_\Lambda)R \\ &+ R_{\mu\nu}h_2(-\Box_\Lambda)R^{\mu\nu} \\ &+ \underbrace{\ldots\ldots\ldots O(R^3)\ldots\ldots\ldots + R^{N+2}}_{\text{finite number of terms}}\Big]. \end{aligned} \tag{95}$$

The last line is a collection of local terms that are renormalized at quantum level. In the action, the couplings and the nonlocal functions have the following dimensions: $[a_n] = [b_n] = M^{d-4}$, $[\kappa^2] = M^{2-d}$, $[\bar\lambda] = M^d$, $[h_2] = [h_0] = M^{d-4}$.

At this point, we are ready to expand the Lagrangian at the second order in the graviton fluctuation. Splitting the spacetime metric in the flat Minkowski background and the fluctuation $h_{\mu\nu}$ defined by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, we get [57]

$$\begin{aligned} \mathcal{L}_{\rm lin} = &-\frac{1}{2}[h^{\mu\nu}\Box h_{\mu\nu} + A_\nu^2 + (A_\nu - \phi_{,\nu})^2] \\ &+\frac{1}{4}\Big[\frac{\kappa^2}{2}\Box h_{\mu\nu}\beta(\Box)\Box h^{\mu\nu} - \frac{\kappa^2}{2}A^\mu_{,\mu}\beta(\Box)A^\nu_{,\nu} \\ &-\frac{\kappa^2}{2}F^{\mu\nu}\beta(\Box)F_{\mu\nu} + \frac{\kappa^2}{2}(A^\alpha_{,\alpha} - \Box\phi)\beta(\Box)(A^\beta_{,\beta} - \Box\phi) \\ &+2\kappa^2(A^\alpha_{,\alpha} - \Box\phi)\alpha(\Box)(A^\beta_{,\beta} - \Box\phi)\Big], \end{aligned} \tag{96}$$

where $A^\mu = h^{\mu\nu}{}_{,\nu}$, $\phi = h$ (the trace of $h_{\mu\nu}$), $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ and the functionals of the D'Alembertian operator $\beta(\Box)$, $\alpha(\Box)$ are defined by

$$\begin{aligned} \alpha(\Box)/2 &:= \sum_{n=0}^{N} a_n(-\Box_\Lambda)^n + h_0(-\Box_\Lambda), \\ \beta(\Box)/2 &:= \sum_{n=0}^{N} b_n(-\Box_\Lambda)^n + h_2(-\Box_\Lambda). \end{aligned} \tag{97}$$

The d'Alembertian operator in (96) and (97) must be conceived on the flat spacetime. The linearized Lagrangian (96) is invariant under infinitesimal coordinate transformations $x^\mu \to x^\mu + \kappa\xi^\mu(x)$, where $\xi^\mu(x)$ is an infinitesimal vector field of dimensions $[\xi(x)] = M^{(d-4)/2}$. Under this transformation, the graviton field turns into

$$h_{\mu\nu} \to h_{\mu\nu} - \xi(x)_{\mu,\nu} - \xi(x)_{\nu,\mu}. \tag{98}$$

The presence of the local gauge simmetry (98) calls for the addition of a gauge-fixing term to the linearized Lagrangian (96). Hence, we choose the following fairly general gauge-fixing operator:

$$\begin{aligned} \mathcal{L}_{\rm GF} = &\lambda_1(A_\nu - \lambda\phi_{,\nu})\omega_1(-\Box_\Lambda)(A^\nu - \lambda\phi^{,\nu}) \\ &+\frac{\lambda_2\kappa^2}{8}(A^\mu_{,\mu} - \lambda\Box\phi)\beta(\Box)\omega_2(-\Box_\Lambda)(A^\nu_{,\nu} - \lambda\Box\phi) \\ &+\frac{\lambda_3\kappa^2}{8}F_{\mu\nu}\beta(\Box)\omega_3(-\Box_\Lambda)F^{\mu\nu}, \end{aligned} \tag{99}$$

where $\omega_i(-\Box_\Lambda)$ are three weight functionals [1]. In the harmonic gauge $\lambda = \lambda_2 = \lambda_3 = 0$ and $\lambda_1 = 1/\xi$. The linearized gauge-fixed Lagrangian reads

$$\mathcal{L}_{\rm lin} + \mathcal{L}_{\rm GF} = \frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma}, \tag{100}$$

where the operator $\mathcal{O}$ is made of two terms, one coming from the linearized Lagrangian (96) and the other from the gauge-fixing term (99). Inverting the operator $\mathcal{O}$ [57], we find the two point function in the harmonic gauge ($\partial^\mu h_{\mu\nu} = 0$),

$$\mathcal{O}^{-1}(k) = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2\omega_1(k^2/\Lambda^2)} + \frac{P^{(2)}}{k^2\left(1 + \frac{k^2\kappa^2\beta(k^2)}{4}\right)} - \frac{P^{(0)}}{2k^2\left(\frac{d-2}{2} - k^2\frac{d\beta(k^2)\kappa^2/4+(d-1)\alpha(k^2)\kappa^2}{2}\right)}. \quad (101)$$

The tensorial indexes for the operator $\mathcal{O}^{-1}$ and the projectors $P^{(0)}$, $P^{(2)}$, $P^{(1)}$, $\bar{P}^{(0)}$ have been omitted and the functions $\alpha(k^2)$ and $\beta(k^2)$ are achieved by replacing $-\Box \to k^2$ in the definitions (97). The projectors are defined by [28,57]

$$\begin{aligned}
P^{(2)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{d-1}\theta_{\mu\nu}\theta_{\rho\sigma},\\
P^{(1)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}),\\
P^{(0)}_{\mu\nu\rho\sigma}(k) &= \frac{1}{d-1}\theta_{\mu\nu}\theta_{\rho\sigma},\\
\bar{P}^{(0)}_{\mu\nu\rho\sigma}(k) &= \omega_{\mu\nu}\omega_{\rho\sigma}, \quad (102)\\
\bar{\bar{P}}^{(0)}_{\mu\nu\rho\sigma} &= \theta_{\mu\nu}\omega_{\rho\sigma} + \omega_{\mu\nu}\theta_{\rho\sigma},\\
\theta_{\mu\nu} &= \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2},\\
\omega_{\mu\nu} &= \frac{k_\mu k_\nu}{k^2}.
\end{aligned}$$

Looking at the last two gauge invariant terms in (101), we deem it convenient to introduce the following definitions:

$$\begin{aligned}
\bar{h}_2(z) &= 1 + \frac{\kappa^2\Lambda^2}{2}z\sum_{n=0}^{N} b_n z^n + \frac{\kappa^2\Lambda^2}{2}zh_2(z),\\
\left[\frac{d-2}{2}\right]\bar{h}_0(z) &= \frac{d-2}{2} - \frac{\kappa^2\Lambda^2 d}{4}z\left[\sum_{n=0}^{N} b_n z^n + h_2(z)\right]\\
&\quad - \kappa^2\Lambda^2(d-1)z\left[\sum_{n=0}^{N} a_n z^n + h_0(z)\right], \quad (103)
\end{aligned}$$

where again $z = -\Box_\Lambda$. Through the above definitions (103), the gauge invariant part of the propagator greatly simplifies to

$$\mathcal{O}^{-1}(k)^{\xi=0} = \frac{1}{k^2}\left(\frac{P^{(2)}}{\bar{h}_2} - \frac{P^{(0)}}{(d-2)\bar{h}_0}\right). \quad (104)$$

As pointed out in Sec. III, we again demand that the general properties (i) and (ii) be met for the transcendental entire functions $h_i(z)$ $(i = 0, 2)$ and/or $\bar{h}_i(z)$ $(i = 0, 2)$, while we replace the requirement (iii) with the following generalization in d dimensions:

(iii) → (iii) There exists $\Theta > 0$ such that

$$\begin{aligned}
&\lim_{|z|\to+\infty}|h_i(z)| \to |z|^{\gamma+N},\\
&\gamma \geqslant d/2 \quad \text{for } d = d_{\text{even}},\\
&\gamma \geqslant (d-1)/2 \quad \text{for } d = d_{\text{odd}}, \quad (105)
\end{aligned}$$

for the argument of z in the following conical regions:

$$C = \{z| -\Theta < \arg z < +\Theta,\ \pi - \Theta < \arg z < \pi + \Theta\},$$
$$\text{for } 0 < \Theta < \pi/2.$$

Let us then examine the ultraviolet behavior of the quantum theory. According the property (iii) in the high energy regime, the propagator in the momentum space goes as

$$\mathcal{O}^{-1}(k) \sim 1/k^{2\gamma+2N+4} \quad \text{for large } k^2$$

[see (95), (103), and (104)]. However, the n-graviton interaction has the same leading scaling of the kinetic term, since it can be written in the following schematic way:

$$\begin{aligned}
\mathcal{L}^{(n)} &\sim h^n \Box h h_i(-\Box_\Lambda)\Box_\eta h\\
&\to h^n\Box h(\Box_\eta + h^m\partial h\partial)^{\gamma+N}\Box_\eta h, \quad (106)
\end{aligned}$$

where the indexes for the graviton fluctuation $h_{\mu\nu}$ are omitted and $h_i(-\Box_\Lambda)$ is the entire function defined by the properties (i)–(iii). From (106), the superficial degree of divergence in a spacetime of "even" dimension is

$$D_{\text{even}} = d_{\text{even}}. \quad (107)$$

On the other hand, in a spacetime of "odd" dimension the upper limit to the degree of divergence is

$$D_{\text{odd}} = d_{\text{odd}} - (2\gamma + 1)(L - 1). \quad (108)$$

In (107) and (108) we used again the topological relation between vertexes V, internal lines I, and number of loops L: $I = V + L - 1$. Thus, if $\gamma > d_{\text{even}}/2$ or $\gamma > (d_{\text{odd}} - 1)/2$, only 1-loop divergences exist and the theory is super-renormalizable. Only a finite number of constants are renormalized in the action (95), i.e. κ, $\bar{\lambda}$, a_n, b_n and the finite number of couplings that multiply the operators in the last line. The renormalized action reads

$$\begin{aligned}
S = \int d^dx\sqrt{|g|}\Big[&2Z_\kappa\kappa^{-2}R + Z_{\bar{\lambda}}\bar{\lambda}\\
&+ \sum_{n=0}^{N}(Z_{a_n}a_nR(-\Box_\Lambda)^nR + Z_{b_n}b_nR_{\mu\nu}(-\Box_\Lambda)^nR^{\mu\nu})\\
&+ Rh_0(-\Box_\Lambda)R + R_{\mu\nu}h_2(-\Box_\Lambda)R^{\mu\nu}\\
&+ Z_{c_1^{(1)}}c_1^{(1)}R^3 + \ldots\ldots\ldots + Z_{c_1^{(N)}}c_1^{(N)}R^{N+2}\Big]. \quad (109)
\end{aligned}$$

All the couplings in (109) must be understood as renormalized at an energy scale μ. Contrarily, the functions $h_i(z)$ are not renormalized. Like in $d = 4$, we can write the entire functions as a series, i.e. $h_i(z) = \sum_{r=0}^{+\infty} a_r z^r$.

Because of the superficial degrees of divergence (107) and (108), there are no counterterms that renormalize a_r for $r > N$. As a matter of fact, the couplings in the second line of (109) already incorporate the renormalizations of the coefficients a_r for $r \leqslant N$. Therefore, the nontrivial dependence of the entire functions $h_i(z)$ on their argument is preserved at quantum level.

Imposing the conditions (i)–(iii) we have the freedom to choose the following form for the functions h_i:

$$h_2(z) = \frac{V(z)^{-1} - 1 - \frac{\kappa^2\Lambda^2}{2} z \sum_{n=0}^{N} \tilde{b}_n z^n}{\frac{\kappa^2\Lambda^2}{2} z}, \qquad h_0(z) = -\frac{V(z)^{-1} - 1 + \kappa^2\Lambda^2 z \sum_{n=0}^{N} \tilde{a}_n z^n}{\kappa^2\Lambda^2 z}, \tag{110}$$

for general parameters $\tilde{a}_n$ and $\tilde{b}_n$. Here $V(z)^{-1} = e^{H(z)}$ and $H(z)$ is an entire function that exhibits logarithmic asymptotic behavior in the conical region C. The form factor $\exp H(z)$ has no zeros in the entire complex plane for $|z| < +\infty$. Furthermore, the nonlocality in the action is actually a "soft" form of nonlocality, because a Taylor expansion of $h_i(z)$ eliminates the denominator $\Box_\Lambda$ at any energy scale.

The entire function $H(z)$, which is compatible with the property (iii), can be defined as

$$H(z) = \int_0^{p_{\gamma+N+1}(z)} \frac{1 - \zeta(\omega)}{\omega} d\omega, \tag{111}$$

where $p_{\gamma+N+1}(z)$ is a real polynomial of degree $\gamma + N + 1$ with $p_{\gamma+N+1}(0) = 0$, and $\zeta(z)$ must satisfy the requirements (b) and (c) of Sec. III.

Let us investigate the unitarity of the theory. We assume that the theory is renormalized at some scale μ_0. If we set

$$\tilde{a}_n = a_n(\mu_0), \qquad \tilde{b}_n = b_n(\mu_0), \tag{112}$$

the bare propagator does not possess other gauge-invariant pole than the physical graviton one. Since the energy scale μ_0 is taken as the renormalization point, we get $\bar{h}_2 = \bar{h}_0 = V(z)^{-1} = \exp H(z)$. Thus, only the physical massless spin-2 graviton pole occurs in the bare propagator and (104) reads

$$\mathcal{O}^{-1}(k)^{\xi=0} = \frac{V(k^2/\Lambda^2)}{k^2}\left(P^{(2)} - \frac{P^{(0)}}{d-2}\right). \tag{113}$$

The momentum or energy scale at which the relation between the quantity computed and the quantity measured is identified is called the subtraction point and is indicated usually by "μ" [58]. The subtraction point is arbitrary and unphysical, so the final answers do not have to depend on the subtraction scale μ. Therefore, the derivative $d/d\mu^2$ of physical quantities has to be zero. In our case, if we choose another renormalization scale μ, then the bare propagator acquires poles. However, these poles cancel in the dressed physical propagator because the shift in the bare part is cancelled by a corresponding shift in the self energy. The renormalized action (109) will produce finite Green's functions to whatever order in the coupling constants we have renormalized the theory to. For example, the 2-point Green's function at the first order in the couplings a_n, b_n can be schematically written as

$$G_{2R}^{-1} = V(k^2/\Lambda^2)(k^2 - \Sigma_R(k^2)), \tag{114}$$

where the renormalization prescription requires that Σ_R satisfies (on shell)

$$\Sigma_R(0) = 0 \quad \text{and} \quad \frac{\partial \Sigma_R}{\partial k^2}\bigg|_{k^2=0} = 0. \tag{115}$$

We did not consider the tensorial structure and the longitudinal components because they project away when attached to a conserved energy tensor.

The subtraction point is arbitrary and therefore we can take the renormalization prescription off shell to $k^2 = \mu^2$. The couplings we wish to renormalize must be dependent on the chosen subtraction point, $a_n(\mu)$ and $b_n(\mu)$, in such a way that the experimentally measured couplings do not vary on shell. The renormalized Green's function G_{2R}^{-1} at μ^2 must produce the same Green function when expressed in terms of bare quantities. Consequently, the scalings Z_{a_n} and Z_{b_n} must also depend on μ^2. The Green's function written in terms of bare quantities can not depend on μ^2, but since μ^2 is arbitrary, the renormalized Green's function must not either. This fact, $\partial_{\mu^2}(G_{2R}^{-1}) = 0$ is well known as the renormalization group invariance.

An explicit example of $H(z)$ can be achieved by replacing $p_{\gamma+1}$ with $p_{\gamma+N+1}$ in (43) as we did in Sec. III. Another possible choice is $H(z) = z^n$ as in Sec. V.

The same results obtained in Secs. V and VII concerning the gravitational potential, spherically symmetric solutions, and black holes also work in d dimensions.

IX. WHY NONLOCALITY?

In this section we suggest two possible interpretations of the nonlocal nature of gravity at short distances: one is linked to the non-ommutativity of the spacetime coodinates, the other to the nonlocal nature of string field theory.

A. Noncommutativity

In [26] the authors find an elegant reason for the nonlocal nature of the action in this article as well as a way to fix uniquely the entire function $H(z)$. The propagator of the theory, for a particular choice of the entire function $H(z)$, has exactly the same form of the propagator we obtain starting from a theory of gravity endowed with θ-Poincaré quantum groups of symmetry. The right choice is much easier than we could think, i.e. $H(z) = z$. Any other entire function gives of course a well-defined super-renormalizable theory of gravity, but is not compatible with the requirement of having a nontrivial Hopf-algebralike symmetry regulating the super-renormalizability of the theory. In particular, the Hopf-algebra underlying the super-renormalizable model discussed in [26] is a quantum group associated to an associative noncommutative spacetime. More specifically, this is the only quantum group of (spacetime) symmetry that can be accounted within the model presented in this paper, if we do not relax the associativity of the spacetime coordinates. What emerges is therefore a new symmetric structure underlying the theory.

Following the Smailagic and Spallucci work [25] we calculate the two point function in the noncommutative theory. Let us consider the noncommutative d-dimensional Euclidean spacetime, in which the coordinates satisfied the following algebra:

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\hat{\theta}^{\mu\nu}, \qquad \mu, \nu = 1, \ldots, d. \tag{116}$$

Assuming Lorentz covariance we can bring the antisymmetric matrix $\theta^{\mu\nu}$ in a block-diagonal form by a suitable rotation [25],

$$\hat{\theta} = \mathrm{diag}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{d/2}), \qquad \hat{\theta}_i \equiv \theta_i \epsilon_{ij}, \qquad i, j = 1, 2. \tag{117}$$

Moreover, if we want to maintain Lorentz invariance, all the θ_i must be equal: $\theta_i := \theta := 4/\Lambda^2$, $\forall i$. The relation (116) prompts us to consider the following "creation" and "annihilation" operators in even dimension,

$$\mathbf{z}_i \equiv \frac{1}{\sqrt{2}}(\mathbf{x}_{1i} + i\mathbf{x}_{2i}), \qquad \mathbf{z}_i^\dagger \equiv \frac{1}{\sqrt{2}}(\mathbf{x}_{1i} - i\mathbf{x}_{2i}), \tag{118}$$

satisfying the relation $[\mathbf{z}_i, \mathbf{z}_j^\dagger] = \delta_{ij}\theta$, with $i = 1, \ldots, d/2$, while the d coordinates are represented by $d/2$ two-vectors $\vec{\mathbf{x}}_i \equiv (\mathbf{x}_{1i}, \mathbf{x}_{2i})$. The advantage to use the coordinates (118) instead of $\vec{\mathbf{x}}_i$ lies in the possibility to have coherent eigenstates

$$\begin{aligned} |\alpha\rangle &\equiv \prod_i \exp\left[\frac{\Lambda^2}{4}(\alpha_i^* \mathbf{z}_i - \alpha_i \mathbf{z}_i^\dagger)\right]|0\rangle, \\ \mathbf{z}_i|\alpha\rangle &= \alpha_i|\alpha\rangle, \qquad \langle\alpha|\mathbf{z}_i^\dagger = \langle\alpha|\alpha_i^*, \end{aligned} \tag{119}$$

where $|0\rangle$ is the vacuum state annihilated by the $\mathbf{z}_i$ operators, with α_i as their eigenvalues. The mean coordinates are

$$\langle\alpha|\mathbf{x}_{1i}|\alpha\rangle = \sqrt{2}\,\mathrm{Re}\alpha_i, \qquad \langle\alpha|\mathbf{x}_{2i}|\alpha\rangle = \sqrt{2}\,\mathrm{Im}\alpha_i. \tag{120}$$

The quantum graviton field on the non commutative plane is defined as follows:

$$h_{\mu\nu}(x) = \int \frac{d^{d-1}p}{2p_0}[\mathbf{a}(p)^\dagger_{\mu\nu}\langle\alpha|e^{i\sum_1^{d/2}\vec{p}_i\cdot\vec{\mathbf{x}}_i}|\alpha\rangle + \mathrm{H.c.}], \tag{121}$$

where $\mathbf{a}(p)_{\mu\nu}$ and $\mathbf{a}(p)^\dagger_{\mu\nu}$ are the lowering and rising operators acting on the graviton Fock states. We now proceed to evaluate the expectation value in (121),

$$\langle\alpha|e^{i\sum_1^{d/2}\vec{p}_i\cdot\vec{\mathbf{x}}_i}|\alpha\rangle = e^{-\sum_1^{d/2}\frac{\vec{p}_i^2}{\Lambda^2} + i\sum_1^{d/2}\vec{p}_i\cdot\vec{x}_i}, \tag{122}$$

where $\vec{x}_i = (\mathrm{Re}\alpha_i, \mathrm{Im}\alpha_i)$, $i = 1, \ldots, d/2$. In (121) the expectation value between coherent states achieves the inverse Weyl map,[7] also known as the Wigner map,

$$\Omega^{-1}(\cdots) = \langle z|\cdots|z\rangle. \tag{124}$$

Given the field expansion (121), the four-dimensional two point function turns out to be

$$\begin{aligned} &\langle 0|T(h_{\mu\nu}(x)h_{\rho\sigma}(x')|0\rangle \\ &\quad\equiv \langle 0|T(\langle\alpha|h_{\mu\nu}(\mathbf{x})|\alpha\rangle\langle\alpha'|h_{\rho\sigma}(\mathbf{x})|\alpha'\rangle)|0\rangle \\ &\quad\propto \int d^4p\,\frac{e^{-\sum_1^d(p_i^2/\Lambda^2)}e^{i\sum_1^d(x_i - x_i')p_i}}{p^2} \times \mathrm{TS}, \end{aligned} \tag{125}$$

where TS means tensorial structure. This propagator coincides exactly with (66) of section V for $n = 1$.

Above we have proved the coincidence of the two point functions in the noncommutative theory and in the super-renormalizable gravity. However, we can only validate such coincidence if applicable to all the n-point functions. This is something we will try to develop in the next future.

B. String field theory

Another possible interpretation could be the following. We can identify some similarities between the class of super-renormalizable theories with $H(z) = z$ and "string field theory." Using the results found at the end of the 1980s [59–66] and several more recent ideas [67,68], the string field theory has the following schematic structure for the spacetime bosonic as well as fermionic fields:

$$\begin{aligned} S &= \int d^dx\left(\frac{1}{2}\phi_i K_{ij}(\Box)\phi_j - v_{ijk}\tilde{\phi}_i\tilde{\phi}_j\tilde{\phi}_k\right), \\ &\text{where: } \tilde{\phi}_i \equiv e^{\alpha'((\ln(3\frac{\sqrt{3}}{4}))/2)\Box}\phi_i := e^{\tilde{\alpha}'\Box}\phi_i. \end{aligned} \tag{126}$$

The kinetic operator is $K_{ij}(\Box) \approx \Box\delta_{ij}$ for open, as well as closed, bosonic strings, and α' is the inverse mass square in string theory. By a field redefinition [67], the action (126) simplifies to

$$S_{\mathrm{SFT}} = \int d^dx\left(\frac{1}{2}\phi_i\Box e^{-\tilde{\alpha}'\Box}\phi_i - v_{ijk}\phi_i\phi_j\phi_k\right). \tag{127}$$

We can immediately observe that the kinetic term in (127) has the same scaling of the linearized theory studied in this paper for the exponential form factor

[7] Let us briefly recall here that the Weyl map Ω associates an auxiliary commutative function $f^{(c)}(x)$ to any function of the operators $\mathbf{x}$, $f(\mathbf{x})$. The easiest way of implementing this map is to consider the Fourier transform $\tilde{f}(p)$ of $f(\mathbf{x})$ and then apply the Weyl map on the Fourier modes, namely,

$$f(\mathbf{x}) = \Omega(f^{(c)}(x)) \equiv \Omega\left(\int d^dp\,\tilde{f}(p)e^{ipx}\right). \tag{123}$$

$V_2(z) = \exp(-z)$ ($n = 1$). If we expand (95) in powers of the graviton field neglecting the exponential factor in the interaction, the three-graviton vertex is quite similar to the one in (127). However, the general covariance in (95) implies the same leading scaling in the kinetic term as well as in the interaction vertexes and we are unable to get a finite theory at any order in the loop expansion. As we already pointed out, one possible loophole to this puzzle could be a supersymmetric extension of the action in (95). About the finiteness of string theory, we are likely to endorse the following ideas. Due to the presence of the exponential factor, the effective string theory in (127) manifests an asymmetry between the kinetic and the interaction terms. Contrary to our covariant action (95), such an asymmetrical state implies that the string theory does not manifest any divergence. The well-known "softness" of the high energy tree-level amplitudes also descends from the same asymmetry. However, the comparison here proposed can only be qualitative and partial because, unlike the effective string field theory, ours is a general covariant theory. Indeed, general coordinate invariance in string theory can only be achieved through cancellations among contributions from infinitely many interactions terms [62]. However, we do not exclude that a supersymmetric extension of our theory (95) can be framed within "*M* theory" as one of its possible vacua.

X. CONCLUSIONS

In this paper we studied a new perturbative quantum gravity theory with a "gentle nonlocal character." This theory was introduced by Tomboulis in 1997 [4]. Here we derived the same theory from a different "semiclassical" point of view as extensively explained in Sec. I. We recalled the Tomboulis theory in Sec. II and we expanded the super-renormalizability analysis in Sec. III. We also studied another class of super-renormalizable theories in Sec. V. In Sec. IV we provided an interpretation of the good ultraviolet behavior as an effective dimensional reduction of the spacetime, and in Sec. VII we studied spherically symmetric/black hole solutions. In Sec. VIII we generalized the theory to a multidimensional spacetime. The theory is super-renormalizable and unitary regardless of the spacetime dimension. In the last Sec. IX we suggested a possible interpretation of the nonlocal nature of gravity at short distance.

In brief, the properties required for the theory expanded and studied in this paper were the following:

(i) the theory should reproduce general relativity in the infrared limit;
(ii) black hole solutions of the classical theory have to be singularity free;
(iii) the theory should be perturbatively renormalizable or super-renormalizable or finite;
(iv) the spectral dimension should decrease at short distances;
(v) the theory has to be unitary with no other degrees of freedom than the graviton.

All the above properties are satisfied by our quite restrictive class of actions, which differ uniquely for the choice of an entire function $H(z)$. This class of theories can not be renormalizable or finite but such theories turn out to be super-renormalizable, since only one loop Feynman diagrams diverge, implying a renormalization of just three coupling constants. The propagator has only one pole in the graviton mass shell. The minimal super-renormalizable theory we built in $d = 4$ has spectral dimension $d_s = 4/5$ in the ultraviolet regime and is four-dimensional in the infrared limit.

We have also considered a truncation of the classical theory, showing that spherically symmetric black hole solutions are singularity-free. As for the solutions in [30–49], we have black holes only if the mass is bigger than the Planck mass (if we assume the fundamental scale in the theory to be the Planck mass). The new black hole solutions are more properly "*multihorizon black holes*," showing a very reach spacetime structure depending on the value of the ADM mass.

Future work will focus on the following subjects:
(i) the cosmological singularity problem at the classical or semiclassical level. Some preliminary work has been already done in [69];
(ii) the high curvature corrections to the black hole solutions presented in this paper.

Future work will also take the following directions:
(i) we will consider more in detail the connection between nonlocality and the fractality of the spacetime (see, in particular, the recent work by Calcagni in [10]);
(ii) we will reconsider super-renormalizable gauge theories, with particular attention to the grand unification of the fundamental interactions with or without gravity. It this work, we will try to put together ideas coming from [4] and the more recent paper [70].

Possible simplifications of the theory are
(i) a scalar field theory with the same nonlocal structure;
(ii) a simplification of the metric to the conformal form $g_{\mu\nu} = \Omega(x)\eta_{\mu\nu}$ and therefore a quantization of the conformal factor $\Omega(x)$.

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[1] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[2] J. W. Moffat, Phys. Rev. D **41**, 1177 (1990); B. J. Hand and J. W. Moffat, Phys. Rev. D **43**, 1896 (1991); D. Evens, J. W. Moffat, G. Kleppe, and R. P. Woodard, Phys. Rev. D **43**, 499 (1991); N. J. Cornish, Mod. Phys. Lett. A **7**, 1895 (1992); Int. J. Mod. Phys. A **7**, 6121 (1992).
[3] N. J. Cornish, Mod. Phys. Lett. A **7**, 631 (1992); J. W. Moffat, Eur. Phys. J. Plus **126**, 43 (2011).
[4] E. T. Tomboulis, arXiv:hep-th/9702146v1.
[5] A. O. Barvinsky, Phys. Lett. B **710**, 12 (2012); Phys. Rev. D **71**, 084007 (2005); Phys. Lett. B **572**, 109 (2003); H. W. Hamber and R. M. Williams, Phys. Rev. D **72**, 044026 (2005); N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and G. Gabadadze, arXiv:hep-th/0209227v1; G. Calcagni and G. Nardelli, J. High Energy Phys. 02 (2010) 093.
[6] L. Crane and L. Smolin, Nucl. Phys. **B267**, 714 (1986).
[7] O. Lauscher and M. Reuter, J. High Energy Phys. 10 (2005) 050; D. Benedetti, Pedro F. Machado, and F. Saueressig, Mod. Phys. Lett. A **24**, 2233 (2009).
[8] D. ben-Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems* (Cambridge University Press, Cambridge, United Kingdom, 2000); J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Lett. B **607**, 205 (2005).
[9] L. Modesto, Classical Quantum Gravity **26**, 242002 (2009).
[10] S. Carlip, in XXV Max Born Symposium, "Physics at the Planck scale," Wroclaw, Poland (2009) (to be published), arXiv:0909.3329.
[11] D. Benedetti, Phys. Rev. Lett. **102**, 111303 (2009).
[12] L. Modesto and Piero Nicolini, Phys. Rev. D **81**, 104040 (2010).
[13] L. Modesto, Classical Quantum Gravity **26**, 242002 (2009).
[14] L. Modesto, arXiv:0905.1665.
[15] F. Caravelli and L. Modesto, arXiv:0905.2170.
[16] E. Magliaro, C. Perini, and L. Modesto, arXiv:0911.0437.
[17] G. Calcagni, Phys. Rev. Lett. **104**, 251301 (2010).
[18] G. Calcagni, J. High Energy Phys. 03 (2010)120.
[19] G. Calcagni, Phys. Lett. B **697**, 251 (2011).
[20] G. Calcagni, Phys. Rev. D **84**, 061501 (2011); arXiv:1106.5787.
[21] L. Anchordoqui, De Chang Dai, M. Fairbairn, G. Landsberg, and D. Stojkovic, Mod. Phys. Lett. A **27**, 1250021-1 (2012); J. R. Mureika and D. Stojkovic, Phys. Rev. Lett. **106**, 101101 (2011).
[22] L. Modesto, J. W. Moffat, and P. Nicolini, Phys. Lett. B **695**, 397 (2011).
[23] A. Smailagic and E. Spallucci, J. Phys. A **36**, L467 (2003).
[24] A. Smailagic and E. Spallucci, J. Phys. A **36**, L517 (2003).
[25] A. Smailagic and E. Spallucci, J. Phys. A **37**, 7169 (2004).
[26] S. Alexander, A. Marciano, and L. Modesto, Phys. Rev. D **85**, 124030 (2012).
[27] I. L. Buchbinder, Sergei D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (Taylor & Francis, London, 1992).
[28] P. Van Nieuwenhuizen, Nucl. Phys. **B60**, 478 (1973).
[29] M. Asorey, J. L. Lopez, and I. L. Shapiro, Int. J. Mod. Phys. A **12**, 5711 (1997).
[30] P. Nicolini, A. Smailagic, and E. Spallucci, in *Proceedings of EPS 13 General Conference—"Beyond Einstein, Physics for the 21st Century," 2005* (University of Bern, Bern, Switzerland) [hep-th/0507226].
[31] P. Nicolini, J. Phys. A **38**, L631 (2005).
[32] P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. B **632**, 547 (2006).
[33] T. G. Rizzo, J. High Energy Phys. 09 (2006) 021.
[34] E. Spallucci, A. Smailagic, and P. Nicolini, Phys. Rev. D **73**, 084004 (2006).
[35] S. Ansoldi, P. Nicolini, A. Smailagic, and E. Spallucci, Phys. Lett. B **645**, 261 (2007).
[36] E. Spallucci, A. Smailagic, and P. Nicolini, Phys. Lett. B **670**, 449 (2009).
[37] P. Nicolini, Int. J. Mod. Phys. A **24**, 1229 (2009).
[38] R. Casadio and P. Nicolini, J. High Energy Phys. 11 (2008) 072.
[39] I. Arraut, D. Batic, and M. Nowakowski, Classical Quantum Gravity **26**, 245006 (2009).
[40] P. Nicolini and E. Spallucci, Classical Quantum Gravity **27**, 015010 (2010).
[41] A. Smailagic and E. Spallucci, Phys. Lett. B **688**, 82 (2010).
[42] D. M. Gingrich, J. High Energy Phys. 05 (2010) 022.
[43] I. Arraut, D. Batic, and M. Nowakowski, J. Math. Phys. (N.Y.) **51**, 022503 (2010).
[44] R. Banerjee, S. Gangopadhyay, and S. K. Modak, Phys. Lett. B **686**, 181 (2010).
[45] P. Nicolini, Phys. Rev. D **82**, 044030 (2010).
[46] L. Modesto and A. Randono, arXiv:1003.1998; F. Caravelli and L. Modesto, Phys. Lett. B **702**, 307 (2011).
[47] L. Modesto and P. Nicolini, Phys. Rev. D **82**, 104035 (2010).
[48] R. B. Mann and P. Nicolini, Phys. Rev. D **84**, 064014 (2011).
[49] J. R. Mureika and P. Nicolini, Phys. Rev. D **84**, 044020 (2011).
[50] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. **95**, 171301 (2005).
[51] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[52] G. V. Efimov, *Nonlocal Interactions of Quantized Fields* (Nauka, Moscow, 1977).
[53] N. V. Krasnikov, Teor. Math. Fiz. **73**, 235 (1986).
[54] N. Barnaby and N. Kamran, J. High Energy Phys. 02 (2008) 008; G. Calcagni, M. Montobbio, and G. Nardelli, Phys. Rev. D **76**, 126001 (2007); Phys. Lett. B **662**, 285 (2008).
[55] A. DeBenedictis, Can. J. Phys. **87**, 255 (2009).
[56] P. Gaete, J. A. Helayel-Neto, and E. Spallucci, Phys. Lett. B **693**, 155 (2010).
[57] A. Acciolyan and A. Azeredo, J. Math. Phys. (N.Y.) **43**, 1 (2002).
[58] B. Hatfield, *Quantum Field Theory of Point Particles and Strings* (Addison-Wesley, Redwood City, CA, 1992).
[59] E. Witten, Nucl. Phys. **B268**, 253 (1986).
[60] L. Brekke, P. G. O. Freund, M. Olson, and E. Witten, Nucl. Phys. **B302**, 365 (1988).
[61] D. J. Gross and P. F. Mende, Phys. Lett. B **197**, 129 (1987).
[62] Pei-Ming Ho, Classical Quantum Gravity **21**, 2641 (2004).
[63] T. Biswas, J. Kapusta, and A. Reddy, arXiv:1201.1580.
[64] N. Barnaby, T. Biswas, and J. M. Cline, J. High Energy Phys. 04 (2007) 056.

[65] T. Biswas, A. Mazumdar, and W. Siegel, J. Cosmol. Astropart. Phys. 03 (2006) 009.
[66] T. Biswas, T. Koivisto, and A. Mazumdar, J. Cosmol. Astropart. Phys. 11 (2010) 008.
[67] V. A. Kostelecky and S. Samuel, Phys. Rev. D **42**, 1289 (1990).
[68] A. S. Koshelev, Proceedings of BW2011 Workshop (to be published), arXiv:1112.6410.
[69] G. Calcagni and G. Nardelli, Phys. Rev. D **82**, 123518 (2010); T. Biswas, T. Koivisto, and A. Mazumdar, J. Cosmol. Astropart. Phys. 11 (2010) 008.
[70] D. V. Shirkov, Phys. Part. Nucl. Lett. **7**, 379 (2010).