

Generalized Galileon model: Cosmological reconstruction and the Vainshtein mechanismNorihito Shirai,¹ Kazuharu Bamba,² Shota Kumekawa,¹ Jiro Matsumoto,¹ and Shin'ichi Nojiri^{1,2}¹*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*²*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan*

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Explicit formulae of the equations in the generalized Galileon models are given. We also develop the formulation of the reconstruction. By using the formulation, we can explicitly construct an action that reproduces an arbitrary development of the expansion of the Universe. We show that we can completely and explicitly separate the action to the part relevant for the expansion and the irrelevant part. The irrelevant part is related to the stability of the reconstructed solution, and we can further separate that part to the part relevant for the stability and the part irrelevant for the stability. The conditions showing how the reconstructed solution becomes stable, and therefore become an attractor solution, are also given. Working in the static and spherically symmetric space-time, we investigate how the Vainshtein mechanism works in the generalized Galileon model and how the correction to Newton law becomes small. It is also shown that any spherically symmetric and static geometry can be realized by properly choosing the form of the action, which may show that the solution has a fourth hair corresponding to the scalar field. We again separate the action to the part relevant for the reconstruction for the spherically symmetric and static geometry and the irrelevant part. We show that by choosing the relevant and irrelevant parts appropriately, we can obtain an action that admits both the solution corresponding to an arbitrarily given spherically symmetric and static geometry and the solution corresponding to an arbitrarily given history of the expansion of the Universe.

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I. INTRODUCTION

The observation of the type Ia supernovae at the end of the last century tells us that the expansion of the present Universe is accelerating [1–3]. In order for the accelerating expansion to occur in Einstein gravity, we need cosmological fluid with negative pressure, and we call that fluid dark energy (for a review, see [4–6]). The simplest model of dark energy is the cosmological term in Einstein gravity, referred to as the Λ CDM model (CDM is cold dark matter). The Λ CDM model, however, suffers from the so-called fine-tuning problem and/or coincidence problem. In order to avoid these problems, many kinds of dynamical models have been proposed, especially dynamical models using the scalar field(s) like the quintessence model [7–11], k -essence models [12–16], or ghost condensation models [17,18]. Such scalar models, however, generate a large correction to Newton law in general by the propagation of the scalar field. In order to make the correction decrease, the so-called chameleon mechanism has been proposed in [19,20]. In the chameleon mechanism, the mass of the scalar field becomes large when the mass density is large and therefore the force propagated by the scalar field becomes very short range, so that the correction can become very small so as not to conflict with the observational or experimental results. Another mechanism to suppress the contribution from the scalar field is the Vainshtein mechanism [21], where the scalar field is suppressed by the nonlinear structure of the scalar field equation. Originally the Vainshtein mechanism was a mechanism

for the decoupling of the longitudinal mode in massive gravity. After that, it was found that a similar mechanism works [22,23] for the bending mode of the Dvali-Gabadadze-Porrati model [24–26]. Then scalar field models where the Vainshtein mechanism works were proposed. The actions of the original models have a symmetry called Galileon symmetry and hence the scalar field is called as the Galileon field [27–29]. The equation of motion for the Galileon field does not include a derivative higher than two, which may prevent the existence of the ghost although the condition is not a necessary nor sufficient condition for no ghost. We also note that the structure of the equation does not have a direct relation with the Galileon symmetry. Until now, there have been a number of activities related to the Galileon scalar field [30–32].

In generalized models of the Galileon field, the field equations of motion are very complicated, however, they could be useful for future research if the explicit forms of the equations are given. In this paper, we explain how the field equations of motion do not include derivatives higher than two. As pointed in [31], the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ plays a crucial role in the structure. In [32], it was shown that the actions obtained in [31] are equivalent to those in [33]. When we consider the curved space-time, in order to preserve the structure of the field equation, the correction terms appear, including the curvatures in the action, which also guarantee that Einstein equations do not include derivatives higher than two, either. We also develop the formulation of the reconstruction, which shows the explicit form of the action reproducing an arbitrary development of

the expansion of the Universe. We show the conditions that explain how the reconstructed solution becomes stable and thus becomes an attractor solution. We also investigate how the Vainshtein mechanism works in the curved space-time for the generalized Galileon model. We also show that any spherically symmetric and static geometry can be realized by properly choosing the form of the action, which may show that the solution has a fourth hair corresponding to the scalar field.

We give explicit forms of the equations as far as we can, although they look very complicated. We believe explicit formulae could be necessary for later applications.

In the next section, we provide a general formulation of the Galileon field and show how derivatives higher than two do not appear in the field equations and the Einstein equation. In Sec. III, we consider the Friedmann-Robertson-Walker (FRW) dynamics and give

the formulae for the reconstruction. The stability of the reconstructed solution is also investigated. In Sec. IV, we examine the spherically symmetric and static space-time and investigate how the Vainshtein mechanism works for the generalized Galileon model. We also realize any spherically symmetric and static geometry in the framework of the Galileon model. The last section is devoted to a summary and discussion.

II. FORMULATION OF GALILEON SCALAR

In this section, we give a general formulation of the Galileon models by using the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ in the flat space-time and the curved one.

We first consider the flat space-time. We now find the contributions from the Galileon scalar π to the equations of motion are expressed as [27–29]

$$\begin{aligned}\mathcal{E}_1 &= 1 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma}, & \mathcal{E}_2 &= -2 \text{tr} \Pi = -\frac{1}{3} \epsilon_{\mu\nu\rho}{}^\delta \epsilon^{\mu\nu\rho\sigma} \partial_\delta \partial_\sigma \pi, \\ \mathcal{E}_3 &= -3\{(\text{tr} \Pi)^2 - \text{tr} \Pi^2\} = -\frac{3}{2} \epsilon_{\mu\nu}{}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\gamma \partial_\rho \pi \partial_\delta \partial_\sigma \pi, \\ \mathcal{E}_4 &= -2\{(\text{tr} \Pi)^3 - 3 \text{tr} \Pi \text{tr} \Pi^2 + 2 \text{tr} \Pi^3\} = -2 \epsilon_\mu{}^{\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\beta \partial_\nu \pi \partial_\gamma \partial_\rho \pi \partial_\delta \partial_\sigma \pi, \\ \mathcal{E}_5 &= -\frac{5}{6} \{(\text{tr} \Pi)^4 - 6(\text{tr} \Pi)^2 \text{tr} \Pi^2 + 8(\text{tr} \Pi) \text{tr} \Pi^3 + 3(\text{tr} \Pi^2)^2 - 6 \text{tr} \Pi^4\} = -\frac{5}{6} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\alpha \partial_\mu \pi \partial_\beta \partial_\nu \pi \partial_\gamma \partial_\rho \pi \partial_\delta \partial_\sigma \pi.\end{aligned}\tag{1}$$

Here $\Pi^\mu{}_\nu \equiv \partial^\mu \partial_\nu \pi$ and $\text{tr} \Pi = \Pi^\mu{}_\mu$. We have also used the following formula:

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} &= \eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\gamma\rho} \eta^{\delta\sigma} - \eta^{\alpha\mu} \eta^{\beta\nu} \eta^{\gamma\sigma} \eta^{\delta\rho} + \eta^{\alpha\mu} \eta^{\beta\rho} \eta^{\gamma\sigma} \eta^{\delta\nu} - \eta^{\alpha\mu} \eta^{\beta\rho} \eta^{\gamma\nu} \eta^{\delta\sigma} + \eta^{\alpha\mu} \eta^{\beta\sigma} \eta^{\gamma\nu} \eta^{\delta\rho} \\ &\quad - \eta^{\alpha\mu} \eta^{\beta\sigma} \eta^{\gamma\rho} \eta^{\delta\nu} - \eta^{\alpha\nu} \eta^{\beta\rho} \eta^{\gamma\sigma} \eta^{\delta\mu} + \eta^{\alpha\nu} \eta^{\beta\rho} \eta^{\gamma\mu} \eta^{\delta\sigma} - \eta^{\alpha\nu} \eta^{\beta\sigma} \eta^{\gamma\mu} \eta^{\delta\rho} + \eta^{\alpha\nu} \eta^{\beta\sigma} \eta^{\gamma\rho} \eta^{\delta\mu} \\ &\quad - \eta^{\alpha\nu} \eta^{\beta\mu} \eta^{\gamma\rho} \eta^{\delta\sigma} + \eta^{\alpha\nu} \eta^{\beta\mu} \eta^{\gamma\sigma} \eta^{\delta\rho} + \eta^{\alpha\rho} \eta^{\beta\sigma} \eta^{\gamma\mu} \eta^{\delta\nu} - \eta^{\alpha\rho} \eta^{\beta\sigma} \eta^{\gamma\nu} \eta^{\delta\mu} + \eta^{\alpha\rho} \eta^{\beta\mu} \eta^{\gamma\nu} \eta^{\delta\sigma} \\ &\quad - \eta^{\alpha\rho} \eta^{\beta\mu} \eta^{\gamma\sigma} \eta^{\delta\nu} + \eta^{\alpha\rho} \eta^{\beta\nu} \eta^{\gamma\sigma} \eta^{\delta\mu} - \eta^{\alpha\rho} \eta^{\beta\nu} \eta^{\gamma\mu} \eta^{\delta\sigma} - \eta^{\alpha\sigma} \eta^{\beta\mu} \eta^{\gamma\nu} \eta^{\delta\rho} + \eta^{\alpha\sigma} \eta^{\beta\mu} \eta^{\gamma\rho} \eta^{\delta\nu} \\ &\quad - \eta^{\alpha\sigma} \eta^{\beta\nu} \eta^{\gamma\rho} \eta^{\delta\mu} + \eta^{\alpha\sigma} \eta^{\beta\nu} \eta^{\gamma\mu} \eta^{\delta\rho} - \eta^{\alpha\sigma} \eta^{\beta\rho} \eta^{\gamma\mu} \eta^{\delta\nu} + \eta^{\alpha\sigma} \eta^{\beta\rho} \eta^{\gamma\nu} \eta^{\delta\mu}.\end{aligned}\tag{2}$$

Then it is easy to find the corresponding Lagrangian densities as follows:

$$\begin{aligned}\mathcal{L}_1 &= \pi, & \mathcal{L}_2 &= \frac{1}{6} \epsilon_{\mu\nu\rho}{}^\delta \epsilon^{\mu\nu\rho\sigma} \partial_\delta \pi \partial_\sigma \pi = \partial^\mu \pi \partial_\mu \pi, \\ \mathcal{L}_3 &= \frac{1}{2} \epsilon_{\mu\nu}{}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\gamma \pi \partial_\rho \pi \partial_\delta \partial_\sigma \pi = \partial^\mu \pi \partial_\mu \pi \square \pi - \partial^\mu \pi \partial^\nu \pi \partial_\mu \partial_\nu \pi, \\ \mathcal{L}_4 &= \frac{1}{2} \epsilon_\mu{}^{\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\beta \pi \partial_\nu \pi \partial_\gamma \partial_\rho \pi \partial_\delta \partial_\sigma \pi \\ &= \frac{1}{2} (\partial_\mu \pi \partial^\mu \pi (\square \pi)^2 - \partial_\mu \pi \partial^\mu \pi \partial^\sigma \partial_\rho \pi \partial^\rho \partial_\sigma \pi - 2 \partial^\mu \pi \partial^\nu \pi \partial_\mu \partial_\nu \pi \pi + 2 \partial^\nu \pi \partial^\mu \pi \partial^\rho \partial_\nu \pi \partial_\mu \partial_\rho \pi), \\ \mathcal{L}_5 &= \frac{1}{6} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_\alpha \pi \partial_\mu \pi \partial_\beta \partial_\nu \pi \partial_\gamma \partial_\rho \pi \partial_\delta \partial_\sigma \pi \\ &= \frac{1}{6} (\partial_\mu \pi \partial^\mu \pi (\square \pi)^3 - 3 \partial_\mu \pi \partial^\mu \pi \square \pi \partial^\nu \partial_\rho \pi \partial^\rho \partial_\nu \pi + 2 \partial_\mu \pi \partial^\mu \pi \partial^\rho \partial_\nu \pi \partial^\sigma \partial_\rho \pi \partial^\nu \partial_\sigma \pi - 3 \partial^\mu \pi \partial^\nu \pi \partial_\mu \partial_\nu \pi (\square \pi)^2 \\ &\quad + 3 \partial^\mu \pi \partial^\nu \pi \partial_\mu \partial_\nu \pi \partial^\sigma \partial_\rho \pi \partial^\rho \partial_\sigma \pi + 6 \partial^\nu \pi \partial^\mu \pi \partial^\rho \partial_\nu \pi \partial_\mu \partial_\rho \pi \square \pi - 6 \partial^\nu \pi \partial^\mu \pi \partial^\sigma \partial_\nu \pi \partial_\mu \partial_\rho \pi \partial^\rho \partial_\sigma \pi).\end{aligned}\tag{3}$$

In the curved space-time, instead of the above Lagrangian densities (3), we may consider the following ones:

$$\begin{aligned}
\mathcal{L}_1 &= \pi, & \mathcal{L}_2 &= \frac{1}{6} \epsilon_{\mu\nu\rho}^{\delta} \epsilon^{\mu\nu\rho\sigma} \partial_{\delta} \pi \partial_{\sigma} \pi = \partial^{\mu} \pi \partial_{\mu} \pi, \\
\mathcal{L}_3 &= \frac{1}{2} \epsilon_{\mu\nu}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_{\gamma} \pi \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi = \partial^{\mu} \pi \partial_{\mu} \pi \square \pi - \partial^{\mu} \pi \partial^{\nu} \pi \nabla_{\mu} \partial_{\nu} \pi, \\
\mathcal{L}_4^{(0)} &= \frac{1}{2} \epsilon_{\mu}^{\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_{\beta} \pi \partial_{\nu} \pi \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi, \\
&= \frac{1}{2} (\partial_{\mu} \pi \partial^{\mu} \pi (\square \pi)^2 - \partial_{\mu} \pi \partial^{\mu} \pi \nabla^{\sigma} \partial_{\rho} \pi \nabla^{\rho} \partial_{\sigma} \pi - 2 \partial^{\mu} \pi \partial^{\nu} \pi \nabla_{\mu} \partial_{\nu} \pi \square \pi + 2 \partial^{\nu} \pi \partial^{\mu} \pi \nabla^{\rho} \partial_{\nu} \pi \nabla_{\mu} \partial_{\rho} \pi), \\
\mathcal{L}_5^{(0)} &= \frac{1}{6} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \partial_{\alpha} \pi \partial_{\mu} \pi \nabla_{\beta} \partial_{\nu} \pi \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi, \\
&= \frac{1}{6} (\partial_{\mu} \pi \partial^{\mu} \pi (\square \pi)^3 - 3 \partial_{\mu} \pi \partial^{\mu} \pi \square \pi \nabla^{\sigma} \partial_{\rho} \pi \nabla^{\rho} \partial_{\sigma} \pi + 2 \partial_{\mu} \pi \partial^{\mu} \pi \nabla^{\rho} \partial_{\nu} \pi \nabla^{\sigma} \partial_{\rho} \pi \nabla_{\nu} \partial_{\sigma} \pi \\
&\quad - 3 \partial^{\mu} \pi \partial^{\nu} \pi \nabla_{\mu} \partial_{\nu} \pi (\square \pi)^2 + 3 \partial^{\mu} \pi \partial^{\nu} \pi \nabla_{\mu} \partial_{\nu} \pi \nabla^{\sigma} \partial_{\rho} \pi \nabla^{\rho} \partial_{\sigma} \pi + 6 \partial^{\nu} \pi \partial^{\mu} \pi \nabla^{\rho} \partial_{\nu} \pi \nabla_{\mu} \partial_{\rho} \pi \square \pi \\
&\quad - 6 \partial^{\nu} \pi \partial^{\mu} \pi \nabla^{\sigma} \partial_{\nu} \pi \nabla_{\mu} \partial_{\rho} \pi \nabla^{\rho} \partial_{\sigma} \pi).
\end{aligned} \tag{4}$$

Here ∇_{μ} is the covariant derivative and the meaning of the suffix “(0)” in $\mathcal{L}_5^{(0)}$ and $\mathcal{L}_4^{(0)}$ will be clarified soon. In (4), $\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma}$ is defined by

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} &= g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} - g^{\alpha\mu} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\rho} + g^{\alpha\mu} g^{\beta\rho} g^{\gamma\sigma} g^{\delta\nu} - g^{\alpha\mu} g^{\beta\rho} g^{\gamma\nu} g^{\delta\sigma} + g^{\alpha\mu} g^{\beta\sigma} g^{\gamma\nu} g^{\delta\rho} \\
&\quad - g^{\alpha\mu} g^{\beta\sigma} g^{\gamma\rho} g^{\delta\nu} - g^{\alpha\nu} g^{\beta\rho} g^{\gamma\sigma} g^{\delta\mu} + g^{\alpha\nu} g^{\beta\rho} g^{\gamma\mu} g^{\delta\sigma} - g^{\alpha\nu} g^{\beta\sigma} g^{\gamma\mu} g^{\delta\rho} + g^{\alpha\nu} g^{\beta\sigma} g^{\gamma\rho} g^{\delta\mu} \\
&\quad - g^{\alpha\nu} g^{\beta\mu} g^{\gamma\rho} g^{\delta\sigma} + g^{\alpha\nu} g^{\beta\mu} g^{\gamma\sigma} g^{\delta\rho} + g^{\alpha\rho} g^{\beta\sigma} g^{\gamma\mu} g^{\delta\nu} - g^{\alpha\rho} g^{\beta\sigma} g^{\gamma\nu} g^{\delta\mu} + g^{\alpha\rho} g^{\beta\mu} g^{\gamma\nu} g^{\delta\sigma} \\
&\quad - g^{\alpha\rho} g^{\beta\mu} g^{\gamma\sigma} g^{\delta\nu} + g^{\alpha\rho} g^{\beta\nu} g^{\gamma\sigma} g^{\delta\mu} - g^{\alpha\rho} g^{\beta\nu} g^{\gamma\mu} g^{\delta\sigma} - g^{\alpha\sigma} g^{\beta\mu} g^{\gamma\nu} g^{\delta\rho} + g^{\alpha\sigma} g^{\beta\mu} g^{\gamma\rho} g^{\delta\nu} \\
&\quad - g^{\alpha\sigma} g^{\beta\nu} g^{\gamma\rho} g^{\delta\mu} + g^{\alpha\sigma} g^{\beta\nu} g^{\gamma\mu} g^{\delta\rho} - g^{\alpha\sigma} g^{\beta\rho} g^{\gamma\mu} g^{\delta\nu} + g^{\alpha\sigma} g^{\beta\rho} g^{\gamma\nu} g^{\delta\mu},
\end{aligned} \tag{5}$$

which gives

$$\begin{aligned}
\epsilon_{\mu}^{\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} &= g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma} - g^{\beta\nu} g^{\gamma\sigma} g^{\delta\rho} + g^{\beta\rho} g^{\gamma\sigma} g^{\delta\nu} - g^{\beta\rho} g^{\gamma\nu} g^{\delta\sigma} + g^{\beta\sigma} g^{\gamma\nu} g^{\delta\rho} - g^{\beta\sigma} g^{\gamma\rho} g^{\delta\nu}, \\
\epsilon_{\mu\nu}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma} &= 2(g^{\gamma\rho} g^{\delta\sigma} - g^{\gamma\sigma} g^{\delta\rho}), & \epsilon_{\mu\nu\rho}^{\delta} \epsilon^{\mu\nu\rho\sigma} &= 6g^{\delta\sigma}.
\end{aligned} \tag{6}$$

Since $\nabla_{\rho} g_{\mu\nu} = 0$, we find $\nabla_{\lambda}(\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma}) = 0$. By the variation of π , instead of (1), we obtain

$$\begin{aligned}
\mathcal{E}_1 &= 1, & \mathcal{E}_2 &= -2\nabla^{\mu} \partial_{\mu} \pi, & \mathcal{E}_3 &= \epsilon_{\mu\nu}^{\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \left\{ -\frac{3}{2} \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi + \frac{3}{4} R^{\lambda}_{\sigma\gamma\delta} \partial_{\rho} \pi \partial_{\lambda} \pi \right\}, \\
\mathcal{E}_4^{(0)} &= \epsilon_{\mu}^{\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \left\{ -2 \nabla_{\beta} \partial_{\nu} \pi \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi + \frac{5}{2} R^{\lambda}_{\rho\beta\gamma} \partial_{\nu} \pi \partial_{\lambda} \pi \nabla_{\delta} \partial_{\sigma} \pi - \frac{1}{2} \partial_{\beta} \pi \partial_{\nu} \pi (\nabla_{\rho} R^{\lambda}_{\sigma\gamma\delta}) \partial_{\lambda} \pi \right. \\
&\quad \left. - \frac{1}{2} \partial_{\beta} \pi \partial_{\nu} \pi R^{\lambda}_{\sigma\gamma\delta} \nabla_{\rho} \partial_{\lambda} \pi \right\}, \\
\mathcal{E}_5^{(0)} &= \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\rho\sigma} \left\{ -\frac{5}{6} \nabla_{\alpha} \partial_{\mu} \pi \nabla_{\beta} \partial_{\nu} \pi \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi + \frac{7}{4} R^{\lambda}_{\nu\alpha\beta} \partial_{\mu} \pi \partial_{\lambda} \pi \nabla_{\gamma} \partial_{\rho} \pi \nabla_{\delta} \partial_{\sigma} \pi + \frac{1}{4} \partial_{\alpha} \pi \partial_{\mu} \pi R^{\lambda}_{\rho\beta\gamma} \partial_{\lambda} \pi R^{\tau}_{\delta\nu\sigma} \partial_{\tau} \pi \right. \\
&\quad \left. - \frac{1}{2} \partial_{\alpha} \pi \partial_{\mu} \pi (\nabla_{\nu} R^{\lambda}_{\rho\beta\gamma}) \partial_{\lambda} \pi \nabla_{\delta} \partial_{\sigma} \pi - \frac{1}{2} \partial_{\alpha} \pi \partial_{\mu} \pi R^{\lambda}_{\rho\beta\gamma} \nabla_{\nu} \partial_{\lambda} \pi \nabla_{\delta} \partial_{\sigma} \pi \right\}.
\end{aligned} \tag{7}$$

Here we have used the Bianchi identity

$$0 = \nabla_{\lambda} R^{\alpha}_{\beta\mu\nu} + \nabla_{\mu} R^{\alpha}_{\beta\nu\lambda} + \nabla_{\nu} R^{\alpha}_{\beta\lambda\mu}, \tag{8}$$

and the definition of the Riemann curvature:

$$[\nabla_{\mu}, \nabla_{\nu}] V_{\rho} = -R^{\lambda}_{\rho\mu\nu} V_{\lambda}. \tag{9}$$

Here V_{λ} is a covariant vector. Note that in $\mathcal{E}_5^{(0)}$ and $\mathcal{E}_4^{(0)}$, the derivatives of the Riemann curvature appear, which include the third derivatives. By using the Bianchi identity (8), we find

$$\begin{aligned}
& -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\alpha\pi\partial_\mu\pi(\nabla_\nu R^\lambda{}_{\rho\beta\gamma})\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\alpha\pi\partial_\mu\pi\nabla^\lambda(R_{\rho\nu\beta\gamma})\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi, \\
& -\frac{1}{2}\epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\beta\pi\partial_\nu\pi(\nabla_\rho R^\lambda{}_{\sigma\gamma\delta})\partial_\lambda\pi = \frac{1}{4}\epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\beta\pi\partial_\nu\pi\nabla^\lambda(R_{\sigma\rho\gamma\delta})\partial_\lambda\pi.
\end{aligned} \tag{10}$$

Then if we consider the following Lagrangian densities:

$$\Delta\mathcal{L}_4 = \frac{1}{8}\epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\beta\pi\partial_\nu\pi R_{\sigma\rho\gamma\delta}\partial_\lambda\pi\partial^\lambda\pi, \quad \Delta\mathcal{L}_5 = \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\partial_\lambda\pi\partial^\lambda\pi \tag{11}$$

by the variation of π , we have

$$\begin{aligned}
\Delta\mathcal{E}_4 &= -\frac{1}{8}\epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\{2\nabla_\beta\partial_\nu\pi R_{\sigma\rho\gamma\delta}\partial_\lambda\pi\partial^\lambda\pi + 8\partial_\nu\pi R_{\sigma\rho\gamma\delta}\nabla_\beta\partial_\lambda\pi\partial^\lambda\pi + 2\partial_\beta\pi\partial_\nu\pi R_{\sigma\rho\gamma\delta}\square\pi \\
&\quad + 2\partial_\beta\pi\partial_\nu\pi\nabla_\lambda R_{\sigma\rho\gamma\delta}\partial^\lambda\pi\}, \\
\Delta\mathcal{E}_5 &= \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left\{-3\nabla_\alpha\partial_\mu\pi\nabla_\delta\partial_\sigma\pi R_{\rho\nu\beta\gamma}\partial_\lambda\pi\partial^\lambda\pi + \frac{3}{2}\partial_\mu\pi R_{\rho\nu\beta\gamma}R^\tau{}_{\sigma\alpha\delta}\partial_\tau\pi\partial_\lambda\pi\partial^\lambda\pi \right. \\
&\quad - 12\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\nabla_\alpha\partial_\lambda\pi\partial^\lambda\pi + 2\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\sigma\partial_\lambda\pi\nabla_\delta\partial^\lambda\pi - 2\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}R^\tau{}_{\delta\sigma\lambda}\partial_\tau\pi\partial^\lambda\pi \\
&\quad \left. - 2\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\square\pi - 2\partial_\alpha\pi\partial_\mu\pi\nabla_\lambda R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\partial^\lambda\pi\right\}.
\end{aligned} \tag{12}$$

Hence, if we define

$$\mathcal{L}_4 \equiv \mathcal{L}_4^{(0)} + \Delta\mathcal{L}_4, \quad \mathcal{L}_5 \equiv \mathcal{L}_5^{(0)} + \Delta\mathcal{L}_5 \tag{13}$$

by the variation of π , we acquire

$$\begin{aligned}
\mathcal{E}_4 &= \mathcal{E}_4^{(0)} + \Delta\mathcal{E}_4 \\
&= \epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left\{-2\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi + \frac{5}{2}R^\lambda{}_{\rho\beta\gamma}\partial_\nu\pi\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi - \frac{1}{2}\partial_\beta\pi\partial_\nu\pi R^\lambda{}_{\sigma\gamma\delta}\nabla_\rho\partial_\lambda\pi \right. \\
&\quad \left. - \frac{1}{4}\nabla_\beta\partial_\nu\pi R_{\sigma\rho\gamma\delta}\partial_\lambda\pi\partial^\lambda\pi - \partial_\nu\pi R_{\sigma\rho\gamma\delta}\nabla_\beta\partial_\lambda\pi\partial^\lambda\pi - \frac{1}{4}\partial_\beta\pi\partial_\nu\pi R_{\sigma\rho\gamma\delta}\square\pi\right\}, \\
\mathcal{E}_5 &= \mathcal{E}_5^{(0)} + \Delta\mathcal{E}_5 \\
&= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left\{-\frac{5}{6}\nabla_\alpha\partial_\mu\pi\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi + \frac{7}{4}R^\lambda{}_{\nu\alpha\beta}\partial_\mu\pi\partial_\lambda\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \right. \\
&\quad + \frac{1}{4}\partial_\alpha\pi\partial_\mu\pi R^\lambda{}_{\rho\beta\gamma}\partial_\lambda\pi R^\tau{}_{\delta\nu\sigma}\partial_\tau\pi - \frac{1}{2}\partial_\alpha\pi\partial_\mu\pi R^\lambda{}_{\rho\beta\gamma}\nabla_\nu\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi - \frac{3}{8}\nabla_\alpha\partial_\mu\pi\nabla_\delta\partial_\sigma\pi R_{\rho\nu\beta\gamma}\partial_\lambda\pi\partial^\lambda\pi \\
&\quad + \frac{3}{16}\partial_\mu\pi R_{\rho\nu\beta\gamma}R^\tau{}_{\sigma\alpha\delta}\partial_\tau\pi\partial_\lambda\pi\partial^\lambda\pi - \frac{3}{2}\partial_\mu R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\nabla_\alpha\partial_\lambda\pi\partial^\lambda\pi + \frac{1}{4}\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\sigma\partial_\lambda\pi\nabla_\delta\partial^\lambda\pi \\
&\quad \left. - \frac{1}{4}\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}R^\tau{}_{\delta\sigma\lambda}\partial_\tau\pi\partial^\lambda\pi - \frac{1}{4}\partial_\alpha\pi\partial_\mu\pi R_{\rho\nu\beta\gamma}\nabla_\delta\partial_\sigma\pi\square\pi\right\}.
\end{aligned} \tag{14}$$

Note that in the expressions in (14), derivatives higher than two do not appear.

In the curved space-time, we may also consider the following Lagrangian densities:

$$\begin{aligned}
\mathcal{L}_6 &= \epsilon_\mu{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\beta\pi\partial_\nu\pi R_{\gamma\delta\rho\sigma} = -4\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\partial^\mu\pi\partial^\nu\pi, \\
\mathcal{L}_7 &= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\alpha\pi\partial_\mu\pi\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} = 2\partial_\mu\pi\partial^\mu\pi\square\pi R - 4\partial_\mu\pi\partial^\mu\pi\nabla^\nu\nabla^\rho\pi R_{\nu\rho} - 2\partial^\nu\pi\partial^\mu\pi\nabla_\nu\partial_\mu\pi R \\
&\quad + 8\partial^\nu\pi\partial^\mu\pi\nabla_\nu\partial^\rho\pi R_{\mu\rho} - 4\partial^\rho\pi\partial^\mu\pi\square\pi R_{\mu\rho} + 4\partial^\rho\pi\partial^\mu\pi\nabla^\sigma\partial_\sigma\pi R_{\mu\nu\rho\sigma}.
\end{aligned} \tag{15}$$

By the variation π , we obtain

$$\begin{aligned}\mathcal{E}_6 &= -2\epsilon_\mu^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma}, \\ \mathcal{E}_7 &= -3\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\nabla_\alpha\partial_\mu\pi\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{3}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\pi\partial_\lambda\pi R^\lambda_{\nu\alpha\beta}R_{\gamma\delta\rho\sigma}.\end{aligned}\quad (16)$$

Even in the curved space-time, $\mathcal{E}_i$'s ($i = 1, 2, \dots, 7$) do not include derivatives higher than the second ones.¹ We should also note that

$$\Delta\mathcal{L}_4 = -\frac{1}{8}\mathcal{L}_6\partial_\lambda\pi\partial^\lambda\pi, \quad \Delta\mathcal{L}_5 = -\frac{1}{8}\mathcal{L}_7\partial_\lambda\pi\partial^\lambda\pi. \quad (17)$$

Now we find that the general Lagrangian has the following form:

$$\begin{aligned}\mathcal{L} &= \frac{R}{2\kappa^2} + G_2(\pi, X) + G_3(\pi, X)\mathcal{L}_3 + \frac{\partial G_4(\pi, X)}{\partial X}\mathcal{L}_4^{(0)} - \frac{1}{8}G_4(\pi, X)\mathcal{L}_6 + \frac{\partial G_5(\pi, X)}{\partial X}\mathcal{L}_5^{(0)} \\ &\quad - \frac{1}{8}G_5(\pi, X)\mathcal{L}_7 + G_6(\pi)\mathcal{L}_6 + G_7(\pi)\mathcal{L}_7 + G_8(\pi)\mathcal{L}_8.\end{aligned}\quad (18)$$

Here $X = \partial_\mu\pi\partial^\mu\pi$ and $\mathcal{L}_8$ is 4 times the Gauss-Bonnet invariant:

$$\mathcal{L}_8 = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}R_{\alpha\beta\mu\nu}R_{\gamma\delta\rho\sigma} = 4(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \quad (19)$$

We now show that even in the generalized action (18), the higher derivative (higher than second order) do not appear in the equations given by the variation of π . In fact, we find

$$\mathcal{E}_2^{(G)} = \frac{\partial G_2}{\partial \pi} - 2\partial_\alpha\frac{\partial G_2}{\partial X}\partial^\alpha\pi - 2\frac{\partial G_2}{\partial X}\square\pi. \quad (20)$$

$$\begin{aligned}\mathcal{E}_3^{(G)} &= \epsilon_\mu^{\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left[2\frac{\partial^2 G_3}{\partial X^2}\partial_\beta\pi\partial_\alpha\pi\partial_\gamma\pi\partial_\rho\pi(\nabla_\sigma\partial^\beta\pi\nabla_\delta\partial^\alpha\pi - \nabla^\alpha\partial^\beta\pi\nabla_\delta\partial_\sigma\pi) \right. \\ &\quad + \frac{\partial^2 G_3}{\partial \pi\partial X}\partial_\gamma\pi\partial_\rho\pi(2\partial_\sigma\pi\partial_\alpha\pi\nabla_\delta\partial^\alpha\pi - \partial^\alpha\pi\partial_\alpha\pi\nabla_\delta\partial_\sigma\pi) + \frac{1}{2}\frac{\partial^2 G_3}{\partial \pi^2}\partial_\sigma\pi\partial_\delta\pi\partial_\gamma\pi\partial_\rho\pi \\ &\quad + \frac{\partial G_3}{\partial X}(\nabla_\sigma\partial_\alpha\pi\partial_\gamma\pi\partial_\rho\pi\nabla_\delta\partial^\alpha\pi + 3\partial_\alpha\pi\nabla_\sigma\partial_\gamma\pi\partial_\rho\pi\nabla_\delta\partial^\alpha\pi - 4\partial_\alpha\pi\nabla^\alpha\partial_\gamma\pi\partial_\rho\pi\nabla_\delta\partial_\sigma\pi - \square\pi\partial_\gamma\pi\partial_\rho\pi\nabla_\delta\partial_\sigma\pi) \\ &\quad + \frac{\partial G_3}{\partial \pi}\left(\partial_\sigma\pi\partial_\gamma\pi\nabla_\delta\partial_\rho\pi - \frac{3}{2}\partial_\gamma\pi\partial_\rho\pi\nabla_\delta\partial_\sigma\pi\right) - \frac{3}{2}G_3\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \\ &\quad \left. + \frac{\partial G_3}{\partial X}\partial^\alpha\pi\partial_\gamma\pi\partial_\rho\pi R_{\sigma\alpha\delta}{}^\beta\partial_\beta\pi - \frac{3}{4}G_3R_{\gamma\delta\sigma}{}^\alpha\partial_\alpha\pi\partial_\rho\pi\right],\end{aligned}\quad (21)$$

$$\begin{aligned}\mathcal{E}_4^{(G,0)} &= \epsilon_\mu^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left[-\partial_\alpha\frac{\partial^2 G_4}{\partial X^2}\partial^\alpha\pi\partial_\beta\pi\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi - \frac{\partial^2 G_4}{\partial X^2}\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi(\square\pi\partial_\beta\pi + 2\partial^\alpha\pi\nabla_\alpha\partial_\beta\pi) \right. \\ &\quad + 2\partial_\rho\frac{\partial^2 G_4}{\partial X^2}\partial_\alpha\pi\nabla_\gamma\partial^\alpha\pi\partial_\beta\pi\partial_\nu\pi\nabla_\delta\partial_\sigma\pi + 2\frac{\partial^2 G_4}{\partial X^2}\nabla_\rho\partial_\alpha\pi\nabla_\gamma\partial^\alpha\pi\partial_\beta\pi\partial_\nu\pi\nabla_\delta\partial_\sigma\pi + \partial_\rho\frac{\partial^2 G_4}{\partial \pi\partial X}\partial_\gamma\pi\partial_\beta\pi\partial_\nu\pi\nabla_\delta\partial_\sigma\pi \\ &\quad + \frac{\partial G_4}{\partial X}\left(-2\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi + \frac{5}{2}R^\lambda_{\rho\beta\gamma}\partial_\nu\pi\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi - \frac{1}{2}\partial_\beta\pi\partial_\nu\pi R^\lambda_{\sigma\gamma\delta}\nabla_\rho\partial_\lambda\pi\right) \\ &\quad + 2\frac{\partial^2 G_4}{\partial X^2}(\partial_\alpha\pi\nabla_\gamma\partial^\alpha\pi\partial_\beta\pi\partial_\nu\pi R_{\rho\sigma\delta}{}^\lambda\partial_\lambda\pi + \partial^\alpha\pi R_{\rho\alpha\gamma}{}^\lambda\partial_\lambda\pi\partial_\beta\pi\partial_\nu\pi\nabla_\delta\partial_\sigma\pi) \\ &\quad \left. + 2\frac{\partial^2 G_4}{\partial X^2}\partial^\alpha\pi R_{\rho\alpha\gamma}{}^\lambda\partial_\lambda\pi\partial_\beta\pi\partial_\nu\pi\nabla_\delta\partial_\sigma\pi + \frac{\partial^2 G}{\partial \pi\partial X}\partial_\gamma\pi\partial_\beta\pi\partial_\nu\pi R_{\rho\sigma\delta}{}^\lambda\partial_\lambda\pi - \frac{1}{2}\frac{\partial G_4}{\partial X}\partial_\beta\pi\partial_\nu\pi(\nabla_\rho R^\lambda_{\sigma\gamma\delta})\partial_\lambda\pi\right].\end{aligned}\quad (22)$$

Only the last term in (22) includes the third derivative of the metric. This term can be canceled by $\Delta\mathcal{E}_4^{(G)}$, which is given by the variation of $-\frac{1}{8}G_4\mathcal{L}_6$,

¹In addition to the Lagrangian densities in (15), the Brans-Dicke type Lagrangian density $\mathcal{L}_{\text{BD}} = f(\pi)R$ does not generate the field equation and the Einstein equation with derivatives higher than the second ones. Here $f(\pi)$ is an arbitrary function of the scalar field π .

$$\Delta\mathcal{E}_4^{(G)} = \epsilon_\mu^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{4}\partial_\lambda\frac{\partial G_4}{\partial X}\partial^\lambda\pi\partial_\beta\pi\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{4}\frac{\partial G_4}{\partial X}\square\pi\partial_\beta\pi\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{2}\frac{\partial G_4}{\partial X}\partial^\lambda\pi\nabla_\lambda\partial_\beta\pi\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{2}\frac{\partial G_4}{\partial X}\partial_\beta\pi\partial_\nu(\nabla_\rho R^\lambda_{\sigma\gamma\delta})\partial_\lambda\pi - \frac{1}{8}\frac{\partial G_4}{\partial\pi}\partial_\beta\pi\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{4}\partial_\beta G_4\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{4}G_4\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma}\right]. \quad (23)$$

The third derivative term in (22) is canceled by the fourth term in (23).

We also have

$$\begin{aligned} \mathcal{E}_5^{(G,0)} = & \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left[-\frac{1}{3}\partial_\lambda\frac{\partial^2 G_5}{\partial X^2}\partial^\lambda\pi\partial_\alpha\pi\partial_\mu\pi\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \right. \\ & -\frac{1}{3}\frac{\partial^2 G_5}{\partial X^2}\partial_\mu\pi\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi(\square\pi\partial_\alpha\pi + 2\partial^\lambda\pi\nabla_\lambda\partial_\alpha\pi) + \frac{1}{6}\frac{\partial^2 G_5}{\partial\pi\partial X}\partial_\alpha\pi\partial_\nu\pi\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \\ & + \left(\partial_\nu\frac{\partial^2 G_5}{\partial X^2}\partial_\lambda\pi + \frac{\partial^2 G_5}{\partial X^2}\nabla_\nu\partial_\lambda\pi\right)\nabla_\beta\partial^\lambda\pi\partial_\alpha\pi\partial_\mu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \\ & + \frac{1}{2}\left(\partial_\nu\frac{\partial^2 G_5}{\partial\pi\partial X}\partial_\beta\pi + \frac{\partial^2 G_5}{\partial\pi\partial X}\nabla_\nu\partial_\beta\pi\right)\partial_\alpha\pi\partial_\mu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi\frac{\partial G_5}{\partial X}\left(-\frac{5}{6}\nabla_\alpha\partial_\mu\pi\nabla_\beta\partial_\nu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi \right. \\ & + \frac{7}{4}R^\lambda_{\nu\alpha\beta}\partial_\mu\pi\partial_\lambda\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi + \frac{1}{4}\partial_\alpha\pi\partial_\mu\pi R^\lambda_{\rho\beta\gamma}\partial_\lambda\pi R^\tau_{\delta\nu\sigma}\partial_\tau\pi - \frac{1}{2}\partial_\alpha\pi\partial_\mu\pi R^\lambda_{\rho\beta\gamma}\nabla_\nu\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi\left.) \right. \\ & + \frac{\partial^2 G_5}{\partial X^2}\partial^\lambda\pi R_{\nu\lambda\beta}{}^\tau\partial_\tau\pi\partial_\alpha\pi\partial_\mu\pi\nabla_\gamma\partial_\rho\pi\nabla_\delta\partial_\sigma\pi + \partial_\beta\frac{\partial G_5}{\partial X}\partial_\alpha\pi\partial_\mu\pi\nabla_\gamma\partial_\rho\pi R_{\nu\sigma\delta}{}^\lambda\partial_\lambda\pi \\ & \left. - \frac{1}{2}\frac{\partial G_5}{\partial X}\partial_\alpha\pi\partial_\mu\pi(\nabla_\nu R^\lambda_{\rho\beta\gamma})\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi\right]. \quad (24) \end{aligned}$$

Only the last term in (24) includes the third derivative term, and this term is canceled by the seventh term in $\Delta\mathcal{E}_5^{(G)}$

$$\begin{aligned} \Delta\mathcal{E}_5^{(G)} = & \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{4}\left(\partial_\lambda\frac{\partial G_5}{\partial X}\partial^\lambda\pi + \frac{\partial G_5}{\partial X}\square\pi\right)\partial_\alpha\pi\partial_\mu\pi\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{2}\frac{\partial G_5}{\partial X}\partial_\lambda\pi\nabla_\lambda\partial_\alpha\pi\partial_\mu\pi\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} \right. \\ & + \frac{1}{4}(\partial_\alpha G_5\partial_\mu\pi + G_5\nabla_\alpha\partial_\mu\pi)\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} - \frac{1}{4}\partial_\beta G_5\nabla_\nu\partial_\alpha\pi\partial_\mu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{8}G_5\nabla_\nu\partial_\alpha\pi\nabla_\beta\partial_\mu\pi R_{\gamma\delta\rho\sigma} \\ & - \frac{1}{4}\frac{\partial G_5}{\partial\pi}\partial_\alpha\pi\partial_\mu\pi\nabla_\beta\partial_\nu\pi R_{\gamma\delta\rho\sigma} + \frac{1}{2}\frac{\partial G_5}{\partial X}\partial_\alpha\pi\partial_\mu\pi(\nabla_\nu R^\lambda_{\rho\beta\gamma})\partial_\lambda\pi\nabla_\delta\partial_\sigma\pi \\ & \left. + \frac{1}{4}\frac{\partial G_5}{\partial X}\partial_\lambda\pi\partial_\alpha\pi\partial_\mu\pi R_{\lambda\nu\beta}{}^\tau\partial_\tau\pi R_{\gamma\delta\rho\sigma} + \frac{3}{8}G_5\partial_\mu\pi R_{\alpha\beta\nu}{}^\tau\partial_\tau\pi R_{\gamma\delta\rho\sigma}\right]. \quad (25) \end{aligned}$$

It is straightforward to see that the equations of motion given by the variation of π from $G_6(\pi)\mathcal{L}_6$, $G_7(\pi)\mathcal{L}_7$, and $G_8(\pi)\mathcal{L}_8$ do not include a derivative higher than two because $G_6(\pi)$, $G_7(\pi)$, and $G_8(\pi)$ do not contain X .

We also investigate the equation given by the variation of the metric. The equation corresponds to the Einstein equation. Since

$$\begin{aligned} \delta g &= g g^{\mu\nu}\delta g_{\mu\nu}, \quad \delta\Gamma^\kappa_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}), \quad \delta R^\mu_{\nu\lambda\sigma} = \nabla_\lambda\delta\Gamma^\mu_{\sigma\nu} - \nabla_\sigma\delta\Gamma^\mu_{\lambda\nu}, \\ \delta R_{\mu\nu\lambda\sigma} &= \frac{1}{2}[\nabla_\lambda\nabla_\nu\delta g_{\sigma\mu} - \nabla_\lambda\nabla_\mu\delta g_{\sigma\nu} - \nabla_\sigma\nabla_\nu\delta g_{\lambda\mu} + \nabla_\sigma\nabla_\mu\delta g_{\lambda\nu} + \delta g_{\mu\rho}R^\rho_{\nu\lambda\sigma} - \delta g_{\nu\rho}R^\rho_{\mu\lambda\sigma}], \quad \delta R_{\mu\nu} \\ &= \frac{1}{2}[\nabla^\rho(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho}) - \nabla^2\delta g_{\mu\nu} - \nabla_\mu\nabla_\nu(g^{\rho\lambda}\delta g_{\rho\lambda})] \\ &= \frac{1}{2}[\nabla_\mu\nabla^\rho\delta g_{\nu\rho} + \nabla_\nu\nabla^\rho\delta g_{\mu\rho} - \nabla^2\delta g_{\mu\nu} - \nabla_\mu\nabla_\nu(g^{\rho\lambda}\delta g_{\rho\lambda}) - 2R^\lambda_{\nu}{}^\rho{}_\mu\delta g_{\lambda\rho} + R^\rho{}_\mu\delta g_{\rho\nu} + R^\rho{}_\nu\delta g_{\rho\mu}], \\ \delta R &= -\delta g_{\mu\nu}R^{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - \nabla^2(g^{\mu\nu}\delta g_{\mu\nu}), \quad (26) \end{aligned}$$

we find

$$0 = \frac{1}{2\kappa^2}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right) + (\mathcal{H}_2)^{\mu\nu} + (\mathcal{H}_3)^{\mu\nu} + (\mathcal{H}_4)^{\mu\nu} + (\mathcal{H}_5)^{\mu\nu} + (\mathcal{H}_6)^{\mu\nu} + (\mathcal{H}_7)^{\mu\nu} + (\mathcal{H}_8)^{\mu\nu}. \quad (27)$$

Here

$$(\mathcal{H}_2)^{\mu\nu} = \frac{1}{2}g^{\mu\nu}G_2(\pi, X) - \frac{\partial G_2(\pi, X)}{\partial X}\partial^\mu\pi\partial^\nu\pi, \quad (28)$$

$$\begin{aligned}
(\mathcal{H}_3)^{\mu\nu} = & \frac{1}{2} g^{\mu\nu} G_3(\pi, X) \mathcal{L}_3 - \frac{\partial G_3(\pi, X)}{\partial X} \mathcal{L}_3 \partial^\mu \pi \partial^\nu \pi - G_3(\pi, X) (\partial^\mu \pi \partial^\nu \pi \square \pi - \partial^\mu \pi \partial^\rho \pi \nabla^\nu \partial_\rho \pi - \partial^\nu \pi \partial^\rho \pi \nabla^\mu \partial_\rho \pi) \\
& + \frac{1}{2} (-g^{\mu\nu} \square \pi + \partial^\nu \pi \partial^\mu + \partial^\mu \pi \partial^\nu - g^{\mu\nu} \partial^\tau \pi \partial_\tau) (G_3(\pi, X) \partial_\rho \pi \partial^\rho \pi) - \frac{1}{2} \nabla_\rho (G_3(\pi, X) \partial^\rho \pi \partial^\mu \pi \partial^\nu \pi), \quad (29)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_4)^{\mu\nu} = & \frac{1}{2} g^{\mu\nu} \left\{ \frac{\partial G_4(\pi, X)}{\partial X} \mathcal{L}_4^{(0)} - \frac{1}{8} G_4(\pi, X) \mathcal{L}_6 \right\} - \left\{ \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \mathcal{L}_4^{(0)} - \frac{1}{8} \frac{\partial G_4(\pi, X)}{\partial X} \mathcal{L}_6 \right\} \partial^\mu \pi \partial^\nu \pi \\
& + \frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \left\{ -\partial^\mu \pi \partial^\nu \pi ((\square \pi)^2 - \nabla^\sigma \partial_\rho \pi \nabla^\rho \partial_\sigma \pi) + 2 \partial_\rho \pi \partial^\rho \pi \nabla^\mu \partial_\sigma \pi \nabla^\nu \partial^\sigma \pi + 2 \partial^\mu \pi \partial^\rho \pi \nabla^\nu \partial_\rho \pi \square \pi \right. \\
& + 2 \partial^\nu \pi \partial^\rho \pi \nabla^\mu \partial_\rho \pi \square \pi - 2 \partial^\mu \pi \partial^\rho \pi \nabla^\sigma \partial^\nu \pi \nabla_\rho \partial_\sigma \pi - 2 \partial^\nu \pi \partial^\rho \pi \nabla^\sigma \partial^\mu \pi \nabla_\rho \partial_\sigma \pi - 2 \partial^\rho \pi \partial^\sigma \pi \nabla^\mu \partial_\rho \pi \nabla^\nu \partial_\sigma \pi \left. \right\} \\
& + \frac{1}{2} (-g^{\mu\nu} \square \pi + \partial^\nu \pi \partial^\mu + \partial^\mu \pi \partial^\nu - g^{\mu\nu} \partial^\tau \pi \partial_\tau) \left[\frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \{ 2 \partial_\rho \pi \partial^\rho \pi \square \pi - 2 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \} \right] \\
& - \nabla_\lambda \left[\frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \{ \partial_\rho \pi \partial^\rho \pi (\nabla^\lambda \partial^\mu \pi \partial^\nu \pi + \nabla^\lambda \partial^\nu \pi \partial^\mu \pi - \partial^\lambda \pi \nabla^\mu \partial^\nu \pi) + \partial^\lambda \pi \partial^\mu \pi \partial^\nu \pi \square \pi \right. \\
& \left. - 2 \nabla_\tau \partial^\lambda \pi \partial^\tau \pi \partial^\mu \pi \partial^\nu \pi \} \right] + \frac{1}{2} G_4(\pi, X) \left\{ \frac{1}{2} (R^{\mu\rho} \partial^\nu \pi \partial_\rho \pi + R^{\nu\rho} \partial^\mu \pi \partial_\rho \pi) - \frac{1}{2} R \partial^\mu \pi \partial^\nu \pi \right\} \\
& + \frac{1}{2} \nabla_\rho \nabla^\nu (G_4(\pi, X) \partial^\rho \pi \partial^\mu \pi) - \frac{1}{4} \square (G_4(\pi, X) \partial^\mu \pi \partial^\nu \pi) - \frac{1}{4} \nabla_\rho \nabla_\sigma (G_4(\pi, X) g^{\mu\nu} \partial^\rho \pi \partial^\sigma \pi) \\
& + \frac{1}{4} G_4(\pi, X) R^{\mu\nu} \partial_\rho \pi \partial^\rho \pi - \frac{1}{4} \nabla^\mu \nabla^\nu (G_4(\pi, X) \partial_\rho \pi \partial^\rho \pi) + \frac{1}{4} \square (G_4(\pi, X) g^{\mu\nu} \partial_\rho \pi \partial^\rho \pi) - \frac{1}{2} \partial_\rho \pi \partial_\sigma R^{\mu\nu\rho\sigma}, \quad (30)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_5)^{\mu\nu} = & \frac{1}{2} g^{\mu\nu} \left\{ \frac{\partial G_5(\pi, X)}{\partial X} \mathcal{L}_5^{(0)} - \frac{1}{8} G_5(\pi, X) \mathcal{L}_7 \right\} - \left\{ \frac{\partial^2 G_5(\pi, X)}{\partial X^2} \mathcal{L}_5^{(0)} - \frac{1}{8} \frac{\partial G_5(\pi, X)}{\partial X} \mathcal{L}_7 \right\} \partial^\mu \pi \partial^\nu \pi + \frac{1}{6} \frac{\partial G_5(\pi, X)}{\partial X} \\
& \times \{ -\partial^\mu \pi \partial^\nu \pi (\square \pi)^3 + 3 \partial^\mu \pi \partial^\nu \pi \square \pi \nabla^\sigma \partial_\rho \pi \nabla^\rho \partial_\sigma \pi + 6 \partial_\rho \pi \partial^\rho \pi \square \pi \nabla^\mu \partial_\sigma \pi \nabla^\nu \partial^\sigma \pi \\
& - 2 \partial^\mu \pi \partial^\nu \pi \nabla^\rho \partial_\tau \pi \nabla^\sigma \partial_\rho \pi \nabla^\tau \partial_\sigma \pi - 6 \partial_\rho \pi \partial^\rho \pi \nabla^\sigma \partial^\nu \pi \nabla^\tau \partial_\sigma \pi \nabla^\mu \partial_\tau \pi + 3 \partial^\mu \pi \partial^\rho \pi \nabla^\nu \partial_\rho \pi (\square \pi)^2 \\
& + 3 \partial^\nu \pi \partial^\rho \pi \nabla^\mu \partial_\rho \pi (\square \pi)^2 - 3 \partial^\mu \pi \partial^\rho \pi \nabla^\nu \partial_\rho \pi \nabla^\sigma \partial_\tau \pi \nabla^\tau \partial_\sigma \pi - 3 \partial^\nu \pi \partial^\rho \pi \nabla^\mu \partial_\rho \pi \nabla^\sigma \partial_\tau \pi \nabla^\tau \partial_\sigma \pi \\
& - 6 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \nabla^\mu \partial_\tau \pi \nabla^\tau \partial^\nu \pi - 6 \partial^\mu \pi \partial^\rho \pi \nabla^\sigma \partial^\nu \pi \nabla_\rho \partial_\sigma \pi \square \pi - 6 \partial^\nu \pi \partial^\rho \pi \nabla^\sigma \partial^\mu \pi \nabla_\rho \partial_\sigma \pi \square \pi \\
& - 6 \partial^\rho \pi \partial^\sigma \pi \nabla^\mu \partial_\rho \pi \nabla_\sigma \partial^\nu \pi \square \pi + 6 \partial^\mu \pi \partial^\rho \pi \nabla^\sigma \partial^\nu \pi \nabla_\rho \partial_\tau \pi \nabla^\tau \partial_\sigma \pi + 6 \partial^\nu \pi \partial^\rho \pi \nabla^\sigma \partial^\mu \pi \nabla_\rho \partial_\tau \pi \nabla^\tau \partial_\sigma \pi \\
& + 6 \partial^\rho \pi \partial^\sigma \pi \nabla^\mu \partial_\rho \pi \nabla_\sigma \partial_\tau \pi \nabla^\tau \partial^\nu \pi + 6 \partial^\rho \pi \partial^\sigma \pi \nabla^\nu \partial_\rho \pi \nabla_\sigma \partial_\tau \pi \nabla^\tau \partial^\mu \pi \} + \frac{1}{2} (-g^{\mu\nu} \square \pi + \partial^\nu \pi \partial^\mu + \partial^\mu \pi \partial^\nu \\
& - g^{\mu\nu} \partial^\tau \pi \partial_\tau) \left\{ \frac{1}{6} \frac{\partial G_5(\pi, X)}{\partial X} (3 \partial_\rho \pi \partial^\rho \pi (\square \pi)^2 - 3 \partial_\rho \pi \partial^\rho \pi \nabla^\sigma \partial_\tau \pi \nabla^\tau \partial_\sigma \pi - 6 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \square \pi \right. \\
& + 6 \partial^\rho \pi \partial^\sigma \pi \nabla^\tau \partial_\rho \pi \nabla_\sigma \partial_\tau \pi) \left. \right\} + \nabla_\lambda \left\{ \frac{1}{6} \frac{\partial G_5(\pi, X)}{\partial X} (-3 \partial_\rho \pi \partial^\rho \pi \square \pi \partial^\nu \pi \nabla^\mu \partial^\lambda \pi - 3 \partial_\rho \pi \partial^\rho \pi \square \pi \partial^\mu \pi \nabla^\nu \partial^\lambda \pi \right. \\
& + 3 \partial_\rho \pi \partial^\rho \pi \square \pi \partial^\lambda \pi \nabla^\nu \partial^\mu \pi + 3 \partial_\rho \pi \partial^\rho \pi \nabla^\sigma \partial^\lambda \pi \nabla^\mu \partial_\sigma \pi \partial^\nu \pi + 3 \partial_\rho \pi \partial^\rho \pi \nabla^\sigma \partial^\lambda \pi \nabla^\nu \partial_\sigma \pi \partial^\mu \pi \\
& - 3 \partial_\rho \pi \partial^\rho \pi \nabla^\sigma \partial^\mu \pi \nabla^\nu \partial_\sigma \pi \partial^\lambda \pi - \frac{3}{2} \partial^\lambda \pi \partial^\mu \pi \partial^\nu \pi (\square \pi)^2 + \frac{3}{2} \partial^\lambda \pi \partial^\mu \pi \partial^\nu \pi \nabla^\rho \partial_\sigma \pi \nabla^\sigma \partial_\rho \pi \\
& + 3 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \nabla^\mu \partial^\lambda \pi \partial^\nu \pi + 3 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \nabla^\nu \partial^\lambda \pi \partial^\mu \pi - 3 \partial^\rho \pi \partial^\sigma \pi \nabla_\rho \partial_\sigma \pi \nabla^\nu \partial^\mu \pi \partial^\lambda \pi \\
& - 6 \partial^\rho \pi \partial^\mu \pi \nabla^\sigma \partial_\rho \pi \partial^\nu \pi \nabla^\lambda \partial_\sigma \pi + 6 \partial^\mu \pi \partial^\rho \pi \partial^\nu \pi \nabla_\rho \partial^\lambda \pi \square \pi - 3 \partial^\rho \pi \partial^\sigma \pi \nabla^\mu \partial_\rho \pi \nabla_\sigma \partial^\lambda \pi \partial^\nu \pi \\
& \left. - 3 \partial^\rho \pi \partial^\sigma \pi \nabla^\nu \partial_\rho \pi \nabla_\sigma \partial^\lambda \pi \partial^\mu \pi + 3 \partial^\rho \pi \partial^\sigma \pi \nabla^\nu \partial_\rho \pi \nabla_\sigma \partial^\mu \pi \partial^\lambda \pi) \right\} - \frac{1}{8} G_5(\pi, X) \mathcal{L}_7 (-2 \partial^\mu \pi \partial^\nu \pi \square \pi R \\
& + 4 \partial^\mu \pi \partial^\nu \pi \nabla^\rho \partial^\sigma \pi R_{\rho\sigma} + 4 \partial_\rho \pi \partial^\rho \pi \nabla^\mu \partial^\sigma \pi R^\nu{}_\sigma + 4 \partial_\rho \pi \partial^\rho \pi \nabla^\nu \partial^\sigma \pi R^\mu{}_\sigma + 2 \partial^\mu \pi \partial^\rho \pi \nabla^\nu \partial_\rho \pi R \\
& + 2 \partial^\nu \pi \partial^\rho \pi \nabla^\mu \partial_\rho \pi R - 4 \partial^\mu \pi \partial_\rho \pi \nabla^\nu \partial_\sigma \pi R^{\rho\sigma} - 4 \partial^\nu \pi \partial_\rho \pi \nabla^\mu \partial_\sigma \pi R^{\rho\sigma} - 4 \partial^\rho \pi \partial^\mu \pi \nabla_\rho \partial_\sigma \pi R^{\nu\sigma} \\
& - 4 \partial^\rho \pi \partial^\nu \pi \nabla_\rho \partial_\sigma \pi R^{\mu\sigma} - 4 \partial^\rho \pi \partial_\sigma \pi \nabla_\rho \partial^\mu \pi R^{\sigma\nu} - 4 \partial^\rho \pi \partial_\sigma \pi \nabla_\rho \partial^\nu \pi R^{\sigma\mu} + 4 \partial^\mu \pi \partial^\rho \pi \square \pi R^\nu{}_\rho \\
& + 4 \partial^\nu \pi \partial^\rho \pi \square \pi R^\mu{}_\rho - 3 \partial^\mu \pi \partial^\rho \pi \nabla_\sigma \partial_\tau R_\rho{}^{\tau\nu\sigma} - 3 \partial^\nu \pi \partial^\rho \pi \nabla_\sigma \partial_\tau R_\rho{}^{\tau\mu\sigma} - 3 \partial^\rho \pi \partial^\tau \pi \nabla_\sigma \partial_\tau R_\rho{}^{\nu\mu\sigma}
\end{aligned}$$

$$\begin{aligned}
& -3\partial^\rho\pi\partial^\tau\pi\nabla_\sigma\partial^\nu\pi R_{\tau\rho}{}^\mu{}^\sigma - 2\partial_\rho\pi\partial^\rho\pi\Box\pi R^{\mu\nu} + 2\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial_\sigma\pi R^{\mu\nu}) \\
& + \frac{1}{2}(-g^{\mu\nu}\Box\pi + \partial^\nu\pi\partial^\mu + \partial^\mu\pi\partial^\nu - g^{\mu\nu}\partial^\tau\pi\partial_\tau)\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(2\partial_\rho\pi\partial^\rho\pi R - 4\partial^\rho\pi\partial^\sigma\pi R_{\rho\sigma})\right\} \\
& + \nabla_\lambda\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(-2\partial_\rho\pi\partial^\rho\pi\partial^\nu\pi R^{\lambda\mu} - 2\partial_\rho\pi\partial^\rho\pi\partial^\mu\pi R^{\lambda\nu} + 2\partial_\rho\pi\partial^\rho\pi\partial^\lambda\pi R^{\mu\nu} - \partial^\lambda\pi\partial^\mu\pi\partial^\nu\pi R \right. \\
& + 4\partial^\mu\pi\partial_\rho\pi\partial^\nu\pi R^{\rho\lambda} + 2\partial^\rho\pi\partial^\sigma\pi\partial^\nu\pi R_{\sigma}{}^\mu{}_\rho{}^\lambda + 2\partial^\rho\pi\partial^\sigma\pi\partial^\mu\pi R_{\sigma}{}^\nu{}_\rho{}^\lambda - 2\partial^\rho\pi\partial^\sigma\pi\partial^\lambda\pi R_{\sigma}{}^\nu{}_\rho{}^\mu)\left. \right\} \\
& + (\nabla^\nu\nabla^\mu - g^{\mu\nu}\nabla^2)\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(2\partial_\rho\pi\partial^\rho\pi\Box\pi - 2\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial_\sigma\pi)\right\} \\
& + \frac{1}{2}\nabla_\sigma\nabla^\nu\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(-8\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\mu\pi + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\mu\pi + 8\partial^\rho\pi\partial^\mu\pi\nabla_\rho\partial^\sigma\pi - 8\partial^\sigma\pi\partial^\mu\pi\Box\pi)\right\} \\
& + \frac{1}{2}\nabla_\sigma\nabla^\mu\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(-8\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\nu\pi + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\nu\pi + 8\partial^\rho\pi\partial^\nu\pi\nabla_\rho\partial^\sigma\pi - 8\partial^\sigma\pi\partial^\nu\pi\Box\pi)\right\} \\
& - \frac{1}{2}\nabla^2\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(-4\partial_\rho\pi\partial^\rho\pi\nabla^\mu\partial^\nu\pi + 4\partial^\rho\pi\partial^\mu\pi\nabla_\rho\partial^\nu\pi + 4\partial^\rho\pi\partial^\nu\pi\nabla_\rho\partial^\mu\pi - 4\partial^\mu\pi\partial^\nu\pi\Box\pi)\right\} \\
& - \frac{1}{2}g^{\mu\nu}\nabla_\tau\nabla_\sigma\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7(-4\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\tau\pi + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\tau\pi - 4\partial^\sigma\pi\partial^\tau\pi\Box\pi)\right\} \\
& + 2\nabla_\rho\nabla_\sigma\left\{-\frac{1}{8}G_5(\pi, X)\mathcal{L}_7\frac{1}{2}(\partial^\sigma\pi\partial^\nu\pi\nabla^\mu\partial^\rho\pi + \partial^\sigma\pi\partial^\mu\pi\nabla^\nu\partial^\rho\pi - 2\partial^\sigma\pi\partial^\rho\pi\nabla^\mu\partial^\nu\pi \right. \\
& - 2\partial^\mu\pi\partial^\nu\pi\nabla^\sigma\partial^\rho\pi + \partial^\mu\pi\partial^\rho\pi\nabla^\sigma\partial^\nu\pi + \partial^\nu\pi\partial^\rho\pi\nabla^\sigma\partial^\mu\pi)\left. \right\} \tag{31}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_6)^{\mu\nu} &= \frac{1}{2}g^{\mu\nu}G_6(\pi) - 4G_6(\pi)\left\{-(R^{\mu\rho}\partial^\nu\pi\partial_\rho\pi + R^{\nu\rho}\partial^\mu\pi\partial_\rho\pi) + \frac{1}{2}R\partial^\mu\pi\partial^\nu\pi\right\} - 4\nabla_\rho\nabla^\nu(G_6(\pi)\partial^\rho\pi\partial^\mu\pi) \\
& + 2\Box(G_6(\pi)\partial^\mu\pi\partial^\nu\pi) + 2\nabla_\rho\nabla_\sigma(G_6(\pi)g^{\mu\nu}\partial^\rho\pi\partial^\sigma\pi) - 2G_6(\pi)R^{\mu\nu}\partial_\rho\pi\partial^\rho\pi \\
& + 2\nabla^\mu\nabla^\nu(G_6(\pi)\partial_\rho\pi\partial^\rho\pi) - 2\Box(G_6(\pi)g^{\mu\nu}\partial_\rho\pi\partial^\rho\pi), \tag{32}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_7)^{\mu\nu} &= \frac{1}{2}g^{\mu\nu}G_7(\pi) + G_7(\pi)\mathcal{L}_7(-2\partial^\mu\pi\partial^\nu\pi\Box\pi R + 4\partial^\mu\pi\partial^\nu\pi\nabla^\rho\partial^\sigma\pi R_{\rho\sigma} + 4\partial_\rho\pi\partial^\rho\pi\nabla^\mu\partial^\sigma\pi R^\nu{}_\sigma \\
& + 4\partial_\rho\pi\partial^\rho\pi\nabla^\nu\partial^\sigma\pi R^\mu{}_\sigma + 2\partial^\mu\pi\partial^\rho\pi\nabla^\nu\partial_\rho\pi R + 2\partial^\nu\pi\partial^\rho\pi\nabla^\mu\partial_\rho\pi R - 4\partial^\mu\pi\partial_\rho\pi\nabla^\nu\partial_\sigma\pi R^{\rho\sigma} \\
& - 4\partial^\nu\pi\partial_\rho\pi\nabla^\mu\partial_\sigma\pi R^{\rho\sigma} - 4\partial^\rho\pi\partial^\mu\pi\nabla_\rho\partial_\sigma\pi R^{\nu\sigma} - 4\partial^\rho\pi\partial^\nu\pi\nabla_\rho\partial_\sigma\pi R^{\mu\sigma} - 4\partial^\rho\pi\partial_\sigma\pi\nabla_\rho\partial^\mu\pi R^{\sigma\nu} \\
& - 4\partial^\rho\pi\partial_\sigma\pi\nabla_\rho\partial^\nu\pi R^{\sigma\mu} + 4\partial^\mu\pi\partial^\rho\pi\Box\pi R^\nu{}_\rho + 4\partial^\nu\pi\partial^\rho\pi\Box\pi R^\mu{}_\rho - 3\partial^\mu\pi\partial^\rho\pi\nabla_\sigma\partial_\tau R^{\tau\nu\sigma} \\
& - 3\partial^\nu\pi\partial^\rho\pi\nabla_\sigma\partial_\tau R^{\tau\mu\sigma} - 3\partial^\rho\pi\partial^\tau\pi\nabla_\sigma\partial^\mu\pi R_{\tau}{}^\nu{}_\rho{}^\sigma - 3\partial^\rho\pi\partial^\tau\pi\nabla_\sigma\partial^\nu\pi R_{\tau}{}^\mu{}_\rho{}^\sigma - 2\partial_\rho\pi\partial^\rho\pi\Box\pi R^{\mu\nu} \\
& + 2\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial_\sigma\pi R^{\mu\nu}) + \frac{1}{2}(-g^{\mu\nu}\Box\pi + \partial^\nu\pi\partial^\mu + \partial^\mu\pi\partial^\nu - g^{\mu\nu}\partial^\tau\pi\partial_\tau)\{G_7(\pi)\mathcal{L}_7(2\partial_\rho\pi\partial^\rho\pi R \\
& - 4\partial^\rho\pi\partial^\sigma\pi R_{\rho\sigma})\} + \nabla_\lambda\{G_7(\pi)\mathcal{L}_7(-2\partial_\rho\pi\partial^\rho\pi\partial^\nu\pi R^{\lambda\mu} - 2\partial_\rho\pi\partial^\rho\pi\partial^\mu\pi R^{\lambda\nu} + 2\partial_\rho\pi\partial^\rho\pi\partial^\lambda\pi R^{\mu\nu} \\
& - \partial^\lambda\pi\partial^\mu\pi\partial^\nu\pi R + 4\partial^\mu\pi\partial_\rho\pi\partial^\nu\pi R^{\rho\lambda} + 2\partial^\rho\pi\partial^\sigma\pi\partial^\nu\pi R_{\sigma}{}^\mu{}_\rho{}^\lambda + 2\partial^\rho\pi\partial^\sigma\pi\partial^\mu\pi R_{\sigma}{}^\nu{}_\rho{}^\lambda - 2\partial^\rho\pi\partial^\sigma\pi\partial^\lambda\pi R_{\sigma}{}^\nu{}_\rho{}^\mu)\} \\
& + (\nabla^\nu\nabla^\mu - g^{\mu\nu}\nabla^2)\{G_7(\pi)\mathcal{L}_7(2\partial_\rho\pi\partial^\rho\pi\Box\pi - 2\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial_\sigma\pi)\} + \frac{1}{2}\nabla_\sigma\nabla^\nu\{G_7(\pi)\mathcal{L}_7(-8\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\mu\pi \\
& + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\mu\pi + 8\partial^\rho\pi\partial^\mu\pi\nabla_\rho\partial^\sigma\pi - 8\partial^\sigma\pi\partial^\mu\pi\Box\pi)\} + \frac{1}{2}\nabla_\sigma\nabla^\mu\{G_7(\pi)\mathcal{L}_7(-8\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\nu\pi \\
& + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\nu\pi + 8\partial^\rho\pi\partial^\nu\pi\nabla_\rho\partial^\sigma\pi - 8\partial^\sigma\pi\partial^\nu\pi\Box\pi)\} - \frac{1}{2}\nabla^2\{G_7(\pi)\mathcal{L}_7(-4\partial_\rho\pi\partial^\rho\pi\nabla^\mu\partial^\nu\pi \\
& + 4\partial^\rho\pi\partial^\mu\pi\nabla_\rho\partial^\nu\pi + 4\partial^\rho\pi\partial^\nu\pi\nabla_\rho\partial^\mu\pi - 4\partial^\mu\pi\partial^\nu\pi\Box\pi)\} - \frac{1}{2}g^{\mu\nu}\nabla_\tau\nabla_\sigma\{G_7(\pi)\mathcal{L}_7(-4\partial_\rho\pi\partial^\rho\pi\nabla^\sigma\partial^\tau\pi \\
& + 8\partial^\rho\pi\partial^\sigma\pi\nabla_\rho\partial^\tau\pi - 4\partial^\sigma\pi\partial^\tau\pi\Box\pi)\} + 2\nabla_\rho\nabla_\sigma\left\{G_7(\pi)\mathcal{L}_7\frac{1}{2}(\partial^\sigma\pi\partial^\nu\pi\nabla^\mu\partial^\rho\pi + \partial^\sigma\pi\partial^\mu\pi\nabla^\nu\partial^\rho\pi \right. \\
& - 2\partial^\sigma\pi\partial^\rho\pi\nabla^\mu\partial^\nu\pi - 2\partial^\mu\pi\partial^\nu\pi\nabla^\sigma\partial^\rho\pi + \partial^\mu\pi\partial^\rho\pi\nabla^\sigma\partial^\nu\pi + \partial^\nu\pi\partial^\rho\pi\nabla^\sigma\partial^\mu\pi)\left. \right\}, \tag{33}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_8)^{\mu\nu} = & 8(\nabla^\mu \nabla^\nu G_8(\pi))R - 8g^{\mu\nu}(\nabla^2 G_8(\pi))R - 16(\nabla_\rho \nabla^\mu G_8(\pi))R^{\nu\rho} - 16(\nabla_\rho \nabla^\nu G_8(\pi))R^{\mu\rho} \\
& + 16(\nabla^2 G_8(\pi))R^{\mu\nu} + 16g^{\mu\nu}(\nabla_\rho \nabla_\sigma G_8(\pi))R^{\rho\sigma} - 16(\nabla_\rho \nabla_\sigma G_8(\pi))R^{\mu\rho\nu\sigma}.
\end{aligned} \quad (34)$$

Thus, we have provided the explicit expressions of the field equations and the equations corresponding to the Einstein equation for the generalized Galileon scalar models.

III. FRW DYNAMICS

In this section, we provide the formulae for the cosmological reconstruction by considering the FRW dynamics, and we investigate the stability of the reconstructed solution. Usually, we start from a theory, which is defined by the action, and solve the equations of motion to define the background dynamics. The reconstruction is the inverse problem, i.e., when an arbitrary development of the expansion of the Universe is given, we construct the explicit form of the action which reproduces the development. For the reconstructed action, the solution describing the development of the history of the expansion is not always stable. In this section, we show the conditions necessary for the solution to become stable or an attractor.

We now take the FRW universe with the flat spatial part,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (35)$$

and we assume π only depends on the time variable.

Let us denote that the energy density and pressure of the matter by ρ and p . Then the first FRW equation is given by

$$-\frac{3}{2\kappa^2}H^2 + \sum_{k=2}^8 (\mathcal{H}_k)_{00} = -\frac{\rho}{2}. \quad (36)$$

We also write $(\mathcal{H}_k)_{ij}$ as $(\mathcal{H}_k)_{ij} = \mathcal{H}_k a^2 \delta_{ij}$. Accordingly, the second FRW equation is given by

$$\frac{1}{2\kappa^2}(3H^2 + 2\dot{H}) + \sum_{k=2}^8 \mathcal{H}_k = -\frac{p}{2}. \quad (37)$$

If ρ and p are given by the sums of the contribution of the matter with a constant equation of state parameter w_l , we find

$$\rho = \sum_l \rho_l a^{-3(1+w_l)}, \quad p = \sum_l w_l \rho_l a^{-3(1+w_l)}. \quad (38)$$

Here ρ_l 's are constants.

We now demonstrate the reconstruction of the history of the expansion of the Universe. That is, for arbitrary development of the Hubble rate $H(t)$ or scale factor $a(t)$, we determine the functions $G_2(\pi, X)$, $G_3(\pi, X)$, $G_4(\pi, X)$, $G_5(\pi, X)$, $G_6(\pi)$, $G_7(\pi)$, and $G_8(\pi)$, which gives the development. Since the redefinition of the scalar field π can be absorbed into the redefinition of G_i ($i = 2, 3, \dots, 8$), we may choose

$$\pi = t. \quad (39)$$

Then we find

$$X = -1, \quad \dot{G}_i(\pi, X) = \frac{\partial G_i(\pi, X)}{\partial \pi} \Big|_{\pi=t, X=-1}, \text{ etc.} \quad (40)$$

We also write $G_i(\pi, X)$ ($i = 2, 3, 4, 5$) as follows:

$$G_i(\pi, X) = \sum_{n=0}^{\infty} G_i^{(n)}(\pi)(1+X)^n. \quad (41)$$

In addition, we suppose that the scale factor $a(t)$ can be given by an appropriate function $g(t)$ as $a(t) = e^{g(t)}$, which gives $H(t) = \dot{g}(t)$. Thus, the first FRW equation (36) has the following form:

$$\begin{aligned}
& -\frac{3}{2\kappa^2}H^2 - \frac{1}{2}G_2^{(0)}(t) - G_2^{(1)}(t) + \left\{3G_3^{(1)}(t) - \frac{9}{2}G_3^{(0)}(t)\right\}\dot{g}(t) + \left\{-6G_4^{(1)}(t) - 6G_4^{(2)}(t) + \frac{9}{4}G_4^{(0)}(t)\right\}\dot{g}(t)^2 \\
& + \left\{-5G_5^{(1)}(t) + 2G_5^{(2)}(t) + \frac{15}{4}G_5^{(0)}(t)\right\}\dot{g}(t)^3 - 18G_6(t)\dot{g}(t)^2 - 30G_7(t)\dot{g}(t)^3 + 48\dot{G}_8(t)\dot{g}(t)^3 = -\frac{1}{2}\sum_l \rho_l e^{-3(1+w_l)g(t)}.
\end{aligned} \quad (42)$$

On the other hand, the second FRW equation (37) has the following form:

$$\begin{aligned}
& \frac{1}{2\kappa^2}(3H^2 + 2\dot{H}) + \frac{1}{2}G_2^{(0)}(t) + \frac{1}{2}\dot{G}_3^{(0)}(t) - \frac{3}{2}G_4^{(1)}(t)\dot{g}(t)^2 - \dot{G}_4^{(1)}(t)\dot{g}(t) - G_4^{(1)}(t)\ddot{g}(t) + \frac{3}{4}G_4^{(0)}(t)\dot{g}(t)^2 \\
& + \frac{1}{2}\dot{G}_4^{(0)}(t)\dot{g}(t) + \frac{1}{2}G_4^{(0)}(t)\ddot{g}(t) + G_5^{(1)}(t)\dot{g}(t)^3 + \frac{1}{2}\dot{G}_5^{(1)}(t)\dot{g}(t)^2 + G_5^{(1)}(t)\dot{g}(t)\ddot{g}(t) - \frac{3}{2}G_5^{(0)}(t)\dot{g}(t)^3 - \frac{3}{4}\dot{G}_5^{(0)}(t)\dot{g}(t)^2 \\
& - \frac{3}{2}G_5^{(0)}(t)\dot{g}(t)\ddot{g}(t) - 6G_6(t)\dot{g}(t)^2 - 4\dot{G}_6(t)\dot{g}(t) - 4G_6(t)\ddot{g}(t) + 12G_7(t)\dot{g}(t)^3 + 6\dot{G}_7(t)\dot{g}(t)^2 + 12G_7(t)\dot{g}(t)\ddot{g}(t) \\
& - 32\dot{G}_8(t)\dot{g}(t)^3 - 16\ddot{G}_8(t)\dot{g}(t)^2 - 32\dot{G}_8(t)\dot{g}(t)\ddot{g}(t) = -\frac{1}{2}\sum_l w_l \rho_l e^{-3(1+w_l)g(t)}.
\end{aligned} \quad (43)$$

Then if we choose $G_2^{(0)}(t)$, $G_2^{(1)}(t)$, $G_3^{(0)}(t)$, $G_3^{(1)}(t)$, $G_4^{(0)}(t)$, $G_4^{(1)}(t)$, $G_5^{(0)}(t)$, $G_5^{(1)}(t)$, $G_5^{(2)}(t)$, $G_6(t)$, $G_7(t)$, and $G_8(t)$ so that the FRW equations (42) and (43) can be satisfied, we have the following solution:

$$H(t) = \dot{g}(t), \quad \pi = t. \quad (44)$$

We should note that $G_2^{(i)}(t)$ ($i = 2, 3, \dots$), $G_3^{(i)}(t)$ ($i = 2, 3, \dots$), $G_4^{(i)}(t)$ ($i = 3, 4, \dots$), and $G_5^{(i)}(t)$ ($i = 3, 4, \dots$) are irrelevant for the expansion of the Universe.

Then we can completely and explicitly separate G_n to the parts relevant for the expansion and the parts that are irrelevant. We should note, however, that some of the irrelevant parts are related to the stability of the reconstructed solution as we see in the following.

For simplicity, we neglect $G_4(\pi, X)$, $G_5(\pi, X)$, $G_6(\pi)$, $G_7(t)$, and $G_8(t)$ in the rest of this section and consider the examples of the reconstruction and the (in)stability. Then, we acquire

$$\begin{aligned} \frac{3}{\kappa^2} H^2 = & \frac{1}{\kappa^2} g''(\pi)(1+X) + \frac{3}{\kappa^2} g'^2(\pi) - \dot{G}_3^{(0)}(\pi) + (X-1) \left\{ -\frac{1}{2} \dot{G}_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g'(\pi) \right\} \\ & - \sum_{n=2} G_2^{(n)}(\pi)(1+X)^n - 2 \sum_{n=2} G_2^{(n)}(\pi)n(1+X)^{n-1} \dot{\pi}^2 + 6 \frac{\partial G_3(\pi, X)}{\partial X} H \dot{\pi}^5 - 9G_3(\pi, X)H \dot{\pi}^3, \end{aligned} \quad (45)$$

$$\begin{aligned} -\frac{1}{\kappa^2} (3H^2 + 2\dot{H}) = & \frac{1}{\kappa^2} g''(\pi)(-1+X) - \frac{3}{\kappa^2} g'^2(\pi) + \dot{G}_3^{(0)}(\pi) + (X+1) \left\{ -\frac{1}{2} \dot{G}_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) \right. \\ & \left. - \frac{9}{2} G_3^{(0)}(\pi)g'(\pi) \right\} + \sum_{n=2} G_2^{(n)}(\pi)(1+X)^n - \dot{G}_3(\pi, X) \dot{\pi}^3 - 3G_3(\pi, X) \dot{\pi}^2 \ddot{\pi}. \end{aligned} \quad (46)$$

By combining the equation of motion for π with (46), we obtain

$$\begin{aligned} \ddot{\pi} = & A^{-1} \left[-18 \sum_{n=0}^{\infty} G_3^{(n)}(\pi)n(1+X)^{n-1} H^2 \dot{\pi}^4 - 6 \sum_{n=0}^{\infty} G_3^{(n)}(\pi)n(1+X)^{n-1} H \dot{\pi}^5 \right. \\ & + \left(-6 \sum_{n=0}^{\infty} G_3^{(n)}n(1+X)^{n-1} \dot{\pi}^4 + 9 \sum_{n=0}^{\infty} G_3^{(n)}(1+X)^n \dot{\pi}^2 \right) \left\{ -\frac{1}{2} g''(-1+X) + \frac{3}{2} g'^2 - \frac{1}{2} \kappa^2 G_3^{(0)}(\pi) - \frac{\kappa^2}{2} (X+1) \right. \\ & \times \left(-\frac{1}{2} G_3^{(0)}(\pi) + 3G_3^{(1)}g'(\pi) - \frac{9}{2} G_3^{(0)}g'(\pi) \right) - \frac{\kappa^2}{2} \sum_{n=2}^{\infty} G_2^{(n)}(1+X)^n + \frac{\kappa^2}{2} \sum_{n=0}^{\infty} G_3^{(n)}(1+X)^n \dot{\pi}^4 - \frac{3}{2} H^2 \left. \right\} \\ & + 27 \sum_{n=0}^{\infty} G_3^{(n)}(1+X)^n H^2 \dot{\pi}^2 + 9 \sum_{n=0}^{\infty} G_3^{(n)}(1+X)^n H \dot{\pi}^3 - 3 \sum_{n=0}^{\infty} G_3^{(n)}(1+X)^n H \dot{\pi}^3 \\ & - 6 \left\{ \frac{1}{\kappa^2} g''(\pi) - \frac{1}{2} G_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g'(\pi) + \sum_{n=2}^{\infty} G_2^{(n)}(\pi)n(1+X)^{n-1} \right\} H \dot{\pi} \\ & - 2 \left\{ \frac{1}{\kappa^2} g'''(\pi) - \frac{1}{2} G_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) + 3G_3^{(1)}(\pi)g''(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g'(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g''(\pi) \right. \\ & + \sum_{n=2}^{\infty} G_2^{(n)}n(1+X)^n \left. \right\} - \left\{ -\frac{6}{\kappa^2} g'(\pi)g''(\pi) - \frac{2}{\kappa^2} g'''(\pi) + \left(\frac{1}{\kappa^2} g'''(\pi) - \frac{1}{2} G_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) \right. \right. \\ & \left. \left. + 3G_3^{(1)}(\pi)g''(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g''(\pi) \right) (1+X) + G_3^{(0)}(\pi) + \sum_{n=2}^{\infty} G_2^{(n)}(1+X)^n \right\} \left. \right], \end{aligned} \quad (47)$$

$$\begin{aligned} \dot{H} = & -\frac{1}{2} g''(\pi)(-1+X) + \frac{3}{2} g'^2(\pi) - \frac{1}{2} \kappa^2 G_3^{(0)}(\pi) - \frac{\kappa^2}{2} (X+1) \left(-\frac{1}{2} G_3^{(0)}(\pi) + 3G_3^{(1)}(\pi)g'(\pi) - \frac{9}{2} G_3^{(0)}(\pi)g'(\pi) \right) \\ & - \frac{\kappa^2}{2} \sum_{n=2}^{\infty} G_2^{(n)}(1+X)^n + \frac{\kappa^2}{2} \sum_{n=0}^{\infty} G_3^{(n)}(\pi)(1+X)^n \dot{\pi}^4 - \kappa^2 \sum_{n=0}^{\infty} G_3^{(n)}(\pi)n(1+X)^{n-1} \dot{\pi}^4 \ddot{\pi} \\ & + \frac{3}{2} \kappa^2 \sum_{n=0}^{\infty} G_3^{(n)}(\pi)(1+X)^n \dot{\pi}^2 \ddot{\pi} - \frac{3}{2} H^2. \end{aligned} \quad (48)$$

We can eliminate $\ddot{\pi}$ by substituting (47) into (48). Here

$$\begin{aligned}
A \equiv & \left(\kappa^2 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) n(1+X)^{n-1} \dot{\pi}^4 - \frac{3}{2} \kappa^2 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) (1+X)^n \dot{\pi}^2 \right) \left(-6 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) n(1+X)^{n-1} \dot{\pi}^4 \right. \\
& + 9 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) (1+X)^n \dot{\pi}^2 \Big) + 42 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) n(1+X)^{n-1} H \dot{\pi}^3 - 18 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) (1+X)^n H \dot{\pi} \\
& - 12 \sum_{n=0}^{\infty} G_3^{(n)}(\pi) n(n-1) (1+X)^{n-2} H \dot{\pi}^5 + \frac{2}{\kappa^2} g''(\pi) - G_3^{(0)}(\pi) + 6G_3^{(1)}(\pi) g'(\pi) - 9G_3^{(0)}(\pi) g'(\pi) \\
& + 2 \sum_{n=2}^{\infty} G_2^{(n)}(\pi) n(1+X)^{n-1} - 4 \sum_{n=2}^{\infty} G_2^{(n)}(\pi) n(n-1) (1+X)^{n-2},
\end{aligned} \tag{49}$$

We now define the variables x and y by $x \equiv \dot{\pi}$ and $y \equiv g'(\pi)/H$. Then for the reconstructed solution (44), we find $x = 1$ and $y = 1$. Since

$$\frac{dx}{dN} = \frac{\ddot{\pi}y}{g'(\pi)}, \quad \frac{dy}{dN} = \frac{g''(\pi)}{g'^2(\pi)} y^2 x - \frac{y^3}{g'^2(\pi)} \dot{H}, \quad \frac{d\pi}{dN} = \frac{yx}{g'(\pi)}, \tag{50}$$

we can examine the perturbation from the reconstructed solution $x = y = 1$ and $\pi = t$ as $x = 1 + \delta x$, $y = 1 + \delta y$, and $\pi = t + \delta \pi$ by using x and y . Equations (47) and (48) are as follows:

$$\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \\ \delta \pi \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta y \\ \delta \pi \end{pmatrix}, \quad M \equiv \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}. \tag{51}$$

Here

$$\begin{aligned}
M_{11} = & A_0^{-1} g'^{-1}(t) \left[72G_3^{(2)}(t) g'^2(t) - 72G_3^{(1)}(t) g'^2(t) + 24\dot{G}_3^{(2)}(t) g'(t) - 30\dot{G}_3^{(1)}(t) g'(t) + (g''(t) + 3g'^2(t) \right. \\
& - \kappa^2 \dot{G}_3^{(0)}(t)) (24G_3^{(2)}(t) - 42G_3^{(1)}(t) + 18G_3^{(0)}(t)) + (-6G_3^{(1)}(t) + 9G_3^{(0)}(t)) \left\{ g''(t) + \kappa^2 \left(\frac{3}{2} \dot{G}_3^{(0)}(t) + 3G_3^{(1)}(t) g'(t) \right. \right. \\
& \left. \left. - \frac{9}{2} G_3^{(0)}(t) g'(t) - \dot{G}_3^{(1)}(t) \right) \right\} + 54G_3^{(1)}(t) g'^2(t) + 54G_3^{(0)}(t) g'^2(t) - 12\dot{G}_3^{(1)}(t) g'(t) + 18\dot{G}_3^{(0)}(t) g'(t) + 24G_2^{(2)}(t) g'(t) \\
& - 6 \left(\frac{1}{\kappa^2} g'(t) g''(t) - \frac{1}{2} \dot{G}_3^{(0)}(t) g'(t) + 3G_3^{(1)}(t) g'^2(t) - \frac{9}{2} G_3^{(0)}(t) g'^2(t) \right) + 8\dot{G}_2^{(2)} \\
& \left. + 2 \left(\frac{1}{\kappa^2} g'''(t) - \frac{1}{2} \ddot{G}_3^{(0)}(t) + 3\dot{G}_3^{(1)}(t) g'(t) + 3G_3^{(1)}(t) g''(t) - \frac{9}{2} \dot{G}_3^{(0)}(t) g'(t) - \frac{9}{2} G_3^{(0)}(t) g''(t) \right) \right],
\end{aligned} \tag{52}$$

$$\begin{aligned}
M_{12} = & A_0^{-1} g'^{-1}(t) \left[36G_3^{(1)}(t) g'^2(t) + 6\dot{G}_3^{(1)}(t) g'(t) + 3(-6G_3^{(1)}(t) + 9G_3^{(0)}(t)) g'^2(t) - 54G_3^{(0)}(t) g'^2(t) - 9\dot{G}_3^{(0)}(t) g'(t) \right. \\
& \left. + 3\dot{G}_3^{(0)}(t) g'(t) + 6g'(t) \left(\frac{1}{\kappa^2} g''(t) - \frac{1}{2} \dot{G}_3^{(0)}(t) + 3G_3^{(1)}(t) g'(t) - \frac{9}{2} G_3^{(0)}(t) g'(t) \right) \right],
\end{aligned} \tag{53}$$

$$\begin{aligned}
M_{13} = & A_0^{-1} g'^{-1}(t) \left[-18\dot{G}_3^{(1)}(t) g'^2(t) - 36G_3^{(1)}(t) g'(t) g''(t) - 6\ddot{G}_3^{(1)}(t) g'(t) - 6G_3^{(1)}(t) g''(t) \left(g''(t) + \frac{3}{2} g'^2(t) - \kappa^2 \dot{G}_3^{(0)} + \frac{3}{2} g'^2(t) \right) \right. \\
& \times (-6\dot{G}_3^{(1)}(t) + 9\dot{G}_3^{(0)}(t)) + (-6G_3^{(1)}(t) + 9G_3^{(0)}(t)) g'''(t) + 27\dot{G}_3^{(0)}(t) g'^2(t) + 54G_3^{(0)}(t) g'(t) g''(t) + 6\ddot{G}_3^{(0)}(t) g'(t) \\
& + 6\dot{G}_3^{(0)}(t) g''(t) - 6g'(t) \left(\frac{1}{\kappa^2} g'''(t) - \frac{1}{2} \ddot{G}_3^{(0)}(t) + 3\dot{G}_3^{(1)}(t) g'(t) + 3G_3^{(1)}(t) g''(t) - \frac{9}{2} \dot{G}_3^{(0)}(t) g'(t) - \frac{9}{2} G_3^{(0)}(t) g''(t) \right) \\
& - 6g''(t) \left(\frac{1}{\kappa^2} g''(t) - \frac{1}{2} \dot{G}_3^{(0)}(t) + 3G_3^{(1)}(t) g'(t) - \frac{9}{2} G_3^{(0)}(t) g'(t) \right) - 2 \left(\frac{1}{\kappa^2} g'''(t) - \frac{1}{2} \ddot{G}_3^{(0)}(t) + 3\dot{G}_3^{(1)}(t) g'(t) \right. \\
& \left. + 6\dot{G}_3^{(1)}(t) g''(t) + 3G_3^{(1)}(t) g'''(t) - \frac{9}{2} \ddot{G}_3^{(0)}(t) g'(t) - 9\dot{G}_3^{(0)}(t) g''(t) \right) + \frac{6}{\kappa^2} g'^2(t) + \frac{6}{\kappa^2} g'(t) g'''(t) - \ddot{G}_3^{(0)} \Big],
\end{aligned} \tag{54}$$

$$M_{21} = \frac{g''(t)}{g'^2(t)} - g'^{-2}(t) \left\{ g''(t) + \kappa^2 \left(\frac{3}{2} \dot{G}_3^{(0)}(t) + 3G_3^{(1)}(t)g'(t) - \frac{9}{2}G_3^{(0)}(t)g'(t) - \dot{G}_3^{(1)}(t) \right) \right\} - g'^{-1}(t) \left(-\kappa^2 G_3^{(1)}(t) + \frac{3}{2} \kappa^2 G_3^{(0)}(t) \right) M_{11}, \quad (55)$$

$$M_{22} = -g'^{-2}(t)g''(t) - 3 - g'^{-1}(t) \left(-\kappa^2 G_3^{(1)}(t) + \frac{3}{2} \kappa^2 G_3^{(0)}(t) \right) M_{12}, \quad (56)$$

$$M_{23} = -g'(t)^{-1} \left(-\kappa^2 G_3^{(1)}(t) + \frac{3}{2} \kappa^2 G_3^{(0)}(t) \right) M_{13} + 3g'^{-1}(t)g''(t) - g'^{-2}(t)g'''(t), \quad (57)$$

where

$$A_0 = -\frac{\kappa^2}{6} (6G_3^{(1)}(t) - 9G_3^{(0)}(t))^2 + 42G_3^{(1)}(t)g'(t) - 27G_3^{(0)}(t)g'(t) - 24G_3^{(2)}(t)g'(t) + \frac{2}{\kappa^2} g''(t) - \dot{G}_3^{(0)}(t) + 6G_3^{(1)}(t)g'(t) - 8G_2^{(2)}(t), \quad (58)$$

and

$$\begin{aligned} M_{31} &= g'^{-1}(t), & M_{32} &= g'^{-1}(t), \\ M_{33} &= -g'^{-2}(t)g''(t). \end{aligned} \quad (59)$$

The eigenvalue equation for the matrix M in (51) has the following form:

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0, \quad (60)$$

where

$$\alpha = -M_{11} - M_{22} + g'^{-2}(t)g''(t), \quad (61)$$

$$\beta = -(M_{11} + M_{22})g'^{-2}(t)g''(t) - (M_{23} + M_{13})g'^{-1}(t) + M_{11}M_{22} - M_{12}M_{21}, \quad (62)$$

$$\gamma = (M_{11}M_{22} - M_{12}M_{21})g'^{-2}(t)g''(t) + (M_{11}M_{23} - M_{13}M_{21} - M_{12}M_{23} + M_{13}M_{22})g'^{-1}(t). \quad (63)$$

When all the eigenvalues λ are negative, the reconstructed solution becomes stable. The Hurwitz theorem shows the condition that all the eigenvalues are negative and is given by

$$(i) \alpha > 0, \quad (ii) \alpha\beta > \gamma, \quad (iii) \alpha\beta\gamma > \gamma^2. \quad (64)$$

For simplicity, we further put $G_3^{(0)}(t) = G_3^{(1)}(t) = 0$. Thus, we find

$$\begin{aligned} M_{11} &= A_0^{-1}g'^{-1}(t) \left\{ 72G_3^{(2)}(t)g'^2(t) + 24\dot{G}_3^{(2)}(t)g'(t) + 24G_3^{(2)}(t)(g''(t) + 3g'^2(t)) + 24G_2^{(2)}(t)g'(t) - \frac{6}{\kappa^2}g'(t)g''(t) + 8\dot{G}_2^{(2)} + \frac{2}{\kappa^2}g'''(t) \right\}, \\ M_{12} &= 6A_0^{-1}\frac{1}{\kappa^2}g''(t), \\ M_{13} &= -2A_0^{-1}g'^{-1}(t)\frac{1}{\kappa^2}g'''(t), & M_{21} &= 0, \\ M_{22} &= -g'^{-2}(t)g''(t) - 3, \\ M_{23} &= 3g'^{-1}(t)g''(t) - g'^{-2}(t)g'''(t), \\ M_{31} &= M_{32} = g'^{-1}(t), & M_{33} &= -g'^{-2}(t)g''(t). \end{aligned} \quad (65)$$

Here A_0 has the following form:

$$A_0 = -24G_3^{(2)}(t)g'(t) + \frac{2}{\kappa^2}g''(t) - 8G_2^{(2)}. \quad (66)$$

The eigenvalue equation becomes

$$(M_{22} - \lambda)\{\lambda^2 - (M_{11} + M_{33})\lambda + M_{11}M_{33} - M_{13}M_{31}\} = 0. \quad (67)$$

Therefore, we obtain

$$\begin{aligned} \lambda &= M_{22}, \\ \frac{1}{2} \left[M_{11} + M_{33} \pm \sqrt{(M_{11} + M_{33})^2 - 4(M_{11}M_{33} - M_{13}M_{31})} \right]. \end{aligned} \quad (68)$$

Consequently, the condition of the stability is given by

$$\begin{aligned} (i) & M_{22} < 0, \\ (ii) & M_{11} + M_{33} < 0, \\ (iii) & M_{11}M_{33} - M_{13}M_{31} > 0. \end{aligned} \quad (69)$$

From the first condition (i) in (69), we find

$$0 > -g'^{-2}(t)g''(t) - 3, \quad (70)$$

and from the second condition (ii),

$$\begin{aligned} 0 &> A_0^{-1} \left\{ -g'^{-2}(t)g''(t)A_0 + 144G_3^{(2)}(t)g'(t) + 24\dot{G}_3^{(2)}(t) + 24G_3^{(2)}(t)g'^{-1}(t)g''(t) + 24G_2^{(2)}(t) - \frac{6}{\kappa^2}g''(t) + 8\dot{G}_2^{(2)}(t)g'^{-1}(t) + \frac{2}{\kappa^2}g'''(t)g'^{-1}(t) \right\}. \end{aligned} \quad (71)$$

The third condition (iii) in (69) presents

$$0 < A_0^{-1} g'^{-1}(t) \left[-g'^{-2}(t) g''(t) \left\{ 72 G_3^{(2)}(t) g'^2(t) + 24 \dot{G}_3^{(2)}(t) g'(t) + 24 G_3^{(2)}(t) (g''(t) + 3 g'^2(t)) \right. \right. \\ \left. \left. + 24 G_2^{(2)}(t) g'(t) - \frac{6}{\kappa^2} g'(t) g''(t) + 8 \dot{G}_2^{(2)}(t) + \frac{2}{\kappa^2} g'''(t) \right\} + 2 g'^{-1}(t) \frac{1}{\kappa^2} g''''(t) \right]. \quad (72)$$

Before going to the nontrivial case, in order to check whether the above formulation could work, we consider the de Sitter space-time, where $g'(t)$ is constant. Then we find

$$M = \begin{pmatrix} M_{11} & 0 & 0 \\ 0 & -3 & 0 \\ g'^{-1}(t) & g'^{-1}(t) & 0 \end{pmatrix}, \quad M_{11} = \frac{144 G_3^{(2)}(t) g'^2(t) + 24 \dot{G}_3^{(2)}(t) g'(t) + 24 G_2^{(2)}(t) g'(t) + 8 \dot{G}_2^{(2)}}{-24 G_3^{(2)}(t) g'^2(t) + \frac{2}{\kappa^2} g'(t) g''(t) - 8 G_2^{(2)}(t) g'(t)}. \quad (73)$$

Then by solving the eigenvalue equation (67), we find the eigenvalues:

$$\lambda = -3, \quad 0, \quad \frac{144 G_3^{(2)}(t) g'^2(t) + 24 \dot{G}_3^{(2)}(t) g'(t) + 24 G_2^{(2)}(t) g'(t) + 8 \dot{G}_2^{(2)}}{-24 G_3^{(2)}(t) g'^2(t) + \frac{2}{\kappa^2} g'(t) g''(t) - 8 G_2^{(2)}(t) g'(t)}. \quad (74)$$

Note that the eigenvector corresponding to the eigenvalue 0 is given by

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta \pi \end{pmatrix} = s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (75)$$

which shows that the eigenvector with the eigenvalue 0 corresponds to the shift of π or the origin of time and therefore the eigenvalue 0 does not correspond to any instability. Especially if we consider the case that $G_3^{(2)}(t) = 0$, the condition that the last eigenvalue in (74) is given by

$$-\frac{\dot{G}_2^{(2)}(t) g'^{-1}(t)}{G_2^{(2)}(t)} - 3 < 0, \quad (76)$$

can be satisfied if

$$G_2^{(2)}(t) = D \exp(-f(t)), \quad (77)$$

where D is a constant and $f(t)$ is an arbitrary function satisfying the condition $0 < f'(t) < 3g'(t)$. We should also note that Eq. (77) can be satisfied if $G_2^{(2)}(t)$ is a constant.

As a somewhat nontrivial example, we study the case that $g'(t)$ is a solution of the Λ CDM model:

$$a(t) = A \sinh^{2/3}(bt), \quad (78)$$

where A and b are positive constants. Equation (78) gives

$$g'(t) = \frac{2}{3} b \coth(bt), \quad g''(t) = -\frac{2b^2}{3 \sinh^2(bt)}. \quad (79)$$

Then the first condition (i) in (69) gives

$$0 > \frac{3}{2 \cosh^2(bt)} - 3, \quad (80)$$

which is trivially satisfied, and the second condition (ii) presents

$$0 > A_0^{-1} \left\{ 24 \dot{G}_3^{(2)} - \frac{48b}{\cosh(bt) \sinh(bt)} G_3^{(2)} \right. \\ \left. + \left(24 - \frac{12}{\cosh^2(bt)} \right) G_2^{(2)}(t) + \frac{12}{b} \tanh(bt) \dot{G}_2^{(2)} \right. \\ \left. + \frac{b^2}{\kappa^2} \left(\frac{8}{\sinh^2(bt)} - \frac{2}{\cosh^2(bt) \sinh^2(bt)} \right) \right\}. \quad (81)$$

Here

$$A_0 = -16 G_3^{(2)} b \coth(bt) - \frac{4b^2}{3 \kappa^2} \frac{1}{\sinh^2(bt)} - 8 G_2^{(2)}. \quad (82)$$

By using the third condition (iii) in (69), we find

$$0 < A_0^{-1} \left\{ 64 b^2 \coth^2(bt) G_3^{(2)} + 16 b \coth(bt) \dot{G}_3^{(2)} \right. \\ \left. - \frac{16 b^2}{\sinh^2(bt)} G_3^{(2)} + 16 b \coth(bt) G_2^{(2)} + 8 \dot{G}_2^{(2)} \right. \\ \left. - \frac{b^3}{\kappa^2} \left(\frac{16}{3} \coth(bt) + \frac{8 \cosh(bt)}{3 \sinh^3(bt)} \right) \right\}. \quad (83)$$

Although the above expressions (81) and (82) are very complicated, these conditions can be satisfied. The simplest example is given by

$$G_2^{(2)}(t) = C, \quad G_3^{(2)}(t) = 0. \quad (84)$$

Here C is a constant. By substituting (84) into (81)–(83), we find

$$A_0 = -\frac{4b^2}{3\kappa^2} \frac{1}{\sinh^2(bt)} - 8C, \quad (85)$$

$$0 > A_0^{-1} \left\{ \left(24 - \frac{12}{\cosh^2(bt)} \right) C + \frac{b^2}{\kappa^2} \left(\frac{8}{\sinh^2(bt)} - \frac{2}{\cosh^2(bt)\sinh^2(bt)} \right) \right\}, \quad (86)$$

$$0 < A_0^{-1} \left\{ 16b \left(C - \frac{b^2}{3\kappa^2} \right) \coth(bt) - \frac{8b^3 \cosh(bt)}{3\kappa^2 \sinh^3(bt)} \right\}. \quad (87)$$

If $C > 0$, we find $A_0 < 0$ in (85) and the inequality (86) is satisfied. The inequality (87) is also satisfied if

$$0 < C < \frac{b^2}{3\kappa^2}. \quad (88)$$

Then stability can be realized. Note that even for the de Sitter space-time, if $G_2^{(2)}(t)$ and $G_3^{(2)}(t)$ are given by (84), the stability is realized as discussed after (77).

As a result, in this section, by exploring the FRW dynamics, we have given the explicit formulae for the reconstruction and we have investigated the condition of the stability for the reconstructed solution.

IV. VAINSHTEIN MECHANISM

In this section, in order to investigate whether the Vainshtein mechanism could work, we consider the behavior of the Galileon scalar field π in the spherically symmetric space-time, especially in the Schwarzschild background.

The Schwarzschild space-time has a spherically symmetric and static metric. The general spherically symmetric and static space-time has the metric of the following form:

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (89)$$

where Φ and Λ only depend on the radial coordinate r . In the Schwarzschild background, the equation derived by the variation of the Galileon scalar field π is given by

$$\begin{aligned} & \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_2(\pi, X)}{\delta \pi} + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_3(\pi, X) \mathcal{L}_3}{\delta \pi} + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \frac{\partial G_4(\pi, X)}{\partial X} \mathcal{L}_4^{(0)}}{\delta \pi} - \frac{1}{8} \frac{1}{\sqrt{-g}} \frac{\delta}{\sqrt{-g} G_4(\pi, X) \mathcal{L}_6} \delta \pi \\ & + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} \frac{\partial G_5(\pi, X)}{\partial X} \mathcal{L}_5^{(0)}}{\delta \pi} + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_5(\pi, X) \mathcal{L}_7}{\delta \pi} + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_6(\pi) \mathcal{L}_6}{\delta \pi} + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_7(\pi) \mathcal{L}_7}{\delta \pi} \\ & + \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g} G_8(\pi) \mathcal{L}_8}{\delta \pi} = 0, \end{aligned} \quad (90)$$

where we have used the notation $\frac{\delta A}{\delta \pi} \equiv \frac{\partial A}{\partial \pi} - \partial_\mu \frac{\partial A}{\partial (\partial_\mu \pi)}$. The explicit forms of $\mathcal{E}_k$ and also $(\mathcal{H}_k)_{\mu\nu}$ are given in Appendix B. We also note that the Einstein tensor is given by

$$\begin{aligned} G_{tt} &= \frac{1}{r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})], & G_{rr} &= -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \frac{d\Phi}{dr}, \\ G_{\theta\theta} &= r^2 e^{-2\Lambda} \left[\frac{d^2\Phi}{dr^2} + \left(\frac{d\Phi}{dr} \right)^2 + \frac{1}{r} \frac{d\Phi}{dr} - \frac{d\Phi}{dr} \frac{d\Lambda}{dr} - \frac{1}{r} \frac{d\Lambda}{dr} \right], & G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}. \end{aligned} \quad (91)$$

The Einstein equation (27) shows that in order for the modification of the Schwarzschild geometry due to the Galileon scalar to be small, we find

$$\sum_{i=2}^8 (\mathcal{H}_i)_{\mu\nu} \sim 0. \quad (92)$$

We now investigate how the Vainshtein mechanism works for the generalized Galileon scalar model. In the limit of $r \gg GM$, the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dr^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (93)$$

for general spherically symmetric and static space-time in (89) behaves as

$$\Phi(r) \simeq -\frac{GM}{r}, \quad \Lambda(r) \simeq \frac{GM}{r}. \quad (94)$$

If the Vainshtein mechanism works, the gravity with the Galileon scalar behaves as the usual Einstein gravity in the

short distance compared with the cosmological scale, and $\Phi(r)$ and $\Lambda(r)$ in (89) behave as those in (94).

Before examining the Vainshtein mechanism in the covariant and generalized Galileon model, we review how the Vainshtein mechanism works when we only include $\mathcal{L}_4$ in (3). In this case, the energy momentum tensors of the Galileon scalar behave as

$$\begin{aligned} T_{00}^\pi &= 8c_4 \frac{(\pi')^3 \pi''}{r} + c_4 \frac{(\pi')^4}{r^2} + \dots, \\ T_{11}^\pi &= -5c_4 \frac{(\pi')^4}{r^2} + \dots. \end{aligned} \quad (95)$$

If we write $c_4 = \tilde{c}_4 \frac{M_{\text{Pl}}^2}{H_0^4}$, we find $\tilde{c}_4 \sim \mathcal{O}(1)$.

We now study the generalized Galileon model in (18) but for simplicity, we only include $\mathcal{L}_2$ and $\mathcal{L}_3$:

$$\mathcal{L}_\pi = -\frac{1}{2}c_2 X + G_3(\pi, X)\mathcal{L}_3. \quad (96)$$

We may assume that the first term in (96) dominates in the short distance and that the second term does in the long distance. Since the field equation has a form like

$$\pi''(r) = f(\pi(r), \pi'(r)) \frac{\pi'(r)}{r}, \quad (97)$$

when $r \ll r_V$, we find

$$\pi'(r) = \pi'(r_V) \exp\left(\int_{r_V}^r \frac{f(\pi(r'), \pi'(r'))}{r'} dr'\right) = \pi'(r_V) \frac{e^{F(r)}}{e^{F(r_V)}}, \quad (98)$$

where

$$F(r) \equiv \int_{r_V}^r \frac{f(\pi(r'), \pi'(r'))}{r'} dr. \quad (99)$$

If we provide $\pi'(r_V) \sim \Phi'_N(r_V)$, $\pi'(r)$ becomes small when we choose $f(\pi(r), \pi'(r))$ so that we can have $F(r) \ll F(r_V)$. If $\pi'(r) \ll \pi'(r_V)$ as required so that the Vainshtein mechanism can work, we may find $\pi(r) \ll \pi(r_V)$. Then if we can choose $G_3(\pi, X)$ so that $f(\pi(r), \pi'(r)) > 0$ when $\pi'(r) < \pi'(r_V)$ and $\pi(r) < \pi(r_V)$, we may surely obtain $\pi'(r) \ll \pi'(r_V)$, consistently. We now have

$$f(\pi(r), \pi'(r)) = \frac{-2r^2 - 2\frac{\partial G_3}{\partial \pi}(\pi')^2 r^2 - 3G_3 \pi' r - 2\frac{\partial G_3}{\partial X}(\pi')^3 r - 2\frac{\partial^2 G_3}{\partial \pi \partial X}(\pi')^4 r^2}{r^2 + 6G_3 \pi' r + 14\frac{\partial G_3}{\partial X}(\pi')^3 r + 4\frac{\partial^2 G_3}{\partial X^2}(\pi')^5 r}. \quad (100)$$

It is easy to find that $G_3(\pi, X)$ exists, which satisfies the condition that $f(\pi(r), \pi'(r)) > 0$ when $\pi'(r) < \pi'(r_V)$ and $\pi(r) < \pi(r_V)$ but $G_3(\pi, X)$ cannot be uniquely determined.

Finally, we show that any spherically symmetric and static geometry given by arbitrary $\Phi(r)$ and $\Lambda(r)$ can be realized by properly choosing G_i 's. Since the redefinition of the scalar field can be absorbed into the redefinition of G_i 's, here we can identify the Galileon scalar field with the radial coordinate, $\pi = r$. Then we find

$$\begin{aligned} 0 &= \frac{1}{2\kappa^2 r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})] - \frac{1}{2} e^{2\Phi} G_2(r, e^{-2\Lambda}) + \frac{e^{2\Phi-2\Lambda}}{2} \frac{d}{dr} (G_3(r, e^{-2\Lambda}) e^{-2\Lambda}) - 5 \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r} \\ &+ \frac{d}{dr} \left(\frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \right) \frac{e^{2\Phi-6\Lambda}}{r} + \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{2\Phi-6\Lambda}}{r^2} - \frac{3}{2} G_4(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r} + \frac{1}{2} \frac{dG_4(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-4\Lambda}}{r} \\ &+ \frac{1}{4} G_4(r, e^{-2\Lambda}) \frac{e^{2\Phi-4\Lambda}}{r^2} + \frac{1}{4} G_4(r, e^{-2\Lambda}) \frac{e^{-4\Phi-2\Lambda}}{r^2} + \frac{1}{2} \frac{d}{dr} \left(\frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \right) \frac{e^{2\Phi-8\Lambda}}{r^2} - \frac{7}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Lambda}{dr} \frac{e^{2\Phi-8\Lambda}}{r^2} \\ &+ \frac{3}{4} G_5(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} - \frac{15}{4} G_5(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} - \frac{1}{4} \frac{dG_5(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} + \frac{3}{4} \frac{dG_5(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} \\ &+ 12G_6(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r} - 4 \frac{dG_6(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r} - 2G_6(r) \frac{e^{2\Phi-4\Lambda}}{r^2} - 2G_6(r) \frac{e^{-4\Phi-2\Lambda}}{r^2} - 6G_7(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} \\ &+ 30G_7(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} + 2 \frac{dG_7(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} - 6 \frac{dG_7(r)}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} - 192 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Phi}{dr} \\ &+ 16 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Lambda}{dr} - 16 \frac{d^2 G_8(r)}{dr^2} \frac{e^{2\Phi-2\Lambda}}{r^2} - 48 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} \frac{d\Lambda}{dr} + 16 \frac{d^2 G_8(r)}{dr^2} \frac{e^{2\Phi-4\Lambda}}{r^2}, \end{aligned} \quad (101)$$

$$\begin{aligned}
0 = & \frac{1}{2\kappa^2} \left\{ -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \frac{d\Phi}{dr} \right\} + \frac{1}{2} e^{2\Lambda} G_2(r, e^{-2\Lambda}) - \frac{\partial G_2(r, e^{-2\Lambda})}{\partial X} + \frac{1}{2} e^{2\Lambda} G_3(r, e^{-2\Lambda}) \left(\frac{e^{-3\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + e^{-4\Lambda} \frac{d\Lambda}{dr} \right) \\
& - \frac{\partial G_3(r, e^{-2\Lambda})}{\partial X} \left(\frac{e^{-3\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + e^{-4\Lambda} \frac{d\Lambda}{dr} \right) - G_3(r, e^{-2\Lambda}) \left(\left(\frac{e^{-\Phi-\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) \right) + 2e^{-4\Lambda} \frac{d\Lambda}{dr} \pi' \right) \\
& + \frac{1}{2} \left(-\frac{e^{-\Phi+\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + \frac{d}{dr} \right) (G_3(r, e^{-2\Lambda}) e^{-2\Lambda}) - \frac{1}{2} e^{-2\Lambda} \frac{d}{dr} (G_3(r, e^{-2\Lambda})) - 5 \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r} \\
& - \frac{5}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-4\Lambda}}{r^2} - 2 \frac{\partial^2 G_4(r, e^{-2\Lambda})}{\partial X^2} \frac{d\Phi}{dr} \frac{e^{-6\Lambda}}{r} - \frac{\partial^2 G_4(r, e^{-2\Lambda})}{\partial X^2} \frac{e^{-6\Lambda}}{r^2} - \frac{3}{2} G_4(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r} \\
& - \frac{3}{4} G_4(r, e^{-2\Lambda}) \frac{e^{-2\Lambda}}{r^2} + \frac{1}{4} \frac{(G_4(r, e^{-2\Lambda}))}{r^2} - \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r} + \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-2\Lambda}}{r^2} - \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-2\Lambda}}{r^2} \\
& - \frac{7}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} - \frac{\partial^2 G_5(r, e^{-2\Lambda})}{\partial X^2} \frac{e^{-8\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{3}{4} G_5(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r^2} - \frac{15}{4} G_5(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} \\
& + \frac{1}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} - \frac{3}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-6\Lambda}}{r^2} + 12 G_6(r) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r} + 6 G_6(r) \frac{e^{-2\Lambda}}{r^2} - \frac{2 G_6(r)}{r^2} \\
& - 6 G_7(r) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r^2} + 30 G_7(r) \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} + 16 \frac{dG_8(r)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} - 48 \frac{dG_8(r)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} e^{-2\Lambda}. \tag{102}
\end{aligned}$$

As in (41), if we wrote

$$G_2(\pi, X) = \sum_{n=0}^{\infty} \tilde{G}_2^{(n)}(\pi) (1 - e^{2\Lambda(r=\phi)} X)^n, \tag{103}$$

we find

$$G_2(r, e^{-2\Lambda}) = \tilde{G}_2^{(0)}(r), \quad \frac{\partial G_2(r, e^{-2\Lambda})}{\partial X} = \tilde{G}_2^{(1)}(r). \tag{104}$$

Then Eqs. (101) and (102) can be rewritten as

$$\begin{aligned}
\tilde{G}_2^{(0)}(r) = & F^{(0)}(r) \\
= & 2e^{-2\Phi} \left[\frac{1}{2\kappa^2 r^2} e^{2\Phi} \frac{d}{dr} [r(1 - e^{-2\Lambda})] \frac{e^{2\Phi-2\Lambda}}{2} \frac{d}{dr} (G_3(r, e^{-2\Lambda}) e^{-2\Lambda}) - 5 \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r} \right. \\
& + \frac{d}{dr} \left(\frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \right) \frac{e^{2\Phi-6\Lambda}}{r} + \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{2\Phi-6\Lambda}}{r^2} - \frac{3}{2} G_4(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r} + \frac{1}{2} \frac{dG_4(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-4\Lambda}}{r} \\
& + \frac{1}{4} G_4(r, e^{-2\Lambda}) \frac{e^{2\Phi-4\Lambda}}{r^2} + \frac{1}{4} G_4(r, e^{-2\Lambda}) \frac{e^{-4\Phi-2\Lambda}}{r^2} + \frac{1}{2} \frac{d}{dr} \left(\frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \right) \frac{e^{2\Phi-8\Lambda}}{r^2} - \frac{7}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Lambda}{dr} \frac{e^{2\Phi-8\Lambda}}{r^2} \\
& + \frac{3}{4} G_5(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} - \frac{15}{4} G_5(r, e^{-2\Lambda}) \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} - \frac{1}{4} \frac{dG_5(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} + \frac{3}{4} \frac{dG_5(r, e^{-2\Lambda})}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} \\
& + 12 G_6(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r} - 4 \frac{dG_6(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r} - 2 G_6(r) \frac{e^{2\Phi-4\Lambda}}{r^2} - 2 G_6(r) \frac{e^{-4\Phi-2\Lambda}}{r^2} - 6 G_7(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} \\
& + 30 G_7(r) \frac{d\Lambda}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} + 2 \frac{dG_7(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} - 6 \frac{dG_7(r)}{dr} \frac{e^{2\Phi-6\Lambda}}{r^2} - 192 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Phi}{dr} \\
& \left. + 16 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Lambda}{dr} - 16 \frac{d^2 G_8(r)}{dr^2} \frac{e^{2\Phi-2\Lambda}}{r^2} - 48 \frac{dG_8(r)}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} \frac{d\Lambda}{dr} + 16 \frac{d^2 G_8(r)}{dr^2} \frac{e^{2\Phi-4\Lambda}}{r^2} \right], \tag{105}
\end{aligned}$$

$$\tilde{G}_2^{(1)}(r) = F^{(1)}(r)$$

$$\begin{aligned} & \equiv \frac{1}{2} e^{2\Lambda} F^{(0)}(r) + \frac{1}{2\kappa^2} \left\{ -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \frac{d\Phi}{dr} \right\} + \frac{1}{2} e^{2\Lambda} G_3(r, e^{-2\Lambda}) \left(\frac{e^{-3\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + e^{-4\Lambda} \frac{d\Lambda}{dr} \right) \\ & - \frac{\partial G_3(r, e^{-2\Lambda})}{\partial X} \left(\frac{e^{-3\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + e^{-4\Lambda} \frac{d\Lambda}{dr} \right) - G_3(r, e^{-2\Lambda}) \left(\left(\frac{e^{-\Phi-\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) \right) + 2e^{-4\Lambda} \frac{d\Lambda}{dr} \pi' \right) \\ & + \frac{1}{2} \left(-\frac{e^{-\Phi+\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda}) \right) + \frac{d}{dr} \right) (G_3(r, e^{-2\Lambda}) e^{-2\Lambda}) - \frac{1}{2} e^{-2\Lambda} \frac{d}{dr} (G_3(r, e^{-2\Lambda})) - 5 \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r} \\ & - \frac{5}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-4\Lambda}}{r^2} - 2 \frac{\partial^2 G_4(r, e^{-2\Lambda})}{\partial X^2} \frac{d\Phi}{dr} \frac{e^{-6\Lambda}}{r} - \frac{\partial^2 G_4(r, e^{-2\Lambda})}{\partial X^2} \frac{e^{-6\Lambda}}{r^2} - \frac{3}{2} G_4(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r} \\ & - \frac{3}{4} G_4(r, e^{-2\Lambda}) \frac{e^{-2\Lambda}}{r^2} + \frac{1}{4} \frac{G_4(r, e^{-2\Lambda})}{r^2} - \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r} + \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-2\Lambda}}{r^2} - \frac{1}{2} \frac{\partial G_4(r, e^{-2\Lambda})}{\partial X} \frac{e^{-2\Lambda}}{r^2} \\ & - \frac{7}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} - \frac{\partial^2 G_5(r, e^{-2\Lambda})}{\partial X^2} \frac{e^{-8\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{3}{4} G_5(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r^2} - \frac{15}{4} G_5(r, e^{-2\Lambda}) \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} \\ & + \frac{1}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} - \frac{3}{2} \frac{\partial G_5(r, e^{-2\Lambda})}{\partial X} \frac{d\Phi}{dr} \frac{e^{-6\Lambda}}{r^2} + 12 G_6(r) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r} + 6 G_6(r) \frac{e^{-2\Lambda}}{r^2} - \frac{2 G_6(r)}{r^2} \\ & - 6 G_7(r) \frac{d\Phi}{dr} \frac{e^{-2\Lambda}}{r^2} + 30 G_7(r) \frac{d\Phi}{dr} \frac{e^{-4\Lambda}}{r^2} + 16 \frac{dG_8(r)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} - 48 \frac{dG_8(r)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} e^{-2\Lambda}. \end{aligned} \quad (106)$$

Then an arbitrary geometry is given by $\Phi(r)$, and $\Lambda(r)$ can be realized for arbitrary functions G_i ($i = 3, 4, \dots, 8$) by choosing $\tilde{G}_2^{(0)}(r)$ and $\tilde{G}_2^{(1)}(r)$ as in (105) and (106). This may also show that the spherical symmetric solution has a fourth hair corresponding to the scalar field, in addition to the usual three hairs corresponding to the mass, angular momentum, and electric charge. Then only $\tilde{G}_2^{(0)}$ and $\tilde{G}_2^{(1)}$ in $G_2(\pi, X)$ are relevant for the reconstruction of the spherically symmetric and static solution and $\tilde{G}_2^{(n)}(\pi)$, $n = 2, 3, 4, \dots$ are irrelevant for the reconstruction, although they may be related to the stability of the reconstructed solution.

As we have succeeded in the reconstruction of arbitrary spherically symmetric and static geometry, we may show the reconstruction is compatible with the cosmological reconstruction in the last section. Let us assume $G_i^{(n)}$, $n = 0, 1, \dots, N_i$ in (41) are relevant for the cosmological reconstruction and $G_i^{(n)}$, $n = N_i + 1, N_i + 2, \dots$ are irrelevant. We also assume $\tilde{G}_i^{(n)}$, $n = 0, 1, \dots, \tilde{N}_i$ as in (103)

$$G_i(\pi, X) = \sum_{n=0}^{\infty} \tilde{G}_i^{(n)}(\pi) (1 - e^{2\Lambda(\phi)} X)^n \quad (107)$$

are relevant for the reconstruction of spherically symmetric and static geometry and $\tilde{G}_i^{(n)}$, $n = \tilde{N}_i + 1, \tilde{N}_i + 2, \dots$ are irrelevant. By expanding (41) as a power series of $1 - e^{2\Lambda(\phi)} X$ by using $1 - X = 1 + e^{-2\Lambda(\phi)} - e^{-2\Lambda(\phi)} \times (1 - e^{2\Lambda(\phi)} X)$, we find

$$\begin{aligned} G_i(\pi, X) &= \sum_{n=0}^{\infty} \sum_{k=0}^n G_i^{(n)}(\pi) C_k (1 + e^{-2\Lambda(\phi)})^{n-k} \\ &\quad \times (-1)^k e^{-2k\Lambda(\phi)} (1 - e^{2\Lambda(\phi)} X)^k \\ &= \sum_{n=0}^{\infty} (-1)^n e^{-2n\Lambda(\phi)} \sum_{k=n}^{\infty} G_i^{(k)}(\pi) C_n \\ &\quad \times (1 + e^{-2\Lambda(\phi)})^{k-n} (1 - e^{2\Lambda(\phi)} X)^n. \end{aligned} \quad (108)$$

By comparing the expression in (107) and (108), we find

$$\tilde{G}_i^{(n)}(\pi) = (-1)^n e^{-2n\Lambda(\phi)} \sum_{k=n}^{\infty} C_k (1 + e^{-2\Lambda(\phi)})^{k-n} G_i^{(k)}(\pi). \quad (109)$$

Then when $n < N_i$, we find

$$\begin{aligned} \tilde{G}_i^{(n)}(\pi) &= (-1)^n e^{-2n\Lambda(\phi)} \sum_{k=n}^N C_k (1 + e^{-2\Lambda(\phi)})^{k-n} G_i^{(k)}(\pi) \\ &= (-1)^n e^{-2n\Lambda(\phi)} \sum_{k=N_i+1}^{\infty} C_k (1 + e^{-2\Lambda(\phi)})^{k-n} G_i^{(k)}(\pi), \end{aligned} \quad (110)$$

and when $n \geq N_i$

$$\tilde{G}_i^{(n)}(\pi) = (-1)^n e^{-2n\Lambda(\phi)} \sum_{k=N_i+1}^{\infty} C_k (1 + e^{-2\Lambda(\phi)})^{k-n} G_i^{(k)}(\pi). \quad (111)$$

For a set given by $G_i^{(n)}(\pi)$, $n = 0, 1, \dots, N_i$ and $\tilde{G}_i^{(n)}(\pi)$, $n = 0, 1, \dots, \tilde{N}_i$, we can always choose $G_i^{(n)}(\pi)$, $n = N_i, N_i + 1, \dots, N_i + \tilde{N}_i$ to satisfy (109) (we may set $G_i^{(n)}(\pi) = 0$, $n = N_i + \tilde{N}_i, N_i + \tilde{N}_i + 1, \dots$ and $\tilde{G}_i^{(n)}(\pi) = 0$, $n = \tilde{N}_i + 1, \tilde{N}_i + 2, \dots$). Therefore, we can obtain an action generating both an arbitrary spherically symmetric and static geometry and an arbitrary history of the expansion of the Universe, simultaneously.

V. SUMMARY

In this paper, we gave explicit formulae of the equations in the generalized Galileon models. Even in generalized Galileon models, derivatives higher than two do not appear in the field equation of motion nor in the Einstein equation. These structures are clearly given by using the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$. We have also developed the formulation of the reconstruction. As a consequence, we can construct an explicit action, which reproduces an arbitrary development of the expansion of the Universe. We should note that the functions G_n 's can be completely and explicitly separate from the parts relevant for the expansion and the irrelevant parts. Some of the irrelevant parts are related to the stability of the reconstructed solution, and we can also identify which parts in G_n 's are relevant for the stability. Then we have succeeded in showing the conditions necessary for the reconstructed solution to become stable and therefore become an attractor solution. Working in the spherically symmetric and static space-time, we have investigated how the Vainshtein mechanism works. It has been also shown that arbitrary spherically symmetric and static geometry can be realized by properly choosing G_i 's, which may show that the solution has a fourth hair corresponding to the scalar field. We again identified the parts in $G_2(\pi, X)$ relevant for the reconstruction. We should note that by choosing $G_i(\pi, X)$ appropriately, we can obtain an action that has both the solution corresponding to an arbitrarily given spherically symmetric and static geometry and the solution corresponding to an arbitrarily given history of the expansion of the Universe.

The generalized Galileon model contains many functions denoted by G_i , which cannot be determined by the history of the expansion of the Universe, the stability of the reconstructed solution, nor the condition that the Vainshtein mechanism works. These functions could be restricted if we consider the development of the several kinds of fluctuations by density perturbation, etc. The analysis by the perturbation could be very complicated in the models but it might become possible with the development of the necessary technologies.

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APPENDIX A: EXPLICIT FORMS OF $\mathcal{E}_k$ AND $(\mathcal{H}_k)_{\mu\nu}$ IN THE FRW UNIVERSE

In this appendix, by assuming the FRW universe (35), we give explicit forms of $\mathcal{E}_k$ in (7), (14), and (16), and $(\mathcal{H}_k)_{\mu\nu}$ in (28).

In the FRW universe (35), we have

$$\begin{aligned} \Gamma_{ij}^t &= a^2 H \delta_{ij}, & \Gamma_{jt}^i &= \Gamma_{ij}^i = H \delta_j^i, \\ R_{ijt} &= -(\dot{H} + H^2) a^2 \delta_{ij}, & R_{ijkl} &= a^4 H^2 (\delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj}). \end{aligned} \quad (\text{A1})$$

Then we find

$$\begin{aligned} \mathcal{E}_1 &= 1, & \mathcal{E}_2 &= 2\ddot{\pi} + 6H\dot{\pi}, \\ \mathcal{E}_3 &= -18H\dot{\pi}\ddot{\pi} - 27H^2\dot{\pi}^2 - 9\dot{H}\dot{\pi}^2, \\ \mathcal{E}_4^{(0)} &= 36H^2\dot{\pi}^2\ddot{\pi} + 36H^3\dot{\pi}^3 + 30H\dot{H}\dot{\pi}^3 \\ &\quad + (\text{third derivative terms}), \\ \Delta\mathcal{E}_4 &= 18H^2\dot{\pi}^2\ddot{\pi} + 18H^3\dot{\pi}^3 + 6H\dot{H}\dot{\pi}^3 \\ &\quad + (\text{third derivative terms}), \\ \mathcal{E}_5^{(0)} &= -20H^3\dot{\pi}^3\ddot{\pi} - 15H^4\dot{\pi}^4 - 21H^2\dot{H}\dot{\pi}^4 \\ &\quad + (\text{third derivative terms}), \\ \Delta\mathcal{E}_5 &= -30H^3\dot{\pi}^3\ddot{\pi} - \frac{45}{2}H^4\dot{\pi}^4 - \frac{33}{2}H^2\dot{H}\dot{\pi}^4 \\ &\quad + (\text{third derivative terms}), \\ \mathcal{E}_6 &= 24H^2\ddot{\pi} + 72H^3\dot{\pi} + 48H\dot{H}\dot{\pi}, \\ \mathcal{E}_7 &= -72H^3\dot{\pi}\ddot{\pi} - 108H^4\dot{\pi}^2 - 108H^2\dot{H}\dot{\pi}^2. \end{aligned} \quad (\text{A2})$$

We also obtain

$$\begin{aligned} (\mathcal{H}_2)_{00} &= -\frac{1}{2}G_2(\pi, X) - \frac{\partial G_2(\pi, X)}{\partial X}\dot{\pi}^2, \\ (\mathcal{H}_2)_{ij} &= \frac{1}{2}G_2(\pi, X)a^2\delta_{ij}, \end{aligned} \quad (\text{A3})$$

$$(\mathcal{H}_3)_{00} = 3 \frac{\partial G_3(\pi, X)}{\partial X} H \dot{\pi}^5 - \frac{9}{2} G_3(\pi, X) H \dot{\pi}^3, \quad (\mathcal{H}_3)_{ij} = \left[-\frac{1}{2} \frac{dG_3(\pi, X)}{dt} \dot{\pi}^3 - \frac{3}{2} G_3(\pi, X) \dot{\pi}^2 \ddot{\pi} \right] a^2 \delta_{ij}, \quad (\text{A4})$$

$$\begin{aligned} (\mathcal{H}_4)_{00} &= -6 \frac{\partial G_4(\pi, X)}{\partial X} H^2 \dot{\pi}^4 - 3 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} H^2 \dot{\pi}^6 + \frac{9}{4} G_4(\pi, X) H^2 \dot{\pi}^2, \\ (\mathcal{H}_4)_{ij} &= \left[-\frac{3}{2} \frac{\partial G_4(\pi, X)}{\partial X} H^2 \dot{\pi}^4 - \frac{d}{dt} \left(\frac{\partial G_4(\pi, X)}{\partial X} \right) H \dot{\pi}^4 - \frac{\partial G_4(\pi, X)}{\partial X} \dot{H} \dot{\pi}^4 - 4 \frac{\partial G_4(\pi, X)}{\partial X} H \dot{\pi}^3 \ddot{\pi} + \frac{3}{4} G_4(\pi, X) H^2 \dot{\pi}^2 \right. \\ &\quad \left. + \frac{1}{2} \frac{dG_4(\pi, X)}{dt} H \dot{\pi}^2 + \frac{1}{2} G_4(\pi, X) \dot{H} \dot{\pi}^2 + G_4(\pi, X) H \dot{\pi} \ddot{\pi} \right] a^2 \delta_{ij}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} (\mathcal{H}_5)_{00} &= -5 \frac{\partial G_5(\pi, X)}{\partial X} H^3 \dot{\pi}^5 + \frac{\partial^2 G_5(\pi, X)}{\partial X^2} H^3 \dot{\pi}^7 + \frac{15}{4} G_5(\pi, X) H^3 \dot{\pi}^3, \\ (\mathcal{H}_5)_{ij} &= \left[\frac{\partial G_5(\pi, X)}{\partial X} H^3 \dot{\pi}^5 + \frac{1}{2} \frac{d}{dt} \left(\frac{\partial G_5(\pi, X)}{\partial X} \right) H^2 \dot{\pi}^5 + \frac{\partial G_5(\pi, X)}{\partial X} H \dot{H} \dot{\pi}^5 + \frac{5}{2} \frac{\partial G_5(\pi, X)}{\partial X} H^2 \dot{\pi}^4 \ddot{\pi} - \frac{3}{2} G_5(\pi, X) H^3 \dot{\pi}^3 \right. \\ &\quad \left. - \frac{3}{4} \frac{dG_5(\pi, X)}{dt} H^2 \dot{\pi}^3 - \frac{3}{2} G_5(\pi, X) H \dot{H} \dot{\pi}^3 - \frac{9}{4} G_5(\pi, X) H^2 \dot{\pi}^2 \ddot{\pi} \right] a^2 \delta_{ij}, \end{aligned} \quad (\text{A6})$$

$$(\mathcal{H}_6)_{00} = -18 G_6(\pi) H^2 \dot{\pi}^2, \quad (\mathcal{H}_6)_{ij} = \left[-6 G_6(\pi) H^2 \dot{\pi}^2 - 4 \frac{dG_6(\pi)}{dt} H \dot{\pi}^2 - 4 G_6(\pi) \dot{H} \dot{\pi}^2 - 8 G_6(\pi) H \dot{\pi} \ddot{\pi} \right] a^2 \delta_{ij}, \quad (\text{A7})$$

$$(\mathcal{H}_7)_{00} = -30 G_7(\pi) H^3 \dot{\pi}^3, \quad (\mathcal{H}_7)_{ij} = \left[12 G_7(\pi) H^3 \dot{\pi}^3 + 6 \frac{dG_7(\pi)}{dt} H^2 \dot{\pi}^3 + 12 G_7(\pi) H \dot{H} \dot{\pi}^3 + 18 G_7(\pi) H^2 \dot{\pi}^2 \ddot{\pi} \right] a^2 \delta_{ij}, \quad (\text{A8})$$

$$(\mathcal{H}_8)_{00} = 48 \frac{dG_8(\pi)}{dt} H^3, \quad (\mathcal{H}_8)_{ij} = \left[-32 \frac{dG_8(\pi)}{dt} H^3 - 16 \frac{d^2 G_8(\pi)}{dt^2} H^2 - 32 \frac{dG_8(\pi)}{dt} \dot{H} H \right] a^2 \delta_{ij}. \quad (\text{A9})$$

APPENDIX B: EXPLICIT FORMS OF $\mathcal{E}_k$ AND $(\mathcal{H}_k)_{\mu\nu}$ IN THE STATIC AND SPHERICALLY SYMMETRIC SPACE-TIME

In this appendix, we give explicit forms of $\mathcal{E}_k$ and $(\mathcal{H}_k)_{\mu\nu}$ in the static and spherically symmetric space-time, especially in the Schwarzschild space-time.

In the static and spherically symmetric space-time in (89), we find

$$\begin{aligned} \mathcal{E}_5^{(0)} &= 7 \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d^2 \Phi}{dr^2} + 5 \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 - 41 \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 2 \frac{(\pi')^4 e^{-6\Lambda}}{r^3} \frac{d\Phi}{dr} + 20 \frac{(\pi')^3 (\pi'') e^{-8\Lambda}}{r^2} \frac{d\Phi}{dr} \\ &\quad - 4 \frac{(\pi')^4 e^{-8\Lambda}}{r^3} \frac{d\Phi}{dr} + (\text{third derivative terms}), \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \mathcal{E}_4^{(0)} &= 10 \frac{(\pi')^3 e^{-6\Lambda}}{r} \frac{d^2 \Phi}{dr^2} + 8 \frac{(\pi')^3 e^{-6\Lambda}}{r} \left(\frac{d\Phi}{dr} \right)^2 - 44 \frac{(\pi')^3 e^{-6\Lambda}}{r} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 24 \frac{(\pi')^2 (\pi'') e^{-6\Lambda}}{r} \frac{d\Phi}{dr} + 10 \frac{(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} \\ &\quad - 22 \frac{(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} + 2 \frac{(\pi')^3 e^{-4\Lambda}}{r^3} + 12 \frac{(\pi')^2 (\pi'') e^{-6\Lambda}}{r^2} - 2 \frac{(\pi')^3 e^{-6\Lambda}}{r^3} + (\text{third derivative terms}), \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \mathcal{E}_3 &= 2 (\pi')^2 e^{-4\Lambda} \frac{d^2 \Phi}{dr^2} + 2 (\pi')^2 e^{-4\Lambda} \left(\frac{d\Phi}{dr} \right)^2 - 8 (\pi')^2 e^{-4\Lambda} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 12 \frac{(\pi')^2 e^{-4\Lambda}}{r} \frac{d\Phi}{dr} + 6 (\pi') (\pi'') e^{-4\Lambda} \frac{d\Phi}{dr} \\ &\quad - 16 \frac{(\pi')^2 e^{-4\Lambda}}{r} \frac{d\Lambda}{dr} + 12 \frac{(\pi') (\pi'') e^{-4\Lambda}}{r} + 6 \frac{(\pi')^2 e^{-4\Lambda}}{r^2}, \end{aligned} \quad (\text{B3})$$

$$\mathcal{E}_2 = -2(\pi'')e^{-2\Lambda} - 2(\pi')e^{-2\Lambda} \frac{d\Phi}{dr} + 2(\pi')e^{-2\Lambda} \frac{d\Lambda}{dr} - 4 \frac{(\pi')e^{-2\Lambda}}{r}, \quad (\text{B4})$$

$$\begin{aligned} \mathcal{E}_6 = & -16 \frac{(\pi')e^{-4\Lambda}}{r} \frac{d^2\Phi}{dr^2} - 16 \frac{(\pi')e^{-4\Lambda}}{r} \left(\frac{d\Phi}{dr} \right)^2 + 48 \frac{(\pi')e^{-4\Lambda}}{r} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 8 \frac{(\pi')e^{-2\Lambda}}{r^2} \frac{d\Phi}{dr} - 16 \frac{(\pi'')e^{-4\Lambda}}{r} \frac{d\Phi}{dr} \\ & - 24 \frac{(\pi')e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} - 8 \frac{(\pi')e^{-2\Lambda}}{r^2} \frac{d\Lambda}{dr} + 24 \frac{(\pi')e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} + 8 \frac{(\pi'')e^{-2\Lambda}}{r^2} - 8 \frac{(\pi'')e^{-4\Lambda}}{r^2}, \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \mathcal{E}_7 = & 12 \frac{(\pi')^2 e^{-4\Lambda}}{r^2} \frac{d^2\Phi}{dr^2} - 36 \frac{(\pi')^2 e^{-6\Lambda}}{r^2} \frac{d^2\Phi}{dr^2} + 12 \frac{(\pi')^2 e^{-4\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 - 36 \frac{(\pi')^2 e^{-6\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 - 36 \frac{(\pi')^2 e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} \\ & + 180 \frac{(\pi')^2 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 24 \frac{(\pi')(\pi'')e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} - 72 \frac{(\pi')(\pi'')e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr}. \end{aligned} \quad (\text{B6})$$

Here $\pi' \equiv \frac{d\pi}{dr}$ and $\pi'' \equiv \frac{d^2\pi}{dr^2}$. Especially for the Schwarzschild metric (93), we have

$$\begin{aligned} \mathcal{E}_5^{(0)} = & \frac{16(\pi')^4 GM(2GM - r)^2(3GM - r)}{r^8} - \frac{20(\pi')^3 GM(2GM - r)^2(2\pi''rGM - \pi'GM - \pi''r^2)}{r^8} \\ & + (\text{third derivative terms}), \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \Delta\mathcal{E}_5 = & -\frac{96(\pi')^3 \pi'' G^4 M^4}{r^7} + \frac{138(\pi')^4 G^4 M^4}{r^8} + \frac{200(\pi')^3 \pi'' G^3 M^3}{r^6} - \frac{182(\pi')^4 G^3 M^3}{r^7} - \frac{116(\pi')^3 \pi'' G^2 M^2}{r^5} \\ & + \frac{117(\pi')^4 G^2 M^2}{2r^6} + \frac{20(\pi')^3 \pi'' GM}{r^4} - \frac{4(\pi')^4 GM}{r^5} + (\text{third derivative terms}), \end{aligned} \quad (\text{B8})$$

$$\mathcal{E}_7 = -\frac{288\pi'\pi''G^3M^3}{r^6} + \frac{576(\pi')^2G^3M^3}{r^7} + \frac{240\pi'\pi''G^2M^2}{r^5} - \frac{360(\pi')^2G^2M^2}{r^6} - \frac{48\pi'\pi''GM}{r^4} + \frac{48(\pi')^2GM}{r^5}, \quad (\text{B9})$$

$$\begin{aligned} \mathcal{E}_4^{(0)} = & \frac{12(\pi')^2(2GM - r)(2\pi'G^2M^2 + 2\pi''r^2GM - 2\pi'rGM - \pi''r^3)}{r^6} - \frac{8(\pi')^3GM(2GM - r)(3GM - r)}{r^6} \\ & + (\text{third derivative terms}), \end{aligned} \quad (\text{B10})$$

$$\Delta\mathcal{E}_4 = 0 + (\text{third derivative terms}), \quad (\text{B11})$$

$$\mathcal{E}_6 = 0, \quad (\text{B12})$$

$$\mathcal{E}_3 = \frac{36\pi'\pi''G^2M^2}{r^3} - \frac{18(\pi')^2G^2M^2}{r^4} - \frac{42\pi'\pi''GM}{r^2} + 12 \frac{\pi'\pi''}{r} + \frac{6(\pi')^2}{r^2}, \quad (\text{B13})$$

$$\mathcal{E}_2 = -2\left(1 - 2\frac{GM}{r}\right)\pi'' + \frac{4}{r^2}(GM - r)\pi'. \quad (\text{B14})$$

Since

$$\begin{aligned} \Delta\mathcal{E}_4 = & \frac{2(\pi')^3 e^{-6\Lambda}}{r} \frac{d^2\Phi}{dr^2} + \frac{4(\pi')^3 e^{-6\Lambda}}{r} \left(\frac{d\Phi}{dr} \right)^2 - \frac{16(\pi')^3 e^{-6\Lambda}}{r} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - 2 \frac{(\pi')^3 e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{12(\pi')^2(\pi'')e^{-6\Lambda}}{r} \frac{d\Phi}{dr} \\ & + \frac{8(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{6(\pi')^3 e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} - \frac{8(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} - \frac{6(\pi')^2(\pi'')e^{-4\Lambda}}{r^2} - \frac{2(\pi')^3 e^{-4\Lambda}}{r^3} + \frac{6(\pi')^2(\pi'')e^{-6\Lambda}}{r^2} \\ & + (\text{third derivative terms}), \end{aligned} \quad (\text{B15})$$

$$\begin{aligned} \Delta \mathcal{E}_5 = & -\frac{7}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \frac{d^2 \Phi}{dr^2} + \frac{19}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d^2 \Phi}{dr^2} - \frac{7}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 + \frac{27}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 + \frac{6(\pi')^4 e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} \\ & - \frac{49}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - \frac{105}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - \frac{6(\pi')^3 (\pi'') e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{14(\pi')^3 (\pi'') e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} - \frac{2(\pi')^4 e^{-6\Lambda}}{r^3} \frac{d\Phi}{dr} \\ & + \frac{12(\pi')^3 (\pi'') e^{-8\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{4(\pi')^4 e^{-8\Lambda}}{r^3} \frac{d\Phi}{dr} + (\text{third derivative terms}), \end{aligned} \quad (\text{B16})$$

we acquire

$$\begin{aligned} \mathcal{E}_5 = & -\frac{7}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \frac{d^2 \Phi}{dr^2} + \frac{47}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d^2 \Phi}{dr^2} - \frac{7}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 + \frac{47}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \left(\frac{d\Phi}{dr} \right)^2 + \frac{6(\pi')^4 e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} \\ & - \frac{49}{4} \frac{(\pi')^4 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - \frac{269}{4} \frac{(\pi')^4 e^{-8\Lambda}}{r^2} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - \frac{6(\pi')^3 (\pi'') e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{14(\pi')^3 (\pi'') e^{-6\Lambda}}{r^3} \frac{d\Phi}{dr} + \frac{32(\pi')^3 (\pi'') e^{-8\Lambda}}{r^2} \frac{d\Phi}{dr}, \end{aligned} \quad (\text{B17})$$

$$\begin{aligned} \mathcal{E}_4 = & \frac{12(\pi')^3 e^{-6\Lambda}}{r} \frac{d^2 \Phi}{dr^2} + \frac{12(\pi')^3 e^{-6\Lambda}}{r} \left(\frac{d\Phi}{dr} \right)^2 - \frac{60(\pi')^3 e^{-6\Lambda}}{r} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} - \frac{2(\pi')^3 e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{36(\pi')^2 (\pi'') e^{-6\Lambda}}{r} \frac{d\Phi}{dr} \\ & + \frac{18(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{6(\pi')^3 e^{-4\Lambda}}{r^2} \frac{d\Lambda}{dr} - \frac{30(\pi')^3 e^{-6\Lambda}}{r^2} \frac{d\Lambda}{dr} - \frac{6(\pi')^2 (\pi'') e^{-4\Lambda}}{r^2} + \frac{18(\pi')^2 (\pi'') e^{-6\Lambda}}{r^2}. \end{aligned} \quad (\text{B18})$$

Thus, we obtain

$$\mathcal{E}_2^{(G)} = \frac{\partial G_2(\pi, X)}{\partial \pi} - 2e^{-2\Lambda} \left(\frac{d\Phi}{dr} - \frac{d\Lambda}{dr} \right) \frac{\partial G_2(\pi, X)}{\partial X} \pi' - 4 \frac{e^{-2\Lambda}}{r} \frac{\partial G_2(\pi, X)}{\partial X} \pi' - 2e^{-2\Lambda} \frac{d}{dr} \left(\frac{\partial G_2(\pi, X)}{\partial X} \right) \pi' - 2e^{-2\Lambda} \frac{\partial G_2(\pi, X)}{\partial X} \pi'', \quad (\text{B19})$$

$$\begin{aligned} \mathcal{E}_3^{(G)} = & \frac{\partial G_3(\pi, X)}{\partial \pi} e^{-4\Lambda} (\pi')^3 \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) - 3G_3(\pi, X) e^{-4\Lambda} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) (\pi')^2 - 6G_3(\pi, X) \frac{e^{-4\Lambda}}{r} (\pi')^2 \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) \\ & - 6G_3(\pi, X) e^{-4\Lambda} \pi' \pi'' \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) - 3G_3(\pi, X) e^{-4\Lambda} (\pi')^2 \left(\frac{d^2 \Phi}{dr^2} - \frac{2}{r^2} \right) - 3 \frac{dG_3(\pi, X)}{dr} e^{-4\Lambda} (\pi')^2 \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) \\ & - 2 \frac{\partial G_3(\pi, X)}{\partial X} e^{-6\Lambda} (\pi')^4 \left(\frac{d\Phi}{dr} - 5 \frac{d\Lambda}{dr} \right) \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) - 4 \frac{\partial G_3(\pi, X)}{\partial X} \frac{e^{-6\Lambda}}{r} (\pi')^4 \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) - 8 \frac{\partial G_3(\pi, X)}{\partial X} \\ & \times e^{-6\Lambda} (\pi')^3 \pi'' \left(\frac{d\Phi}{dr} + \frac{2}{r} \right) - 2 \frac{\partial G_3(\pi, X)}{\partial X} e^{-6\Lambda} (\pi')^4 \left(\frac{d^2 \Phi}{dr^2} - \frac{2}{r^2} \right) - 2 \frac{d}{dr} \left(\frac{\partial G_3(\pi, X)}{\partial X} \right) e^{-6\Lambda} (\pi')^4 \left(\frac{d\Phi}{dr} + \frac{2}{r} \right), \end{aligned} \quad (\text{B20})$$

$$\begin{aligned} \mathcal{E}_4^{(G,0)} = & -\frac{\partial^2 G_4(\pi, X)}{\partial \pi \partial X} \frac{e^{-6\Lambda}}{r} (\pi')^4 \left(2 \frac{d\Phi}{dr} + \frac{1}{r} \right) + \left(2 \frac{d^2 \Phi}{dr^2} - \frac{1}{r^2} \right) \left(4 \frac{\partial G_4(\pi, X)}{\partial X} \frac{e^{-6\Lambda}}{r} (\pi')^3 + 2 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{e^{-8\Lambda}}{r} (\pi')^5 \right) \\ & + \left(2 \frac{d\Phi}{dr} + \frac{1}{r} \right) \left[4 \frac{\partial G_4(\pi, X)}{\partial X} \frac{e^{-6\Lambda}}{r^2} (\pi')^3 + 12 \frac{\partial G_4(\pi, X)}{\partial X} \frac{e^{-6\Lambda}}{r} (\pi')^2 \pi'' + 4 \frac{d}{dr} \left(\frac{\partial G_4(\pi, X)}{\partial X} \right) \frac{e^{-6\Lambda}}{r} (\pi')^3 + 4 \frac{\partial G_4(\pi, X)}{\partial X} \right. \\ & \times \frac{e^{-6\Lambda}}{r} (\pi')^3 \left(\frac{d\Phi}{dr} - 5 \frac{d\Lambda}{dr} \right) \left. \right] + \left(2 \frac{d\Phi}{dr} + \frac{1}{r} \right) \left[2 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{e^{-8\Lambda}}{r^2} (\pi')^5 + 10 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{e^{-8\Lambda}}{r} (\pi')^4 \pi'' \right. \\ & \left. + 2 \frac{d}{dr} \left(\frac{\partial^2 G_4(\pi, X)}{\partial X^2} \right) \frac{e^{-8\Lambda}}{r} (\pi')^5 + 2 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{e^{-8\Lambda}}{r} (\pi')^5 \left(\frac{d\Phi}{dr} - 7 \frac{d\Lambda}{dr} \right) \right], \end{aligned} \quad (\text{B21})$$

$$\begin{aligned}
\mathcal{E}_5^{(G,0)} = & -\frac{\partial^2 G_5(\pi, X)}{\partial \pi \partial X} \frac{e^{-8\Lambda}}{r^2} (\pi')^5 \frac{d\Phi}{dr} + 5 \frac{\partial G_5(\pi, X)}{\partial X} \frac{e^{-8\Lambda}}{r^2} (\pi')^4 \left(\frac{d\Phi}{dr} - 7 \frac{d\Lambda}{dr} \right) \frac{d\Phi}{dr} + 20 \frac{\partial G_5(\pi, X)}{\partial X} \frac{e^{-8\Lambda}}{r^2} (\pi')^3 \pi'' \frac{d\Phi}{dr} \\
& + 5 \frac{\partial G_5(\pi, X)}{\partial X} \frac{e^{-8\Lambda}}{r^2} (\pi')^4 \frac{d^2 \Phi}{dr^2} + 5 \frac{d}{dr} \left(\frac{\partial G_5(\pi, X)}{\partial X} \right) \frac{e^{-8\Lambda}}{r^2} (\pi')^4 \frac{d\Phi}{dr} + 2 \frac{\partial^2 G_5(\pi, X)}{\partial X^2} \frac{e^{-10\Lambda}}{r^2} (\pi')^6 \left(\frac{d\Phi}{dr} - 9 \frac{d\Lambda}{dr} \right) \frac{d\Phi}{dr} \\
& + 12 \frac{\partial^2 G_5(\pi, X)}{\partial X^2} \frac{e^{-10\Lambda}}{r^2} (\pi')^5 \pi'' \frac{d\Phi}{dr} + 2 \frac{\partial^2 G_5(\pi, X)}{\partial X^2} \frac{e^{-10\Lambda}}{r^2} (\pi')^6 \frac{d^2 \Phi}{dr^2} + 2 \frac{d}{dr} \left(\frac{\partial^2 G_5(\pi, X)}{\partial X^2} \right) \frac{e^{-10\Lambda}}{r^2} (\pi')^6 \frac{d\Phi}{dr}, \quad (B22)
\end{aligned}$$

$$\begin{aligned}
\Delta \mathcal{E}_4^{(G)} = & -\frac{\partial G_4(\pi, X)}{\partial \pi} \frac{(\pi')^2}{r} \left(e^{-4\Lambda} \frac{d\Phi}{dr} - \frac{1}{2} \frac{e^{-2\Lambda}}{r} + \frac{1}{2} \frac{e^{-4\Lambda}}{r} \right) + 2 \left[\frac{e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{e^{-4\Lambda}}{r} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) \frac{d\Phi}{dr} + \frac{e^{-4\Lambda}}{r} \frac{d^2 \Phi}{dr^2} \right. \\
& - \frac{1}{2} \frac{e^{-2\Lambda}}{r^2} \left(\frac{d\Phi}{dr} - \frac{d\Lambda}{dr} \right) + \frac{1}{2} \frac{e^{-4\Lambda}}{r^2} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) \left. \right] \left(\frac{\partial G_4(\pi, X)}{\partial X} e^{-2\Lambda} (\pi')^3 + G_4(\pi, X) \pi' \right) + 2 \left[\frac{e^{-4\Lambda}}{r} \frac{d\Phi}{dr} - \frac{1}{2} \frac{e^{-2\Lambda}}{r^2} \right. \\
& + \frac{1}{2} \frac{e^{-4\Lambda}}{r^2} \left. \right] \left[\frac{d}{dr} \left(\frac{\partial G_4(\pi, X)}{\partial X} \right) e^{-2\Lambda} (\pi')^3 - 2 \frac{\partial G_4(\pi, X)}{\partial X} e^{-2\Lambda} \frac{d\Lambda}{dr} (\pi')^3 + 3 \frac{\partial G_4(\pi, X)}{\partial X} e^{-2\Lambda} (\pi')^2 \pi'' + \frac{dG_4(\pi, X)}{dr} \pi' \right. \\
& \left. + G_4(\pi, X) \pi'' \right], \quad (B23)
\end{aligned}$$

$$\begin{aligned}
\Delta \mathcal{E}_5^{(G)} = & \frac{\partial G_5(\pi, X)}{\partial \pi} \frac{(\pi')^3}{2r^2} (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} - 3\pi' \pi'' G_5(\pi, X) (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} - \frac{3}{2} (\pi')^2 \frac{dG_5(\pi, X)}{dr} (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} \\
& - \frac{3}{2} (\pi')^2 G_5(\pi, X) \left[e^{-4\Lambda} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) - 3e^{-6\Lambda} \left(\frac{d\Phi}{dr} - 5 \frac{d\Lambda}{dr} \right) \right] \frac{d\Phi}{dr} - \frac{3}{2} (\pi')^2 G_5(\pi, X) (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d^2 \Phi}{dr^2} \\
& - 4 \frac{\partial G_5(\pi, X)}{\partial X} (\pi')^3 \pi'' (e^{-6\Lambda} - 3e^{-8\Lambda}) \frac{d\Phi}{dr} - \frac{d}{dr} \left(\frac{\partial G_5(\pi, X)}{\partial X} \right) (\pi')^4 (e^{-6\Lambda} - 3e^{-8\Lambda}) \frac{d\Phi}{dr} \frac{\partial G_5(\pi, X)}{\partial X} (\pi')^4 \\
& \times \left[e^{-6\Lambda} \left(\frac{d\Phi}{dr} - 5 \frac{d\Lambda}{dr} \right) - 3e^{-8\Lambda} \left(\frac{d\Phi}{dr} - 7 \frac{d\Lambda}{dr} \right) \right] \frac{d\Phi}{dr} - \frac{\partial G_5(\pi, X)}{\partial X} (\pi')^4 (e^{-6\Lambda} - 3e^{-8\Lambda}) \frac{d^2 \Phi}{dr^2}, \quad (B24)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_6^{(G)} = & 8 \frac{\partial G_6(\pi)}{\partial \pi} \frac{(\pi')^2}{r} \left(e^{-4\Lambda} \frac{d\Phi}{dr} - \frac{1}{2} \frac{e^{-2\Lambda}}{r} + \frac{1}{2} \frac{e^{-4\Lambda}}{r} \right) - 16 \left[\frac{e^{-4\Lambda}}{r^2} \frac{d\Phi}{dr} + \frac{e^{-4\Lambda}}{r} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) \frac{d\Phi}{dr} + \frac{e^{-4\Lambda}}{r} \frac{d^2 \Phi}{dr^2} \right. \\
& - \frac{1}{2} \frac{e^{-2\Lambda}}{r^2} \left(\frac{d\Phi}{dr} - \frac{d\Lambda}{dr} \right) + \frac{1}{2} \frac{e^{-4\Lambda}}{r^2} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) \left. \right] G_6(\pi) \pi' - 16 \left[\frac{e^{-4\Lambda}}{r} \frac{d\Phi}{dr} - \frac{1}{2} \frac{e^{-2\Lambda}}{r^2} + \frac{1}{2} \frac{e^{-4\Lambda}}{r^2} \right] \left(\frac{dG_6(\pi)}{dr} \pi' + G_6(\pi) \pi'' \right), \quad (B25)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_7^{(G)} = & -8 \frac{\partial G_7(\pi)}{\partial \pi} \frac{(\pi')^3}{2r^2} (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} + 24 \pi' \pi'' G_7(\pi) (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} + 12 (\pi')^2 \frac{dG_7(\pi)}{dr} (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d\Phi}{dr} \\
& + 12 (\pi')^2 G_7(\pi) \left[e^{-4\Lambda} \left(\frac{d\Phi}{dr} - 3 \frac{d\Lambda}{dr} \right) - 3e^{-6\Lambda} \left(\frac{d\Phi}{dr} - 5 \frac{d\Lambda}{dr} \right) \right] \frac{d\Phi}{dr} + 12 (\pi')^2 G_7(\pi) (e^{-4\Lambda} - 3e^{-6\Lambda}) \frac{d^2 \Phi}{dr^2}, \quad (B26)
\end{aligned}$$

$$\mathcal{E}_8^{(G)} = -\frac{32e^{-4\Lambda}}{r^2} \frac{\partial G_8(\pi)}{\partial \pi} \left[e^{2\Lambda} \frac{d^2 \Phi}{dr^2} - \frac{d^2 \Phi}{dr^2} + e^{2\Lambda} \left(\frac{d\Phi}{dr} \right)^2 - \left(\frac{d\Phi}{dr} \right)^2 - e^{2\Lambda} \frac{d\Lambda}{dr} \frac{d\Phi}{dr} + 3 \frac{d\Lambda}{dr} \frac{d\Phi}{dr} \right]. \quad (B27)$$

In the static and spherically symmetric space-time in (89), we obtain

$$(\mathcal{H}_2)_{tt} = -\frac{1}{2} e^{2\Phi} G_2(\pi, X), \quad (B28)$$

$$(\mathcal{H}_2)_{rr} = \frac{1}{2} e^{2\Lambda} G_2(\pi, X) - \frac{\partial G_2(\pi, X)}{\partial X} (\pi')^2, \quad (B29)$$

$$(\mathcal{H}_3)_{tt} = \frac{1}{2} G_3(\pi, X) e^{-4\Lambda} (\pi')^2 \left(\pi'' - \frac{d\Lambda}{dr} \pi' \right) + \frac{1}{2} e^{2\Phi-2\Lambda} \pi' \frac{d}{dr} (G_3(\pi, X) e^{-2\Lambda} (\pi')^2), \quad (B30)$$

$$\begin{aligned}
(\mathcal{H}_3)_{rr} = & \frac{1}{2} e^{2\Lambda} G_3(\pi, X) \left(\frac{e^{-3\Lambda}}{r^2} (\pi')^2 \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda} \pi') \right) - e^{-4\Lambda} (\pi'')^2 \left(\pi'' - \frac{d\Lambda}{dr} \pi' \right) \right) - \frac{\partial G_3(\pi, X)}{\partial X} \left(\frac{e^{-3\Lambda}}{r^2} (\pi')^2 \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda} \pi') \right) \right. \\
& \left. - e^{-4\Lambda} (\pi'')^2 \left(\pi'' - \frac{d\Lambda}{dr} \pi' \right) \right) (\pi')^2 - G_3(\pi, X) \left((\pi')^2 \left(\frac{e^{-\Phi-\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda} \pi') \right) \right) - 2e^{-4\Lambda} (\pi')^2 \left(\pi'' - \frac{d\Lambda}{dr} \pi' \right) \right) \\
& + \frac{1}{2} \left(-\frac{e^{-\Phi+\Lambda}}{r^2} \left(\frac{d}{dr} (r^2 e^{\Phi-\Lambda} \pi') \right) + \pi' \frac{d}{dr} \right) (G_3(\pi, X) e^{-2\Lambda} (\pi')^2) - \frac{1}{2} e^{-2\Lambda} \frac{d}{dr} (G_3(\pi, X) (\pi')^3), \tag{B31}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_4)_t = & -5 \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r} \frac{d\Lambda}{dr} e^{2\Phi-6\Lambda} + \frac{d}{dr} \left(\frac{\partial G_4(\pi, X)}{\partial X} \right) \frac{(\pi')^4}{r} e^{2\Phi-6\Lambda} + 4 \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4 \pi''}{r} e^{2\Phi-6\Lambda} \\
& + \frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r^2} e^{2\Phi-6\Lambda} - \frac{3}{2} G_4(\pi, X) \frac{(\pi')^2}{r} \frac{d\Lambda}{dr} e^{2\Phi-4\Lambda} + \frac{1}{2} \frac{dG_4(\pi, X)}{dr} \frac{(\pi')^2}{r} e^{2\Phi-4\Lambda} + G_4(\pi, X) \frac{\pi \pi''}{r} e^{2\Phi-4\Lambda} \\
& + \frac{1}{4} G_4(\pi, X) \frac{(\pi')^2}{r^2} e^{2\pi-4\Lambda} + \frac{1}{4} G_4(\pi, X) \frac{(\pi')^2}{r^2} e^{-4\Phi-2\Lambda}, \tag{B32}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_4)_{rr} = & -5 \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r} \frac{d\Phi}{dr} e^{-4\Lambda} - \frac{5}{2} \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r^2} e^{-4\Lambda} - 2 \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{(\pi')^6}{r} \frac{d\Phi}{dr} e^{-6\Lambda} - \frac{\partial^2 G_4(\pi, X)}{\partial X^2} \frac{(\pi')^6}{r^2} e^{-6\Lambda} \\
& - \frac{3}{2} G_4(\pi, X) \frac{(\pi')^2}{r} \frac{d\Phi}{dr} e^{-2\Lambda} - \frac{3}{4} G_4(\pi, X) \frac{(\pi')^2}{r^2} e^{-2\Lambda} + \frac{1}{4} G_4(\pi, X) \frac{(\pi')^2}{r^2} - \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r} \frac{d\Phi}{dr} e^{-4\Lambda} \\
& + \frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r^2} e^{-2\Lambda} - \frac{1}{2} \frac{\partial G_4(\pi, X)}{\partial X} \frac{(\pi')^4}{r^2} e^{-2\Lambda}, \tag{B33}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_5)_t = & \frac{1}{2} \frac{d}{dr} \left(\frac{\partial G_5(\pi, X)}{\partial X} \right) \frac{(\pi')^5}{r^2} e^{2\Phi-8\Lambda} + \frac{5}{2} \frac{\partial G_5(\pi, X)}{\partial X} \frac{(\pi')^4 (\pi'')}{r^2} e^{2\Phi-8\Lambda} - \frac{7}{2} \frac{\partial G_5(\pi, X)}{\partial X} \frac{(\pi')^5}{r^2} \frac{d\Lambda}{dr} e^{2\Phi-8\Lambda} \\
& + \frac{3}{4} G_5(\pi, X) \frac{(\pi')^3}{r^2} \frac{d\Lambda}{dr} e^{2\Phi-4\Lambda} - \frac{15}{4} G_5(\pi, X) \frac{(\pi')^3}{r^2} \frac{d\Lambda}{dr} e^{2\Phi-6\Lambda} - \frac{1}{4} \frac{dG_5(\pi, X)}{dr} \frac{(\pi')^3}{r^2} e^{2\Phi-4\Lambda} \\
& - \frac{3}{4} G_5(\pi, X) \frac{(\pi')^2 \pi''}{r^2} e^{2\Phi-4\Lambda} + \frac{3}{4} \frac{dG_5(\pi, X)}{dr} \frac{(\pi')^3}{r^2} e^{2\Phi-6\Lambda} + \frac{9}{4} G_5(\pi, X) \frac{(\pi')^2 \pi''}{r^2} e^{2\Phi-6\Lambda}, \tag{B34}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_5)_{rr} = & -\frac{7}{2} \frac{\partial G_5(\pi, X)}{\partial X} \frac{(\pi')^5}{r^2} e^{-6\Lambda} \frac{d\Phi}{dr} - \frac{\partial^2 G_5(\pi, X)}{\partial X^2} \frac{(\pi')^7}{r^2} e^{-8\Lambda} \frac{d\Phi}{dr} + \frac{3}{4} G_5(\pi, X) \frac{(\pi')^3}{r^2} \frac{d\Phi}{dr} e^{-2\Lambda} \\
& - \frac{15}{4} G_5(\pi, X) \frac{(\pi')^3}{r^2} \frac{d\Phi}{dr} e^{-4\Lambda} + \frac{1}{2} \frac{\partial G_5(\pi, X)}{\partial X} \frac{(\pi')^5}{r^2} \frac{d\Phi}{dr} e^{-4\Lambda} - \frac{3}{2} \frac{\partial G_5(\pi, X)}{\partial X} \frac{(\pi')^5}{r^2} \frac{d\Phi}{dr} e^{-6\Lambda}, \tag{B35}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_6)_t = & 12G_6(\pi) \frac{(\pi')^2}{r} \frac{d\Lambda}{dr} e^{2\Phi-4\Lambda} - 4 \frac{dG_6(\pi)}{dr} \frac{(\pi')^2}{r} e^{2\Phi-4\Lambda} - 8G_6(\pi) \frac{\pi \pi''}{r} e^{2\Phi-4\Lambda} - 2G_6(\pi) \frac{(\pi')^2}{r^2} e^{2\Phi-4\Lambda} \\
& - 2G_6(\pi) \frac{(\pi')^2}{r^2} e^{-4\Phi-2\Lambda}, \tag{B36}
\end{aligned}$$

$$(\mathcal{H}_6)_{rr} = 12G_6(\pi) \frac{(\pi')^2}{r} \frac{d\Phi}{dr} e^{-2\Lambda} + 6G_6(\pi) \frac{(\pi')^2}{r^2} e^{-2\Lambda} - 2G_6(\pi) \frac{(\pi')^2}{r^2}, \tag{B37}$$

$$\begin{aligned}
(\mathcal{H}_7)_t = & -6G_7(\pi) \frac{(\pi')^3}{r^2} \frac{d\Lambda}{dr} e^{2\Phi-4\Lambda} + 30G_7(\pi) \frac{(\pi')^3}{r^2} \frac{d\Lambda}{dr} e^{2\Phi-6\Lambda} + 2 \frac{dG_7(\pi)}{dr} \frac{(\pi')^3}{r^2} e^{2\Phi-4\Lambda} + 6G_7(\pi) \frac{(\pi')^2 \pi''}{r^2} e^{2\Phi-4\Lambda} \\
& - 6 \frac{dG_7(\pi)}{dr} \frac{(\pi')^3}{r^2} e^{2\Phi-6\Lambda} - 18G_7(\pi) \frac{(\pi')^2 \pi''}{r^2} e^{2\Phi-6\Lambda}, \tag{B38}
\end{aligned}$$

$$(\mathcal{H}_7)_{rr} = -6G_7(\pi) \frac{(\pi')^3}{r^2} \frac{d\Phi}{dr} e^{-2\Lambda} + 30G_7(\pi) \frac{(\pi')^3}{r^2} \frac{d\Phi}{dr} e^{-4\Lambda}, \tag{B39}$$

$$\begin{aligned}
(\mathcal{H}_8)_{tt} = & -192 \frac{dG_8(\pi)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Phi}{dr} + 16 \frac{dG_8(\pi)}{dr} \frac{e^{2\Phi-2\Lambda}}{r^2} \frac{d\Lambda}{dr} - 16 \frac{d^2 G_8(\pi)}{dr^2} \frac{e^{2\Phi-2\Lambda}}{r^2} \\
& - 48 \frac{dG_8(\pi)}{dr} \frac{e^{2\Phi-4\Lambda}}{r^2} \frac{d\Lambda}{dr} + 16 \frac{d^2 G_8(\pi)}{dr^2} \frac{e^{2\Phi-4\Lambda}}{r^2},
\end{aligned} \tag{B40}$$

$$(\mathcal{H}_8)_{rr} = 16 \frac{dG_8(\pi)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} - 48 \frac{dG_8(\pi)}{dr} \frac{1}{r^2} \frac{d\Phi}{dr} e^{-2\Lambda}. \tag{B41}$$

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