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Resistive relativistic magnetohydrodynamics from a charged multifluids perspective

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We consider general relativistic magnetohydrodynamics from a charged multifluids point of view, taking a variational approach as our starting point. We develop the case of two charged components in detail, accounting for a phenomenological resistivity, providing specific examples for pair plasmas and proton-electron systems. We discuss both cold, low-velocity, plasmas and hot systems where we account for a dynamical entropy component. The results for the cold case (which accord with recent work in the literature) provide a complete model for resistive relativistic magnetohydrodynamics, clarifying the assumptions that lead to various models that have been used in astrophysical applications. The analysis of the hot case is (as far as we are aware) novel, accounting for the relaxation times that are required to ensure causality and demonstrating the explicit coupling between fluxes of heat and charge.

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I. INTRODUCTION

Magnetic fields are ubiquitous in the Universe, affecting physics across a vast range of scales. The relevance of electromagnetism for our everyday experience is obvious. Electromagnetic effects are also central to many processes in astrophysics and cosmology. The strongest known magnetic fields (above 10^{14} G) are found in a subclass of neutron stars aptly referred to as magnetars [1,2], systems that also form the largest (and hottest!) known superconductors [3–5]. Magnetic fields are equally relevant on the vastly larger scale of entire galaxies, and are likely to have played a role in the early Universe as well [6–8]. Understanding the origin and evolution of electromagnetic fields in their many different guises remains a fundamental question for modern science. The literature on the subject is (understandably) vast [9], yet some problems remain relatively unexplored. This paper concerns one such problem.

Our aim is to develop a model for resistivity in general relativistic magnetohydrodynamics. By necessity this forces us to consider a charged multifluid system (we obviously need charged components in relative motion in order to have a charge current!). This part of the problem is quite straightforward; we make progress by marrying the standard variational model for electromagnetism [10] to the charged fluid version of Carter's convective variational description of relativistic fluids [11,12]. Adding a phenomenological resistivity to the mix is not difficult, either. Combining these ingredients we follow the textbook strategy [13,14] and derive the simplified equations of magnetohydrodynamics. As long as we limit the analysis to low velocities (cold plasmas) the results follow readily. We demonstrate this for the particular problem of a two-component system, composed either of protons and electrons or a pair plasma with positrons and electrons, and compare our results to the recent literature [15–17].

The complexity of the problem increases significantly if we turn our attention to high velocities and hot plasmas. One reason for this is obvious: In order to describe a hot system we need to allow for the presence of heat flow. However, the problem of heat in relativistic systems is known to be difficult, as a naive implementation inevitably leads to causality violation and unwanted instabilities [18–21]. We avoid falling into this trap by building on a recent model that treats the entropy as an additional "fluid," which couples to the substantial matter components through entrainment [22–24]. This effect represents the inertia of heat, and leads to the thermal relaxation that is required to ensure causality and stability. The presence of this coupling makes the analysis less straightforward, and the final results are (obviously) less transparent than in the low-velocity case. However, they are also more "interesting." The more complicated setting allows for a number of additional features, most notably a coupling between the heat flow and the charge current. From a fundamental point of view one would expect such a thermo-electric coupling [25], but this effect has nevertheless not been previously discussed in a relativistic context.

The analysis of this problem requires a number of conventions. We assume that spacetime has a metric g_{ab} with signature +2 and represent the associated indices by italicized letters from the beginning of the alphabet, $a, b, c, \ldots$. Spatial indices, with respect to the frame moving with the four-velocity u^a associated with the specific observer that measures the electromagnetic field (the chosen spacetime fibration), are given by italicized letters $i, j, k, \ldots$. In order to distinguish between different fluid components, we label these by roman letters x, y, z, ... The Einstein summation convention does not apply to these constituent indices. The inclusion of electromagnetism is complicated by the fact that there are different conventions regarding units, signs etc. Our discussion, that follows [26], differs from alternatives like [6] in a few subtle ways. First of all the sign of the magnetic field B^a is different, but this is later compensated for by a difference in the definition of ϵ_{abc} which is used to represent the spatial curl. These differences mean that any comparison with the literature must be carried out with care. The model we develop is completely self-consistent and natural in that it leads to the anticipated weak-field, low-velocity results.

II. CHARGED RELATIVISTIC MULTIFLUID SYSTEMS

This section sets the stage for the discussion by bringing together and adapting established results from the literature. The key building blocks are obvious: We need a framework for discussing electromagnetism in general relativity, and in order to understand the nature of the associated current we also need a multifluid formulation for charged components. The first part can be found in many standard textbooks (see for example [10]). The multifluid part is less mainstream fare, but the required formalism (mainly designed by Carter and colleagues, see [11,12] for reviews) has been developed to the required level. The marriage of the two systems has not been discussed extensively in the literature but, as we will see, it is comfortable.

A. Variational multifluid dynamics

Multifluid dynamics arise whenever a system has several components, each in the "fluid regime," which retain their identity. The archetypal such system, known to be well-described by a two-fluid model, is superfluid ^{4}He [27]. In principle, similar systems arise whenever the mean-free path due to interspecies scattering is much larger than that for intraspecies scattering [28]. On intermediate scales one can then meaningfully discuss different fluid components. This setup may seem somewhat artificial, but there clearly are systems in nature where this separation of scales occurs. One reason why superfluid systems tend to require a multifluid approach is that the relevant scale deciding the "size of the fluid elements" is not the mean-free path (since particle scattering is suppressed in a superfluid) but the coherence length of the relevant condensate. This scale is usually much smaller than the mean-free path in the corresponding system at temperatures above the superfluid transition, so the system ends up acting as a fluid on much smaller length scales than usual. In an astrophysical context, the modelling of mature neutron-star cores must account for superfluidity (and superconductivity!). Indeed, most applications of the general relativistic multifluid formalism have been in that problem area, see for example [29].

The model we consider builds on the convective variational principle developed by Carter [11]. This method deals, in a natural way, with the fact that a variational derivation of the equations of fluid dynamics must be constrained. The development takes as starting point a Lagrangian for the matter, Λ, which is built from all relevant fluxes $n_{\rm x}^a$ in the system. In the variational approach, the conservation of the individual fluxes,

$$\nabla_a n_{\rm x}^a = 0, \tag{1}$$

is ensured by means of a pullback construction based on the notion of a three-dimensional matter space. This exercise identifies the spacetime displacements $\xi_{\rm x}^a$ that guarantee (1), and with respect to which the variation of the Lagrangian is carried out. The detailed procedure is discussed in [12]. For later convenience we simply note that the final result is

$$\delta n_{\rm x}^a = n_{\rm x}^b \nabla_b \xi_{\rm x}^a - \xi_{\rm x}^b \nabla_b n_{\rm x}^a - n_{\rm x}^a \left(\nabla_b \xi_{\rm x}^b + \frac{1}{2} g^{bc} \delta g_{bc} \right), \tag{2}$$

where g_{ab} is the spacetime metric and δg_{ab} is the induced variation.

A key strength of the variational approach is that it correctly identifies the momentum $\mu_a^{\rm x}$ that is conjugate to each flux. This is crucial in a multifluid system since the momenta should encode the so-called entrainment effect [12]. As an illustration of how this effect arises, consider a general isotropic Lagrangian. Taking the view that the fluxes are the fundamental variables in the problem, we can build this Lagrangian from the different scalars that we can construct. This means that we should consider both

$$n_{\rm x}^2 = -n_{\rm x}^a n_a^{\rm x}, \tag{3}$$

which defines the number density of the x component, and

$$n_{\rm xy}^2 = -n_{\rm x}^a n_a^{\rm y}, \qquad {\rm y} \neq {\rm x}. \tag{4}$$

An unconstrained variation of Λ with respect to the independent vectors $n_{\rm x}^a$ and the metric g_{ab} then leads to

$$\delta\Lambda = \sum_{\rm x} \mu_a^{\rm x} \delta n_{\rm x}^a + \frac{1}{2} g^{cb} \left(\sum_{\rm x} n_{\rm x}^a \mu_c^{\rm x} \right) \delta g_{ab}, \tag{5}$$

where the momenta are given by

$$\mu_a^{\rm x} = g_{ab} \left(\mathcal{B}^{\rm x} n_{\rm x}^b + \sum_{{\rm y}\neq{\rm x}} \mathcal{A}^{\rm xy} n_{\rm y}^b \right), \tag{6}$$

with coefficients

$$\mathcal{B}^{\rm x} = -2 \frac{\partial \Lambda}{\partial n_{\rm x}^2}, \tag{7}$$

and

$$\mathcal{A}^{\rm xy} = \mathcal{A}^{\rm yx} = -\frac{\partial \Lambda}{\partial n_{\rm xy}^2}, \qquad {\rm x} \neq {\rm y}. \tag{8}$$

The momenta are dynamically, and thermodynamically, conjugate to their respective number density fluxes, and their magnitudes are the chemical potentials (as we will see later). The $\mathcal{A}^{\rm xy}$ coefficients represent the fact that each fluid momentum $\mu_a^{\rm x}$ may, in general, be given by a linear combination of the individual currents $n_{\rm x}^a$. That is, the current and momentum for a particular fluid do not have to be parallel. This is the entrainment effect. In the limit where all the currents are parallel, e.g. when the fluids are comoving, $-\Lambda$ corresponds to the local thermodynamic energy density, but in the general case this is not so.

In terms of the constrained Lagrangian displacements, $\xi^a_{\rm x}$, a variation of Λ yields [30]

$$\delta(\sqrt{-g}\Lambda) = \frac{1}{2}\sqrt{-g}\left(\Psi\delta^a{}_b + \sum_{\rm x} n^a_{\rm x}\mu^{\rm x}_b\right)g^{bc}\delta g_{ac} - \sqrt{-g}\sum_{\rm x} f^{\rm x}_a \xi^a_{\rm x}, \tag{9}$$

where we have defined

$$f^{\rm x}_a = 2n^b_{\rm x}\nabla_{[b}\mu^{\rm x}_{a]}, \tag{10}$$

(and the square brackets indicate antisymmetrization, as usual). It follows immediately that the equations of motion for the individual fluids are expressed as an integrability condition on the vorticity (associated with the momentum not the flux!)

$$f^{\rm x}_a = 0. \tag{11}$$

In (9) we have also introduced the generalized pressure Ψ, defined by

$$\Psi = \Lambda - \sum_{\rm x} n^a_{\rm x}\mu^{\rm x}_a. \tag{12}$$

Finally, we want to account for the coupling between the matter flow and the dynamics of spacetime [31]. The coupling to gravity follows readily from the fact that the stress-energy tensor is obtained as the variation of the matter Lagrangian with respect to the spacetime metric. Basically, we know that the geometry side of the problem is obtained from the Einstein-Hilbert action, expressed in terms of the Ricci scalar R,

$$I_{\rm EH} = \int R\sqrt{-g}d^4x. \tag{13}$$

Following the standard procedure, this leads to

$$\delta I_{\rm EH} = \int G_{ab}\delta g^{ab}\sqrt{-g}d^4x = \int\left(R_{ab} - \frac{1}{2}g_{ab}R\right)\delta g^{ab}\sqrt{-g}d^4x, \tag{14}$$

where G_{ab} is the Einstein tensor and R_{ab} is the Ricci tensor. Now, in the coupled matter-gravity system we have

$$I = I_{\rm EH} + I_{\rm M} = \int\left(\frac{1}{2\kappa}R + \Lambda\right)\sqrt{-g}d^4x, \tag{15}$$

where the coupling constant $\kappa(= 8\pi G/c^4)$ is determined from the correspondence with Newtonian gravity in the appropriate limit. This system leads to the usual Einstein field equations

$$G_{ab} = \kappa T_{ab}, \tag{16}$$

provided that

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\Lambda)}{\delta g^{ab}}, \tag{17}$$

or, equivalently,

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\Lambda)}{\delta g_{ab}}. \tag{18}$$

Returning to the fluid problem, we see from (9) that the multifluid stress-energy tensor takes the form

$$T^{ab}_{\rm M} = \Psi g^{ab} + \sum_{\rm x} n^a_{\rm x}\mu^b_{\rm x}. \tag{19}$$

It is worth noting that when the set of fluid equations, (1) and (11), is satisfied then it is automatically true that $\nabla_a T^{ab}_{\rm M} = 0$.

Provided we are given the appropriate matter Lagrangian (a far from trivial problem as a realistic model should build on microphysics including the relevant interactions) we now have all the equations we need to describe the dynamics of the fluid system, its effect on the gravitational field and vice versa.

As an aside, it is worth noting that the variational model is more general than the typical multicomponent models considered in the literature (especially in cosmology) as they tend to assume the existence of partial pressures (see [32] for a relevant discussion).

B. Electromagnetism

Let us now consider electromagnetism in Einstein's theory. As usual [10], we construct the relativistic version of Maxwell's equations by means of a variational argument with respect to the vector potential A^a. The corresponding Lagrangian is built from the antisymmetric Faraday tensor

$$F_{ab} = 2\nabla_{[a}A_{b]}. \tag{20}$$

We also need to couple the electromagnetic field to the matter flow, represented by the charge current j^a. Letting the relevant coupling constant be μ_0, the action takes the form [33]

$$I_{\rm EM} = \int L_{\rm EM}\sqrt{-g}d^4x, \tag{21}$$

with

$$L_{\rm EM} = -\frac{1}{4\mu_0}F_{ab}F^{ab} + j^a A_a. \tag{22}$$

However, the current term in this expression is not gauge-invariant. Under a gauge transformation of the vector potential, i.e. exercising the freedom to add the gradient of an arbitrary scalar field ψ, the second term in (22) transforms as

$$j^a A_a \to j^a A_a + j^a \nabla_a \psi = j^a A_a + \nabla_a(\psi j^a) - \psi(\nabla_a j^a). \quad (23)$$

The second term on the right-hand side will contribute a surface term to the action integral, and hence can be "ignored" in the usual way. The third term is different. In order to ensure that the action is gauge-invariant, we must demand that the current is conserved, i.e. that

$$\nabla_a j^a = 0. \quad (24)$$

The field equations that we derive require that this constraint be satisfied.

With an action in hand it is straightforward to work out the variation with respect to the vector potential (keeping j^a fixed!), and we arrive at the standard result

$$\nabla_b F^{ab} = \mu_0 j^a. \quad (25)$$

The relativistic Maxwell equations are completed by

$$\nabla_{[a} F_{bc]} = 0, \quad (26)$$

which is automatically satisfied given the antisymmetry of F_{ab}.

Finally, a variation with respect to the metric leads to the electromagnetic stress-energy tensor being given by

$$T^{\rm EM}_{ab} = \frac{1}{\mu_0}\left[g^{cd} F_{ac} F_{bd} - \frac{1}{4} g_{ab}(F_{cd}F^{cd})\right]. \quad (27)$$

It is worth noting that this leads to

$$\nabla_a T^{ab}_{\rm EM} = j_a F^{ab} \equiv -f^b_{\rm L}, \quad (28)$$

which (as we will see later) defines the Lorentz force $f^a_{\rm L}$.

In principle, the electromagnetic dynamics is now fully specified, as we can solve the system for the vector potential A^a. However, in most applications it is more intuitive to work with the electric and magnetic fields E^a and B^a. The downside to this is that these are observer-dependent quantities. This is obvious since varying electric fields generate magnetic fields and vice versa, and the induced variation depends on the motion of the observer.

According to an observer moving with four-velocity u^a, the Faraday tensor can be expressed as

$$F_{ab} = 2u_{[a}E_{b]} + \epsilon_{abcd} u^c B^d, \quad (29)$$

(where round brackets indicate symmetrization). This defines the electric and magnetic fields as

$$E_a = -u^b F_{ba}, \quad (30)$$

and

$$B_a = -u^b\left(\frac{1}{2}\epsilon_{abcd} F^{cd}\right). \quad (31)$$

The physical fields are both orthogonal to u^a, so each field has three components, just as in nonrelativistic physics. We also need an expression for the current, and it is natural to decompose this in a similar way;

$$j^a = \sigma u^a + J^a, \quad \text{where } J^a u_a = 0. \quad (32)$$

C. A comfortable marriage

So far, we have done quite a lot of preparatory work, going over standard territory without adding any real new insight. Our patience with this exercise is about to pay off, as we will now be able to make swift progress. This illustrates the advantage of having a well-grounded action principle for coupled fluids, and an identification of the true momenta, and shows how easy it is to incorporate electromagnetism into the multifluid system [34]. We simply need to consider multiple charge carriers with identifiable fluxes, $n^a_{\rm x}$, and individual charges, $q_{\rm x}$, such that the charge current associated with each flow is

$$j^a_{\rm x} = q_{\rm x} n^a_{\rm x}, \quad (33)$$

and the total current, that sources the electromagnetic field, is given by the sum

$$j^a = \sum_{\rm x} j^a_{\rm x}. \quad (34)$$

It is worth recalling that the variational derivation in Sec. II B requires that the current is conserved. However, this constraint is *automatically* satisfied if each individual current is conserved, as assumed in the variational multifluid model. Hence, we simply have to change the electromagnetic Lagrangian to

$$L_{\rm EM} = -\frac{1}{4\mu_0} F_{ab}F^{ab} + A_a \sum_{\rm x} j^a_{\rm x}, \quad (35)$$

to combine the two models.

It is easy to see that the equations that govern the electromagnetic field remain exactly as before. However, the coupling to the current leads to modified fluid momenta

$$\bar{\mu}^{\rm x}_a = \mu^{\rm x}_a + q^{\rm x} A_a, \quad (36)$$

which satisfy the equations of motion

$$2n^a_{\rm x} \nabla_{[a} \bar{\mu}^{\rm x}_{b]} = 0. \quad (37)$$

As an alternative, we can write this as an explicit force-balance relation. Moving the electromagnetic contribution to the right-hand side, we get

$$f^{\rm x}_a = 2n^b_{\rm x} \nabla_{[b} \mu^{\rm x}_{a]} = q^{\rm x} n^b_{\rm x} F_{ab} = j^b_{\rm x} F_{ab}. \quad (38)$$

To see that this result makes sense, note that the total energy-momentum tensor is easily obtained as the sum of the two previous expressions

$$T^{ab} = T^{ab}_{\rm M} + T^{ab}_{\rm EM}. \quad (39)$$

This means that we must have

$$\nabla_a T^{ab}_{\rm M} = -\nabla_a T^{ab}_{\rm EM} = -j_a F^{ab} = f^b_{\rm L}. \quad (40)$$

In other words, the combined system is such that

$$f_b^{\mathrm{L}} = \sum_{\mathrm{x}} f_b^{\mathrm{x}}. \tag{41}$$

The variational formalism naturally lends itself to a consideration of conserved quantities, like the magnetic helicity [11]. The discussion becomes particularly elegant if carried out using the language of differential forms [35]. We will not discuss conservation laws in this paper, but the interested reader can find relevant recent discussions in [36,37].

Before we proceed, it is worth digressing on the fact that the charge current does not enter the electromagnetic stress-energy tensor (27). As this is a key (albeit somewhat technical) point, it is worth demonstrating the result in detail. To do this, let us focus on the contribution to the total action from the matter-field coupling

$$I_{\mathrm{C}} = \int j^a A_a \sqrt{-g} d^4 x. \tag{42}$$

Variation of the integrand then leads to a sum of terms of the form

$$\delta(n_{\mathrm{x}}^a A_a \sqrt{-g}) = \sqrt{-g}[A_a \delta n_{\mathrm{x}}^a + n_{\mathrm{x}}^a \delta A_a] + n_{\mathrm{x}}^a A_a \delta\sqrt{-g}. \tag{43}$$

Naively, the first term affects the Euler equation, the second leads to the current term in the Maxwell equations and the final term should enter the stress-energy tensor. However, the last contribution is canceled by a term originating from the variation of the matter flux. Using (2) in (43) (ignoring surface terms) we arrive at

$$\delta(n_{\mathrm{x}}^a A_a \sqrt{-g}) = \sqrt{-g}(2\xi_{\mathrm{x}}^a n^b \nabla_{[a} A_{b]} + n_{\mathrm{x}}^a \delta A_a). \tag{44}$$

The first term enters the Euler Eqs. (38) and the second leads to the current term in the Maxwell equations. The electromagnetic contribution to the stress-energy tensor is completely determined by the first term in (22), leading to (27). It is interesting to note that this result is obtained in a natural way in the constrained variational approach.

III. RESISTIVE MAGNETOHYDRODYNAMICS

The formalism developed in the previous section provides a general framework for describing the dynamics of charged multifluid systems in relativity. However, as the model arises from a variational analysis it does not account for dissipative mechanisms. Hence, we need to amend it if we want to model, for example, resistivity. This is obviously a key aspect if we want to be able to model the evolution of electromagnetic fields in various astrophysical and cosmological settings. However, we know that the general dissipative problem is a severe challenge in relativity. We also know that many different dissipation channels may affect a generic multifluid system [25,38]. Hence, we set a more modest target and explore the role of a simple, phenomenological, resistivity. As it turns out, the problem involves tricky issues already at this level. In general, any dissipative mechanism will generate heat, so a realistic model must account for the associated heat flux. However, this problem is known to be associated with both causality and stability issues in relativity [18–21]. These problems can be resolved [23,24], but we must proceed carefully.

Given the various issues involved, we consider the resistive problem at two different levels of sophistication. First (in this section) we consider a cold plasma, where the various relative velocities are sufficiently low that the problem simplifies. Having understood this problem we proceed (in the next section) to consider the general problem, with arbitrary velocities and the presence of a heat flow. In each case, we consider a system with two charge carriers, with individual particles carrying a single unit of charge. This means that the models apply to both pair plasmas with positrons and electrons and proton-electron plasmas. These examples provide useful illustrations and the discussion highlights the differences between the two systems.

Throughout the discussion, we assume that the system is fully ionized. That is, we do not allow for the presence of a charge-neutral component, as would be required if we wanted to model a magnetized neutron-star core, for example. The inclusion of such a component is, in principle, straightforward although the algebra obviously gets more involved (especially if one accounts for entrainment).

A. Choice of frame

We focus on a two-component system with one component, labeled p, carrying a single positive unit of charge $q_{\mathrm{p}} = e$ while the other component, labeled e, carries a single negative unit of charge $q_{\mathrm{e}} = -e$. The associated charge currents are $j_{\mathrm{p}}^a = e n_{\mathrm{p}}^a$ and $j_{\mathrm{e}}^a = -e n_{\mathrm{e}}^a$, respectively. We will not, initially, make any assumptions regarding the relative masses of the two components. This means that the model applies to both pair plasmas and proton-electron systems (indeed, any two-component system with electrons and single charged ions).

A key aim of the exercise we are embarking on is to derive the relativistic version of Ohm's law. Basically, we want to start from the charged two-fluid system and arrive at a model from which the assumptions associated with standard relativistic magnetohydrodynamics become clear. This discussion will obviously involve both the electric and the magnetic field, as well as the charge currents. Now, we know that E^a and B^a are observer-dependent quantities. Hence, the model involves a judicious choice of observer. It is natural to begin by considering this issue.

Given an observer with four-velocity u^a (normalized such that $u^a u_a = -1$) we can decompose the two fluxes

$$n_{\mathrm{x}}^a = n_{\mathrm{x}} u_{\mathrm{x}}^a, \quad \text{where } n_{\mathrm{x}}^2 = -n_{\mathrm{x}}^a n_a^{\mathrm{x}} \quad \text{and} \quad u_{\mathrm{x}}^a u_a^{\mathrm{x}} = -1, \tag{45}$$

using

$$u^a_{\rm x} = \gamma_{\rm x}(u^a + v^a_{\rm x}),$$
$$\text{where } u^a v^{\rm x}_a = 0, \quad \text{and} \quad \gamma_{\rm x} = (1 - v^2_{\rm x})^{-1/2}. \tag{46}$$

In the first instance, we will assume that the "drift" velocities $v^a_{\rm x}$ are small enough that we can linearize the model, i.e. assume that $\gamma_{\rm x} \approx 1$. This model should be relevant for cold plasmas [39].

We also need the fluid momenta, which would generally involve entrainment between the two components. However, as we are not aware of a physical argument for the presence of entrainment between protons and electrons (or, indeed, positrons and electrons), we do not account for this effect here (although we will consider it later when we discuss heat flux and entropy). This means that we have

$$\mu^{\rm x}_a = \mathcal{B}^{\rm x} n^{\rm x}_a = \mathcal{B}^{\rm x}\gamma_{\rm x} n^{\rm x}(u_a + v^{\rm x}_a). \tag{47}$$

The chemical potential of each component is generally defined by

$$\mu_{\rm x} = -u^a_{\rm x}\mu^{\rm x}_a = n_{\rm x}\mathcal{B}^{\rm x}, \tag{48}$$

which means that

$$\mu^{\rm x}_a = \mu_{\rm x}\gamma_{\rm x}(u_a + v^{\rm x}_a) \approx \mu_{\rm x}(u_a + v^{\rm x}_a). \tag{49}$$

With these definitions we can write the (linearized) fluid stress-energy tensor as

$$T^{\rm M}_{ab} = \Psi g_{ab} + (n_{\rm p}\mu_{\rm p} + n_{\rm e}\mu_{\rm e})u_a u_b + (n_{\rm p}\mu_{\rm p} v^{\rm p}_b + n_{\rm e}\mu_{\rm e} v^{\rm e}_b)u_a + (n_{\rm p}\mu_{\rm p} v^{\rm p}_a + n_{\rm e}\mu_{\rm e} v^{\rm e}_a)u_b. \tag{50}$$

Contracting this with u^a we get an expression for the momentum flux

$$u^a T^{\rm M}_{ab} = (\Psi - n_{\rm p}\mu_{\rm p} - n_{\rm e}\mu_{\rm e})u_b - (n_{\rm p}\mu_{\rm p} v^{\rm p}_b + n_{\rm e}\mu_{\rm e} v^{\rm e}_b). \tag{51}$$

Contracting with u^b again, we find that the energy density measured by the observer is

$$\rho = u^a u^b T^{\rm M}_{ab} = -\Psi + n_{\rm p}\mu_{\rm p} + n_{\rm e}\mu_{\rm e}. \tag{52}$$

We see that, in the linear model Ψ is the pressure. Hence, we replace it with P in the following, leading to the anticipated thermodynamic relation (the integrated first law)

$$P + \rho = n_{\rm p}\mu_{\rm p} + n_{\rm e}\mu_{\rm e}. \tag{53}$$

Note that, as we are only considering the fluid contribution here, our definitions of P and ρ do not include electromagnetic effects (i.e. the magnetic pressure is not accounted for yet).

From (51) we see that we can choose observers such that there is no relative (fluid) momentum flux by setting [40]

$$n_{\rm p}\mu_{\rm p} v^{\rm p}_b + n_{\rm e}\mu_{\rm e} v^{\rm e}_b = 0. \tag{54}$$

This leads us to define a velocity v^a such that

$$(P + \rho)v^a = n_{\rm p}\mu_{\rm p} v^a_{\rm p} + n_{\rm e}\mu_{\rm e} v^a_{\rm e} \tag{55}$$

and highlights the relevance of the frame in which $v^a = 0$. We express the second degree of freedom in terms of the relative velocity

$$w^a = v^a_{\rm p} - v^a_{\rm e}. \tag{56}$$

With these definitions we have

$$v^a_{\rm p} = v^a + \frac{n_{\rm e}\mu_{\rm e}}{P + \rho} w^a \tag{57}$$

and

$$v^a_{\rm e} = v^a - \frac{n_{\rm p}\mu_{\rm p}}{P + \rho} w^a \tag{58}$$

and the charge current takes the form

$$j^a = e(n_{\rm p} - n_{\rm e})(u^a + v^a) + e\frac{n_{\rm p} n_{\rm e}}{P + \rho}(\mu_{\rm p} + \mu_{\rm e})w^a. \tag{59}$$

From this result we read off the charge density $\sigma = e(n_{\rm p} - n_{\rm e})$ in the observer's frame. If we assume that the system is charge-neutral on macroscopic scales, a natural assertion for systems where the charge carriers (like the electrons) are highly mobile and one of the key assumption in standard magnetohydrodynamics, then the current simplifies to

$$j^a = J^a = e\frac{n_{\rm p} n_{\rm e}}{P + \rho}(\mu_{\rm p} + \mu_{\rm e})w^a. \tag{60}$$

Moreover, in the case of a charge-neutral plasma we have $P + \rho = n_{\rm e}(\mu_{\rm p} + \mu_{\rm e})$ which means that the current takes the final form

$$J^a = e n_{\rm e} w^a. \tag{61}$$

B. The resistivity

In order to account for the resistivity, we need to add a phenomenological "force" term to (38). This additional term should represent the dissipative interaction between the two components, and from nonrelativistic intuition [13,14], we expect it to be linear in the relative velocity between the two components. We also see from (38) that the required force must be orthogonal to each respective flux (note that this condition must be relaxed if we want to allow for particle creation/destruction). Based on these points, we let the resistive forces [41] take the form

$$\tilde{f}^{\,a}_{\rm p} = e\mathcal{R} \perp^{ab}_{\rm p} n^{\rm e}_b = -\mathcal{R} \perp^{ab}_{\rm p} j_b \tag{62}$$

and

$$\tilde{f}^{\,a}_{\rm e} = e\mathcal{R} \perp^{ab}_{\rm e} n^{\rm p}_b = \mathcal{R} \perp^{ab}_{\rm e} j_b \tag{63}$$

where we have introduced the projections

$$\perp^{ab}_{\rm x} = g^{ab} + u^a_{\rm x} u^b_{\rm x}. \tag{64}$$

These expressions represent linear scattering of the two components. The resistivity experienced by one component is proportional to the number of particles of the other kind that flow relative to it.

The resistivity is further constrained by the fact that the sum of the forces must vanish (essentially Newton's third law). This follows immediately from the fact that the divergence of the nondissipative stress-energy tensor [which arises from the sum of (38)] must vanish. For the suggested forces, we have

$$\tilde{f}^{\,a}_{\rm p} + \tilde{f}^{a}_{\rm e} = e\mathcal{R}(\perp^{ab}_{\rm p} n^{\rm e}_b + \perp^{ab}_{\rm e} n^{\rm p}_b) \approx e\mathcal{R}(n_{\rm p} - n_{\rm e})w^a. \tag{65}$$

This shows that the linearized model is only consistent as long as the system is charge-neutral. If there is charge imbalance, we need to alter the model. At first sight, this may seem surprising but it is actually quite natural. The model only accounts for the two charged components, whereas the general system would also have the heat generated by the dissipation. The correct interpretation of (65) is that, for a charge-neutral system, there is no heat generated at the linear level. In order to consider a more general system, we need to account for the heat. Then the force balance is ensured by introducing an additional component, which we will take to be the entropy, with a corresponding force of the required form. We will discuss this extended system in the next section. For now, we simply assume charge neutrality and note that the corresponding low-velocity model describes a "cold plasma" in the sense that there is no heat generated in the system.

C. Generalized Ohm's law

The problem under consideration has two fluid degrees of freedom, represented by (38) with the added resistivity terms (on the right-hand side). One can (obviously) combine these two equations in different ways. It seems natural to adapt the standard strategy from nonrelativistic plasma physics [13,14] and consider a "total momentum" equation alongside a suitably weighted difference. The first of these equations follows by adding (38), and from the discussion in the previous section we know that this leads to

$$\nabla_a T^{ab} = 0 \tag{66}$$

as the sum of the resistive forces vanishes (to linear order). In order to represent the second degree of freedom, we divide the two equations from (38) by $n_{\rm x}\mu_{\rm x}$ and then take the difference. The weighting (different from that used in other recent discussions of the problem [17]) is motivated by the Newtonian limit, where $\mu_{\rm x} \rightarrow m_{\rm x}$ (the rest mass), and represents the "center-of-mass" frame. With this weighting the difference equation simplifies considerably. One may obtain the same final result with a different weighting, but the analysis would then have to make explicit use of the total momentum Eq. (66) in simplifying the expressions. Our route is more direct.

The difference equation that we require is made up of three pieces. Considering first the fluid contribution to (38) and the definition of the chemical potentials, we have

$$2n^a_{\rm x}\nabla_{[a}\mu^{\rm x}_{b]} = n_{\rm x} \perp^{a}_{{\rm x}b} \nabla_a\mu_{\rm x} + n_{\rm x}\mu_{\rm x}u^a_{\rm x}\nabla_a u^{\rm x}_b. \tag{67}$$

We also have

$$\perp^{ab}_{\rm x} \approx \perp^{ab} + 2u^{(a}v^{b)}_{\rm x}, \tag{68}$$

where $\perp^{ab} = g^{ab} + u^a u^b$ is the projection orthogonal to u^a. Using these results, we find that the weighted difference (let us call it $f^{\rm D}_a$) in the linear model takes the form;

$$\begin{aligned} f^{\rm D}_b &= \frac{2}{n_{\rm p}\mu_{\rm p}} n^a_{\rm p}\nabla_{[a}\mu^{\rm p}_{b]} - \frac{2}{n_{\rm e}\mu_{\rm e}} n^a_{\rm e}\nabla_{[a}\mu^{\rm e}_{b]} \\ &= u^a_{\rm p}\nabla_a u^{\rm p}_b - u^a_{\rm e}\nabla_a u^{\rm e}_b + \frac{1}{\mu_{\rm p}} \perp^{a}_{{\rm p}b} \nabla_a\mu_{\rm p} \\ &\quad - \frac{1}{\mu_{\rm e}} \perp^{a}_{{\rm e}b} \nabla_a\mu_{\rm e} \approx u^a\nabla_a w_b + w^a\nabla_a u_b \\ &\quad + \frac{1}{\mu_{\rm p}} \perp^{a}_{{\rm p}b} \nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}} \perp^{a}_{{\rm e}b} \nabla_a\mu_{\rm e}. \end{aligned} \tag{69}$$

The last two terms expand to

$$\begin{aligned} &\frac{1}{\mu_{\rm p}} \perp^{a}_{{\rm p}b} \nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}} \perp^{a}_{{\rm e}b} \nabla_a\mu_{\rm e} \\ &\quad\approx (\perp^a{}_b + u^a v_b + u_b v^a)\left(\frac{1}{\mu_{\rm p}}\nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right) \\ &\qquad + \frac{1}{(P+\rho)\mu_{\rm p}\mu_{\rm e}}(u^a w_b + u_b w^a) \\ &\qquad\quad \times [n_{\rm e}\mu^2_{\rm e}\nabla_a\mu_{\rm p} + n_{\rm p}\mu^2_{\rm p}\nabla_a\mu_{\rm e}]. \end{aligned} \tag{70}$$

This expression obviously simplifies somewhat in the frame where $v^a = 0$. The result also simplifies for the two specific examples we are considering. For pair plasmas we have $\mu_{\rm p} = \mu_{\rm e} \equiv \mu$ so in the chosen frame the final result would be

$$f^{\rm D}_b = u^a\nabla_a w_b + w^a\nabla_a u_b + \frac{2}{\mu} u_{(a}w_{b)}\nabla^a\mu. \tag{71}$$

Meanwhile, for a proton-electron plasma we may assume that $\mu_{\rm e} \ll \mu_{\rm p}$ in which case (when $v^a = 0$) we are left with [42]

$$\begin{aligned} f^{\rm D}_b &\approx u^a\nabla_a w_b + w^a\nabla_a u_b + \perp^a{}_b\left(\frac{1}{\mu_{\rm p}}\nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right) \\ &\quad + \frac{2}{\mu_{\rm e}} u_{(a}w_{b)}\nabla^a\mu_{\rm e}. \end{aligned} \tag{72}$$

In each case we can replace the relative velocity w^a with the charge current via Eq. (61).

The expressions we have obtained represent the left-hand side of the equation that we are developing. The right-hand side is made up of two pieces. The first is the weighted difference of the two magnetic forces from (38). That is, we have

$$
\begin{aligned}
f_b^{\mathrm{M}} &= \frac{e}{n_{\mathrm{p}}\mu_{\mathrm{p}}} n_{\mathrm{p}}^a F_{ba} + \frac{e}{n_{\mathrm{e}}\mu_{\mathrm{e}}} n_{\mathrm{e}}^a F_{ba} \\
&= e\left(\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}\right)E_b + e\left(\frac{1}{\mu_{\mathrm{p}}} v_{\mathrm{p}}^a + \frac{1}{\mu_{\mathrm{e}}} v_{\mathrm{e}}^a\right)F_{ba} \\
&= e\left(\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}\right)(E_b + v^a F_{ba}) \\
&\quad - \frac{e(n_{\mathrm{p}}\mu_{\mathrm{p}}^2 - n_{\mathrm{e}}\mu_{\mathrm{e}}^2)}{(P+\rho)\mu_{\mathrm{p}}\mu_{\mathrm{e}}} w^a F_{ba}. \qquad (73)
\end{aligned}
$$

As before, this simplifies in the frame where $v^a = 0$. In addition, for (charge-neutral) pair plasmas the second term vanishes identically and we are left with

$$
f_b^{\mathrm{M}} = e\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}} E_b = \frac{2e}{\mu} E_b. \qquad (74)
$$

Meanwhile, for a proton-electron system we would have

$$
f_b^{\mathrm{M}} \approx \frac{e}{\mu_{\mathrm{e}}}(E_b - w^a F_{ba}). \qquad (75)
$$

In this case we need to consider the remaining term involving the Faraday tensor in more detail. From (29) it is easy to see that we will have

$$
w^a F_{ba} = u_b(w^a E_a) + \epsilon_{bcd} w^c B^d, \qquad (76)
$$

where we have defined $\epsilon_{bcd} = \epsilon_{abcd} u^a$ in order to make the final term resemble the standard three-dimensional cross product [43]. Since w^a is proportional to the charge current, we recognize the two terms as the Joule heating and the Hall effect, respectively. These effects are notably absent in the linear model for a pair plasma [15–17].

Finally, we need the weighted difference between the two resistivities. That is,

$$
\begin{aligned}
f_b^{\mathrm{R}} &= \frac{1}{n_{\mathrm{p}}\mu_{\mathrm{p}}} \tilde{f}_b^{\mathrm{p}} - \frac{1}{n_{\mathrm{e}}\mu_{\mathrm{e}}} \tilde{f}_b^{\mathrm{e}} \\
&= \frac{e\mathcal{R}}{n_{\mathrm{p}}\mu_{\mathrm{p}}} \perp^{\mathrm{p}}_{ab} n_{\mathrm{e}}^a - \frac{e\mathcal{R}}{n_{\mathrm{e}}\mu_{\mathrm{e}}} \perp^{\mathrm{e}}_{ab} n_{\mathrm{p}}^a \\
&= -\mathcal{R}\left(\frac{1}{n_{\mathrm{p}}\mu_{\mathrm{p}}} \perp^{\mathrm{p}}_{ab} + \frac{1}{n_{\mathrm{e}}\mu_{\mathrm{e}}} \perp^{\mathrm{e}}_{ab}\right) j^a \\
&\approx -\mathcal{R}\frac{P+\rho}{(n_{\mathrm{p}}\mu_{\mathrm{p}})(n_{\mathrm{e}}\mu_{\mathrm{e}})} J_b, \qquad (77)
\end{aligned}
$$

where the fact that the current is proportional to w^a in the charge-neutral case has allowed us to neglect the various v_{x}^a terms in the projections.

The final result now follows from the combination

$$
f_a^{\mathrm{D}} = f_a^{\mathrm{M}} + f_a^{\mathrm{R}}. \qquad (78)
$$

In the general charge-neutral case we have, in the frame where $v^a = 0$,

$$
\begin{aligned}
& e\frac{(P+\rho)}{n_{\mathrm{e}}\mu_{\mathrm{p}}\mu_{\mathrm{e}}} E_b - \frac{e(\mu_{\mathrm{p}}-\mu_{\mathrm{e}})}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}[u_b(w^a E_a) + \epsilon_{bcd} w^c B^d] \\
&\quad - \mathcal{R}\frac{P+\rho}{(n_{\mathrm{p}}\mu_{\mathrm{p}})(n_{\mathrm{e}}\mu_{\mathrm{e}})} J_b \\
&= u^a\nabla_a w_b + w^a\nabla_a u_b + \perp^a{}_b\left(\frac{1}{\mu_{\mathrm{p}}}\nabla_a\mu_{\mathrm{p}} - \frac{1}{\mu_{\mathrm{e}}}\nabla_a\mu_{\mathrm{e}}\right) \\
&\quad + \frac{2}{(P+\rho)\mu_{\mathrm{p}}\mu_{\mathrm{e}}} u_{(a}w_{b)}[n_{\mathrm{e}}\mu_{\mathrm{e}}^2\nabla^a\mu_{\mathrm{p}} + n_{\mathrm{p}}\mu_{\mathrm{p}}^2\nabla^a\mu_{\mathrm{e}}]. \qquad (79)
\end{aligned}
$$

This result can be simplified by projecting out the contribution along u^a (which will not affect the expression for the electric field). This leads to the "final" result

$$
\begin{aligned}
& e\frac{(P+\rho)}{n_{\mathrm{e}}\mu_{\mathrm{p}}\mu_{\mathrm{e}}} E_b - \frac{e(\mu_p-\mu_e)}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}\epsilon_{bcd} w^c B^d \\
&\quad - \mathcal{R}\frac{P+\rho}{(n_{\mathrm{p}}\mu_{\mathrm{p}})(n_{\mathrm{e}}\mu_{\mathrm{e}})} J_b \\
&= \perp^a{}_b\left[u^c\nabla_c w_a + \frac{1}{\mu_{\mathrm{p}}}\nabla_a\mu_{\mathrm{p}} - \frac{1}{\mu_{\mathrm{e}}}\nabla_a\mu_{\mathrm{e}}\right] \\
&\quad + w^a\nabla_a u_b + \frac{n_{\mathrm{e}}}{(P+\rho)\mu_{\mathrm{p}}\mu_{\mathrm{e}}} w_b u_a[\mu_{\mathrm{e}}^2\nabla_a\mu_{\mathrm{p}} + \mu_{\mathrm{p}}^2\nabla_a\mu_{\mathrm{e}}]. \qquad (80)
\end{aligned}
$$

In these expressions, it may be useful to decompose $\nabla_a u_b$ in the standard way, see for example [6]. That is, we use

$$
\nabla_a u_b = \sigma_{ab} + \omega_{ab} - u_a\dot{u}_b + \frac{1}{3}\theta\perp_{ab} \qquad (81)
$$

where the dot represents the commoving time derivative $u^c\nabla_c$, in terms of the expansion scalar

$$
\theta = \nabla_a u^a, \qquad (82)
$$

the shear

$$
\sigma_{ab} = D_{\langle a}u_{b\rangle}, \qquad (83)
$$

where the angle brackets indicate symmetrization and trace removal, and

$$
D_a u_b = \perp_a{}^c \perp_b{}^d \nabla_c u_d. \qquad (84)
$$

The merit of using this (totally projected) derivative is that the individual terms in (81) are perpendicular to u^a. We have also defined the vorticity [44]

$$
\omega_{ab} = D_{[a}u_{b]}. \qquad (85)
$$

The decomposition (81) makes the coupling between the charge current J^a and the nature of the fluid motion more explicit.

The final relation for pair plasmas can now be written [45]

$$E_b - \frac{\mathcal{R}}{en_e} J_b = \frac{\mu}{2e^2 n_e}\left[\perp_{ab} \dot{J}^a + J^a\left(\sigma_{ab} + \omega_{ab} + \frac{4}{3}\theta \perp_{ab}\right)\right]. \quad (86)$$

As already mentioned, this expression is notable for the absence of the Hall effect, i.e. there is no term proportional to $\epsilon_{abc} J^b B^c$.

The case of a proton-electron plasma is only slightly more complicated. After neglecting μ_e compared to μ_p, we end up with

$$E_b - \frac{1}{en_e}\epsilon_{bcd} J^c B^d - \frac{\mathcal{R}}{en_e} J_b = \frac{\mu_e}{e^2 n_e}\left[\perp_{ab} \dot{J}^a + J^a\left(\sigma_{ab} + \omega_{ab} + \frac{4}{3}\theta \perp_{ab}\right)\right] - \frac{1}{e}\perp^a{}_b \nabla_a \mu_e. \quad (87)$$

In this case, the Hall term is obviously present. We also have a "Biermann battery" term, $\perp^a{}_b \nabla_a \mu_e$, which would serve to generate a magnetic field even if there was no field initially [17].

It is easy to show that our final results agree perfectly with the results obtained in [17].

Before moving on, it is useful to consider the relation between our results and the common starting point for discussions of resistive effects in numerical simulations [46–48]. Much of the relevant literature builds on the work by Bekenstein and Oron [49]. Ignoring the right-hand side of Ohm's law in the proton-electron case we have

$$E_b = \frac{1}{en_e}\epsilon_{bcd} J^c B^d + \frac{\mathcal{R}}{en_e} J_b = \frac{\mathcal{R}}{n_e e}\left(\perp_{ab} + \frac{1}{\mathcal{R}}\epsilon_{bacd} u^c B^d\right) J^a = S_{ba} J^a. \quad (88)$$

Define $\zeta = 1/\mathcal{R}$ and $\tilde{\sigma} = \mathcal{R}/n_e e$ to get

$$S_{ba} = \frac{1}{\tilde{\sigma}}(\perp_{ba} + \zeta \epsilon_{bacd} u^c B^d). \quad (89)$$

Inverting this, we arrive at

$$J_b = \sigma^{ab} E_b, \quad (90)$$

with

$$\sigma^{ab} = \frac{\sigma}{1 + \zeta^2 B^2}(\perp^{ab} + \zeta^2 B^a B^b - \zeta \epsilon^{abcd} u_c B_d). \quad (91)$$

This is the result stated in [49], once we account for the different sign conventions. At this point, we can make an important observation. It is easy to identify the Hall effect in the initial expression (88), but its presence is more convoluted in the alternative expression (91). That this can lead to conceptual confusion is evidenced by [50], where numerical evolutions for a truncated form of (91) are carried out. The considered model includes a peculiarly amputated Hall effect, the actual meaning of which is unclear. This lesson tells us that an understanding of the physical origin of the model is imperative.

D. Towards ideal MHD

We are now in a position where we can assess the relative importance of the different terms in the generalized version of Ohm's law (80). Let us first consider under what conditions we can neglect the inertia of the charge current compared to the resistivity. In order to do this we need

$$\frac{\mathcal{R}}{en_e} J \gg \frac{\mu_e}{e^2 n_e}\dot{J} \approx \frac{m_e}{e^2 n_e}\dot{J}. \quad (92)$$

It is natural [49] to associate the resistivity with a relaxation time scale τ_r such that

$$\mathcal{R} = \frac{m_e}{e\tau_r}. \quad (93)$$

If we also assume that the dynamics has a characteristic time scale τ_d, such that $\dot{J} \sim J/\tau_d$, then it is easy to see that we can neglect the inertia of the charge current as long as

$$\tau_d \gg \tau_r. \quad (94)$$

When this condition holds, i.e. for sufficiently slow dynamics, the system is essentially oblivious of its plasma physics origins. This condition shows why ideal magnetohydrodynamics is a good model for slowly evolving, or stationary, systems.

Next let us compare the Hall term to the resistivity. The former dominates (in magnitude) if

$$B \gg \mathcal{R}, \quad (95)$$

which, if we introduce the electron cyclotron frequency

$$\omega_c = \frac{eB}{m_e}, \quad (96)$$

leads to the condition

$$\omega_c \tau_r \gg 1. \quad (97)$$

This means that the electron executes many cyclotron "oscillations" before the motion is damped.

Finally, we need to establish when the resistivity can be neglected compared to the electric field. This requires

$$E \gg \frac{\mathcal{R}}{en_e} J = \frac{m_e}{n_e e^2 \tau_r} J. \quad (98)$$

Here we need to make use of Maxwell's equations (see below), which lead to (in Gaussian units!)

$$E \sim \frac{V}{c} B, \quad (99)$$

where we assume that the dynamics has a characteristic length scale $L$and an associated velocity $V = L/\tau_d$. This leads to the final condition

$$J \ll n_e e\left(\frac{V}{c}\right)\omega_c\tau_r \ll n_e e\left(\frac{V}{c}\right). \tag{100}$$

The last condition is required since we also want to be able to neglect the Hall term. We are essentially left with a low-velocity constraint.

These rough estimates provide useful insight into the applicability of "ideal" magnetohydrodynamics, which corresponds to the assumption that $E^a \approx 0$. The usual argument for this is that the medium is a perfect conductor, i.e. $\mathcal{R} \to 0$. However, this limit only affects the resistive term in (80). We still have to argue that the remaining terms are unimportant. This is not quite as easy. At the end of the day, ideal magnetohydrodynamics is more an assumption than an approximation [13] (the interested reader may want to compare the present discussion to the variational derivation of ideal magnetohydrodynamics in [51]).

E. The remaining fluid equation

So far, we have focused on the weighted difference between the two momentum equations in the plasma. To complete the "single-fluid" model we need to express the remaining degree of freedom in terms of our chosen variables. It is natural to obtain the required equation from (66).

As a first step, we consider the nonmagnetic contributions. As the individual number fluxes are conserved in the variational approach, we see that

$$\begin{aligned}\nabla_a T^{ab}_{\rm M} &= \nabla^b\Psi + u^b_{\rm p} n^a_{\rm p}\nabla_a\mu_{\rm p} + u^b_{\rm e} n^a_{\rm e}\nabla_a\mu_{\rm e} \\ &\quad + \mu_{\rm p} n^a_{\rm p}\nabla_a u^b_{\rm p} + \mu_{\rm e} n^a_{\rm e}\nabla_a u^b_{\rm e} \\ &\approx \perp^{ab}\nabla_a P + (P+\rho)\dot{u}^b + 2u^{(a}v^{b)}_{\rm p} n_{\rm p}\nabla_a\mu_{\rm p} \\ &\quad + 2u^{(a}v^{b)}_{\rm e} n_{\rm p}\nabla_a\mu_{\rm e} + n_{\rm p}\mu_{\rm p}(\dot{v}^b_{\rm p} + v^a_{\rm p}\nabla_a u^b) \\ &\quad + n_{\rm e}\mu_{\rm e}(\dot{v}^b_{\rm e} + v^a_{\rm e}\nabla_a u^b). \end{aligned} \tag{101}$$

The first equality is exact, while the second line holds at the level of linearized relative velocities. Expressing this result in terms of v^a and w^a, we have

$$\begin{aligned}\nabla_a T^{ab}_{\rm M} &\approx (P+\rho)\dot{u}^b + \perp^{ab}\nabla_a P + 2u^{(a}v^{b)}\nabla_a\Psi \\ &\quad + \frac{n^2_{\rm e}\mu_{\rm p}\mu_{\rm e}}{P+\rho}u^b w^a\left(\frac{1}{\mu_{\rm p}}\nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right) \\ &\quad + (P+\rho)(\dot{v}^b + v^a\nabla_a u^b). \end{aligned} \tag{102}$$

Here we have used the fact that we are considering a charge-neutral system. This result provides the left-hand side of the final equation. As discussed in Sec. II C the matter contribution is balanced by the electromagnetic stresses, which provides the right-hand side for the equation we are interested in. This takes the form

$$-\nabla_a T^{ab}_{\rm EM} = u^b(j_a E^a) + \epsilon^{bac} j_a B_c. \tag{103}$$

The final equation will only have spatial components with respect to u^a, but the relations we have written down so far also have a parallel contribution. However, if we contract the combined equation with u_b and compare to what we get if we contract our generalized Ohm's law (80) with the current, then we see that the two results agree (at the linearized level, of course). Hence, only the orthogonal component contains new information. In the frame where $v^a = 0$ the final fluid equation takes the form

$$(P+\rho)\dot{u}^b + \perp^{ab}\nabla_a P = \epsilon^{bac} J_a B_c. \tag{104}$$

This is simply the perfect fluid equation of motion augmented by the Lorentz force.

To complete the model, we may also consider the two conservation laws (1). It is straightforward to show that the difference between these corresponds to the required conservation law for the charge current. Meanwhile, after making use of the component aligned with u^a from the weighted difference equation that leads to (80), the sum of the two conservation laws can be written

$$J_a E^a = \frac{P+\rho}{n_{\rm e}}(\dot{n}_{\rm e} + n_{\rm e}\theta). \tag{105}$$

Moreover, one can show that (at the linear level) this expression also follows from the component of the total momentum equation, Eq. (66) that is aligned with u^a.

At the end of the day we have two scalar equations and two equations governing velocity components that are spatial with respect to u^a. Thus, we have explicitly accounted for the degrees of freedom of the original two-fluid system. To complete the system we also need Maxwell's equations. Before discussing these, let us make a brief diversion and touch upon a model that is common for neutron-star magnetospheres.

The conditions in the magnetosphere of a neutron star (or, indeed, a black hole) are expected to be such that there is sufficient plasma present to support a charge current, but the associated inertia can be neglected [52,53]. Thus, we may neglect the inertia in (104), essentially decoupling the matter from the magnetic problem. This leads to what is known as force-free electrodynamics [54,55]. In these circumstances we would have

$$\epsilon^{bac} J_a B_c \approx 0, \tag{106}$$

which implies that charges may only flow along the magnetic field lines.

The force-free assumption can obviously be used independently of the assumptions that lead to ideal magnetohydrodynamics. Basically, one may envisage a range of different "approximations" depending on the circumstances. The force-free model simplifies magnetosphere modelling, but one must apply it with care since it breaks down near magnetic neutral points. In the context of the present discussion, it is also worth noting that (104) may be extended to include various dissipation channels (like shear viscosity) other than the pure collisional resistivity that we have accounted for. If the multifluid aspects are taken seriously [28] this may lead to a much more complex problem.

F. Maxwell's equations

Given an observer moving with u^a, representing a fibration of spacetime, the decomposition of Maxwell's equations is standard. Nevertheless, we list the results here for completeness. For a more detailed discussion, see, for example, [6].

First of all,

$$\nabla_a F^{ba} = \mu_0 j^b \tag{107}$$

leads to

$$\begin{aligned}\perp^{ab} \nabla_b E_a &= \nabla_a E^a - E^a \dot{u}_a \\ &= \mu_0 \sigma + \epsilon^{abc} \omega_{ab} B_c \\ &= \mu_0 \sigma + 2W^a B_a,\end{aligned} \tag{108}$$

where we have defined the vorticity vector as

$$W^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}, \quad \text{so that } \omega_{ab} = \epsilon_{abc} W^c, \quad \text{and} \quad u^a W_a = 0. \tag{109}$$

We also get

$$\begin{aligned}&\perp_{ab} \dot{E}^b - \epsilon_{abc}\nabla^b B^c + \mu_0 J_a \\ &= \left(\sigma_{ab} - \omega_{ab} - \frac{2}{3}\theta \perp_{ab}\right) E^b + \epsilon_{abc}\dot{u}^b B^c.\end{aligned} \tag{110}$$

Secondly,

$$\nabla_{[a} F_{bc]} = 0 \tag{111}$$

leads to

$$\perp^{ab} \nabla_b B_a = -2W^a E_a \tag{112}$$

and

$$\begin{aligned}&\perp_{ab} \dot{B}^b + \epsilon_{abc}\nabla^b E^c \\ &= -\epsilon_{abc}\dot{u}^b E^c + \left(\sigma_{ab} - \omega_{ab} - \frac{2}{3}\theta \perp_{ab}\right) B^b.\end{aligned} \tag{113}$$

It is easy to see that, if we consider an inertial observer, these results reduce to the standard textbook form of Maxwell's equations. The complete expressions given here are, of course, useful if we are interested in more general settings. In particular, they highlight the coupling between the electromagnetic field and a given fluid flow (with shear, vorticity and expansion).

IV. ADDING ENTROPY: HOT PLASMAS

The low-velocity model we have discussed so far is consistent and applicable to many situations of interest. It also provides a number of potentially important extensions of the ideal magnetohydrodynamics that tends to be used in relativistic astrophysics. However, as we have already hinted at, the model does not account for the presence of heat. This is an unfortunate omission since resistivity is a dissipative process and hence will be associated with entropy variations constrained by the second law of thermodynamics. This effect turns out to be quadratic in the relative velocities, which is why we got away with neglecting it in the linear model. In a more general setting we need to account for the induced heat flow. The problem of heat in relativity is, however, known to be thorny. A model needs to be constructed carefully in order to avoid unwanted instabilities and causality violation [18–21]. As recently demonstrated, one can construct a satisfactory model by treating the entropy as an additional fluid component [56], accounting for entrainment between the entropy and the other components in the system [23,24]. This entropy entrainment is closely associated with the inertia of heat and the finite thermal relaxation time scale that is required in order to avoid superluminal signal propagation. We develop our model with these key points in mind.

A. Setting the stage: A three-fluid system

We consider a hot plasma consisting of the two charged components from the cold model, labeled p and e as before, and an additional entropy, which we label s. As the entropy plays a special role, being constrained by the second law, we single out this component by letting its flux be given by $s^a = n^a_{\rm s}$ while the corresponding chemical potential is $\Theta_a = \mu^{\rm s}_a$. The latter determines the temperature measured by a given observer. With these definitions we have the total stress-energy tensor [12]

$$T_{ab} = \Psi g_{ab} + n^{\rm p}_a \mu^{\rm p}_b + n^{\rm e}_a \mu^{\rm e}_b + s_a \Theta_b, \tag{114}$$

where the generalized pressure Ψ is defined as

$$\Psi = \Lambda - n^a_{\rm p}\mu^{\rm p}_a - n^a_{\rm e}\mu^{\rm e}_e - s^a \Theta_a. \tag{115}$$

Following [23,24] we account for entrainment between the entropy and each of the material components (encoded in coefficients $\mathcal{A}^{\rm xs}$). Thus, we have the momenta

$$\mu^{\rm x}_a = \mathcal{B}^{\rm x} n^{\rm x}_a + \mathcal{A}^{\rm xs} s_a, \qquad {\rm x} = {\rm p}, {\rm e}, \tag{116}$$

and

$$\Theta_a = \mathcal{B}^{\rm s} s_a + \mathcal{A}^{\rm ps} n^{\rm p}_a + \mathcal{A}^{\rm es} n^{\rm e}_a. \tag{117}$$

As in the low-velocity model, we introduce a family of observers that allow us to define the electric and magnetic field components. We now have a number of different options. The strategy that we adopt provides a natural extension of the Eckart frame for a single component matter model, cf. [23,24]. To be specific, we choose the observer frame to be such that the only relative momentum flow is due to the heat.

Defining first of all the number densities as measured in the respective fluid frames, we have [57]

$$\hat{n}_{\rm x}^2 = -n_{\rm x}^a n^{\rm x}_a, \quad \text{and} \quad \hat{s}^2 = -s^a s_a. \tag{118}$$

Decomposing the velocities with respect to a specific observer moving with u^a we then have

$$\begin{aligned} n_{\rm x}^a &= \hat{n}_{\rm x}\gamma_{\rm x}(u^a + v_{\rm x}^a), \\ u^a v^{\rm x}_a &= 0, \\ \gamma_{\rm x} &= (1 - v_{\rm x}^2)^{-1/2}, \end{aligned} \tag{119}$$

leading to the number density measured by the observer being given by

$$n_{\rm x} = -u_a n_{\rm x}^a = \hat{n}_{\rm x}\gamma_{\rm x}. \tag{120}$$

Similarly, we have

$$s^a = \hat{s}\gamma_{\rm s}(u^a + v_{\rm s}^a), \qquad u^a v^{\rm s}_a = 0, \qquad \gamma_{\rm s} = (1 - v_{\rm s}^2)^{-1/2}, \tag{121}$$

and

$$s = -u_a a^a = \hat{s}\gamma_{\rm s}. \tag{122}$$

It is also natural to introduce the chemical potentials inferred by the observer

$$\mu_{\rm x} = -u^a \mu^{\rm x}_a = n_{\rm x}\mathcal{B}^{\rm x} + s\mathcal{A}^{\rm xs} \tag{123}$$

and

$$\Theta = s\mathcal{B}^{\rm s} + n_{\rm p}\mathcal{A}^{\rm ps} + n_{\rm e}\mathcal{A}^{\rm es}. \tag{124}$$

With these definitions it is straightforward to show that the total energy density measured by the observer will be

$$\rho = u^a u^b T_{ab} = -\Psi + n_{\rm p}\mu_{\rm p} + n_{\rm e}\mu_{\rm e} + s\Theta, \tag{125}$$

corresponding to the (integrated) first law of thermodynamics once we identify Ψ as the generalized pressure and Θ as the temperature. Meanwhile, the momentum flux relative to the observer's frame is given by

$$u^a T_{ab} = -\rho u_b - n_{\rm p}\mu_{\rm p} v_b^{\rm p} - n_{\rm e}\mu_{\rm e} v_b^{\rm e} - s\Theta v_b^{\rm s}. \tag{126}$$

Here it is, first of all, natural to identify the heat flux as [23,24]

$$q^a = s\Theta v_{\rm s}^a. \tag{127}$$

We also see that we can choose the observer frame in such a way that this is the only relative momentum flux. To do this, we let

$$(\rho + \Psi)v^a = n_{\rm p}\mu_{\rm p} v_{\rm p}^a + n_{\rm e}\mu_{\rm e} v_{\rm e}^a = 0. \tag{128}$$

This is the natural extension of the " center of mass" frame we used in the low-velocity model, cf. Eq. (55). We also define the velocity difference (as before)

$$w^a = v_{\rm p}^a - v_{\rm e}^a. \tag{129}$$

In the frame where $v^a = 0$ (which will be assumed from now on) we have

$$v_{\rm p}^a = \frac{n_{\rm e}\mu_{\rm e}}{\rho + \Psi} w^a, \tag{130}$$

and

$$v_{\rm e}^a = -\frac{n_{\rm p}\mu_{\rm p}}{\rho + \Psi} w^a, \tag{131}$$

which means that the charge current can be written

$$j^a = e(n_{\rm p} - n_{\rm e})u^a + e\frac{n_{\rm p}n_{\rm e}}{\rho + \Psi}(\mu_{\rm p} + \mu_{\rm e})w^a. \tag{132}$$

At this point we note an important difference with respect to the low-velocity discussion. While we were naturally led to the assumption of charge neutrality in that case, the situation is much less clear now. This is immediately obvious from (132) once we recall that the densities in the first term are measured by the chosen observer, not in the respective rest frames. Hence, it would not be appropriate to assume that $n_{\rm p} = n_{\rm e}$ at this point.

B. Friction and causal heat flow

Having introduced the various ingredients, let us move on to the new aspect of the problem; the equation that governs the heat propagation. As we are treating the entropy as an additional fluid, it follows from the general analysis in [12] that the thermal dynamics will be governed by its own momentum equation. We already know from [23,24] that this will lead to an equation that contains the thermal relaxation that is required to ensure causality. However, as the entropy need not be conserved this momentum equation takes a slightly different form from those that govern the (individually conserved) material components. We have [23,24]

$$2s^a\nabla_{[a}\Theta_{b]} + \Theta_b\Gamma_{\rm s} = \tilde{f}^{\rm s}_b, \tag{133}$$

where

$$\nabla_a s^a = \Gamma_{\rm s} \geq 0, \tag{134}$$

in accordance with the second law.

Building on the analysis of the low-velocity case, we know that the overall conservation of energy and momentum requires

$$\sum_{\rm x}\tilde{f}^a_{\rm x} = 0 \rightarrow \tilde{f}^a_{\rm s} = -\tilde{f}^a_{\rm p} - \tilde{f}^a_{\rm e}. \tag{135}$$

The form of the force that acts on the entropy thus follows immediately from the forces on the charged components. Extending the low-velocity model, we will allow for resistivity due to scattering between both charged species ($\mathcal{R}$) and entropy ($\mathcal{S}_{\rm x}$). Thus, we let the forces take the (still phenomenological) form

$$\tilde{f}^{\,a}_{\rm p} = \perp^{ab}_{\rm p} (e\mathcal{R} n^{\rm e}_b + \mathcal{S}_{\rm p} u^{\rm s}_b), \tag{136}$$

and

$$\tilde{f}^{\,a}_{\rm e} = \perp^{ab}_{\rm e} (e\mathcal{R} n^{\rm p}_b + \mathcal{S}_{\rm e} u^{\rm s}_b), \tag{137}$$

where we have used $s^a = \hat{s} u^a_{\rm s}$. Combining these expressions and expanding the projections, we arrive at

$$\begin{aligned}\tilde{f}^a_{\rm s} &= -e\mathcal{R}[n_{\rm e} - n_{\rm p}\gamma^2_{\rm e}(1 - v^b_{\rm p} v^{\rm e}_b)](u^a + v^a_{\rm e}) \\ &\quad - e\mathcal{R}[n_{\rm p} - n_{\rm e}\gamma^2_{\rm p}(1 - v^b_{\rm p} v^{\rm e}_b)](u^a + v^a_{\rm p}) \\ &\quad - \mathcal{S}_{\rm p}[u^a_{\rm s} - \gamma_{\rm s}\gamma_{\rm p} u^a_{\rm p}(1 - v^b_{\rm p} v^{\rm s}_b)] \\ &\quad - \mathcal{S}_{\rm e}[u^a_{\rm s} - \gamma_{\rm s}\gamma_{\rm e} u^a_{\rm e}(1 - v^b_{\rm e} v^{\rm s}_b)].\end{aligned} \tag{138}$$

Let us now return to (133), focusing on the entropy creation rate. Contracting the equation with the observer's four-velocity we easily arrive at

$$\Theta\Gamma_{\rm s} = -u^b \tilde{f}^{\rm s}_b + 2 s u^b v^a_{\rm s} \nabla_{[a}\Theta_{b]}. \tag{139}$$

We need to constrain this in such a way that the right-hand side is non-negative. To do this, we first need the contraction between the entropy force and the four-velocity. This leads to

$$\begin{aligned}-u^b \tilde{f}^{\rm s}_b &= \frac{e\mathcal{R}}{\rho + \Psi}(n^2_{\rm e}\mu_{\rm e}\gamma^2_{\rm p} + n^2_{\rm p}\mu_{\rm p}\gamma^2_{\rm e})w^2 \\ &\quad + \frac{\gamma_s}{(\rho + \Psi)^2}[\mathcal{S}_{\rm p}\gamma^2_{\rm p}(n_{\rm e}\mu_{\rm e})^2 + \mathcal{S}_{\rm e}\gamma^2_{\rm e}(n_{\rm p}\mu_{\rm p})^2]w^2 \\ &\quad + \frac{\gamma_{\rm s}}{\rho + \Psi}(\mathcal{S}_{\rm e}\gamma^2_{\rm e} n_{\rm p}\mu_{\rm p} - \mathcal{S}_{\rm p}\gamma^2_{\rm p} n_{\rm e}\mu_{\rm e})\frac{w^b q_b}{s\Theta}.\end{aligned} \tag{140}$$

In this expression, the first two terms on the right-hand side will be positive as long as $\mathcal{R} \geq 0$ and $\mathcal{S}_{\rm x} \geq 0$. The sign of the third term is not so clear.

Moving on to the final term in (139), we first of all note that it is proportional to $v^a_{\rm s}$ (in turn proportional to q^a). Defining

$$\beta_1 = \frac{1}{s\Theta}(\Theta - n_{\rm p}\mathcal{A}^{\rm ps} - n_{\rm e}\mathcal{A}^{\rm es}) \tag{141}$$

and

$$\beta_2 = \frac{n_{\rm p} n_{\rm e}}{\rho + \Psi}(\mu_{\rm e}\mathcal{A}^{\rm ps} - \mu_{\rm p}\mathcal{A}^{\rm es}), \tag{142}$$

we have

$$\Theta_a = \Theta u_a + \beta_1 q_a + \beta_2 w_a, \tag{143}$$

and we find that

$$\begin{aligned}2u^b\nabla_{[a}\Theta_{b]} &= -\perp^b_a \nabla_b\Theta - \Theta\dot{u}_a - \beta_1\dot{q}_a - \beta_2\dot{w}_a - \dot{\beta}_1 q_a \\ &\quad - \dot{\beta}_2 w_a - (\beta_1 q^b + \beta_2 w^b)\nabla_a u_b.\end{aligned} \tag{144}$$

Combining (140) and (144) we see that $\Gamma_{\rm s}$ satisfies the required constraint provided that [58]

$$\begin{aligned}&\kappa\beta_1\dot{q}_a + (1 + \kappa\dot{\beta}_1)q_a \\ &\quad = -\kappa\Big[\perp^b_a \nabla_b\Theta + \Theta\dot{u}_a + \beta_2\dot{w}_a + \dot{\beta}_2 w_a \\ &\qquad + (\beta_1 q^b + \beta_2 w^b)\nabla_a u_b \\ &\qquad - \frac{\gamma_{\rm s}}{s(\rho + \Psi)}(\mathcal{S}_{\rm e}\gamma^2_{\rm e} n_{\rm p}\mu_{\rm p} - \mathcal{S}_{\rm p}\gamma^2_{\rm p} n_{\rm e}\mu_{\rm e})w_a\Big],\end{aligned} \tag{145}$$

with $\kappa \geq 0$. This has the form of a Cattaneo-type equation [23], and a comparison of the q_a and $\dot{q}_a$ terms suggest that the thermal relaxation time is

$$\tau = \frac{\kappa\beta_1}{1 + \kappa\dot{\beta}_1}. \tag{146}$$

However, the equation is also coupled to the four-acceleration $\dot{u}^a$ and the variation of the charge current, in terms of $\dot{w}^a$, so if we want to infer the actual relaxation times in the problem we need to consider the coupled system.

Combining the relevant contributions, we find that the total entropy creation rate is given by

$$\begin{aligned}\Gamma_{\rm s} &= \frac{1}{\Theta}\Big[\frac{q^2}{\kappa\Theta} + \frac{e\mathcal{R}}{\rho + \Psi}(n^2_{\rm e}\mu_{\rm e}\gamma^2_{\rm p} + n^2_{\rm p}\mu_{\rm p}\gamma^2_{\rm e})w^2 \\ &\quad + \frac{\gamma_{\rm s}}{(\rho + \Psi)^2}[\mathcal{S}_{\rm p}\gamma^2_{\rm p}(n_{\rm e}\mu_{\rm e})^2 + \mathcal{S}_{\rm e}\gamma^2_{\rm e}(n_{\rm p}\mu_{\rm p})^2]w^2\Big] \geq 0.\end{aligned} \tag{147}$$

C. Ohm's law

The derivation of the generalized form of Ohm's law follows the same steps as in the linear model, although now we need to keep careful track of the different redshift factors, etc. Basically, we want to construct the weighted difference between the momentum equations for the two charged components, but the two momenta now depend also on the entropy flux. In the frame associated with our chosen observer, we have

$$\begin{aligned}\mu^{\rm x}_a &= \mu_{\rm x}(u_a + v^{\rm x}_a) + s\mathcal{A}^{\rm xs}(v^{\rm s}_a - v^{\rm x}_a) \\ &\equiv \mu_{\rm x}(u_a + v^{\rm x}_a) + \mathcal{W}^{\rm x}_a,\end{aligned} \tag{148}$$

where we have introduced the combinations

$$\mathcal{W}^{\rm p}_a = s\mathcal{A}^{\rm ps}\Big(\frac{q_a}{s\Theta} - \frac{n_{\rm e}\mu_{\rm e}}{\rho + \Psi}w_a\Big), \tag{149}$$

and

$$\mathcal{W}^{\rm e}_a = s\mathcal{A}^{\rm es}\Big(\frac{q_a}{s\Theta} + \frac{n_{\rm p}\mu_{\rm p}}{\rho + \Psi}w_a\Big), \tag{150}$$

(obviously expressed in the chosen frame).

Given these expressions it follows that

$$\tilde{f}_b^{\rm x} = 2n_{\rm x}^a \nabla_{[a}\mu^{\rm x}_{b]} = n_{\rm x}\mu_{\rm x}(\dot{u}_b + \dot{v}_b^{\rm x} + v_{\rm x}^a \nabla_b u_a) + n_{\rm x} \perp_b^a \nabla_a \mu_{\rm x} + n_{\rm x} u^a v_b^{\rm x} \nabla_a \mu_{\rm x} + 2 n_{\rm x} u^a \nabla_{[a} \mathcal{W}^{\rm x}_{b]} + 2 n_{\rm x} v_{\rm x}^a \nabla_{[a}\mu^{\rm x}_{b]}. \quad (151)$$

The weighted difference equation (inevitably rather complicated) combines three pieces. On the left-hand side we have

$$f_b^{\rm D} = \frac{1}{n_{\rm p}\mu_{\rm p}}\tilde{f}_b^p - \frac{1}{n_{\rm e}\mu_{\rm e}}\tilde{f}_b^e = \dot{w}_b + w^a\nabla_b u_a + \perp_b^a \left(\frac{1}{\mu_{\rm p}}\nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right) + u^a\left(\frac{1}{\mu_{\rm p}} v_{\rm p}^b \nabla_a \mu_{\rm p} - \frac{1}{\mu_{\rm e}} v_{\rm e}^b \nabla_a\mu_{\rm e}\right)$$
$$+ \frac{2}{\mu_{\rm p}} u^a \nabla_{[a}\mathcal{W}^{\rm p}_{b]} - \frac{2}{\mu_{\rm e}} u^a\nabla_{[a}\mathcal{W}^{\rm e}_{b]} + \frac{2}{\mu_{\rm p}} v_{\rm p}^a \nabla_{[a}\mu^{\rm p}_{b]} - \frac{2}{\mu_{\rm e}} v_{\rm e}^a\nabla_{[a}\mu^{\rm e}_{b]}. \quad (152)$$

In the frame associated with the chosen observer, this expression takes the form [59]

$$f_b^{\rm D} = \dot{w}_b + w^a\nabla_a u_b + \perp_b^a \left(\frac{1}{\mu_{\rm p}}\nabla_a\mu_{\rm p} - \frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right) + \frac{1}{\mu_{\rm p}\mu_{\rm e}(\rho+\Psi)} 2u_{(a}w_{b)}[n_{\rm e}\mu_{\rm e}^2\nabla^a\mu_{\rm p} - n_{\rm p}\mu_{\rm p}^2\nabla^a\mu_{\rm e}]$$
$$- \perp_{ab}^w \left[\left(\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\right)^2 \frac{1}{\mu_{\rm p}}\nabla^a\mu_{\rm p} - \left(\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\right)^2\frac{1}{\mu_{\rm e}}\nabla_a\mu_{\rm e}\right] + u^a\left[2\nabla_{[a}\left(\frac{\beta_4}{s\Theta}q_{b]}\right) - 2\nabla_{[a}(\beta_3 w_{b]}) - \mathcal{W}_b^{\rm p}\nabla_a\left(\frac{1}{\mu_{\rm p}}\right)\right.$$
$$\left. + \mathcal{W}_b^{\rm e}\nabla_a\left(\frac{1}{\mu_{\rm e}}\right)\right] + 2\mathcal{D}w^a\nabla_{[a}w_{b]} - 2\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi} w^a w_{[b}\nabla_{a]}\left(\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\right) + 2\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi} w^a w_{[b}\nabla_{a]}\left(\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\right)$$
$$+ 2\frac{n_{\rm e}\mu_{\rm e}}{\mu_{\rm p}(\rho+\Psi)} w^a\nabla_{[a}\mathcal{W}^{\rm p}_{b]} - 2\frac{n_{\rm p}\mu_{\rm p}}{\mu_{\rm e}(\rho+\Psi)} w^a\nabla_{[a}\mathcal{W}^{\rm e}_{b]}. \quad (153)$$

Here we have introduced

$$\beta_3 = \frac{s}{\mu_{\rm e}\mu_{\rm p}(\rho+\Psi)}(n_{\rm e}\mu_e^2\mathcal{A}^{\rm ps} + n_{\rm p}\mu_{\rm p}^2\mathcal{A}^{\rm es}) = \frac{s}{n_{\rm p}n_{\rm e}}\left[\frac{\Theta}{\rho+\Psi}(s\beta_1 - 1) + \frac{n_{\rm e}\mu_{\rm e} - n_{\rm p}\mu_{\rm p}}{n_{\rm p}\mu_{\rm p}n_{\rm e}\mu_e}\beta_2\right], \quad (154)$$

$$\beta_4 = \frac{s(\rho+\Psi)}{n_{\rm e}\mu_{\rm e}n_{\rm p}\mu_{\rm p}}\beta_2. \quad (155)$$

and

$$\mathcal{D} = \left(\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\right)^2 - \left(\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\right)^2 \quad (156)$$

We have also used the projection orthogonal to w^a, given by

$$\perp_{ab}^w = w^2 g_{ab} - w_a w_b. \quad (157)$$

Meanwhile, the right-hand side is made up of, first of all, the combined friction forces

$$f_b^{\rm R} = \frac{1}{n_{\rm p}\mu_{\rm p}} \perp^{\rm p}_{ab} (e\mathcal{R}n_{\rm e}^a + \mathcal{S}_{\rm p}u_{\rm s}^a) - \frac{1}{n_{\rm e}\mu_{\rm e}}\perp^{\rm e}_{ab}(e\mathcal{R}n_{\rm p}^a + \mathcal{S}_{\rm e}u_{\rm s}^a)$$
$$= \frac{e\mathcal{R}}{n_{\rm p}\mu_{\rm p}n_{\rm e}\mu_{\rm e}}\left[\frac{1}{\rho+\Psi}(n_{\rm p}^3\mu_{\rm p}^2\gamma_{\rm e}^2 - n_{\rm e}^3\mu_{\rm e}^2\gamma_{\rm p}^2)w^2u_b - (n_{\rm e}^2\mu_{\rm e}\gamma_{\rm p}^2 + n_{\rm p}^2\mu_{\rm p}\gamma_{\rm e}^2)\left(1 - \frac{s\Theta}{\rho+\Psi}\right)w_b\right]$$
$$+ \mathcal{S}_{\rm p}\frac{\gamma_{\rm s}\gamma_{\rm p}^2}{n_{\rm p}\mu_{\rm p}}\left\{\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\left(\frac{w^aq_a}{s\Theta} - \frac{n_{\rm e}\mu_{\rm e}w^2}{\rho+\Psi}\right)u_b + \left[1 - \left(\frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\right)^2 w^2\right]\frac{q_b}{s\Theta} - \frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\left(1 - \frac{n_{\rm e}\mu_{\rm e}}{\rho+\Psi}\frac{w^aq_a}{s\Theta}\right)w_b\right\}$$
$$+ \mathcal{S}_{\rm e}\frac{\gamma_{\rm s}\gamma_{\rm e}^2}{n_{\rm e}\mu_{\rm e}}\left\{\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\left(\frac{w^aq_a}{s\Theta} + \frac{n_{\rm p}\mu_{\rm p}w^2}{\rho+\Psi}\right)u_b - \left[1 - \left(\frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\right)^2 w^2\right]\frac{q_b}{s\Theta} - \frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\left(1 + \frac{n_{\rm p}\mu_{\rm p}}{\rho+\Psi}\frac{w^aq_a}{s\Theta}\right)w_b\right\}, \quad (158)$$

where the second equality holds in the frame where $v^a = 0$.

The final part accounts for the electromagnetic field. We need

$$f_b^{\mathrm{M}} = \frac{1}{n_{\mathrm{p}}\mu_{\mathrm{p}}} e n_{\mathrm{p}}^a F_{ba} + \frac{1}{n_{\mathrm{e}}\mu_{\mathrm{e}}} e n_{\mathrm{e}}^a F_{ba} = e\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}} u^a F_{ba} + e\left(\frac{1}{\mu_p} v_{\mathrm{p}}^a + \frac{1}{\mu_{\mathrm{e}}} v_{\mathrm{e}}^a\right) F_{ba}$$
$$= e\left(\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}\right) E_b + e\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}^2 - n_{\mathrm{p}}\mu_{\mathrm{p}}^2}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}(\rho+\Psi)}[u_b(w^a E_a) + \epsilon_{bac} w^a B^c]. \tag{159}$$

Again the second equality holds only in the chosen frame.

The final relation follows from the combination (78) after projecting out the component orthogonal to u^a. The result is (obviously) rather complex and may not be particularly instructive. Yet, we provide it in the interest of completeness. The generalized form for Ohm's law for a hot two-component plasma can be written (expressed in terms of w^a rather than the spatial component of the charge current J^a, for convenience)

$$\dot{w}_b + w^a\nabla_a u_b + \perp_b^a\left(\frac{1}{\mu_{\mathrm{p}}}\nabla_a\mu_{\mathrm{p}} - \frac{1}{\mu_{\mathrm{e}}}\nabla_a\mu_{\mathrm{e}}\right) + \frac{1}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}(\rho+\Psi)} w_b u^a [n_{\mathrm{e}}\mu_{\mathrm{e}}^2\nabla_a\mu_{\mathrm{p}} + n_{\mathrm{p}}\mu_{\mathrm{p}}^2\nabla_a\mu_{\mathrm{e}}]$$
$$- (w^2\perp_b^a - w^a w_b)\left[\left(\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\rho+\Psi}\right)^2\frac{1}{\mu_{\mathrm{p}}}\nabla_a\mu_{\mathrm{p}} - \left(\frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\right)^2\frac{1}{\mu_{\mathrm{e}}}\nabla_a\mu_{\mathrm{e}}\right] + u^a\left[2\nabla_{[a}\mathcal{Q}_{b]} - \mathcal{W}_b^{\mathrm{p}}\nabla_a\left(\frac{1}{\mu_{\mathrm{p}}}\right) + \mathcal{W}_b^{\mathrm{e}}\nabla_a\left(\frac{1}{\mu_{\mathrm{e}}}\right)\right]$$
$$+ \perp_b^c[\mathcal{D}w^a\nabla_a w_c - \nabla_c(\mathcal{D}w^2)] - \frac{1}{2}w^a w_b\nabla_a\mathcal{D} + \frac{2}{\rho+\Psi}\perp_b^c\left(\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}} w^a\nabla_{[a}\mathcal{W}_{c]}^{\mathrm{p}} - \frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\mu_{\mathrm{e}}} w^a\nabla_{[a}W_{c]}^{\mathrm{e}}\right)$$
$$= -\frac{e\mathcal{R}}{n_{\mathrm{p}}\mu_{\mathrm{p}}n_{\mathrm{e}}\mu_{\mathrm{e}}}(n_{\mathrm{e}}^2\mu_{\mathrm{e}}\gamma_{\mathrm{p}}^2 + n_{\mathrm{p}}^2\mu_{\mathrm{p}}\gamma_{\mathrm{e}}^2)\left(1 - \frac{s\Theta}{\rho+\Psi}\right)w_b + \mathcal{S}_{\mathrm{p}}\frac{\gamma_{\mathrm{s}}\gamma_{\mathrm{p}}^2}{n_{\mathrm{p}}\mu_{\mathrm{p}}}\left\{\left[1 - \left(\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\rho+\Psi}\right)^2 w^2\right]\frac{q_b}{s\Theta}\right.$$
$$\left. - \frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\rho+\Psi}\left(1 - \frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\rho+\Psi}\frac{w^a q_a}{s\Theta}\right)w_b\right\} - \mathcal{S}_{\mathrm{e}}\frac{\gamma_{\mathrm{s}}\gamma_{\mathrm{e}}^2}{n_{\mathrm{e}}\mu_{\mathrm{e}}}\left\{\left[1 - \left(\frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\right)^2 w^2\right]\frac{q_b}{s\Theta} + \frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\left(1 + \frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\frac{w^a q_a}{s\Theta}\right)w_b\right\}$$
$$+ e\left(\frac{\mu_{\mathrm{p}}+\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}}\right)E_b + e\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}^2 - n_{\mathrm{p}}\mu_{\mathrm{p}}^2}{\mu_{\mathrm{p}}\mu_{\mathrm{e}}(\rho+\Psi)}\epsilon_{bac}w^a B^c, \tag{160}$$

where we have defined

$$\mathcal{Q}_b = \frac{\beta_4}{s\Theta} q_b - \beta_3 w_b. \tag{161}$$

D. The total momentum equation

As in the two-component system, the model is completed by the total momentum equation, which follows (more or less) immediately from the divergence of the stress-energy tensor. In the frame moving with u^a (where $v^a = 0$), the matter stress-energy tensor takes the form

$$T_{ab}^{\mathrm{M}} = \rho u_a u_b + \perp_{ab}\Psi + 2u_{(a}q_{b)} + \alpha w_a w_b + \frac{2\beta_2}{\Theta} w_{(a}q_{b)} + \frac{\beta_1}{\Theta} q_a q_b, \tag{162}$$

with

$$\alpha = \frac{n_{\mathrm{e}}\mu_{\mathrm{e}}n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\left(1 - \frac{s\Theta}{\rho+\Psi}\right) - \left(\frac{n_{\mathrm{e}}\mu_{\mathrm{e}}}{\rho+\Psi}\right)^2 s n_{\mathrm{p}}\mathcal{A}^{\mathrm{ps}} - \left(\frac{n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\right)^2 s n_{\mathrm{e}}\mathcal{A}^{\mathrm{es}}$$
$$= \frac{n_{\mathrm{e}}\mu_{\mathrm{e}}n_{\mathrm{p}}\mu_{\mathrm{p}}}{\rho+\Psi}\left(1 - \frac{s^2\beta_1\Theta}{\rho+\Psi}\right) - \frac{s(n_{\mathrm{e}}\mu_{\mathrm{e}} - n_{\mathrm{p}}\mu_{\mathrm{p}})}{\rho+\Psi}\beta_2. \tag{163}$$

As usual, $\nabla_a T^{ab} = 0$ can be divided into a component along the four-velocity and an orthogonal piece. After a bit of algebra, recalling that the electromagnetic contribution is given by Eq. (28), we find that the former can be written

$$\dot{\rho} + (\rho+\Psi)\nabla_a u^a + \nabla_a q^a - u_b\left[\dot{q}^b + \alpha w^a\nabla_a w^b + \frac{\beta_2}{\Theta}(w^a\nabla_a q^b + q^a\nabla_a w^b) + \frac{\beta_1}{\Theta} q^a\nabla_a q^b\right]$$
$$= e\frac{n_{\mathrm{p}}n_{\mathrm{e}}}{\rho+\Psi}(\mu_{\mathrm{p}}+\mu_{\mathrm{e}})(w_a E^a). \tag{164}$$

Meanwhile, the orthogonal projection leads to the momentum equation

$$(\rho+\Psi)\dot{u}^b + \perp^{ab}\nabla_a\Psi + q^a\nabla_a u^b + q^b\nabla_a(u^a + V_1^a) + \perp_c^b(\dot{q}^c + V_2^a\nabla_a w^c + V_1^a\nabla_a q^c) + w^b\nabla_a V_2^a$$
$$= e(n_{\mathrm{p}} - n_{\mathrm{e}})E^b + e\frac{n_{\mathrm{p}}n_{\mathrm{e}}}{\rho+\Psi}(\mu_{\mathrm{p}}+\mu_{\mathrm{e}})\epsilon^{bac}w_a B_c, \tag{165}$$

where we have defined

$$V_1^a = \frac{1}{\Theta}(\beta_2 w^a + \beta_1 q^a), \tag{166}$$

and

$$V_2^a = \alpha w^a + \frac{\beta_2}{\Theta} q^a. \tag{167}$$

E Linearized model

The final equations for the coupled three-component model are obviously rather complex. This is not surprising since, apart from working in a specific observer frame, we did not make any simplifications. Hence, the model is quite general, including the relevant nonlinearities and redshift factors. The main take-home message should be that the steps involved in the derivation are natural and intuitive, but the expressions involved will be messy. One may query the immediate usefulness of the analysis, as it takes us far beyond what is currently considered in applications. However, the argument on behalf of the defense is clear. Once we have worked our way through the general case it is relatively straightforward to reduce the complexity by considering specific models. This is, in fact, a valuable exercise as it provides a clearer insight into the key features of the hot system.

A natural, and in many cases of interest reasonable, assumption is that we only need to retain the linear relative velocities. As in the cold model, we neglect higher-order terms in w^a and q^a, and we also ignore all the redshift factors by taking $\gamma_x \approx 1$. It then follows that the pressure is $\Psi = P$ and the temperature is $\Theta = T$. In the interest of clarity we will also ignore the resistive scattering between entropy (phonons) and the material components, i.e. we set $\mathcal{S}_x = 0$. These assumptions lead to, i) the heat equation

$$\kappa\beta_1\dot{q}_a + (1+\kappa\dot{\beta}_1)q_a = -\kappa[\perp^a_b \nabla_b T + T\dot{u}_a + \beta_2\dot{w}_a + \dot{\beta}_2 w_a + (\beta_1 q^b + \beta_2 w^b)\nabla_a u_b], \tag{168}$$

ii) the generalized version of Ohm's law

$$e\left(\frac{\mu_p+\mu_e}{\mu_p\mu_e}\right)E_b - e\frac{n_p\mu_p^2 - n_e\mu_e^2}{\mu_p\mu_e(P+\rho)}\epsilon_{bac}w^a B^c - \frac{e\mathcal{R}}{n_p\mu_p n_e\mu_e}(n_e^2\mu_e + n_p^2\mu_p)\left(1-\frac{sT}{P+\rho}\right)w_b$$
$$= \dot{w}_b + w^a\nabla_a u_b + \perp^a_b\left(\frac{1}{\mu_p}\nabla_a\mu_p - \frac{1}{\mu_e}\nabla_a\mu_e\right) + \frac{1}{\mu_p\mu_e(P+\rho)}w_b u^a[n_e\mu_e^2\nabla_a\mu_p + n_p\mu_p^2\nabla_a\mu_e]$$
$$+ u^a\left[2\nabla_{[a}Q_{b]} - \mathcal{W}^p_b\nabla_a\left(\frac{1}{\mu_p}\right) + \mathcal{W}^e_b\nabla_a\left(\frac{1}{\mu_e}\right)\right], \tag{169}$$

and, iii) the total momentum conservation equation

$$(P+\rho)\dot{u}^b + \perp^{ab}\nabla_a P + q^a\nabla_a u^b + q^b\nabla_a u^a + \perp^b_c \dot{q}^c = e(n_p - n_e)E^b + e\frac{n_p n_e}{P+\rho}(\mu_p+\mu_e)\epsilon^{bac}w_a B_c. \tag{170}$$

We simplify these relations further by noting that we can reinstate the assumption of charge neutrality, as the issues alluded to after Eq. (132) originate from the redshift factors. Thus we let $n_p = n_e$, which leads to a number of simplifications (the arguments are the same as in the cold case). Focussing on the proton-electron plasma, we also assume that $\mu_e \ll \mu_p$. It follows that the charge current is (again) given by

$$J^a = en_e w^a. \tag{171}$$

We can use this to write (168) in the elegant form

$$q_a = -\kappa(\perp^b_a \nabla_b T + T\dot{u}_a + 2u^b\nabla_{[b}\tilde{Q}_{a]}), \tag{172}$$

where

$$\tilde{Q}_a = \beta_1 q_a + \frac{\beta_2}{en_e}J_a. \tag{173}$$

The thermal relaxation is encoded in this quantity.

Turning to the momentum Eq. (170), we have

$$(P+\rho)\dot{u}^b + \perp^{ab}\nabla_a P + q^a\nabla_a u^b + q^b\nabla_a u^a + \perp^b_c \dot{q}^c = \epsilon^{bac}J_a B_c. \tag{174}$$

Finally, we find that Ohm's law simplifies to [60]

$$E_b - \frac{1}{en_e}\epsilon_{bcd}J^c B^d - \frac{\mathcal{R}}{en_e}J_b = \frac{\mu_e}{e^2 n_e}\left[\perp_{ab}\dot{J}^a + J^a\left(\sigma_{ab} + \omega_{ab} + \frac{4}{3}\theta\perp_{ab}\right)\right] - \frac{1}{e}\perp^a{}_b\nabla_a\mu_e + 2u^a\nabla_{[a}Q_{b]}. \tag{175}$$

At this point, we have stripped the hot plasma model down to the level where it is easy to compare the final expressions to those of the cold model. At the linear level, the only difference is the presence of couplings that arise due to the entropy entrainment (expressed in terms of the different β coefficients) and the explicit presence of the heat flux q^a in the momentum equation. These may seem like minor adjustments, but they are significant. In particular, we need to retain the relevant relaxation times in order to ensure that the model is causal.

Of course, the main differences between the two models we have developed enters at the nonlinear level (in the relative velocities). At quadratic order, the problem is nonadiabatic (as $\Gamma_s \neq 0$) and it is no longer natural to assume charge neutrality. Given these effects, it would be very interesting to study a quadratic model in more detail. However, the corresponding problem is somewhat involved so we prefer to postpone discussion of it for the future.

The model is completed by three scalar relations (whose origin are the conservation laws for the fluxes). In the linear case, these take the simple form

$$\dot{\rho} + (P + \rho)\theta + \nabla_a q^a - u_b \dot{q}^b = 0, \tag{176}$$

$$\dot{s} + s\theta + \nabla_a\left(\frac{q^a}{T}\right) = 0, \tag{177}$$

and

$$\nabla_a J^a = 0, \tag{178}$$

which can be replaced by

$$\dot{n}_e + n_e\theta = 0. \tag{179}$$

V. CONCLUDING REMARKS

We have developed the theory for charged fluids coupled to an electromagnetic field in the framework of general relativity, accounting for both a phenomenological resistivity and the relaxation times (associated with the charge current and the heat flux) that are required to ensure causality. The final formalism can be applied to a range of interesting problems in astrophysics and cosmology. The cold two-component plasma model (from Sec. III) extends the ideal magnetohydrodynamics framework in several directions, and the hot model (from Sec. IV) adds dimensions that come into play when thermal aspects of the problem cannot be neglected. These developments are important as a number of interesting problems may require "nonideal" aspects for their solution. Of most obvious relevance are problems involving not only electromagnetic fields but the live spacetime of general relativity. Several key gravitational-wave sources come to mind, like core-collapse supernovae [61] and compact binary mergers [62,63]. Both cases involve strong gravity, a significant thermal component and magnetic fields. To apply a resistive framework to these problems is, of course, seriously challenging but this does not mean that we should not have aspirations in this direction [46–48]. Actual multifluid simulations [64] are also of obvious relevance.

Focusing on relativistic stars, one can think of a number of unresolved problems, ranging from the dynamics of the magnetosphere and the pulsar emission mechanism to the formation and evolution of the star's interior magnetic field. These are problems where there has been significant progress, but further effort is required. In the case of the magnetosphere, the main focus has been on force-free models, but recent arguments [65] point to the need to include resistivity in the discussion. In the case of the formation and evolution of a compact star's global magnetic field, we need a better understanding of dynamo effects that may come into operation (see [1] and also [66] for a recent review) and we also need to understand the coupled evolution of the star's spin, temperature and magnetic field [67]. There are some very difficult issues to resolve here.

In fact, the suggested examples highlight the need to develop the theory further. Typical questions that would need to be addressed involve (i) the dynamics predicted by the model, e.g. causality and stability of wave propagation and relation to issues like pulsar emission or the launch of outflows and jets, (ii) transitions between spatial regions where different simplifying assumptions are valid, such as a region in the magnetosphere where the fluid model applies and a low-density region where the description breaks down and one would need to fall back on a kinetic theory model [32,68,69], the transition from magnetosphere to interior field at the star's surface or, indeed, accreting systems where an ion-electron plasma describes the inflowing matter while regions in the magnetosphere may still be appropriately modeled as a pair plasma, (iii) the role of more complex physics, like the superconductor that is expected to be present in the star's core [70] or regions where the assumption that the medium is electromagnetically "passive" does not apply, possibly in the pasta region near the crust-core transition. The present work provides a foundation for developments in all these directions, but each problem is associated with specific challenges that will need to be addressed if we want to make further progress.

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[31] A significant part of the literature on astrophysical magnetic fields has focussed on fixed spacetimes, e.g. in the context of black-hole or neutron-star magnetospheres or jet dynamics. In the present discussion we do not assume that the spacetime dynamics is frozen, even though we do not explicitly discuss the Einstein field equations and the metric degrees of freedom. The aim is to keep the discussion at a sufficiently general level that it can be applied to problems where the live gravitational field plays a relevant role.
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[40] This "center-of-mass" frame provides the natural relativistic extension of the standard plasma physics analysis [13,14]. A common alternative strategy [17] is to use a frame in which there is no net particle flux, i.e. where

$$n_p v_p^a + n_e v_e^a = 0.$$

The final results should be the same in the two cases, but as is evidenced by the relatively straightforward route to the final result, our chosen strategy is the most natural. It is also readily extended to the hot case, where heat flux is added to the problem, as in Sec. IV.

[41] The tildes are added as a reminder that the friction forces are phenomenological. In principle, one can add more complicated dissipation terms that may, or may not, be important in a practical situation. We will consider the general problem elsewhere.

[42] It is worth being a little bit more precise here. In general, one can extract the rest-mass contribution (m_x) to the chemical potential to get

$$\mu_x = m_x c^2 + \mu_x^N.$$

At low velocities, where the second term can be neglected, it is obviously the case that $\mu_e \ll \mu_p$ for proton-electron plasmas. It is also clear that we have no reasons to assume that the gradients

$$\nabla_a \mu_x = \nabla_a \mu_x^N$$

obey a similar ordering.

[43] Note different conventions here, with some work defining $\epsilon_{bcd} = \epsilon_{bcda} u^a$ together with the opposite sign of the magnetic field, see e.g. [6].

[44] Note that we define the vorticity tensor to have the opposite sign compared to [6]. This is obviously just convention, but it is important to keep it in mind if one wants to compare the various final relations.

[45] We have neglected a term,

$$(\dot{\mu}/\mu) J_b \approx (\dot{\mu}_e^N / m_e c^2) J_b,$$

in the bracket on the right-hand side. This should be a valid approximation at low velocities. We have also used

$$(\dot{n}_e / n_e) J_b \approx -\theta_b,$$

which is true at the linear (in the current) level since $\nabla_a n_e^a = 0$.

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[57] In order to avoid the notation becoming unnecessarily cluttered we opt to distinguish quantities measured in the various rest frames by hats. This means that the notation differs from that used in, say, [12], which is unfortunate. However, the main focus will be on quantities measured by the selected observer. As these quantities appear much more frequently than the rest-frame components, the chosen convention is more convenient.

[58] The strategy is to construct terms that are explicitly positive definite, e.g. quadratic in the relevant fluxes. The procedure is not necessarily unique, but the combinations that we have opted to use seem natural.

[59] For reasons of clarity we will not expand the terms that are quadratic in the relative velocities. It is straightforward to do so, but the final expressions are messy and not very instructive. The linear terms, on the other hand, highlight the explicit coupling between the different fluxes in the problem.

[60] Here we have neglected the thermal pressure, i.e. we have assumed that $s\Theta \ll P + \rho$. It is natural to make this assumption along with $\mu_e \ll \mu_p$, and moreover the thermal pressure is likely to be negligible in many realistic situations.

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