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# Collisions of protons with light nuclei shed new light on nucleon structure

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The high rates of multiparton interactions at the LHC can provide a unique opportunity to study the multiparton structure of the hadron. To this purpose high-energy collisions of protons with nuclei are particularly suitable. The rates of multiparton interactions depend, in fact, both on the partonic multiplicities and on the distributions of partons in transverse space, which produce different effects on the cross section in $pA$ collisions, as a function of the atomic mass number $A$. Differently with respect to the case of multiparton interactions in $pp$ collisions, the possibility of changing the atomic mass number thus provides an additional handle to distinguish the diverse contributions. Some relevant features of double parton interactions in $pD$ collisions have been discussed in a previous paper. In the present paper we show how the effects of double and triple correlation terms of the multiparton structure can be disentangled, by comparing the rates of multiple parton interactions in collisions of protons with $D$, $^3$H, and $^3$He.

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## I. INTRODUCTION

The experimental evidence [1–6] and the beginning of the operations at the LHC have recently triggered a lot of attention about the problem of multiple parton interactions (MPI) in high-energy $pp$ collisions. Several papers have been written on the topic in the last few months and four international workshops have been organized on the theme [7–12]. Issues discussed in the literature range from estimates of the contributions of MPI in various reaction channels of particular interest for the LHC physics [13–24] to the effects on the global features of the inelastic event and of the underlying event [25–33], the QCD evolution of the double parton distributions [34–36] and the general formulation of MPI within QCD [37–50]. Somewhat less attention has been devoted to the study of MPI in hadron-nucleus collisions, although all effects of MPI are sizably enhanced in that case as a consequence of the much larger parton flux [51–53]. In our opinion, a good reason to pay more attention to $pA$ collisions in this context is that, when studied jointly with $pp$, MPI in $pA$ collisions can provide a unique handle to study some aspects of the multiparton structure of the hadron [54].

In spite of being directly related to the multiparton distribution functions, MPI in $pp$ collisions can in fact provide only partial information on the multiparton distributions. Because of the localization of the large momentum transfer processes, the incoming parton flux and thus the multiparton distribution functions depend explicitly on the relative transverse distances between the interacting parton pairs [55,56]. At the same time, correlations in the hadron structure will prevent expressing the multiparton distributions as an uncorrelated product of one-body distribution functions [57]. The rates of MPI will therefore depend both on the typical relative transverse parton distances and also on the moments of the multiparton distribution in multiplicity. In the case of MPI in $pp$ collisions, the two features are unavoidably linked in the measured cross section [54,58] and, as a consequence, only partial information on the multiparton distributions can be obtained by measuring MPI in $pp$ collisions.

On the other hand, MPI in $pA$ collisions can provide a further handle for a deeper insight into the correlated multiparton structure [54,58]. In $pA$ collisions the MPI cross section is a function of the multiplicity of the target nucleons. In the case of two or more target nucleons, the dimensional scale factor, characterizing each MPI event, is provided both by the hadronic parameters, radius and partonic correlation length, and by the nuclear size. The hadronic scale measured with the generalized parton distributions [59] is rather small compared to the nuclear scale, which thus acquires a dominant role even in the case of light nuclei. When two or more target nucleons take part in the hard interaction, the contributions to the double parton scattering cross section in $pA$ collisions therefore depend only weakly on the hadronic dimensions. By studying MPI in $pA$ collisions, one can thus single out the effects of the moments in multiplicity of the multiparton distributions from the effects due to the correlations in the transverse parton coordinates.

Studying MPI in $pA$ is simpler in the case of light nuclei, where the binding is not very strong: in this case the structure of the nucleon is not much affected by the

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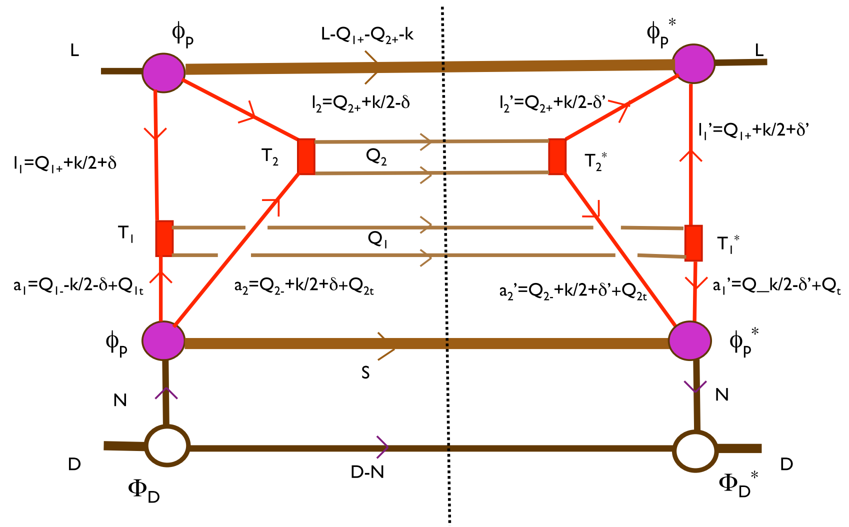


FIG. 1 (color online). Double parton scattering in $pD$ interactions. Only single target nucleons interact with large momentum exchange.

binding, and the nonrelativistic form of the wave function in the rest frame of the nucleus is appropriate. In such a case it is simple enough to construct the boost, which allows us to move from the rest frame of the nucleus, where the wave function is given, to the hadron-nucleon rest frame, which is most suitable to describe the collision and where the relativistic expression of the nuclear wave function is compulsory [58,60]. Some relevant features of double parton collisions in proton-deuteron interactions have been discussed in a previous article [58]. In the present paper we will extend the analysis to the case of double and triple parton collisions of protons with deuterium, $^3$H, and $^3$He; the main results presented in [58] will be reviewed and completed.

The paper is organized as follows: the processes are described in a covariant way, using the formalism of Feynman graphs supplemented by effective vertices for the nonperturbative dynamics. The resulting expression is then reduced to a form containing the fractional longitudinal momenta and the transverse coordinates. The use of the nonrelativistic nuclear wave function in the relativistic process is justified by the same argument as in our previous study of double parton interactions in $pD$ collisions [58]. The only difference concerns the technicalities, which are heavier when considering the three-body dynamics of a nonrelativistic nuclear bound state (the problem is discussion in detail in Appendix A). Concerning the double and triple parton distributions, to remain as general as possible, we have not introduced any explicit expression with correlation parameters. Rather, we have limited our discussion to the actual relations between the observables (namely, the MPI cross sections) and nonperturbative quantities, characterizing the double and triple parton distributions, directly related to the correlated multiparton structure: namely, the various overlap integrals, with a strong dependence on the partonic correlations in transverse space, and the two functions of fractional momenta (one for the double and one for the triple multiparton distributions) representing the deviation of the parton population from an uncorrelated, i.e. Poissonian, distribution. A simple and fully explicit correlated Gaussian model, where all quantities are worked out in detail, is presented in Appendix B.

We anticipate a feature that has no analogy in the case of MPI in $pp$ collisions. The spread of the momenta of the bound nucleons will allow one to produce the same initial partonic configuration in different ways. MPI in $pA$ collisions is thus characterized by quantum interferences between initial state configurations, which differ in the nuclear fractional momenta and in the transverse parton coordinates.

Since our main interest is to recognize the most important features of MPI in collisions of protons with light nuclei, we introduce a drastic simplification in treating all the particles entering the game as spinless bosons and the nuclear wave functions as spherical symmetric. In addition other finer details are neglected from the beginning: the proton and neutron masses are considered equal, as are the binding energies of $^3$H and $^3$He, so $p$ $^3$H and $p$ $^3$He collisions will be considered as equal.

In Sec. II the double parton cross sections are worked out, distinguishing the cases with a different number of spectator nucleons. In Sec. III the triple parton cross sections are worked out, distinguishing the cases with the same procedure. In Sec. IV we discuss the relations between the accessible experimental information, namely, the different cross sections, and the unknown quantities most directly related to the multiparton correlations. The main points examined and the results obtained are finally summarized in the last part of the paper.

## II. DOUBLE SCATTERING ON DEUTERON OR TRITIUM

### A. Only one bound nucleon interacts with large momentum transfer

Some aspects of double parton scattering of protons with deuterium have already been discussed in [58]. The process will be reviewed here and the analysis will be extended to the case of double parton collisions of protons with $^3$H and $^3$He. In double parton scattering on a deuteron one has two possibilities: either only one nucleon interacts with large transverse-momentum exchange, or there are two interacting nucleons. Analogously, in tritium one may have either one or two spectator nucleons. With minor adjustments, the case of tritium can thus be reduced to the case of a deuteron.

The analytical expression for the hard scattering, when one of the component nucleons interacts twice and there are one (deuteron) or two (tritium) spectators, is conveniently expressed through the discontinuity of the forward scattering amplitude (see Fig. 1). We start with the discontinuity of the amplitude $\mathcal{F}_2$ of double scattering between two free nucleons:

$$\begin{aligned}\mathrm{Disc}\mathcal{F}_2(L,L',N,N') = {}& \frac{1}{(2\pi)^{18}}\int \frac{\hat\phi_p}{l_1{}^2 l_2{}^2}\frac{\hat\phi_p^*}{l_1'{}^2 l_2'{}^2}\frac{\hat\phi_p}{a_1{}^2 a_2{}^2}\frac{\hat\phi_p^*}{a_1'{}^2 a_2'{}^2} T_2(l_2,a_2\to q_2,q_2')T_2^*(l_2',a_2'\to q_2,q_2')\\ &\times T_1(l_1,a_1\to q_1,q_1')T_1^*(l_1',a_1'\to q_1,q_1')\delta(L-l_1-l_2-F_3)\delta(L'-l_1'-l_2'-F_3)\\ &\times\delta(N-a_1-a_2-F_1)\delta(N'-a_1'-a_2'-F_1)\delta(l_1+a_1-Q_1)\delta(l_1'+a_1'-Q_1)\\ &\times\delta(l_2+a_2-Q_2)\delta(l_2'+a_2'-Q_2)\prod_{i,j} d(\Omega_i/8)d^4a_i d^4a_i' d^4l_i d^4l_i' d^4F_j\delta(F_j{}^2-M_j{}^2)d^4Q_i dM_j{}^2. \end{aligned} \quad (1)$$

Here and later on, we use the following notation: $\phi_i$ is the effective vertex for one-parton emission, $\hat\phi_i$ is the effective vertex for two-parton emission, $i=p$ when the nucleon is the projectile proton, and $i=1,2$ labels the bound nucleons of the target nucleons. $T_i$ is the $T$ matrix for the parton scattering (the index $i$ labels the corresponding bound nucleon). The final state momenta are $q_i$, $q_i'$, while $Q_i = q_i + q_i'$ is the overall four-momentum of the final state $i$. The final directions are embodied in the angle $\Omega_i$; the factor $1/8$ relates the invariant relative phase space with the solid angle. We use the fact that the momentum variables have large and small components, so the *plus* components of $L$, $l_i$, $F_3$ are large and the corresponding *minus* variables are small. On the contrary, the minus components of $N$, $a_1$, $F_1$ are large and the plus components are small. The four-momenta of the produced particles can have both plus and minus large components. More explicitly, large means $\propto\sqrt{s}$, small means $\propto 1/\sqrt{s}$, the transverse variables are constant with respect to center-of-mass energy $s$, and the general attitude, as in [58], is to integrate over the small components.

It is useful to introduce now the fractional plus or minus momenta in the following way:

$$x_1 = l_{1+}/L_+,\qquad x_2 = l_{2+}/L_+,$$
$$z_1 = a_{1-}/N_-,\qquad z_2 = a_{2-}/N_-.$$

$L$ is the four-momentum of the free proton, $N$ the four-momentum of the other nucleon, $D$ the four-momentum of the deuteron, and $T$ the four-momentum of tritium. The $T$ matrix amplitudes are related to the partonic cross section by

$$\begin{aligned}|T(l+a\to qq')|^2 &= (8\pi)^2(l_+a_-)d\hat\sigma_Q/d\Omega\\ &= (8\pi)^2 xz(L_+N_-)d\hat\sigma_Q/d\Omega. \end{aligned}\quad (2)$$

Hadronization is not included. The cross section is obtained from the discontinuity of the forward amplitude ($L=L'$, $N=N'$), removing the overall four-momentum conservation and dividing by the incoming flux $2s$.

If one of the colliding nucleons is bound, then we define the fractional momentum of the nucleon as $Z = 2N_-/D_-$, in the case of the deuteron, and $Z = 3N_-/T_-$, in the case of tritium. It is useful to also introduce the variables $\bar x_i$, defined as $\bar x_i = 2a_{i-}/D_-$ for the deuteron and $\bar x_i = 3a_{i-}/T_-$ for tritium. The fractional momenta of partons with respect to the parent nucleons are thus $z_1 = \bar x_1/(2-Z)$, $z_2 = \bar x_2/Z$ for the deuteron and analogously for tritium. In the expression of the flux $N_-$ is substituted by $D_-$ and $T_-$, respectively.

If both interacting nucleons are free, one obviously has $L^2 = m^2$, $N_1^2 = m^2$. In the case of a deuteron with a spectator nucleon, the relevant discontinuity is

$$\begin{aligned}\mathrm{Disc}\mathcal{A}_{(2;0)} = {}&\int dN dN' \mathrm{Disc}\mathcal{F}_2(L,L',N,N')\\ &\times\delta((D-N)-(D'-N'))\frac{\Phi_D(N)}{[N^2-m^2]}\frac{\Phi_D{}^*(N')}{[N'^2-m^2]}\\ &\times\delta((D-N)^2-m^2)/(2\pi)^3 \end{aligned}\quad (3)$$

where the deuteron effective vertex $\Phi_D$ is introduced and the condition $N_1^2 = m^2$ is substituted by $D^2 = D'^2 = M_D^2$.

In the case of a $^3$H or $^3$He target with two spectator nucleons, the corresponding discontinuity is

$$\mathrm{Disc}\mathcal{B}_{(2,0,0)} = \int dN dN' dN_3 \mathrm{Disc}\mathcal{F}_2(L, L', N, N')\delta(T - N - T' + N')\frac{\Phi_T(N, N_3)}{[N^2 - m^2]}\frac{\Phi_T{}^*(N', N_3)}{[N'^2 - m^2]} \times \delta(N_3^2 - m^2)\delta((T - N - N_3)^2 - m^2)/(2\pi)^6 \quad (4)$$

where tritium (or $^3$He) effective vertex $\Phi_T$ is introduced and the mass-shell condition is $T^2 = M_T^2$.

Following [58], we proceed by defining the amplitude for finding one or two partons in the projectile when the remnant of the parent nucleon has mass $M$. The integrated variable is $\lambda_- = \frac{1}{2}(l_1 - l_2)_-$ and $M_\perp$ is the transverse mass.

$$\psi_{1,M} = \frac{\phi_p}{l^2} = \frac{\phi_p}{x[m^2 - M_\perp^2/(1-x)] - l_\perp^2}, \qquad \psi_{2,M} = \frac{1}{\sqrt{2}}\int \frac{\hat{\phi}_p}{l_1^2 l_2^2}\frac{d\lambda_-}{2\pi i} = \frac{1}{\sqrt{2}L_-}\frac{\hat{\phi}}{l_{1\perp}{}^2 x_2 + l_{2\perp}{}^2 x_1 - x_1, x_2[m^2 - M_\perp^2/(1 - x_1 - x_2)]}. \quad (5)$$

The one-parton and two-parton amplitudes in the bound nucleon are defined in the same way. The only difference is that in the case of the bound nucleon one needs to replace $m^2$ with $m^2 + N_\perp^2$. The covariant amplitude for finding a nucleon in the deuteron is defined in an analogous way:

$$\frac{1}{\sqrt{2}}\int \frac{\Phi_D}{[(D-N)^2 - m^2]\cdot[N^2 - m^2]}\frac{dN_+}{2\pi i} = \frac{1}{\sqrt{2}}\frac{1}{N_-}\frac{\Phi_D}{[(D-N)^2 - m^2]}\bigg|_{N^2=m^2} = \frac{\Psi_D(N_-)}{N_-} = \frac{1}{\sqrt{2}}\frac{1}{(D-N)_-}\frac{\Phi_D}{[N^2 - m^2]}\bigg|_{(D-N)^2=m^2} = \frac{\Psi_D((D-N)_-)}{(D-N)_-} \quad (6)$$

with these definitions: $\Psi_D(N_-)/N_- = \Psi_D((D-N)_-)/(D-N)_-$. We also have

$$\frac{\Psi_D(N_-)}{N_-} = \frac{\Phi}{\sqrt{2}}\frac{1}{D_-[M_D^2 Z_1 Z_2/4 - m_\perp^2]}, \qquad Z_1 + Z_2 = 2 \quad m_\perp^2 = m^2 + N_\perp^2.$$

Finally we define the covariant amplitude for finding two nucleons in tritium,

$$\frac{1}{2(2\pi i)^2}\int \frac{\Phi_T}{[N_1^2 - m^2]\cdot[N_2^2 - m^2]\cdot[N_3^2 - m^2]} \times \prod_j dN_{j+}\delta\Big(T_+ - \sum_j N_{j+}\Big) = \frac{9}{2T_-^2}\frac{\Phi_T}{M_T^2 Z_1 Z_2 Z_3/3 - m_{\perp,1}^2 Z_2 Z_3 - m_{\perp,2}^2 Z_3 Z_1 - m_{\perp,3}^2 Z_1 Z_2}. \quad (7)$$

The expression is evidently symmetrical in (1, 2, 3) and can thus be identified with $\Psi_T(N_{1-}, N_{2-})/(N_{1-}N_{2-})$, or with $\Psi_T(N_{2-}, N_{3-})/(N_{2-}N_{3-})$, or with $\Psi_T(N_{3-}, N_{1-})/(N_{3-}N_{1-})$.

We proceed now with the integration on the transverse variables, in the frame where the external transverse momenta $L_\perp$ and $D_\perp$ are equal to zero. We take the two-dimensional Fourier transforms ($b_i$ is conjugated to $a_i$, $B_j$ to $N_j$, $\beta_i$ to $l_i$, and all the variables are two-dimensional vectors).

$$\psi_1 = (2\pi)^{-1}\int \tilde{\psi}_1 \exp[ilb]db, \qquad \psi_2 = (2\pi)^{-2}\int \tilde{\psi}_2(b_1, b_2)\exp[il_1 b_1 + il_2 b_2]db_1 db_2, \qquad \Psi = (2\pi)^{-1}\int \tilde{\Psi}(B)\exp[iNB]dB \quad (8)$$

and analogously for the complex conjugated functions, with the variables $b_1'$, $b_2'$, $B'$.

The integration over the transverse-momentum variables gives the diagonal property $b_1 = b_1'$ and so on. Moreover, one obtains the geometrical relation: $b_1 - b_2 = \beta_1 - \beta_2$.

The one-body and two-body parton densities are defined by the following integrals on the invariant mass of the residual hadron fragments:

$$\Gamma(z; b) = \frac{1}{2(2\pi)^3}\int |\tilde{\psi}_M(z; b)|^2 \frac{z}{1-z}dM^2, \qquad \Gamma(x_1, x_2; b_1, b_2) = \frac{1}{2(2\pi)^6}\int |\tilde{\psi}_M(x_1, x_2; b_1, b_2)|^2 \frac{x_1, x_2}{1 - x_1 - x_2}L_+^2 dM^2. \quad (9)$$

The residual dependence on $q_{1\perp}$, $q'_{1\perp}$ is transformed into an angular dependence on $\Omega_1$, $\Omega_2$.

As an effect of the nucleon motion, $|\tilde{\Psi}_D(z; b)|^2$ is coupled to the interactions by the integration on the fractional momentum $Z$, while the integration on the transverse variable $B$ is decoupled from the other transverse variables. So the cross section is readily expressed as

$$\sigma_{2,1}^{pD} = \frac{2}{(2\pi)^3} \int \Gamma(x_1, x_2; \beta_1, \beta_2) \frac{d\sigma(x_1 x_1')}{d\Omega_1} \frac{d\sigma(x_2 x_2')}{d\Omega_2} \Gamma(x_1'/Z, x_2'/Z; b_1, b_2) |\tilde{\Psi}_D(2-Z; B)|^2/(2-Z) dB db_1 db_2 d\beta_1 d\beta_2 \times \delta(b_1 - b_2 - \beta_1 + \beta_2) dx_1 dx_2 dx_1' dx_2' dZ d\Omega_1 d\Omega_2. \quad (10)$$

It is, however, useful to slightly transform the above expression. From the properties of $\Psi$ we have

$$|\Psi_D(2-Z)|^2/(2-Z) = [1 + (1-Z)]|\Psi_D(Z)|^2/Z^2.$$

The second addendum is odd for the substitution $Z \to (2-Z)$ so that the integration in $Z$ which runs from 0 to 2 gives zero and the cross section is more conveniently expressed by

$$\sigma_{2,1}^{pD} = \frac{2}{(2\pi)^3} \int \Gamma(x_1, x_2; \beta_1, \beta_2) \frac{d\sigma(x_1 x_1')}{d\Omega_1} \frac{d\sigma(x_2 x_2')}{d\Omega_2} \Gamma(x_1'/Z, x_2'/Z; b_1, b_2) |\tilde{\Psi}_D(Z; B)|^2/Z^2 dB db_1 db_2 d\beta_1 d\beta_2 \times \delta(b_1 - b_2 - \beta_1 + \beta_2) dx_1 dx_2 dx_1' dx_2' dZ d\Omega_1 d\Omega_2. \quad (11)$$

### B. Two bound nucleons interact with large momentum transfer

In collisions of protons with $D$ or $^3H/^3He$, the presence of the nuclear wave function induces the presence of two kinds of contributions: "diagonal terms" in direct correspondence with the processes taking place when the nucleons are free, and a number of nondiagonal or "interference" terms, which are due to the presence of the nuclear wave function.[1]

The simplest case, where diagonal and nondiagonal terms appear, is the double scattering on a deuteron affecting both bound nucleons. In this case the "diagonal" discontinuity (see Fig. 2(a)) has the form already given:

$$\begin{aligned}\mathrm{Disc}\mathcal{A}_d^{(2)} = {}& \frac{1}{(2\pi)^{21}} \int \frac{\hat{\phi}_p}{l_1^2 l_2^2} \frac{\hat{\phi}_p^*}{l_1'^2 l_2'^2} \frac{\phi_p}{a_1^2} \frac{\phi_p^*}{a_1'^2} \frac{\phi_n}{a_2^2} \frac{\phi_n^*}{a_2'^2} T_2(l_2, a_2 \to q_2, q_2') T_2^*(l_2', a_2' \to q_2, q_2') T_1(l_1, a_1 \to q_1, q_1') \\ & \times T_1^*(l_1', a_1' \to q_1, q_1') \frac{\Phi_D(D; N)}{[(D-N)^2 - m^2][N^2 - m^2]} \frac{\Phi_D^*(D; N')}{[(D-N')^2 - m^2][N'^2 - m^2]} \delta(L - l_1 - l_2 - F_3) \\ & \times \delta(L - l_1' - l_2' - F_3) \delta(N - a_2 - F_2) \delta(N' - a_2' - F_2) \delta(D - N - a_1 - F_1) \delta(D - N' - a_1' - F_1) \\ & \times \delta(l_1 + a_1 - Q_1) \delta(l_1' + a_1' - Q_1) \delta(l_2 + a_2 - Q_2) \delta(l_2' + a_2' - Q_2) \\ & \times \prod_{i,j} d(\Omega_i/8) d^4 a_i d^4 a_i' d^4 l_i d^4 l_i' d^4 F_j \delta(F_j{}^2 - M_j{}^2) d^4 N d^4 N' d^4 Q_i dM_j{}^2 \end{aligned} \quad (12)$$

whereas the interference term (Fig. 2(b)) has the form

$$\begin{aligned}\mathrm{Disc}\mathcal{A}_i^{(2)} = {}& \frac{1}{(2\pi)^{21}} \int \frac{\hat{\phi}_p}{l_1^2 l_2^2} \frac{\hat{\phi}_p^*}{l_1'^2 l_2'^2} \frac{\phi_p}{a_1^2} \frac{\phi_p^*}{a_1'^2} \frac{\phi_n}{a_2^2} \frac{\phi_n^*}{a_2'^2} T_2(l_2, a_2 \to q_2, q_2') T_1^*(l_1', a_1' \to q_2, q_2') T_1(l_1, a_1 \to q_1, q_1') \\ & \times T_2^*(l_2', a_2' \to q_2, q_2') \frac{\Phi_D(D; N)}{[(D-N)^2 - m^2][N^2 - m^2]} \frac{\Phi_D^*(D; N')}{[(D-N')^2 - m^2][N'^2 - m^2]} \delta(L - l_1 - l_2 - F_3) \\ & \times \delta(L - l_1' - l_2' - F_3) \delta(N - a_2 - F_2) \delta(N' - a_1' - F_2) \delta(D - N - a_1 - F_1) \delta(D - N' - a_2' - F_1) \\ & \times \delta(l_1 + a_1 - Q_1) \delta(l_1' + a_2' - Q_1) \delta(l_2 + a_2 - Q_2) \delta(l_2' + a_1' - Q_2) \\ & \times \prod_{i,j} d(\Omega_i/8) d^4 a_i d^4 a_i' d^4 l_i d^4 l_i' d^4 F_j \delta(F_j{}^2 - M_j{}^2) d^4 N d^4 N' d^4 Q_i dM_j{}^2. \end{aligned} \quad (13)$$

The diagonal term was already elaborated in [58], so we are interested in the differences between the two cases. In the diagonal case the conservation of the large components of momenta implies that they are equal on the two sides of the diagram $l_+ = l'_+$, $a_- = a'_-$, $N_- = N'_-$; the transverse variables become diagonal through the Fourier transformation. In this way the whole expression of the cross section is expressible in terms of densities i.e. the square of the partonic wave function and the square of the nuclear wave function. The cross section is thus expressed again through the one-body and two-body partonic densities: $\Gamma(z; b)$, $\Gamma(x_1, x_2; \beta_1, \beta_2)$, obtained from the effective vertices $\phi$, $\hat{\phi}$. The nuclear density is simply given by $|\Psi(Z; B)|^2$.

In the interference case the conservation of the large components of momenta still implies the equality on the two sides of the diagram $l_+ = l'_+$, $a_- = a'_-$ but for the nuclear variables $(N - a_2)_- = (N' - a_1)_-$; moreover, the transverse

[1]The discussion in [58] is limited to the cases where the contribution of the interference term is not relevant.

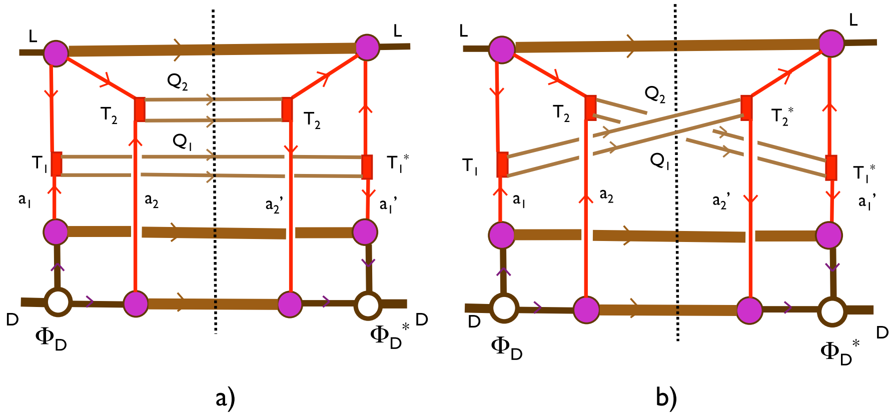


FIG. 2 (color online). Diagonal (a) and off-diagonal (b) contributions to double parton scattering in $pD$ interactions, when both target nucleons interact with large momentum exchange.

variable $b_\perp$, conjugated to $a_\perp$, does not become diagonal through the Fourier transformation. The interference term cannot be expressed only in terms of partonic densities; we need to introduce a more complicated expression:

$$W_1(Z, Z'; \bar{x}_1, \bar{x}_2; b_1, b_2, B) = \frac{1}{4(2\pi)^6} \int dM_1^2 dM_2^2 \frac{\bar{x}_1 \bar{x}_2}{(Z - \bar{x}_1)(2 - Z' - \bar{x}_2)} \psi_{M_1}(\bar{x}_1/Z; b_1) \psi^*_{M_2}(\bar{x}_1/(2 - Z'); b_1 - B) \times \psi_{M_2}(\bar{x}_2/(2 - Z); b_2) \psi^*_{M_1}(\bar{x}_2/Z'; b_2 + B). \tag{14}$$

The previous relation for the nuclear variables can also be written as $Z - Z' = \bar{x}_2 - \bar{x}_1$. It is possible to factor the expression $W$ symmetrically into two parts, $W_1 = H(Z, Z'; \bar{x}_1, \bar{x}_2; b_1, b_2 + B) \times H(2 - Z, 2 - Z'; \bar{x}_2, \bar{x}_1; b_2, b_1 - B)$ with

$$H(Z, Z'; \bar{x}_1, \bar{x}_2; b_1, b_2) = \frac{1}{2(2\pi)^3} \int dM_1^2 \frac{\sqrt{\bar{x}_1 \bar{x}_2}}{\sqrt{(Z - \bar{x}_1)(2 - Z' - \bar{x}_2)}} \psi_{M_1}(\bar{x}_1/Z; b_1) \psi^*_{M_1}(\bar{x}_2/Z'; b_2).$$

The part of the cross section coming from the direct term is (as given in [58])

$$\sigma_{2,2}^{pD}|_d = \frac{1}{(2\pi)^3} \int \Gamma(x_1, x_2; \beta_1, \beta_2) \frac{d\hat{\sigma}(x_1, \bar{x}_1)}{d\Omega_1} \frac{d\hat{\sigma}(x_2, \bar{x}_2)}{d\Omega_2} \Gamma(\bar{x}_1/Z; b_1) \Gamma(\bar{x}_2/(2 - Z); b_2) |\tilde{\Psi}_D(Z; B)|^2 dB db_1 db_2 d\beta_1 d\beta_2 \times \delta(B - b_1 + b_2 - \beta_1 + \beta_2) Z^{-2} dx_1 dx_2 d\bar{x}_1 d\bar{x}_2 dZ d\Omega_1 d\Omega_2. \tag{15}$$

In Fig. 3 we show the corresponding configuration in transverse space.

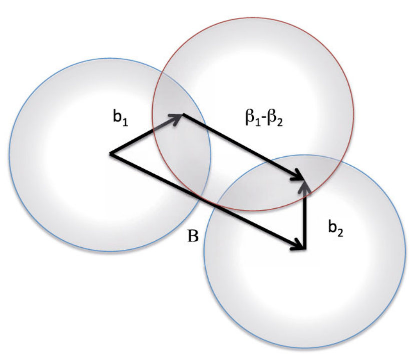


FIG. 3 (color online). $pD$ interactions with two target nucleons involved. We show configurations in transverse space of the diagonal term.

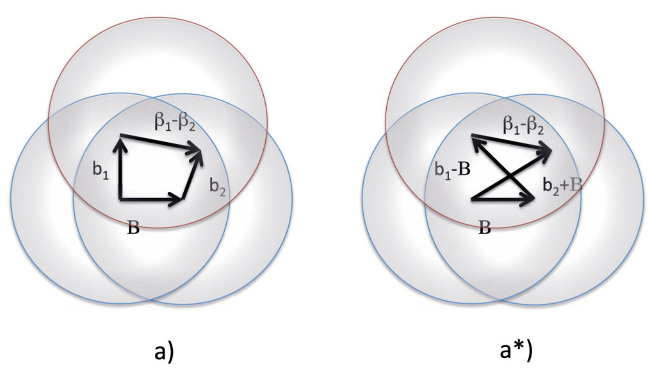


FIG. 4 (color online). $pD$ interactions with two target nucleons involved. We show configurations in transverse space of the two interfering amplitudes. Both configurations generate the same partonic initial state.

The part of the cross section coming from the interference term is

$$\sigma_{2,2}^{pD}|_i = \frac{1}{(2\pi)^3}\int \Gamma(x_1, x_2; \beta_1, \beta_2)\frac{d\hat{\sigma}(x_1, \bar{x}_1)}{d\Omega_1}\frac{d\hat{\sigma}(x_2, \bar{x}_2)}{d\Omega_2}W_1(Z, Z'; \bar{x}_1, \bar{x}_2; b_1, b_2, B)\tilde{\Psi}_D(Z; B)\tilde{\Psi}_D^*(Z'; b)[ZZ']^{-1} \times \delta(B - b_1 + b_2 - \beta_1 + \beta_2)\delta(Z - Z' - \bar{x}_1 + \bar{x}_2)d\Omega_1 d\Omega_2 dB db_1 db_2 d\beta_1 d\beta_2 dx_1 dx_2 d\bar{x}_1 d\bar{x}_2 dZ dZ'. \quad (16)$$

In Fig. 4 we show the configurations in transverse space of the two interfering amplitudes.

Here we give a comparison between the two expressions: As far as the longitudinal variables are concerned the interference term requires the nuclear wave function to be taken at different values of $Z$, so it is depressed with respect to the diagonal term. It must be taken into account that the width of the nuclear wave function is determined by the binding energy, while the mismatch of the $Z$ terms depends on the difference in fractional momentum of the partons. At large total energies (beyond the TeV) the values of $\bar{x}$ can be small, still maintaining the process within the limits of perturbative dynamics, so the depression is not necessarily very strong. For what concerns the transverse variables one has two different nonperturbative scales, the hadron scale, provided by the generalized parton distributions, and the nuclear size. The first scale sets the size of the variables $b_i$, $\beta_i$; the second characterizes the size of $B$. After integrating over $B$ with the $\delta$ function, one has $B = (b_1 - b_2 + \beta_1 - \beta_2)$. A simplified form for the interference term is thus obtained when neglecting the hadronic size as compared with the nuclear size by setting $B = 0$ in nuclear wave function $\Psi$. Note that the effect depends on the transverse degrees of freedom, while the dependence on the total energy is weak once the region of perturbative dynamics is reached. A quantitative but model dependent discussion of this feature can be found in Appendix B.

Conversely, the denominators $Z, Z'$ could be set equal to 1 in the factors $\Gamma$ since the range of variation of $\bar{x}_i$ is large, as compared with the variation of $Z$, allowed by the nuclear function $\Psi$. From the previous treatment we learned that the presence of a spectator has a modest influence on the process, so the double scattering on tritium has only minor differences in comparison with the scattering on a deuteron. The property holds also in the next cases to be considered, with the exception of those processes which are possible in the presence of tritium but not of deuteron.

## III. TRIPLE SCATTERING ON DEUTERON OR TRITIUM

### A. Only one bound nucleon interacts with large momentum transfer

The analytical expression for the hard scattering, where one of the component nucleons interacts three times and there are one (deuteron) or two (tritium) spectators, is again related to the amplitude $\mathcal{F}_3$ for the triple hard scattering between two free nucleons:

$$\begin{aligned}\mathrm{Disc}\mathcal{F}_3 = {}& \frac{1}{(2\pi)^{32}}\int \frac{\check{\phi}_p}{l_1^2 l_2^2 l_3^2}\frac{\check{\phi}_p^*}{l'^2{}_1 l'^2{}_2 l'^2{}_3}\frac{\check{\phi}_1}{a_1^2 a_2^2 a_3^2}\frac{\check{\phi}_1^*}{a'^2{}_1 a'^2{}_2 a'^2{}_3}T_1(l_1 a_1 \to q_1 q_1')T_1{}^*(l'{}_1 a'{}_1 \to q_1 q_2)T_2(l_2 a_2 \to q_2 q_2') \\ & \times T_2{}^*(l'{}_2 a'{}_2 \to q_2 q_2')T_3(l_3 a_3 \to q_3 q_3')T_1{}^*(l'{}_3 a'{}_3 \to q_3 q_3')\delta\Big(L - \sum l - F_4\Big)\delta\Big(L - \sum l' - F_4\Big) \\ & \times \delta\Big(N_1 - \sum a - F_1\Big)\delta\Big(N'{}_1 - \sum a' - F_1\Big)\delta(l_1 + a_1 - Q_1)\delta(l'{}_1 + a'{}_1 - Q_1) \\ & \times \delta(l_2 + a_2 - Q_2)\delta(l'{}_2 + a'{}_2 - Q_2)\delta(l_3 + a_3 - Q_3)\delta(l'{}_3 + a'{}_3 - Q_3)\delta(N_1 - N'{}_1) \\ & \times \prod_{i,j} d(\Omega_i/8)d^4 a_i d^4 a_i' d^4 l_i d^4 l_i' d^4 F_j \delta(F_j{}^2 - M_j{}^2)d^4 N_1 d^4 Q_i dM_j{}^2. \end{aligned} \quad (17)$$

Analogously to the case of double scattering, for the calculations of $\mathcal{F}_3$ we also use the property that, in the regime of interest, the momentum variables have large and small components; the term $\check{\phi}$ represents the vertex for emission of three partons from a nucleon.

The factor $1/(l_1^2 l_2^2 l_3^2)$ is thus integrated in $l_{1-}$ and $l_{2-}$ independently from the rest of the diagram; in fact in all other conservation relations, the small terms $l_{1-}$ and $l_{2-}$ enter together with large components e.g. of $a_i$ and can thus be neglected. Implementing the conservation $(L - F_4) = l_1 + l_2 + l_3$, the integration gives

$$\psi_{3,M} = \frac{1}{(2\pi i)^2}\int \frac{1}{2} dl_{1-} dl_{2-}\frac{\check{\phi}}{l_1^2 l_2^2 l_3^2} = -\frac{1}{2}\frac{\check{\phi}}{l_{1+} l_{2+} l_{3+}(L - F_4)_- - l_{1+} l_{2+} l_{3\perp}{}^2 - l_{2+} l_{3+} l_{1\perp}{}^2 - l_{3+} l_{1+} l_{2\perp}{}^2}.$$

Here we also use the fractional momenta $x_i = l_{i+}/L_+ \simeq Q_{i+}/L_+$ and, through the conservation for the plus components, we obtain

$$\psi_{3,M}(l) = \frac{1}{2L_+{}^2}\frac{\check{\phi}}{x_1x_2l_{3\perp}{}^2 + x_2x_3l_{1\perp}{}^2 + x_3x_1l_{2\perp}{}^2 + [M_\perp^2/(1-\sum x_i) - m^2]x_1, x_2, x_3}. \tag{18}$$

$M_\perp$ is the transverse mass of the remnants: $M_\perp{}^2 = M_F{}^2 + F_\perp{}^2$, $F_\perp = -\sum l_{i\perp}$.

The factor $\check{\phi}/(a_1^2a_2^2a_3^2)$ can be integrated in the same way. One needs only to take into account that we are interested in a situation where $N_1$ is a generic timelike four-vector, with positive energy, no longer subjected to the condition $N_1^2 = m^2$. Even when $N_i$ enters in a wave function, it is still almost on shell, as we are dealing with a weakly bound systems (it nevertheless has a transverse momentum $N_{i\perp}$). Let $F$ be the four-momentum of the remnants; then in strict analogy with the previous result we get

$$\psi_{3,M}(a) = \frac{1}{2N_-{}^2}\frac{\check{\phi}}{z_1z_2a_{3\perp}{}^2 + z_2z_3a_{1\perp}{}^2 + z_3z_1a_{2\perp}{}^2 + [M_\perp^2/(1-\sum z_i) - m_N^2]z_1z_2z_3}. \tag{19}$$

$M_\perp$ is again the transverse mass of the remnants; here the conservation of the *minus* component is used. The integration on the remnants $F$ can be treated as in the double scattering case: $\int d^4F\delta(F^2 - M^2)dM^2 = \int dF_\pm/F_\pm d^2F_\perp$. The longitudinal integration is then performed by means of the $\delta$ functions with the results $1/F_{1-} = 1/[N_-(1-\sum z)]$, $1/F_{4+} = 1/[L_+(1-\sum x)]$.

Looking at the cut diagram in Fig. 5, we see that the equality of $N_2$ and of $N_3$ also forces $N_1$ to be the same on the right- and left-hand sides of the diagram. Concerning the produced pairs, a quick inspection of the kinematics shows that the component $Q_+$ comes from $l_+$ and the component $Q_-$ comes from $a_-$, which, neglecting terms of order $1/\sqrt{s}$, implies $l_{i+} = l'_{i+}$ and $a_{i-} = a'_{i-}$. In terms of the fractional momenta $x_i = x_i'$, $z_i = z_i'$ and, according to the definitions,

$$\prod dl_+da_- = (L_+N_-)^3\prod dxdz.$$

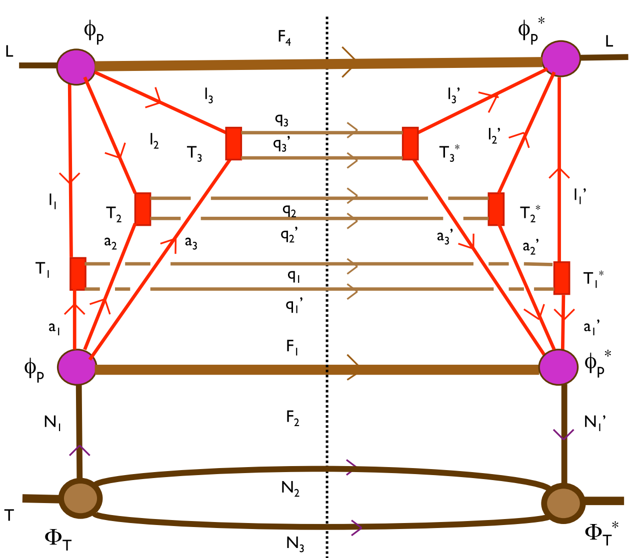


FIG. 5 (color online). Triple parton scattering in $pT$ or $p\,^3$He interactions. Only a single target nucleon interacts with large momentum exchange.

Using the already defined $\psi_3(l)\psi_3(a)$ and analogously for the factors depending on $l'_i$, $a'_i$, we perform a Fourier transform on the transverse momenta; $\beta$ is the conjugate of $l$ and $b$ is the conjugate of $a$.

The subsequent integrations involve the nuclear variables. The longitudinal variables $Z$ are common to both sides of the cut diagram; the transverse variables are different. In the case of the deuteron one has to integrate over one spectator (which is on shell), so we have integrations in $dZ_i/Z_i$ and in $dN_{i\perp}$ (note that $N_{1\perp} = -N_{2\perp}$ and $Z_1 = 2 - Z_2$). In tritium ($^3$He) case we have two transverse variables and the longitudinal integration, which may be expressed as $\int\delta(Z_1 + Z_2 + Z_3 - 3)\times dZ_1dZ_2/Z_2dZ_3/Z_3$. In conclusion, in both cases only one nuclear variable survives, and we need to perform the Fourier transform of the transverse components: we call the corresponding coordinates $B$, $B'$.

The integrations which take care of the conservation conditions give two kinds of results: the diagonalization in the impact parameters given as $B = B'$, $\beta_i = \beta_i'$, $b_i = b_i'$, and the geometrical conditions $\beta_1 - \beta_2 = b_1 - b_2$, $\beta_2 - \beta_3 = b_2 - b_3$, which also imply $\beta_3 - \beta_1 = b_3 - b_1$.

In more detail, the derivation of the condition on the transverse variables is obtained as follows (to simplify the notation, the transverse index, like $l_\perp$, is understood everywhere).

The Fourier transform of the wave function is $\psi_3(l_1, l_2, l_3) = \int\tilde{\psi}_3(\beta_i)\exp[i\sum l_i\beta_i]\prod d\beta$ and analogously for the functions in $a$ and for the conjugate. The conservation relations involving the produced pairs, integrated over the final states, give the two-dimensional constraints

$$\delta(l_i + a_i - l'_i - a'_i) = \frac{1}{(2\pi)^2}\int d\theta_i\exp[i\theta_i(l_i + a_i - l'_i - a'_i)].$$

The conservation between the incoming momenta and the momenta of the remnants gives a factor $\delta(\sum l - \sum l')$. A similar condition holds for the $a_i$'s. The two $\delta$-functions can be transformed into a $\delta$-function with the sum and one with the difference of the arguments. The condition given from the sum is redundant, as it is already contained in the previous relations; the difference gives a new condition $\delta(\sum l - \sum l' - \sum a + \sum a')$. In exponential form,

$$\frac{1}{(2\pi)^2}\int d\zeta \exp\Big[-i\zeta\Big(\sum l-\sum l'-\sum a+\sum a'\Big)\Big].$$

The integrations over the internal variables $l$, $l'$, $a$, $a'$ give

$$\delta(\beta_i+\theta_i-\zeta),\qquad \delta(\beta_i'+\theta_i-\zeta),$$
$$\delta(b_i+\theta_i-\zeta),\qquad \delta(b_i'+\theta_i-\zeta)$$

which in turn implies $\beta_i=\beta_i'$, $b_i=b_i'$. One thus obtains the diagonalization in the impact parameter. Moreover, one is left with $\delta(\beta_i-b_i-2\zeta)$, which represents a geometrical constraint: the difference $\beta_i-b_i$ is independent of the index $i$.

In analogy with the previous definitions [55,58], the three-body densities are defined by partially reabsorbing the factors $L_+$, $N_-$,

$$\Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)=\frac{1}{2(2\pi)^9}\int|\tilde\psi(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)|^2\frac{x_1,x_2,x_3}{1-x_1-x_2-x_3}L_+{}^4dM_4^2,$$
$$\Gamma(z_1,z_2,z_3;b_1,b_2,b_3)=\frac{1}{2(2\pi)^9}\int|\tilde\psi(z_1,z_2,z_3;b_1,b_2,b_3)|^2\frac{z_1z_2z_3}{1-z_1-z_2-z_3}N_-{}^4dM_1^2. \tag{20}$$

In this way the densities $\Gamma$ are such that they are neither vanishing nor growing indefinitely when $L_+$, $N_-\to\infty$. The last expression could be recast in terms of the external variables $\bar x_i$ as

$$\Gamma(\bar x_1/Z,\bar x_2/Z,\bar x_3/Z;b_1,b_2,b_3)=\frac{1}{2^5(2\pi)^9}\frac{1}{Z^2}\int|\tilde\psi(\bar x_1/Z,\bar x_2/Z,\bar x_3/Z;b_1,b_2,b_3)|^2\frac{\bar x_1\bar x_2\bar x_3}{Z-\bar x_1-\bar x_2-\bar x_3}D_-{}^4dM_1^2. \tag{21}$$

The contribution to the triple scattering cross section on a deuteron, where one of the bound nucleons suffers three hard collisions, while the other is a spectator, is hence given by

$$\sigma_{3,1}^{pD}=\frac{2}{(2\pi)^3}\int\Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)\Gamma(x_1'/Z,x_2'/Z,x_3'/Z;b_1,b_2,b_3)\frac{d\sigma}{d\Omega_1}\frac{d\sigma}{d\Omega_2}\frac{d\sigma}{d\Omega_3}|\tilde\Psi_D(Z;B)|^2$$
$$\times Z^{-2}dBdb_1db_2db_3d\beta_1d\beta_2d\beta_3\delta(b_1-b_2-\beta_1+\beta_2)\delta(b_1-b_3-\beta_1+\beta_3)dx_1dx_2dx_3dx_1'dx_2'dx_3'dZd\Omega_1d\Omega_2d\Omega_3. \tag{22}$$

When only a single target nucleon interacts with large momentum exchange, nuclear dynamics completely takes care of the difference between a deuteron and tritium (or $^3$He). There are some minor differences: the total four-momentum of the interacting nucleons can have a transverse component.

### B. Two different target nucleons interact with large transverse-momentum exchange

#### 1. General features and diagonal terms

As already seen, when two or more target nucleons interact with large transverse-momentum exchange, nuclear and partonic dynamics are interconnected. The presence of the nuclear wave function induces, in fact, the presence of two kinds of contributions: a diagonal term and a number of nondiagonal or interference terms. The diagonal discontinuity for the triple scattering is shown in Fig. 6(a), and its analytical expression is

$$\mathrm{Disc}\mathcal{A}^{(2,1)}|_d=\frac{1}{(2\pi)^{35}}\int\frac{\check\phi_p}{l_1^2l_2^2l_3^2}\frac{\check\phi_p^*}{l'^2{}_1l'^2{}_2l'^2{}_3}\frac{\hat\phi_1}{a_1^2a_2^2}\frac{\phi_2}{a_3^2}\frac{\hat\phi_1^*}{a'^2{}_1a'^2{}_2}\frac{\phi_2{}^*}{a_3^2}T_1(l_1a_1\to q_1q_1')T_2(l_2a_2\to q_2q_2')T_3(l_3a_3\to q_3q_3')$$
$$\times T_1{}^*(l'{}_1a'{}_1\to q_1q_1')T_2^*(l_2a_2\to q_2q_2')T_3{}^*(l'{}_3a'{}_3\to q_3q_3')\frac{\Phi_D(N_1,N_2)}{[N_1^2-m^2][N_2^2-m^2]}\frac{\Phi_D{}^*(N'{}_1,N'{}_2)}{[N'^2{}_1-m^2][N'^2{}_2-m^2]}$$
$$\times\delta\Big(L-\sum l-F_4\Big)\delta(N_1-a_3-F_1)\delta(N_2-a_1-a_2-F_2)\delta\Big(L-\sum l'-F_4\Big)$$
$$\times\delta(N'{}_1-a'{}_3-F_1)\delta(N'{}_2-a'{}_1-a'{}_2-F_2)\delta(l_1+a_1-Q_1)\delta(l'{}_1+a'{}_1-Q_1)$$
$$\times\delta(l_2+a_2-Q_2)\delta(l'{}_2+a'{}_2-Q_2)\delta(l_3+a_3-Q_3)\delta(l'{}_3+a'{}_3-Q_3)$$
$$\times\delta(D-N_1-N_2)\delta(D-N'{}_1-N'{}_2)\prod d(\Omega/8)dada'dldl'dFdNdN'dQ_i \tag{23}$$

with the mass-shell condition $D^2=M_D^2$.

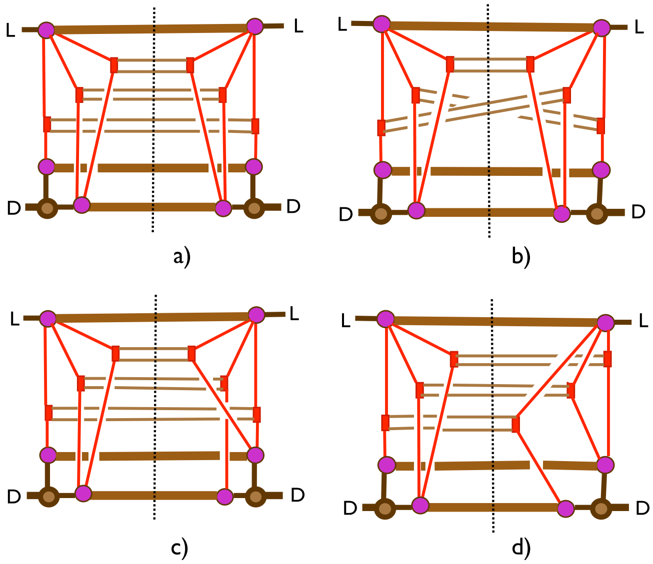


FIG. 6 (color online). Different contributions to the triple parton scattering in $pD$ interactions. Both target nucleons interact with large transverse-momentum exchange.

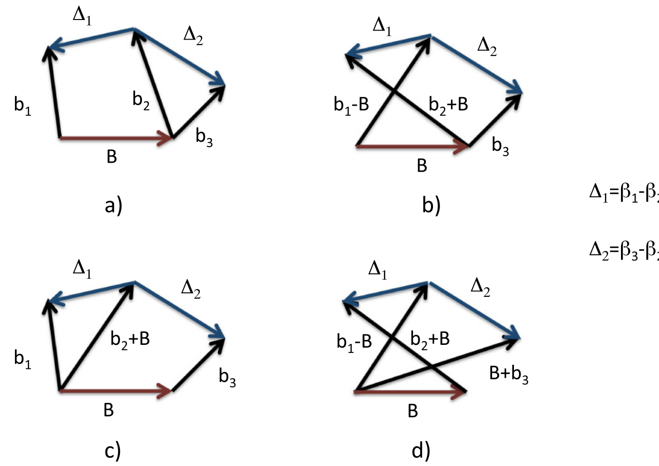


FIG. 7 (color online). Configurations in transverse space of the four amplitudes in Fig. 6, in the right-hand side of the cut.

In the case of tritium or $^3$He (with four-momentum $T$) the corresponding discontinuities Disc$\mathcal{B}^{(2,1,0)}$ are obtained from Disc$\mathcal{A}^{(2,1)}$ by substituting the factor

$$\frac{\Phi_D(N_1, N_2)}{[N_1^2 - m^2][N_2^2 - m^2]} \frac{\Phi_D{}^*(N'_1, N'_2)}{[N'^2{}_1 - m^2][N'^2{}_2 - m^2]} \times \delta(D - N_1 - N_2)\delta(D - N'_1 - N'_2) \tag{24}$$

with

$$\frac{\Phi_T(N_1, N_2, N_3)}{[N_1^2 - m^2][N_2^2 - m^2][N_3^2 - m^2]} \times \frac{\Phi_T{}^*(N'_1, N'_2, N_3)}{[N'^2{}_1 - m^2][N'^2{}_2 - m^2][N'^2{}_3 - m^2]} \times \delta(T - N_1 - N_2 - N_3)\delta(T - N'_1 - N'_2 - N_3) \times \delta(N_3^2 - m^2). \tag{25}$$

The integration runs over $dN_i$, with $i = 1, 2, 3$, and the mass-shell condition is $T^2 = M_T^2$.

For the incoming proton the situation and the subsequent manipulations are the same as in the previous case; they give rise to the factor $\psi_3$. For the nucleus we find different structures: the one-body and two-body parton vertices and the singularities which put a nucleon on mass shell or the partons on mass shell.

Here we need the one-parton and the two-parton wave functions; they were already defined and an explicit form was given in Eq. (5). The longitudinal integration of the remnants is performed as in the previous case. We now have three $F_\pm$, and we get

$$1/F_{1-} = 1/[N_-(1 - z_1 - z_2)],$$
$$1/F_{2-} = 1/[N_-(1 - z_3)],$$
$$1/F_{4+} = 1/\Big[L_+\Big(1 - \sum x_i\Big)\Big].$$

Concerning the nuclear variables, since the binding energy is small, the most important singularities are those corresponding to the nucleons' mass-shell condition. For the deuteron one must thus evaluate $1/(N_1^2 - m^2)$ with the other propagator on mass shell, i.e. $N_2^2 = m^2$, with $N_1 + N_2 = D$. One obtains the expression of $\Psi_D(N_-)/N_-$ as given in Eq. (6). For tritium or $^3$He, the condition $N_1 + N_2 + N_3 = T$ and the mass-shell constraint for the spectator lead to the expression in Eq. (7). The multiparton densities, together with their Fourier transform in the transverse plane $\Gamma(x_i; b_i)$, have already been defined and discussed.

Neglecting terms of order $1/\sqrt{s}$, the conservation of the large components gives again $l_{i+} = l'_{i+}$, $a_{1-} = a'_{1-}$, and the complete expression is brought into a diagonal form by Fourier transforming the transverse variables. In this case, however, the geometrical relations are different. One finds $\beta_2 - \beta_3 = b_2 - b_3$, $\beta_1 - \beta_2 = b_1 - b_2 - B_1 + B_2$. The corresponding configuration in transverse space is shown in Fig. 7(a).

The diagonal contribution to the triple parton scattering cross sections on a deuteron and on tritium or $^3$He, when one nucleon interacts twice and another once, setting $B = B_1 - B_2$, $B' = B_1 - B_3$, $Z = Z_1 - Z_2$, is thus expressed as

$$\sigma_{3,2}^{pD}|_d = \frac{2}{(2\pi)^3}\int \Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)\Gamma(z_1,z_2;b_1,b_2)\Gamma(z_3;b_3)\frac{d\hat\sigma}{d\Omega_1}\frac{d\hat\sigma}{d\Omega_2}$$
$$\times\frac{d\hat\sigma}{d\Omega_3}|\Psi_D(Z;B)|^2 dBd\beta_1 d\beta_2 d\beta_3 db_1 db_2 db_3\delta(b_3-b_2+\beta_2-\beta_3)\delta(b_1-b_2-\beta_1+\beta_2-B)$$
$$\times dx_1dx_2dx_3dz_1dz_2dz_3d\Omega_1d\Omega_2d\Omega_3\delta(Z_1+Z_2-2)dZ_1dZ_2/(Z_1Z_2),$$
$$\sigma_{3,2}^{pT}|_d = \frac{6}{(2\pi)^3}\int \Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)\Gamma(z_1,z_2;b_1,b_2)\Gamma(z_3;b_3)\frac{d\hat\sigma}{d\Omega_1}\frac{d\hat\sigma}{d\Omega_2}\frac{d\hat\sigma}{d\Omega_3}$$
$$\times|\Psi_T(Z_i;B,B')|^2 dBdB'd\beta_1 d\beta_2 d\beta_3 db_1 db_2 db_3\delta(b_3-b_2+\beta_2-\beta_3)\delta(b_1-b_2-\beta_1+\beta_2-B)$$
$$\times dx_1dx_2dx_3dz_1dz_2dz_3d\Omega_1d\Omega_2d\Omega_3\delta(Z_1+Z_2+Z_3-3)dZ_1dZ_2dZ_3/(Z_1Z_2Z_3). \quad (26)$$

### *2. Interference terms*

As it appears from the graphs in Fig. 6, there are three kinds of interference terms which differ from one another for the different relations between the partons and the parent nucleon.

This difference takes, therefore, the form of the difference in the $\delta$ functions. The discontinuities corresponding to three terms are explicitly given for the case of the deuteron; they can be summarized in the following form:

$$\mathrm{Disc}\mathcal{A}^{(2,1)}|_{\mathrm{in}} = \frac{1}{(2\pi)^{35}}\int\frac{\check\phi_p}{l_1^2l_2^2l_3^2}\frac{\check\phi_p^*}{l'^2{}_1l'^2{}_2l'^2{}_3}\frac{\hat\phi_1}{a_1^2a_2^2}\frac{\phi_2}{a_3^2}\frac{\phi_1{}^*}{a'^2{}_1}\frac{\hat\phi_2^*}{a'^2{}_2a'^2{}_3}T_1(l_1a_1\to q_1q_1')T_2(l_2a_2\to q_2q_2')T_3(l_3a_3\to q_3q_3')$$
$$\times T_1{}^*(l'{}_1a'{}_1\to q_1q_1')T_2^*(l_2a_2\to q_2q_2')T_3{}^*(l'{}_3a'{}_3\to q_3q_3')\frac{\Phi_D(N_1,N_2)}{[N_1^2-m^2][N_2^2-m^2]}\frac{\Phi_D{}^*(N'{}_1,N'{}_2)}{[N'^2{}_1-m^2][N'^2{}_2-m^2]}$$
$$\times\delta\Big(L-\sum l-F_4\Big)\delta\Big(L-\sum l'-F_4\Big)\times\eta(N_i,a_j)\delta(l_1+a_1-Q_1)\delta(l'{}_1+a'{}_1-Q_1)\delta(l_2+a_2-Q_2)$$
$$\times\delta(l'{}_2+a'{}_2-Q_2)\delta(l_3+a_3-Q_3)\delta(l'{}_3+a'{}_3-Q_3)\delta(D-N_1-N_2)\delta(D-N'{}_1-N'{}_2)$$
$$\times\prod d\Omega dada'dldl'dFdNdN'dQ. \quad (27)$$

The three relevant realizations of the factor $\eta$, as can be seen from the graphs, are

$$\eta_{2,1}(N_i,a_j)=\delta(N_1-a_3-F_1)\delta(N_2-a_1-a_2-F_2)\delta(N_1'-a_2'-F_1)\delta(N_2'-a_1'-a_3'-F_2),$$
$$\eta_{2,2}(N_i,a_j)=\delta(N_1-a_3-F_1)\delta(N_2-a_1-a_2-F_2)\delta(N_1'-a_1'-a_3'-F_1)\delta(N_2'-a_2'-F_2),\quad (28)$$
$$\eta_{2,3}(N_i,a_j)=\delta(N_1-a_3-F_1)\delta(N_2-a_1-a_2-F_2)\delta(N_1'-a_1'-a_2'-F_1)\delta(N_2'-a_3'-F_2).$$

The corresponding expressions for the case of tritium are obtained by the same substitutions that were used in the diagonal term.

The configurations produced by the factors $\eta_{2,j}$ are similar to the configuration described by the crossed diagram in the double scattering; however, the factors are more strictly interlocked so that it is necessary to introduce other auxiliary terms,

$$W_{2,1}(\bar x_1,\bar x_2,\bar x_3;Z_1,Z_2;b_1,b_2,b_3;B)=\frac{1}{4(2\pi)^9}\frac{\bar x_1\bar x_2\bar x_3}{(Z_1-\bar x_1)(Z_2-\bar x_2-\bar x_3)}\int\tilde\psi_{1,M_1}(\bar x_1/Z_1;b_1)\tilde\psi_{2,M_2}(\bar x_2/Z_2,\bar x_3/Z_2;b_2,b_3)$$
$$\times\tilde\psi^*_{2,M_2}(\bar x_1/Z_2',\bar x_3/Z_2';b_3,b_2+B)\tilde\psi^*_{1,M_1}(\bar x_2/Z_1';B-b_1)dM_1^2dM_2^2. \quad (29)$$

The nuclear factors formally have the same expression as in the diagonal case, but the values of $Z$ are different on the two sides, while keeping the constraints $Z_1+Z_2=Z_1'+Z_2'=2$. We have, in fact, $Z_1-\bar x_1=Z_1'-\bar x_2$, $Z_2-\bar x_2=Z_2'-\bar x_1$.

$$W_{2,2}(\bar x_1,\bar x_2,\bar x_3;Z_1,Z_2;b_1,b_2,b_3;B)=\frac{1}{4(2\pi)^9}\frac{\bar x_1\bar x_2\bar x_3}{(Z_1-\bar x_1)(Z_2-\bar x_2-\bar x_3)}\int\tilde\psi_{1,M_1}(\bar x_1/Z_1;b_1)\tilde\psi_{2,M_2}(\bar x_2/Z_2,\bar x_3/Z_2;b_2,b_3)$$
$$\times\tilde\psi^*_{2,M_1}(\bar x_1/Z_1',\bar x_3/Z_1';b_2+B,b_1)\tilde\psi^*_{1,M_2}(\bar x_2/Z_2';b_3)dM_1^2dM_2^2. \quad (30)$$

Beyond the constraints $Z_1+Z_2=Z_1'+Z_2'=2$ we have the relations $Z_1=Z_1'-\bar x_3$, $Z_2=Z_2'+\bar x_3$.

$$W_{2,3}(\bar{x}_1,\bar{x}_2,\bar{x}_3;Z_1,Z_2;b_1,b_2,b_3;B)=\frac{1}{4(2\pi)^9}\frac{\bar{x}_1\bar{x}_2\bar{x}_3}{(Z_1-\bar{x}_1)(Z_2-\bar{x}_2-\bar{x}_3)}\int\tilde{\psi}_{1,M_1}(\bar{x}_1/Z_1;b_1)\tilde{\psi}_{2,M_2}(\bar{x}_2/Z_2,\bar{x}_3/Z_2;b_3)\times\tilde{\psi}^*_{2,M_1}(\bar{x}_2/Z'_1,\bar{x}_3/Z'_1;b_2+B,b_3+B)\tilde{\psi}^*_{1,M_1}(\bar{x}_1/Z'_2;B-b_1)dM_1^2dM_2^2. \quad (31)$$

Beyond the constraints $Z_1+Z_2=Z'_1+Z'_2=2$ we have the relations $Z_1-\bar{x}_1=Z'_1-\bar{x}_2-\bar{x}_3$, $Z_2-\bar{x}_2-\bar{x}_3=Z'_2-\bar{x}_1$.

We see that the nuclear factors formally have the same expression as in the diagonal part, but the values of $Z$ are different on the two sides. This feature was already found in the cross diagram for double scattering, and so the qualitative considerations are also of the same kind. Precisely, the longitudinal variables on one side are $Z_1$ and $Z_2$, whereas on the other side they are $Z'_1$ and $Z'_2$. In the nonrelativistic conditions of the internal motion, in particular, $N_\perp^2\ll m^2$, which are the actual conditions in the deuteron, the typical width of the nuclear wave function, in dimensionless variables, is of the order $\sqrt{(4m^2-M_D^2)/(M_D^2)}$, while the differences $Z-Z'$ are of the order of the fractional momenta $\bar{x}$.

Now we see that the cross sections for these particular processes can be obtained from the expression of the diagonal term [Eq. (25)] by substituting the factors $\Gamma(z_1,z_2;b_1,b_2)\Gamma(z_3;b_3)$ with the corresponding $W_{2,j}$ term, after solving the constraints which give $Z'_1$, $Z'_2$ in terms of $Z_1$, $Z_2$.

In comparing the diagonal term with the interference terms, one can repeat the same qualitative considerations made for the double scattering case; i.e. in the partonic amplitudes the denominators $Z$, $Z'$ could be set equal to 1, while the scale of the transverse variables is provided by the generalized parton distributions and is relatively small as compared with the nuclear size.

### C. Three different target nucleons interact with large transverse-momentum exchange

#### *1. General features and diagonal terms*

This kind of process can evidently happen only with a three-body nucleus (at least). Also, here the presence of the nuclear wave function induces the presence of two kinds of contributions: a diagonal term and a number of nondiagonal or interference terms.

The diagonal discontinuity (Fig. 8(a)) for the triple scattering is

$$\begin{aligned}\mathrm{Disc}\mathcal{B}^{(1,1,1)}|_d=&\frac{1}{(2\pi)^{38}}\int\frac{\check{\phi}_p}{l_1^2l_2^2l_3^2}\frac{\check{\phi}_p^*}{l'^2{}_1l'^2{}_2l'^2{}_3}\frac{\phi_1}{a_1^2}\frac{\phi_2}{a_2^2}\frac{\phi_3}{a_3^2}\frac{\hat{\phi}_1^*}{a'^2{}_1a'^2{}_2}\frac{\phi_3{}^*}{a_3^2}T_1(l_1a_1\to q_1q'_1)T_2(l_2a_2\to q_2q'_2)T_3(l_3a_3\to q_3q'_3)\\&\times T_1{}^*(l'{}_1a'{}_1\to q_1q'_1)T_2^*(l_2a_2\to q_2q'_2)T_3{}^*(l'{}_3a'{}_3\to q_3q'_3)\\&\times\frac{\Phi_T(N_1,N_2,N_3)}{[N_1^2-m^2][N_2^2-m^2][N_3^2-m^2]}\frac{\Phi_T{}^*(N'{}_1,N'{}_2,N'{}_3)}{[N'^2{}_1-m^2][N'^2{}_2-m^2][N'^2{}_3-m^2]}\delta\Big(L-\sum l-F_4\Big)\\&\times\delta(N_1-a_1-F_1)\delta(N_2-a_2-F_2)\delta(N_3-a_3-F_3)\delta\Big(L-\sum l'-F_4\Big)\delta(N'{}_1-a'{}_1-F_1)\\&\times\delta(N'{}_2-a'{}_2-F_2)\delta(N'{}_3-a'{}_3-F_3)\delta(l_1+a_1-Q_1)\delta(l'{}_1+a'{}_1-Q_1)\delta(l_2+a_2-Q_2)\\&\times\delta(l'{}_2+a'{}_2-Q_2)\delta(l_3+a_3-Q_3)\delta(l'{}_3+a'{}_3-Q_3)\\&\times\delta(T-N_1-N_2-N_3)\delta(T-N'{}_1-N'{}_2-N'{}_3)\prod d(\Omega/8)dada'dldl'dFdNdN'dQ\end{aligned}\quad (32)$$

and the mass-shell condition is $T^2=M_T^2$.

From the projectile side we have the three-parton densities, which have already been defined and used. On the tritium side we must use the full three-body structure of the nuclear wave function; the conservation $N_i=a_i+F_i$ gives $a_i^2=(N_i-F_i)_+a_{i-}-a_{i\perp}{}^2$. The integral to be performed may thus be written as

$$\mathcal{Y}_a=\int dN_{1+}dN_{2+}dN_{3+}\delta\Big(\sum_{i=1,2,3}N_{i+}-T_+\Big)\prod_{i=1,2,3}\frac{1}{N_{i+}N_{i-}m_{\perp i}{}^2}\frac{1}{(N-F)_{i+}a_{i-}-a_{i\perp}{}^2}.$$

As in the previous cases there are two kinds of singularities in the integrand: one kind puts a nucleon on mass shell, and the other one puts the parton on mass shell. Also, here the singularities putting the nucleons on mass shell are the most important ones. So we approximate $\mathcal{Y}_a$ as

$$\mathcal{Y}_a \approx -(2\pi)^2 \frac{1}{T_+ - I_1 - I_2 - I_3} \prod_{i=1,2,3} \frac{1}{[(m_\perp^2 - N_- F_+)_i a_{i-} - (N_- a_\perp^2)_i]}, \qquad I_j = m_{\perp j}{}^2/N_{-j}. \tag{33}$$

When considering the complete cut diagram we find that the relation between $l$, $a$, $Q$ again yields the equalities $l_+ = l'_+$, $a_- = a'_-$, whereas the equality of the remnants yields $N_- = N'_-$; these can be converted into the fractional momenta $Z$, with the constraint $Z_1 + Z_2 + Z_3 = 3$. Here it is also convenient to go from transverse momenta to transverse coordinates.

The factors $\psi$ and $\Gamma$ are defined as before. Since in $\mathcal{Y}_a$ the dependences on $a_\perp$ and on $N_\perp$ are interlocked, the Fourier transform must be performed with respect to both sets of transverse variables $a_\perp$ and $N_\perp$.

As before, $\beta_i$ are conjugated to $l_i$, $b_i$ are conjugated to $a_i$, and $B_i$ are conjugated to $N_i$. The conservations, which will again be conveniently expressed in exponential form, are $l_i + a_i = l'_i + a'_i$, $\sum l_i = \sum l'_i$, $N_i + a_i = N'_1 + a'_i$, $\sum N_i = \sum N'_i = T$. The integrations over the transverse momenta yield the equalities $\beta_i = \beta'_i$, $b_i = b'_i$, $B_i = B'_i$ and the geometrical conditions $b_i + B_i - \beta_i = \text{const}$ that can be expressed also as $b_i + B_i - \beta_i = b_j + B_j - \beta_j$; $i \neq j$.

We can write, for the diagonal term,

$$\begin{aligned}\sigma_{3,3}^{pT}|_d = {} & \frac{3}{(2\pi)^3} \int \Gamma(x_1, x_2, x_3; \beta_1, \beta_2, \beta_3) \Gamma(z_1; b_1) \Gamma(z_2; b_2) \Gamma(z_3; b_3) \frac{d\hat{\sigma}}{d\Omega_1} \frac{d\hat{\sigma}}{d\Omega_2} \frac{d\hat{\sigma}}{d\Omega_3} |\Psi_T(Z_i; B, \underline{B})|^2 \\ & \times dBd\underline{B}d\beta_1 d\beta_2 d\beta_3 db_1 db_2 db_3 \delta(b_1 - b_3 - \beta_1 + \beta_3 + \underline{B}) \delta(b_1 - b_2 - \beta_1 + \beta_2 + B) \\ & \times dx_1 dx_2 dx_3 dz_1 dz_2 dz_3 d\Omega_1 d\Omega_2 d\Omega_3 \delta(Z_1 + Z_2 + Z_3 - 3) dZ_1 dZ_2 dZ_3/(Z_1 Z_2 Z_3).\end{aligned} \tag{34}$$

In the vertex function only the differences $B_i - B_j$ are relevant; therefore the integration variables $B$, $\underline{B}$ represent a pair of these differences, e.g. $B = B_1 - B_2$, $\underline{B} = B_1 - B_3$.

### *2. Interference terms*

The discontinuity for the interference terms in triple scattering can be written as in the previous section:

$$\begin{aligned}\mathrm{Disc}\mathcal{B}^{(1,1,1)}|_i = {} & \frac{1}{(2\pi)^{38}} \int \frac{\check{\phi}_p}{l_1^2 l_2^2 l_3^2} \frac{\check{\phi}_p^*}{l'^2{}_1 l'^2{}_2 l'^2{}_3} \frac{\phi_1}{a_1^2} \frac{\phi_2}{a_2^2} \frac{\phi_3}{a_3^2} \frac{\hat{\phi}_1^*}{a'^2{}_1 a'^2{}_2} \frac{\phi_3{}^*}{a_3^2} T_1(l_1 a_1 \to q_1 q'_1) T_2(l_2 a_2 \to q_2 q'_2) T_3(l_3 a_3 \to q_3 q'_3) \\ & \times T_1{}^*(l'_1 a'_1 \to q_1 q'_1) T_2^*(l_2 a_2 \to q_2 q'_2) T_3{}^*(l'_3 a'_3 \to q_3 q'_3) \\ & \times \frac{\Phi_T(N_1, N_2, N_3)}{[N_1^2 - m^2][N_2^2 - m^2][N_3^2 - m^2]} \frac{\Phi_T{}^*(N'_1, N'_2, N'_3)}{[N'^2{}_1 - m^2][N'^2{}_2 - m^2][N'^2{}_3 - m^2]} \delta\Big(L - \sum l - F_4\Big) \\ & \times \delta\Big(L - \sum l' - F_4\Big) \eta(N_i, a_i) \delta(l_1 + a_1 - Q_1) \delta(l'_1 + a'_1 - Q_1) \delta(l_2 + a_2 - Q_2) \delta(l'_2 + a'_2 - Q_2) \\ & \times \delta(l_3 + a_3 - Q_3) \delta(l'_3 + a'_3 - Q_3) \delta(T - N_1 - N_2 - N_3) \delta(T - N'_1 - N'_2 - N'_3) \\ & \times \prod d(\Omega/8) da da' dl dl' dF dN dN' dQ_i.\end{aligned} \tag{35}$$

Here we find two essentially different realizations of the factors $\eta$,

$$\begin{aligned}\eta_{3,1}(N_i, a_i) &= \delta(N_1 - a_1 - F_1)\delta(N_2 - a_2 - F_2)\delta(N_3 - a_3 - F_3)\delta(N'_1 - a'_3 - F_1)\delta(N'_2 - a'_2 - F_2)\delta(N'_3 - a'_1 - F_3), \\ \eta_{3,2}(N_i, a_i) &= \delta(N_1 - a_1 - F_1)\delta(N_2 - a_2 - F_2)\delta(N_3 - a_3 - F_3)\delta(N'_1 - a'_2 - F_1)\delta(N'_2 - a'_3 - F_2)\delta(N'_3 - a'_1 - F_3),\end{aligned} \tag{36}$$

and this leads to the definition of two more auxiliary functions $W$.

$$\begin{aligned}W_{3,1}(\bar{x}_1, \bar{x}_2, \bar{x}_3; Z_1, Z_2, Z_3; b_1, b_2, b_3; B, \underline{B}) = {} & \frac{1}{4(2\pi)^9} \frac{\bar{x}_1 \bar{x}_2 \bar{x}_3}{(Z_1 - \bar{x}_1)(Z_2 - \bar{x}_2 - \bar{x}_3)} \tilde{\psi}_{1,M_1}(\bar{x}_1/Z_1; b_1) \tilde{\psi}_{1,M_2}(\bar{x}_2/Z_2; b_2) \\ & \times \tilde{\psi}_{1,M_3}(\bar{x}_3/Z_3; b_3) \tilde{\psi}^*_{1,M_3}(\bar{x}_1/Z_1; b_1 + \underline{B}) \tilde{\psi}^*_{1,M_2}(\bar{x}_2/Z_2; b_2) \tilde{\psi}^*_{1,M_1}(\bar{x}_3/Z_3; b_3 - \underline{B}).\end{aligned} \tag{37}$$

Beyond the constraints $Z_1 + Z_2 + Z_3 = Z'_1 + Z'_2 + Z'_3 = 3$ we have the relations $Z_1 - \bar{x}_1 = Z'_1 - \bar{x}_3$, $Z_2 = Z'_2$, $Z_3 - \bar{x}_3 = Z'_3 - \bar{x}_1$.

$$W_{3,2}(\bar{x}_1, \bar{x}_2, \bar{x}_3; Z_1, Z_2, Z_3; b_1, b_2, b_3; B, \underline{B}) = \frac{1}{4(2\pi)^9} \frac{\bar{x}_1 \bar{x}_2 \bar{x}_3}{(Z_1 - \bar{x}_1)(Z_2 - \bar{x}_2 - \bar{x}_3)} \tilde{\psi}_{1,M_1}(\bar{x}_1/Z_1; b_1) \tilde{\psi}_{1,M_2}(\bar{x}_2/Z_2; b_2) \times \tilde{\psi}_{1,M_3}(\bar{x}_3/Z_3; b_3) \tilde{\psi}^*_{1,M_3}(\bar{x}_1/Z_1; b_3 - \Delta_1) \tilde{\psi}^*_{1,M_2}(\bar{x}_2/Z_2; b_2 - \Delta_2) \times \tilde{\psi}^*_{1,M_1}(\bar{x}_3/Z_3; b_1 + \Delta_2 + \Delta_1). \quad (38)$$

Beyond the constraints $Z_1 + Z_2 + Z_3 = Z'_1 + Z'_2 + Z'_3 = 3$ we have the relations $Z_1 - \bar{x}_1 = Z'_1 - \bar{x}_2$, $Z_2 - \bar{x}_2 = Z'_2 - \bar{x}_3$, $Z_3 - \bar{x}_3 = Z'_3 - \bar{x}_1$.

Finally, the cross sections for these particular processes can be obtained from the expression of the diagonal term, Eq. (77), by substituting the factors $\Gamma(z_1; b_1)\Gamma(z_2; b_2)\Gamma(z_3; b_3)$ by the corresponding $W_{3,j}$ term, after solving the constraints which give $Z'_1$, $Z'_2$, $Z'_3$ in terms of $Z_1$, $Z_2$, $Z_3$. The transverse configurations are shown in Fig. 9.

Clearly, the qualitative considerations made previously for the ratio of the diagonal and the interference terms hold here also, since they depend on the existence of two scales (hadronic and nuclear) always playing the same role.

## IV. SIMPLEST ESTIMATES OF THE DOMINANT CONTRIBUTIONS

The nonperturbative component of MPI in $pA$ collisions is characterized by the hadronic and nuclear scales. In MPI, the relevant hadronic scale is the transverse dimension $R$ of the generalized parton distributions, which is smaller as compared with the hadron radius; it may be roughly a factor 4 smaller as compared with the radii of $D$, $^3$H, and $^3$He. Even with light nuclei, one may thus obtain a simple estimate of the dominant contributions to the MPI cross sections by neglecting the hadronic scale when compared to the nuclear scale. In the same spirit one may obtain a further simplification by evaluating the integrals on the fractional momenta of the bound nucleons, $Z_i$, by taking into account the dependence on $Z_i$ only in the nuclear wave function and replacing $Z_i$ with 1 everywhere else.

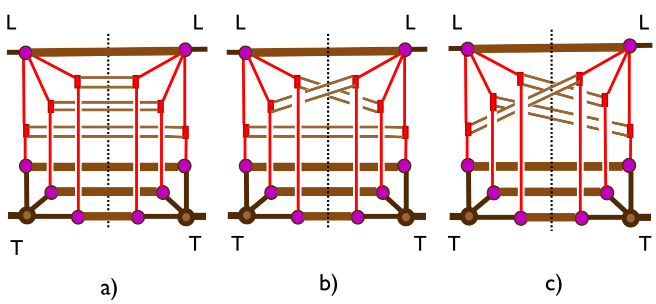


FIG. 8 (color online). Different contributions to triple parton scattering in $pT$ or $p\ ^3$He interactions. All target nucleons interact with large transverse-momentum exchange.

### A. Double scattering

The double parton scattering cross sections, for $pD$ and $p\ ^3$H (or $p\ ^3$He) collisions, are given by the sum of two contributions. In the first one only a single bound nucleon participates and in the second two bound nucleons participate in the hard interaction:

$$\begin{aligned} \sigma_2^{pD}(x_i, \bar{x}_i) &= \sigma_{2,1}^{pD}(x_i, \bar{x}_i) + \sigma_{2,2}^{pD}(x_i, \bar{x}_i), \\ \sigma_2^{pT}(x_i, \bar{x}_i) &= \sigma_{2,1}^{pT}(x_i, \bar{x}_i) + \sigma_{2,2}^{pT}(x_i, \bar{x}_i). \end{aligned} \quad (39)$$

With the simplifying assumptions above, the contributions where only a single bound nucleon participates are given by

$$\sigma_{2,1}^{pD} \simeq 2\sigma_D \quad \text{and} \quad \sigma_{2,1}^{pT} \simeq 3\sigma_D \quad (40)$$

where $\sigma_D$ is the double parton scattering inclusive cross section on an isolated nucleon. One may define

$$\Gamma(x_1, x_2; \beta_1, \beta_2) \equiv K_{x_1,x_2} G(x_1) G(x_2) f_{x_1,x_2}(\beta_1, \beta_2) \quad \text{with} \quad \int f_{x_1,x_2}(\beta_1, \beta_2) d\beta_1 d\beta_2 = 1 \quad (41)$$

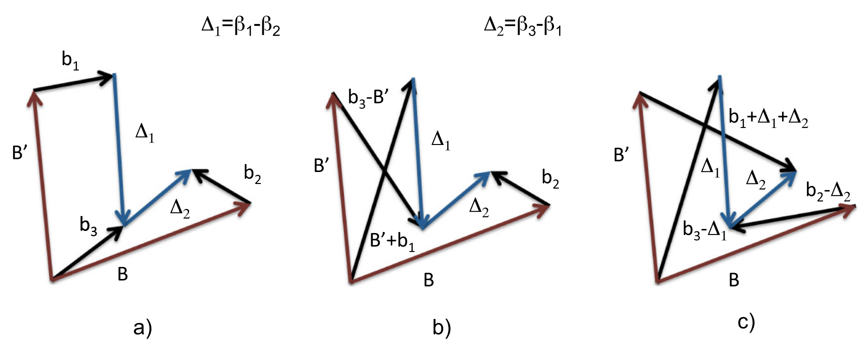


FIG. 9 (color online). Configurations in transverse space of the three amplitudes in Fig. 8, in the right-hand side of the cut.

where $G(x)$ is the one-body inclusive parton distribution, such that $G(x) = \int \Gamma(x; b) db$. In the case of identical interactions, one may thus express the double parton scattering cross section on an isolated nucleon as

$$\sigma_D(x_1, \bar{x}_1, x_2, \bar{x}_2) = \frac{1}{2} K_{x_1,x_2} K_{\bar{x}_1,\bar{x}_2} \int f_{x_1,x_2}(\beta_1, \beta_2) f_{\bar{x}_1,\bar{x}_2}(b_1, b_2) \sigma_S(x_1, \bar{x}_1) \sigma_S(x_2, \bar{x}_2) \delta(\beta_1 - \beta_2 - b_1 + b_2) d\beta_1 d\beta_2 db_1 db_2 \tag{42}$$

where $\sigma_S = \int G(x) \hat{\sigma}(x, x') G(x') dx dx'$ is the single parton scattering inclusive cross section on an isolated nucleon. The effective cross section, namely, the accessible experimental information in nucleon-nucleon collisions, is thus given by

$$\frac{1}{\sigma_{\rm eff}(x_1, \bar{x}_1, x_2, \bar{x}_2)} = K_{x_1,x_2} K_{\bar{x}_1,\bar{x}_2} \int f_{x_1,x_2}(\beta_1, \beta_1 - \Delta_1) f_{\bar{x}_1,\bar{x}_2}(b_1, b_1 - \Delta_1) d\beta_1 db_1 d\Delta_1 \tag{43}$$

where $\Delta_1 = \beta_1 - \beta_2$. In $pD$ collisions, when two nucleons participate in the hard interaction, one has contributions from a diagonal and from an off-diagonal term. The dominant contribution to the diagonal term is

$$\begin{aligned} \sigma_{2,2}^{pD}|_d(x_1, \bar{x}_1, x_2, \bar{x}_2) &= \frac{1}{(2\pi)^3} \int \Gamma(x_1, x_2; \beta_1, \beta_2) \hat{\sigma}(x_1, \bar{x}_1) \hat{\sigma}(x_2, \bar{x}_2) \Gamma(\bar{x}_1/Z; b_1) \Gamma(\bar{x}_2/(2-Z); b_2) |\tilde{\Psi}_D(Z; B)|^2 \\ &\quad \times dZ/Z^2 dB db_1 db_2 d\beta_1 d\beta_2 \delta(B - b_1 + b_2 - \Delta_1) \\ &\simeq K_{x_1,x_2} \sigma_S(x_1, \bar{x}_1) \sigma_S(x_2, \bar{x}_2) \mathcal{I}_D(0) \end{aligned} \tag{44}$$

where the general form of $\mathcal{I}_D(x)$ is

$$\mathcal{I}_D(x) \equiv \frac{1}{(2\pi)^3} \int \tilde{\Psi}_D(Z; 0) \tilde{\Psi}_D^*(Z'; 0) \frac{dZ dZ'}{ZZ'} \delta(Z - Z' - x). \tag{45}$$

The range of the variables $b_i$ and $\beta_j$, defined by the partonic distributions $\Gamma$, is much narrower as compared with the nuclear range of $B$. The $\delta$ function in Eq. (44) thus forces, in $\tilde{\Psi}_D$, $B \approx 0$.

Notice that, differently from the case of double parton scattering in nucleon-nucleon collisions, the scale factor in $\sigma_{2,2}^{pD}|_d$ is given by the value of $\mathcal{I}_D(0)$, which is determined by the radius of the deuteron, while the cross section is proportional to $K_{x_1,x_2}$, which gives the partonic correlation in fractional momenta. As already noticed in [58], $\sigma_{2,2}^{pD}|_d$ thus depends weakly on the correlation between partons in the transverse coordinates; on the contrary, it may provide rather direct information on the size of $K_{x_1,x_2}$.

The contribution of the interference term to the cross section is

$$\begin{aligned} \sigma_{2,2}^{pD}|_i(x_1, \bar{x}_1, x_2, \bar{x}_2) &= \frac{1}{(2\pi)^3} \int \Gamma(x_1, x_2; \beta_1, \beta_2) \hat{\sigma}(x_1, \bar{x}_1) \hat{\sigma}(x_2, \bar{x}_2) W_1(Z, Z'; \bar{x}_1, \bar{x}_2; b_1, b_2, B) \tilde{\Psi}_D(Z; B) \tilde{\Psi}_D^*(Z'; B) \\ &\quad \times [ZZ']^{-1} dB db_1 db_2 d\beta_1 d\beta_2 dZ dZ' \delta(B - b_1 + b_2 - \Delta_1) \delta(Z - Z' - \bar{x}_1 + \bar{x}_2). \end{aligned} \tag{46}$$

By neglecting the hadron scale with respect to the nuclear scale and keeping $Z \neq 1$ only in the deuteron wave function, the integrations in $b_1$ and $b_2$ in Eq. (46) are

$$\begin{aligned} \int W_1(1, 1; \bar{x}_1, \bar{x}_2; b_1, b_2, b_1 - b_2 + \Delta_1) db_1 db_2 &= \frac{1}{4(2\pi)^6} \int dM_1^2 dM_2^2 db_1 db_2 \frac{\bar{x}_1 \bar{x}_2}{(1 - \bar{x}_1)(1 - \bar{x}_2)} \psi_{M_1}(\bar{x}_1; b_1) \\ &\quad \times \psi_{M_2}^*(\bar{x}_2; b_2 - \Delta_1) \psi_{M_2}(\bar{x}_1; b_2) \psi_{M_1}^*(\bar{x}_2; b_1 + \Delta_1) \\ &= \tilde{H}(\bar{x}_1, \bar{x}_2; \Delta_1) \tilde{H}(\bar{x}_2, \bar{x}_1; -\Delta_1) \end{aligned} \tag{47}$$

where the generalized parton distributions $\tilde{H}$ have been introduced:

$$\tilde{H}(\bar{x}_1, \bar{x}_2; \Delta_1) \equiv \int H(1, 1; \bar{x}_1, \bar{x}_2; b_1, b_2) \delta(b_1 - b_2 - \Delta_1) db_1 db_2, \tag{48}$$

and $H$ has been defined in Sec. II B. Notice that the normalization is $\tilde{H}(x, x; 0) = G(x)$. One may thus define

$$C_1(x_1, \bar{x}_1, x_2, \bar{x}_2) = \frac{\int f_{x_1,x_2}(\Delta_1)\tilde{H}(\bar{x}_1, \bar{x}_2; \Delta_1)\tilde{H}(\bar{x}_2, \bar{x}_1; -\Delta_1)d\Delta_1}{G(\bar{x}_1)G(\bar{x}_2)} \tag{49}$$

which is dimensionless and weakly dependent on $\bar{x}_1, \bar{x}_2$ as compared to $I_D(\bar{x}_1 - \bar{x}_2)$ since $C_1$ originates from the partonic structure of the hadron while $I_D$ originates from the nuclear structure. The contribution of the interference term to the cross section may thus be expressed as

$$\sigma_{2,2}^{pD}|_i(x_1, \bar{x}_1, x_2, \bar{x}_2) \simeq K_{x_1,x_2}\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)C_1(x_1, \bar{x}_1, x_2, \bar{x}_2)I_D(\bar{x}_1 - \bar{x}_2). \tag{50}$$

Notice that both $\sigma_{2,2}^{pD}|_d$ and $\sigma_{2,2}^{pD}|_i$ depend linearly on $K_{x_1,x_2}$, and both terms are proportional to the inverse of the square of the deuteron radius, the latter term through the nuclear off-diagonal factor $I_D(\bar{x}_1 - \bar{x}_2)$, which induces a much stronger dependence of $\sigma_{2,2}^{pD}|_i$ on $\bar{x}_1 - \bar{x}_2$ as compared with $\sigma_{2,2}^{pD}|_d$.

In the case of double parton interactions in $p\ {}^3$H or $p\ {}^3$He collisions, with two target nucleons taking part in the hard interaction, one obtains the same expressions for the dominant contributions to the cross sections as in the case of $pD$ interactions. The only difference is in the multiplicity factors and in the terms $I_D(0)$ and $I_D(\bar{x}_1 - \bar{x}_2)$, which are replaced by the corresponding quantities with ${}^3$H or ${}^3$He, actually $I_T(0)$ and $I_T(\bar{x}_1 - \bar{x}_2)$.

The leading contributions to the double parton scattering cross sections in $pD$ and $p\ {}^3$H, $p\ {}^3$He are thus given by

$$\begin{aligned} \sigma_2^{pD}(x_i, \bar{x}_i) &= \sigma_{2,1}^{pD}(x_i, \bar{x}_i) + \sigma_{2,2}^{pD}|_d(x_i, \bar{x}_i) + \sigma_{2,2}^{pD}|_i(x_i, \bar{x}_i) \\ &\simeq 2\sigma_D(x_i, \bar{x}_i) + K_{x_1,x_2}\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)[I_D(0) + C_1(x_i, \bar{x}_i)I_D(\bar{x}_1 - \bar{x}_2)], \\ \sigma_2^{pT}(x_i, \bar{x}_i) &= \sigma_{2,1}^{pT}(x_i, \bar{x}_i) + \sigma_{2,2}^{pT}|_d(x_i, \bar{x}_i) + \sigma_{2,2}^{pT}|_i(x_i, \bar{x}_i) \\ &\simeq 3\sigma_D(x_i, \bar{x}_i) + 3K_{x_1,x_2}\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)[I_T(0) + C_1(x_i, \bar{x}_i)I_T(\bar{x}_1 - \bar{x}_2)]. \end{aligned} \tag{51}$$

The contributions $\sigma_{2,1}^{pD}$ and $\sigma_{2,1}^{pT}$ are well approximated by $2\sigma_D$ and by $3\sigma_D$. The actual values can be evaluated with great accuracy, once the double parton scattering cross sections in $pp$ and in $pn$ collisions are known as a function of fractional momenta. Also the nuclear terms $I_D(0)$, $I_D(\bar{x}_1 - \bar{x}_2)$, $I_T(0)$, and $I_T(\bar{x}_1 - \bar{x}_2)$ can be evaluated very accurately. By measuring the double parton scattering cross sections in $pD$ and $p\ {}^3$H (or $p\ {}^3$He) one may thus obtain accurate estimates of the differences $\sigma_2^{pD} - \sigma_{2,1}^{pD}$ and $\sigma_2^{pT} - \sigma_{2,1}^{pT}$ and, as a consequence, of the rations $\mathcal{R}_D$ and $\mathcal{R}_T$, defined as

$$\begin{aligned} \mathcal{R}_D(x_i, \bar{x}_i) &\equiv \frac{\sigma_2^{pD}(x_i, \bar{x}_i) - \sigma_{2,1}^{pD}(x_i, \bar{x}_i)}{\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)} \simeq K_{x_1,x_2}[I_D(0) + C_1(x_i, \bar{x}_i)I_D(\bar{x}_1 - \bar{x}_2)], \\ \mathcal{R}_T(x_i, \bar{x}_i) &\equiv \frac{\sigma_2^{pT}(x_i, \bar{x}_i) - \sigma_{2,1}^{pT}(x_i, \bar{x}_i)}{\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)} \simeq 3K_{x_1,x_2}[I_T(0) + C_1(x_i, \bar{x}_i)I_T(\bar{x}_1 - \bar{x}_2)]. \end{aligned} \tag{52}$$

One should point out that the contribution of the interference term is not always present. As an example, in the case of the production of $W$ + jets, through double parton collisions, the interference term is absent. In such a case, the second term in square brackets in (52) is missing and Eq. (52) allows a direct estimate of $K_{x_1,x_2}$, namely, of the importance of the correlations in $x$ in the double parton distributions. When the interference term is present, the cross section depends on the additional unknown quantity, $C_1(x_i, \bar{x}_i)$, which multiplies the nuclear overlap integrals $I_D(\bar{x}_1 - \bar{x}_2)$ or $I_T(\bar{x}_1 - \bar{x}_2)$. In this way the main dependence of the cross section on $X \equiv \bar{x}_1 - \bar{x}_2$ is originated. By measuring the cross section at different values of $X$, one may construct the fraction

$$\frac{\Delta\mathcal{R}_{D,T}(x_i, \bar{x}_i)}{\Delta X} \simeq K_{x_1,x_2}C_1(x_i, \bar{x}_i)\frac{\Delta I_{D,T}(X)}{\Delta X}. \tag{53}$$

By studying the dependence of $\mathcal{R}_D$ and of $\mathcal{R}_T$ on $X = \bar{x}_1 - \bar{x}_2$, with the help of Eqs. (52) and (53), one may obtain information both on $C_1(x_i, \bar{x}_i)$ and on $K_{x_1,x_2}$. Notice that the values of $\Delta X$ needed in Eq. (53) may not be too small. The natural scale of $X = \bar{x}_1 - \bar{x}_2$ is in fact $(E_B/m)^{1/2} \approx 5 \times 10^{-2}$, where $E_B$ is the nuclear binding energy.

The indication of the value of $K_{x_1,x_2}$, together with the measure of the effective cross section in nucleon-nucleon collisions, allows us to obtain an indication of the value of the integral $\int f_{x_1,x_2}(\Delta_1)f_{\bar{x}_1,\bar{x}_2}(\Delta_1)d\Delta_1$ [cf. Eq. (43)]. With the help of Eqs. (52) and (53), one may estimate $C_1(x_i, \bar{x}_i)$ and obtain an indication of the value of the integral $\int f_{x_1,x_2}(\Delta_1)\tilde{H}(\bar{x}_1,\bar{x}_2;\Delta_1)\tilde{H}(\bar{x}_2,\bar{x}_1;-\Delta_1)d\Delta_1$ [cf. Eq. (49)]. The information on the two integrals will provide important constraints on the correlation length between partons in transverse space, which is explicit in $f_{x_1,x_2}(\Delta_1)$. By measuring the double parton scattering cross section in $pD$ and $p\ ^3\mathrm{H}$ (or $p\ ^3\mathrm{He}$) one may thus learn both about correlations between partons in fractional momenta, through the factor $K_{x_1,x_2}$, and about correlations between partons in the transverse coordinates.

### B. Triple scattering

As in the case of double parton scattering, the triple parton scattering cross section in $pD$ and $p\ ^3\mathrm{H}$ or $p\ ^3\mathrm{He}$ collisions can be written as

$$\sigma_3^{pD}(x_i,\bar{x}_i)=\sigma_{3,1}^{pD}(x_i,\bar{x}_i)+\sigma_{3,2}^{pD}(x_i,\bar{x}_i)$$
$$\sigma_3^{pT}(x_i,\bar{x}_i)=\sigma_{3,1}^{pT}(x_i,\bar{x}_i)+\sigma_{3,2}^{pT}(x_i,\bar{x}_i)+\sigma_{3,3}^{pT}(x_i,\bar{x}_i). \tag{54}$$

With the simplifying assumptions discussed in the previous section, the dominant contribution to the terms where only a single bound nucleon undergoes a triple parton interaction is given by

$$\sigma_{3,1}^{pD}\simeq 2\sigma_T, \qquad \sigma_{3,1}^{pT}\simeq 3\sigma_T \tag{55}$$

where $\sigma_T$ is the triple parton scattering inclusive cross section on an isolated nucleon. By making the positions

$$\Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3) \equiv K_{x_1,x_2,x_3}G(x_1)G(x_2)G(x_3)f_{x_1,x_2,x_3}(\beta_1,\beta_2,\beta_3) \tag{56}$$

with

$$\int f_{x_1,x_2,x_3}(\beta_1,\beta_2,\beta_3)d\beta_1 d\beta_2 d\beta_3=1 \tag{57}$$

one may express, in the case of identical interactions, the triple parton scattering cross section on an isolated nucleon as

$$\sigma_T(x_1,\bar{x}_1,x_2,\bar{x}_2,x_3,\bar{x}_3)=\frac{1}{6}K_{x_1,x_2,x_3}K_{\bar{x}_1,\bar{x}_2,\bar{x}_3}\int f_{x_1,x_2,x_3}(\beta_i)f_{\bar{x}_1,\bar{x}_2,\bar{x}_3}(b_i)\delta(\beta_1-\beta_2-b_1+b_2)\delta(\beta_1-\beta_3-b_1+b_3)\times\prod\sigma_S(x_i,\bar{x}_i)d\beta_i db_i. \tag{58}$$

#### *1. Two different target nucleons interact with large transverse-momentum exchange*

The contribution to the triple parton scattering cross section, where two target nucleons undergo hard interactions, is the process $\mathcal{O}(1/(S^2R^2))$, to be compared with triple scattering on a single nucleon, which is of $\mathcal{O}(1/R^4)$; $R$ and $S$ are the hadronic and nuclear scales. In the case of $pD$ collisions, the different contributions to the cross section are summarized by the expression

$$\sigma_{3,2}^{pD}|_j=\frac{2}{(2\pi)^3}\mathcal{N}_j\int\Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)\frac{d\hat{\sigma}}{d\Omega_1}\frac{d\hat{\sigma}}{d\Omega_2}\frac{d\hat{\sigma}}{d\Omega_3}W_{2,j}(\bar{x}_1,\bar{x}_2,\bar{x}_3;Z_1,Z_2;b_1,b_2,b_3;B)\tilde{\Psi}_D(Z_1,Z_2;B)\tilde{\Psi}_D^*(Z_1',Z_2';B)$$
$$\times dBdZ_1dZ_1'dZ_2dZ_2'/(Z_1Z_1')\prod_i dx_id\bar{x}_idb_id\beta_id\Omega_idZ_i\delta(Z_1+Z_2-2)\delta(Z_1'+Z_2'-2)\delta(Z_1-Z_1'-X_j)$$
$$\times\delta(Z_2-Z_2'+X_j)\delta(b_2-b_1+\Delta_1)\delta(B-b_1+b_3-\Delta_2) \tag{59}$$

where $\Delta_1=\beta_1-\beta_2$ and $\Delta_2=\beta_3-\beta_1$, while the index $j$ corresponds to the diagonal case, when $j=0$, and to the three different interference terms, when $j=1, 2, 3$ (cf. Figs. 6 and 7). By neglecting the hadronic scale $R$ as compared to the nuclear scale $S$, in the diagonal case one obtains

$$W_{2,0}(\bar{x}_1,\bar{x}_2,\bar{x}_3;Z_1,Z_2;b_1,b_2,b_3;B)\equiv\tfrac{1}{2}\Gamma(z_1,z_2;b_1,b_2)\Gamma(z_3;b_3). \tag{60}$$

The quantity $X_j$ assumes the following values:

$$X_0=0,\qquad X_1=\bar{x}_1-\bar{x}_2,\qquad X_2=-\bar{x}_2,\qquad X_3=\bar{x}_1-\bar{x}_2-\bar{x}_3, \tag{61}$$

and the multiplicity factors $\mathcal{N}_j$ are $\mathcal{N}_0=2$, $\mathcal{N}_1=4$, $\mathcal{N}_2=4$, $\mathcal{N}_3=2$.

Analogously to the case previously discussed, the dominant contributions may be estimated by

$$\sigma_{3,2}^{pD}|_j\simeq\mathcal{N}_j\int\Gamma(x_1,x_2,x_3;\beta_1,\beta_2,\beta_3)\hat{\sigma}(x_1,\bar{x}_1)\hat{\sigma}(x_2,\bar{x}_2)\hat{\sigma}(x_3,\bar{x}_3)$$
$$\times W_{2,j}(\bar{x}_1,\bar{x}_2,\bar{x}_3;1,1;b_1,b_2,b_3;b_1-b_3+\Delta_2)I_D(X_j)\delta(b_2-b_1+\Delta_1)\prod_i dx_id\bar{x}_idb_id\beta_i \tag{62}$$

where all effects of the deuteron wave function are summarized in the terms $I_D(X_j)$:

$$I_D(X_j)=\frac{1}{(2\pi)^3}\int\tilde{\Psi}_D(Z;0)\tilde{\Psi}_D^*(Z';0)\delta(Z-Z'-X_j)\frac{dZdZ'}{ZZ'} \tag{63}$$

as already defined in Eq. (45). By introducing

$$C_{2,j}(x_i, \bar{x}_i) = \frac{\int f_{x_1,x_2,x_3}(\beta_i) W_{2,j}(\bar{x}_i; 1, 1; b_i; b_1 - b_3 + \Delta_2)\delta(b_2 - b_1 + \Delta_1)\prod_i db_i \beta_i}{G(\bar{x}_1)G(\bar{x}_2)G(\bar{x}_3)} \quad (64)$$

one obtains

$$\sigma^{pD}_{3,2}|_j(x_i, \bar{x}_i) \simeq \mathcal{N}_j K_{x_i} \sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2) \times \sigma_S(x_3, \bar{x}_3) C_{2,j}(x_i, \bar{x}_i) I_D(X_j). \quad (65)$$

The expression of the diagonal contribution is

$$\sigma^{pD}_{3,2}|_0(x_i, \bar{x}_i) \simeq \tfrac{1}{2} K_{x_1,x_2,x_3} K_{\bar{x}_1\bar{x}_2} \sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2) \times \sigma_S(x_3, \bar{x}_3)\mathcal{G}_{x_i\bar{x}_k} I_D(0) \quad (66)$$

where

$$\mathcal{G}_{x_i\bar{x}_k} = \int f_{x_1,x_2,x_3}(\beta_1, \beta_2, \beta_3) f_{\bar{x}_1\bar{x}_2}(b_1, b_2)\delta(b_2 - b_1 + \Delta_1) \times \prod d\beta_i db_k. \quad (67)$$

As in the case of double collisions one may introduce the ratio

$$\mathcal{R}'_D(x_i, \bar{x}_i) \equiv \frac{\sigma^{pD}_{3,2}(x_i, \bar{x}_i)}{\prod \sigma_S(x_i, \bar{x}_i)} = \frac{\sum_j \sigma^{pD}_{3,2}|_j(x_i, \bar{x}_i)}{\prod \sigma_S(x_i, \bar{x}_i)} \simeq K_{x_i}\sum_j \mathcal{N}_j C_{2,j}(x_i, \bar{x}_i) I_D(X_j) \quad (68)$$

and, for $j > 0$, construct the fraction

$$\frac{\Delta \mathcal{R}'_D(x_i, \bar{x}_i)}{\Delta X_j} \simeq K_{x_i}\mathcal{N}_j C_{2,j}(x_i, \bar{x}_i)\frac{\Delta I'_D(X_j)}{\Delta X_j}. \quad (69)$$

Differently with respect to the case of the double collisions, Eqs. (68) and (69) do not allow us to disentangle $K_{x_i}$ from $C_{2,j}(x_i, \bar{x}_i)$. Disentangling the effects of longitudinal and transverse correlations is possible in the case of double collisions because, in that case, the dominant contribution to the diagonal term depends only on $K_{x_1,x_2}$. In the actual case, on the contrary, $C_{2,0}(x_i, \bar{x}_i)$ is proportional to the product $K_{x_i}\mathcal{G}_{x_i\bar{x}_k}$. By studying triple scattering on a deuteron, one may only obtain an estimate of the products $K_{x_i}C_{2,j}(x_i, \bar{x}_i)$. To gain further insight into longitudinal and transverse three-body correlations, one needs additional information, which can be provided by triple parton interactions in collisions of protons with $^3$H or with $^3$He.

When two target nucleons participate in the hard interaction, after integrating the spectator nucleon, $p$ $^3$H (or $p$ $^3$He) gives results very similar to $pD$ collisions. The two dominant contributions to the diagonal term differ in fact only by an overall multiplicity factor (which is actually 3) and in the factors $I_T(X_j)$, which substitute for the factors $I_D(X_j)$. Analogously to the case of $D$, $I_T(X_j)$ represents the square of the $^3$H (or $^3$He) wave function in the mixed representation, integrated in the fractional momenta $Z_i$ with the constraints given in Eq. (59), and in the relative transverse distance $B'$, while the transverse distance $B$ has been set equal to zero. One thus obtains the relation

$$\sigma^{pT}_{3,2}|_j(x_i, \bar{x}_i) \simeq \sigma^{pD}_{3,2}|_j(x_i, \bar{x}_i)\frac{I_T(X_j)}{I_D(X_j)}. \quad (70)$$

Equation (70) is a consequence of Eq. (68), which holds in the limit $R^2/S_D^2 \to 0$. If $S_T$ is the $^3$H (or $^3$He) radius, the two terms in Eq. (70) are thus of $\mathcal{O}(1/(R^2S_T^2))$ and the relation is exact in the limit $R^2/S_D^2 \to 0$. Finite values of $R$ contribute, in the left-hand side, with terms of $\mathcal{O}(1/(R^2S_T^2) \times R^2/S_T^2)$ and with terms of $\mathcal{O}(1/(R^2S_T^2) \times R^2/S_D^2)$ in the right-hand side of the equation. One may thus estimate that Eq. (70) is valid up to terms of $\mathcal{O}(1/S_T^2 \times (1/S_T^2 - 1/S_D^2) \approx (S_D^2 - S_T^2)/S_T^6)$.

### *2. Three different target nucleons interact with large transverse-momentum exchange*

In the case of the contribution to the triple parton scattering cross section, $\sigma^{pT}_{3,3}$, where three different target nucleons interact with large transverse-momentum exchange, in $p$ $^3$H or $p$ $^3$He collisions one has three different terms, one diagonal and two off diagonal, which are labeled with the index $j$ in the expression below (cf. Figs. 8 and 9). As in the previous section, the label $j = 0$ corresponds to the diagonal case.

$$\sigma^{pT}_{3,3}|_j = \frac{2}{(2\pi)^3}\mathcal{N}'_j \int \Gamma(x_1, x_2, x_3; \beta_1, \beta_2, \beta_3)\frac{d\hat{\sigma}}{d\Omega_1}\frac{d\hat{\sigma}}{d\Omega_2}\frac{d\hat{\sigma}}{d\Omega_3} W_{3,j}(\bar{x}_1, \bar{x}_2, \bar{x}_3; Z_1, Z_2, Z_3; b_1, b_2, b_3; B, B')\tilde{\Psi}_T(Z_1, Z_2, Z_3; B, B') \times \tilde{\Psi}^*_T(Z_1,' Z'_2, Z'_3; B, B') dB dZ_1 dZ'_1 dZ_2 dZ'_2 dZ_3 dZ'_3 [Z_1Z'_1Z_2Z'_2Z_3Z'_3]^{-1/2}\delta(Z_1 + Z_2 + Z_3 - 3) \times \delta(Z'_1 + Z'_2 + Z'_3 - 3)\delta(Z_1 - Z'_1 - Y_{j,1})\delta(Z_2 - Z'_2 + Y_{j,2})\delta(Z_3 - Z'_3 + Y_{j,3})\delta(B' + b_1 - b_3 + \Delta_1) \times \delta(b_3 - b_2 - \Delta_2 - B)\prod_i dx_i d\bar{x}_i db_i d\beta_i d\Omega_i dB dB'. \quad (71)$$

With the approximations previously discussed one obtains

$$\sigma_{3,3}^{pT}|_j \simeq \frac{2}{(2\pi)^3}\mathcal{N}'_j \int \Gamma(x_1, x_2, x_3; \beta_1, \beta_2, \beta_3)\hat{\sigma}(x_1, \bar{x}_1)\hat{\sigma}(x_2, \bar{x}_2)\hat{\sigma}(x_3, \bar{x}_3) \times W_{3,j}(\bar{x}_1, \bar{x}_2, \bar{x}_3; 1, 1, 1; b_1, b_2, b_3; b_3 - b_2 - \Delta_2, -b_1 + b_3 - \Delta_1)\mathcal{J}(Y_{j,i=1,3})\prod_i dx_i d\bar{x}_i db_i d\beta_i \quad (72)$$

where

$$\mathcal{J}(Y_{j,i=1,3}) = \frac{2}{(2\pi)^3}\mathcal{N}'_j \int \tilde{\Psi}_T(Z_1, Z_2, Z_3; 0, 0)\tilde{\Psi}_T^*(Z'_1, Z'_2, Z'_3; 0, 0)dZ_1 dZ'_1 dZ_2 dZ'_2 dZ_3 dZ'_3[Z_1Z'_1Z_2Z'_2Z_3Z'_3]^{-1/2} \times \delta(Z_1 + Z_2 + Z_3 - 3)\delta(Z'_1 + Z'_2 + Z'_3 - 3)\delta(Z_1 - Z'_1 - Y_{j,1})\delta(Z_2 - Z'_2 + Y_{j,2}) \quad (73)$$

and

| | | | |
|---|---|---|---|
| $Y_{j=0,i=1,3}$: | $Y_{0,1} = 0$; | $Y_{0,2} = 0$; | $Y_{0,3} = 0$ |
| $Y_{j=1,i=1,3}$: | $Y_{1,1} = \bar{x}_1 - \bar{x}_3$; | $Y_{1,2} = 0$; | $Y_{1,3} = \bar{x}_3 - \bar{x}_1$ |
| $Y_{j=2,i=1,3}$: | $Y_{2,1} = \bar{x}_1 - \bar{x}_2$; | $Y_{2,2} = \bar{x}_2 - \bar{x}_3$; | $Y_{2,3} = \bar{x}_3 - \bar{x}_1$ |

while the multiplicity factors are $\mathcal{N}'_0 = 1$, $\mathcal{N}'_1 = 3$, $\mathcal{N}'_1 = 2$. Introducing

$$C_{3,j}(x_i, \bar{x}_i) = \frac{\int f_{x_1,x_2,x_3}(\beta_i)W_{3,j}(\bar{x}_i; 1, 1, 1; b_i; b_3 - b_2 - \Delta_2, -b_1 + b_3 - \Delta_1)\prod_i db_i d\beta_i}{G(\bar{x}_1)G(\bar{x}_2)G(\bar{x}_3)} \quad (74)$$

one obtains

$$\sigma_{3,3}^{pT}(x_i, \bar{x}_i)|_j \simeq \mathcal{N}'_j K_{x_1,x_2,x_3}\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)\sigma_S(x_3, \bar{x}_3)C_{3,j}(x_i, \bar{x}_i)\mathcal{J}(Y_{j,i=1,3}). \quad (75)$$

In the case $j = 0$ one has

$$C_{3,0}(x_i, \bar{x}_i) = 1 \quad (76)$$

and one obtains

$$\sigma_{3,3}^{pT}|_0(x_i, \bar{x}_i) \simeq \tfrac{1}{6}K_{x_1,x_2,x_3}\sigma_S(x_1, \bar{x}_1)\sigma_S(x_2, \bar{x}_2)\sigma_S(x_3, \bar{x}_3)\mathcal{J}(0). \quad (77)$$

For $j = 1$ one has

$$\int W_{3,1}(\bar{x}_i; 1, 1, 1; b_i; b_3 - b_2 - \Delta_2, -b_1 + b_3 - \Delta_1)\prod db_i = \tilde{H}(\bar{x}_1, \bar{x}_3; \Delta_1)G(x_2)\tilde{H}(\bar{x}_3, \bar{x}_1; -\Delta_1) \quad (78)$$

in such a way that

$$C_{3,1}(x_i, \bar{x}_i) = \frac{\int f_{x_1,x_2,x_3}(\Delta_1)\tilde{H}(\bar{x}_1, \bar{x}_3; \Delta_1)\tilde{H}(\bar{x}_3, \bar{x}_1; -\Delta_1)d\Delta_1}{G(\bar{x}_1)G(\bar{x}_3)} \quad (79)$$

where

$$f_{x_1,x_2,x_3}(\Delta_1) = \int f_{x_1,x_2,x_3}(\beta_i)\delta(\Delta_1 - \beta_1 + \beta_2)\prod d\beta_i. \quad (80)$$

For $j = 2$ one has

$$\int W_{3,2}(\bar{x}_i; 1, 1, 1; b_i; b_3 - b_2 - \Delta_2, -b_1 + b_3 - \Delta_1)\prod db_i = \tilde{H}(\bar{x}_1, \bar{x}_3; \Delta_1 + \Delta_2)\tilde{H}(\bar{x}_2, \bar{x}_1; -\Delta_2)\tilde{H}(\bar{x}_3, \bar{x}_2; -\Delta_1) \quad (81)$$

and

$$C_{3,2}(x_i, \bar{x}_i) = \frac{\int f_{x_1,x_2,x_3}(\Delta_1, \Delta_2)\tilde{H}(\bar{x}_1, \bar{x}_3; \Delta_1 + \Delta_2)\tilde{H}(\bar{x}_2, \bar{x}_1; -\Delta_2)\tilde{H}(\bar{x}_3, \bar{x}_2; -\Delta_1)d\Delta_1 d\Delta_2}{G(\bar{x}_1)G(\bar{x}_2)G(\bar{x}_3)} \quad (82)$$

where

$$f_{x_1,x_2,x_3}(\Delta_1, \Delta_2) = \int f_{x_1,x_2,x_3}(\beta_i)\delta(\Delta_1 - \beta_1 + \beta_2)\delta(\Delta_2 + \beta_1 - \beta_3)\prod d\beta_i. \quad (83)$$

As in the case of double parton collisions, discussed previously, the contribution to the triple parton scattering cross section $\sigma_{3,1}^{pT}$ is given to a good approximation by $3\sigma_T$, where $\sigma_T$ is the triple parton scattering cross section on an isolated nucleon. Once the triple parton scattering cross sections in $pp$ and in $pn$ collisions are known as a function of fractional momenta, the smearing effects of the nuclear wave function can be taken into account and $\sigma_{3,1}^{pT}$ can be evaluated with great accuracy. Also, the off-diagonal nuclear terms $\mathcal{J}(Y_{j,i=1,3})$ can be evaluated with great accuracy. By measuring $\sigma_3^{pT}$, one may thus obtain an accurate estimate of the difference $\sigma_3^{pT} - \sigma_{3,1}^{pT}$. Equation (70) allows one to estimate $\sigma_{3,2}^{pT}$. One may thus define

$$\mathcal{R}'_T(x_i, \bar{x}_i) \equiv \frac{\sigma_{3,3}^{pT}(x_i, \bar{x}_i)}{\prod \sigma_S(x_i, \bar{x}_i)} \tag{84}$$

where

$$\sigma_3^{pT}(x_i, \bar{x}_i) = \sigma_{3,1}^{pT}(x_i, \bar{x}_i) + \sigma_{3,2}^{pT}(x_i, \bar{x}_i) + \sigma_{3,3}^{pT}(x_i, \bar{x}_i),$$
$$\sigma_{3,2}^{pT}(x_i, \bar{x}_i) = \sum_j \sigma_{3,2}^{pT}(x_i, \bar{x}_i)|_j \simeq \sum_j \sigma_{3,2}^{pD}(x_i, \bar{x}_i)|_j \frac{I_T(X_j)}{I_D(X_j)}, \tag{85}$$

and then relate the "known" quantity $\mathcal{R}'_T$ to the unknown properties of the hadron structure, represented by $K_{x_i}$ and $C_{3,j}(x_i, \bar{x}_i)$:

$$\mathcal{R}'_T(x_i, \bar{x}_i) = \frac{\sum_j \sigma_{3,3}^{pT}|_j(x_i, \bar{x}_i)}{\prod \sigma_S(x_i, \bar{x}_i)} \simeq K_{x_i} \sum_j \mathcal{N}'_j C_{3,j}(x_i, \bar{x}_i) \mathcal{J}(Y_{j,i=1,3}). \tag{86}$$

Analogously to the case of double parton scattering, an indication of triple correlations in fractional momenta and in the transverse coordinates can then be obtained by looking at the variation of $\mathcal{R}'_T$ as a function of $Y_{j,i}$ and using the property that $C_{3,0} = 1$.

Notice that Eq. (86) is a consequence of Eq. (70), which holds up to terms of $\mathcal{O}((S_D^2 - S_T^2)/S_T^6)$. Since the right-hand side of Eq. (86) is of $\mathcal{O}(1/S_T^4)$, one may estimate that the relative correction to the dominant terms in Eq. (86) is only of $\mathcal{O}((S_D^2 - S_T^2)/S_T^2 \approx 1/5)$. Equation (86) can therefore provide only a semiquantitative indication of the size of triple correlations, while a better determination requires a dedicated study.

## V. FINAL DISCUSSION

MPI in $pA$ collisions allow one to obtain information on multiparton correlations, which cannot be provided by studying MPI in $pp$ collisions [54]. Relevant features of the simplest case, double parton interactions in $pD$ collisions, were pointed out in [58]. In the present paper, we have extended the study of MPI in $pA$ collisions to the cases of double and triple parton interactions in collisions of protons with $D$, $^3$H, and $^3$He, also including the effects of interference terms in the discussion.

Double parton interactions in collisions of protons with $D$, $^3$H, and $^3$He are discussed in Sec. II. When only a single nucleon takes part in the hard process (Sec. II A), the integrations on the relative transverse coordinates of the spectator nucleons are decoupled from all other transverse variables and the cross section is the same as that measured in nucleon-nucleon collisions, apart from the proper multiplicity factor and the smearing corrections in the longitudinal variables, which, as discussed in Sec. IV, are, however, rather small. The explicit expressions of the cross sections for $pD$, $p\ ^3$H and $p\ ^3$He collisions are given in Eq. (10). Similar considerations hold in the case of triple parton collisions on a single nucleon, which is discussed in Sec. III A. The corresponding contribution to the triple parton scattering cross section is given in Eq. (22). Notice that the spectator nucleons are on mass shell. As already discussed in [58], in spite of that, one may still claim that final state interactions of the spectators are approximately taken into account. The statement is supported by unitarity: If a nucleon is produced on mass shell and undergoes a final state interaction with the remnants of another nucleon, the final state interaction does not modify the inclusive cross section, since the spectators are not observed. If a nucleon is produced off mass shell, its virtuality is rather small and it may not be unreasonable to extend the unitarity relation $SS^\dagger = 1$ to the actual kinematical domain. Unitarity hence allows the replacement of all final state interactions with *cut* nucleon lines, i.e. with on mass shell nucleons.

In Sec. II B, we discuss the case of double parton collisions, with two nucleons taking part in the hard process. In addition to the diagonal contribution in Fig. 2(a) discussed in [58], which leads to the geometrical picture of the interaction in transverse space shown in Fig. 3, one has a nondiagonal contribution from Fig. 2(b). The geometrical picture in transverse space, of the corresponding interfering configurations (a) and (a*), is shown in Fig. 4. Notice that, in both interfering configurations, the hard interactions are localized at the same points and are well separated in transverse space. As a consequence, the argument for the suppression of the interference terms in MPI, discussed in [61], does not apply in this case. Differently from $pp$ collisions, the additional degrees of freedom provided by the nucleus, namely, the possibility of having different nucleons involved in the hard process, can in fact produce the same partonic initial state in different ways, which can thus interfere in the process. As discussed in Sec. II B, the contribution of the interference term is important in the region where the fractional momenta of the interacting partons are of order $\sqrt{E_B/m}$, where $E_B$ and $m$ are the nuclear binding energy and nucleon mass, respectively. Differently from the diagonal contributions, which are dominated by the most probable nuclear configuration, where all nucleons' fractional momenta are equal, in the off-diagonal term of Fig. 2(b) the fractional momenta $Z_1$, $Z_1'$ and $Z_2$, $Z_2'$, of the two nucleons taking part in the hard process, are forced by kinematics to be different: $Z_1 - Z_1' = -(Z_2 - Z_2') = \bar{x}_1 - \bar{x}_2$. Here $\bar{x}_1$, $\bar{x}_2$ are the fractional momenta of the two target partons undergoing the double collisions.

As discussed in Sec. IVA, the dominant contribution to the interference term can be expressed in terms of off-diagonal parton distributions, Eqs. (47) and (48). The contribution of the interference term can be singled out by looking at the dependence of the double parton scattering cross section on the difference $\bar{x}_1 - \bar{x}_2$ [cf. Eq. (53)]. Taking into account that the scale which characterizes the dependence of the nuclear wave function on $Z$ is $\sqrt{E_B/m}$, one may roughly estimate that, to single out the contribution of the interference term, one needs to measure the double parton scattering cross section in an interval $\bar{x}_1 - \bar{x}_2 \approx 5 \times 10^{-2}$ with an accuracy greater than 10%. The whole discussion assumes that each couple of scattered partons, and the resulting observed particles, can be identified as a definite pair, which requires that each couple is sufficiently separated in phase space from the other couples. The quantitative amount of this separation depends on the detailed properties of the final state.

As mentioned in Sec. IVA, by studying the ratios in Eqs. (52) and (53), using the information on double parton interactions in $pp$ collisions, and taking into account that the dominant contribution to the diagonal term depends only on $K_{x_1,x_2}$, one may obtain information on parton correlations in fractional momenta and, through the overlap integrals in the transverse parton coordinates, which characterize $\sigma_{\mathrm{eff}}$ and the interference term, also on parton correlations in the transverse coordinates. When the interference term is absent, as for production of $W$ + jets, the task of estimating parton correlations may be simpler. In that case, as discussed in [58], all information concerning correlations may be obtained directly from Eq. (52).

The case of triple parton interactions has many features similar to the case of double parton interactions. The main difference is in the sizably larger number of contributing terms. As already pointed out, the contribution where only a single nucleon participates in the hard process is well approximated by the cross section on an isolated nucleon, multiplied by the multiplicity of target nucleons, while nuclear smearing effects can give only minor corrections. This description becomes complex when two or three nucleons participate in the hard process. The general features of the contributions where two nucleons participate in the hard process are discussed in Sec. III B while, in Sec. IV B 1, one may find a simplified estimate of the different terms. In the case of two participating nucleons, one finds a diagonal and three different off-diagonal contributions. Differently from the case of double parton collisions, in the case of triple parton collisions on two different target nucleons, the dominant contribution to the diagonal term depends both on the correlations in fractional momenta, through $K_{x_1,x_2,x_3}$, and on the correlations in the transverse coordinates, through the overlap function $\mathcal{G}_{x_i,\bar{x}_k}$ [defined by Eq. (67)]. A consequence is that one cannot disentangle the effects of longitudinal and transverse correlations in triple parton collisions by studying the ratios in Eqs. (68) and (69) in $pD$ interactions only.

The information on triple parton collisions in $pD$ interactions can, nevertheless, be utilized to estimate the contribution to triple parton collisions with two participating nucleons, in the case of $p\ ^3$H or $p\ ^3$He collisions. By measuring the triple parton scattering cross section in $p\ ^3$H or $p\ ^3$He collisions, one may thus estimate, using Eq. (70), the contribution to the cross section where all three target nucleons are involved in the hard process and thus figure out the value of the ratio $\mathcal{R}_T'$ in Eq. (86). A relevant feature of $\sigma_{3,3}^{pT}$, the component of the triple parton scattering cross section with three participating target nucleons, is that the leading contribution to the diagonal term, as given by Eq. (77), is proportional to $K_{x_1,x_2,x_3}$ and does not depend on the correlations in the transverse coordinates. By studying the dependence of $\mathcal{R}_T'$ on $Y_{i,j}$ one can thus obtain an estimate both of $K_{x_1,x_2,x_3}$ and of the different overlap integrals in the transverse coordinates, which characterize the different interference terms. As discussed in the last part of Sec. IV B 2, the uncertainties in the determination of $\sigma_{3,2}^{pT}$, by means of Eq. (70), can, however, allow only a qualitative estimate of triple parton correlations, while a better determination requires a dedicated study. Some preliminary results of these investigations were presented in [62].

## VI. CONCLUSIONS

The aim of the present paper is to study the possibility of obtaining model-independent information on multiparton correlations, by measuring MPI in high-energy hadron-nucleus collisions. Two different kinds of correlations, in fractional momenta and in the transverse coordinates, are in fact unavoidably linked and cannot be disentangled, when studying MPI in $pp$ collisions only. The simplest case, namely, double parton interactions (DPI) in $pD$ collisions, is characterized by novel and nontrivial features, as compared to DPI in $pp$ collisions. The component of the cross section, where both target nucleons contribute to the process, depends in fact only weakly on the correlations in transverse space. In addition, one also has a contribution from an interference term. All the different contributions to the cross section can be disentangled, and all the new unknown quantities appearing in the reaction and directly related to parton correlations, in fractional momenta and in the transverse coordinates, can be isolated in an essentially model-independent way. An interesting feature is that the interference term is expressed through the off-diagonal parton distributions. By studying the interference term one may thus also gain information on the off-diagonal parton distributions, in kinematical ranges not easily accessible through other processes.

In order to disentangle triple parton correlations in fractional momenta and in the transverse coordinates, one needs to measure triple parton interactions in $p\,^3$H or $p\,^3$He collisions. In fact, the knowledge of the cross section of double and of triple parton interactions in $pp$ and in $pD$ collisions is not sufficient to isolate all the unknown quantities which appear in the reaction. On the other hand, taking into account that the radii of $D$ and of $^3$H and $^3$He are not very different, we can learn much from triple parton interactions in $pD$ collisions about the contribution to triple parton interactions in $p\,^3$H or $p\,^3$He collisions when only two target nucleons play an active role in the process. As discussed in Sec. IV B, taking advantage of the information on triple parton interactions in $pp$ and $pD$ collisions, it is in fact rather simple to figure out how to obtain, from the cross section of triple parton interactions in $p\,^3$H or $p\,^3$He collisions, a model-independent, although only semiquantitative, indication of the different components of the cross section, with a direct link either to the triple parton correlations in fractional momenta or to the triple parton correlations in the transverse coordinates.

Our conclusion is that MPI of hadrons with light nuclei have a great potential to provide information on the multiparton structure of the hadron. Following the lines outlined in the present paper one can obtain model-independent indications on multiparton correlations, albeit a detailed quantitative evaluation of the effects of multiparton correlations on the cross sections will require a model-dependent numerical study. By measuring the cross sections with a given number of MPI on various nuclear targets, one may in fact identify different features of the incoming parton flux, allowing one to isolate diverse terms of the correlated multiparton structure. To our knowledge, this result cannot be accomplished by other means. The option of studying MPI in collisions of protons with light nuclei at the Relativistic Heavy Ion Collider and to run, at some stage, light nuclear beams at the LHC could therefore be highly rewarding, offering the possibility to exploit the remarkable potential of MPI in $pA$ collisions to yield information on the many-body parton correlations and thus to provide unprecedented insight into the three-dimensional structure of the hadron.

## APPENDIX A: THE NONRELATIVISTIC THREE-BODY WAVE FUNCTION

The nuclear systems we have considered ($^3$H, $^3$He) can be treated with a nonrelativistic dynamics in their center-of-momentum frame, but since they are involved in a highly relativistic process it is necessary to match this internal nonrelativistic dynamics with the overall relativistic treatment. To this end the original procedure used by Salpeter [63,64] to reduce the Bethe-Salpeter equation to the Schrödinger equation will be followed as strictly as possible. We are only able to set a correspondence between nonrelativistic and relativistic wave functions; we are not able to build up a wholly deductive procedure as in the quoted references. For simplicity, we treat both the constituent and the bound state as spinless bosons. Our final aim is to use the nonrelativistic wave functions (as they are known from nuclear physics) in our relativistic calculation with the correct factors and the correct kinematical transformation.

The starting point is given by homogeneous equations in relativistic forms as obtained by a Feynman graph representation, in term of the two-body scattering matrices as suggested by the Faddeev [65,66] treatment of three-body scattering:

$$\begin{aligned} U_3(q_1, q_2, q_3) &= \Delta(q_1)\Delta(q_2)\sum_{J\neq 3}\int i\mathsf{t}_3(q_1+q_2, k) \\ &\quad \times U_J(q_1-k, q_2+k, q_3)dk, \\ \mathsf{t}_3 &= V_3 + G^o{}_3 V_3 \mathsf{t}_3, \\ G_3^o &= \Delta(q_1)\Delta(q_2), \end{aligned} \tag{A1}$$

together with the two analogous terms for $U_1$ and $U_2$. An iteration of the above equation shows that all three terms $U_J$ are preceded by the product of the free propagators of the constituent particles $\Delta(q_1)\Delta(q_2)\Delta(q_3)$, as is explicit in the graphical description in Fig. 10.

Defining $T = q_1 + q_2 + q_3$ the system is nonrelativistic in the frame $\mathbf{T} = 0$.

When the two-body scattering matrices $\mathsf{t}_J$ do not depend on the relative energies but only on the three-momenta, as it happens in nonrelativistic dynamics, we can integrate $U_J$ in $k_o$. A generalization of this procedure suggests the following Ansatz:

$$U_J(q_1, q_2, q_3) = i\Delta(q_1)\Delta(q_2)\Delta(q_3)\Phi_J(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}). \tag{A2}$$

This form means that, in the nonrelativistic limit, the bound particles are near the mass shell, so the singularities of the $\Delta$ factors are the most important; moreover, in this limit, antiparticles are not relevant, so the contribution of the antiparticle poles does not need to be taken into account in the integrations. This can be explicitly written as

$$\begin{aligned} \Delta(q) &= \frac{i}{q^2 - m^2} = \frac{i}{2\omega}\left[\frac{1}{q_o - \omega} - \frac{1}{q_o + \omega}\right] \\ &\approx \frac{i}{2\omega}\frac{1}{q_o - \omega}. \end{aligned}$$

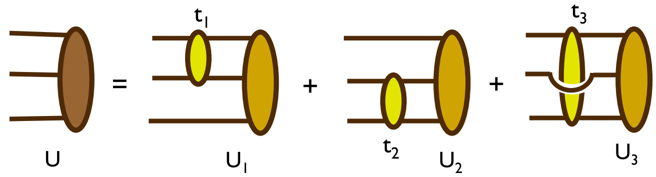


FIG. 10 (color online). Faddeev equation.

By inserting Eq. (A2) into Eq. (A1), dropping the common factors $\Delta(q_1)\Delta(q_2)$ and integrating in $k_o$, we obtain

$$\Delta(q_3)\Phi_3(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = \Delta(q_3)\sum_{J\neq 3}\int i\mathrm{t}_3(q_1+q_2,\mathbf{k})\Delta(q_1-k)\Delta(q_2+k)\Phi_J(\mathbf{q_1}-\mathbf{k}, \mathbf{q_2}+\mathbf{k}, \mathbf{q_3})dk$$
$$= \Delta(q_3)\sum_{J\neq 3}\int \frac{2\pi}{4\tilde{\omega}_1\tilde{\omega}_2}\frac{1}{(T-q_3)_o-\tilde{\omega}_1-\tilde{\omega}_2+i\epsilon}\mathrm{t}_3(q_1+q_2,\mathbf{k})\Phi_J(\mathbf{q_1}-\mathbf{k}, \mathbf{q_2}+\mathbf{k}, \mathbf{q_3})d^3k. \quad \text{(A3)}$$

The subenergies are $\omega_i = [(\mathbf{q_i})^2 + m^2]^{1/2}$, $\tilde{\omega}_1 = [(\mathbf{q_1}-\mathbf{k})^2+m^2]^{1/2}$, $\tilde{\omega}_2 = [(\mathbf{q_2}+\mathbf{k})^2+m^2]^{1/2}$. We proceed by integrating over $(q_3)_o$. Three sources of singularities need to be considered:

(i) Singularities of the propagators where $(q_3)_o$ appears directly.

(ii) Singularities of the propagators containing $(q_1)_o$, $(q_2)_o$, where $(q_3)_o$ enters because we work at constant $T_o$. [Actually it is convenient to define $q_1 = (T-q_3)/2 + l$, $q_2 = (T-q_3)/2 - l$.]

(iii) Singularities of $\mathrm{t}_3$, possibly originating from the two-body subsystem (1 + 2), which can be either poles like $C/[(T-q_3)_o - \eta + i\epsilon]$ (two-body bound states) or cuts like $\int \rho(\eta)d\eta/[(T-q_3)_o - \eta + i\epsilon]$ (two-body scattering states).

A direct inspection shows that the pole $(q_3)_o = \omega_3 - i\epsilon$ has an imaginary part with an opposite sign from all other poles in $(q_3)_o$, so its contribution gives the whole result of the integration. One obtains

$$\Phi_3(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = 2\pi\sum_{J\neq 3}\int d^3k\frac{1}{4\tilde{\omega}_1\tilde{\omega}_2}\mathrm{t}_3(T-q_3,\mathbf{k}) \times \frac{1}{T_o-\tilde{\omega}_1-\tilde{\omega}_2-\omega_3+i\epsilon}\Phi_J(\mathbf{q_1}-\mathbf{k}, \mathbf{q_2}+\mathbf{k}, \mathbf{q_3}). \quad \text{(A4)}$$

It is useful to define $T_o = M_T = 3m + E_B$, $\omega_J = m + \kappa_J$. As a consequence, $E_B < 0$, $\kappa_J > 0$. For the function defined as

$$\varphi_3(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = \frac{\mathcal{N}}{E_B-\kappa_1-\kappa_2-\kappa_3}\Phi_3(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) \quad \text{(A5)}$$

one obtains the equation

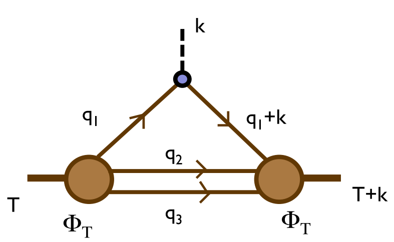


FIG. 11 (color online). Photon tritium vertex.

$$\varphi_3 = 2\pi G_o(E_B)\sum_{J\neq 3}\int \mathrm{t}_3\varphi_J d^3k \quad \text{with}$$
$$G_o(E_B) = \frac{1}{E_B-\kappa_1-\kappa_2-\kappa_3},$$

which, together with the two analogous equations for $\varphi_1$, $\varphi_2$, represents the usual Faddeev equation for a nonrelativistic three-body system [65,66]; in fact, $E_B$ is the binding energy and $\kappa_J$ are the nonrelativistic kinetic energies.

Defining as usual $\mathbf{q_1} = \mathbf{T}/3 + \mathbf{p_s}/2 + \boldsymbol{l}$, $\mathbf{q_2} = \mathbf{T}/3 + \mathbf{p_s}/2 - \boldsymbol{l}$, $\mathbf{q_3} = \mathbf{T}/3 - \mathbf{p_s}$, in the $\mathbf{T} = 0$ frame, the wave function depends on two three-vectors. We set

$$\sum_J \varphi_J(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) = \varphi(\mathbf{p_s}, \boldsymbol{l})$$

where $\varphi$ is the complete nonrelativistic bound-state wave function, which can be made real; we must satisfy the normalization condition

$$\int \varphi^2 d^3p_s d^3l = 1. \quad \text{(A6)}$$

The relations (A1), (A4), and (A5) are linear and cannot give the normalization constant $\mathcal{N}$. A normalization condition may be obtained by considering the total charge of the bound state.

The coupling of tritium with an external electromagnetic field (see Fig. 11) is written as

$$\mathcal{J}^\mu(k) = 2T^\mu f(k^2) = \int \Phi(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3})\Delta(q_1)2q_1^\mu\Delta(q_1+k) \times \Delta(q_2)\Delta(q_3)\Phi(\mathbf{q_1}+\mathbf{k}, \mathbf{q_2}, \mathbf{q_3}) \times \delta\Big(T-\sum q\Big)\prod dq. \quad \text{(A7)}$$

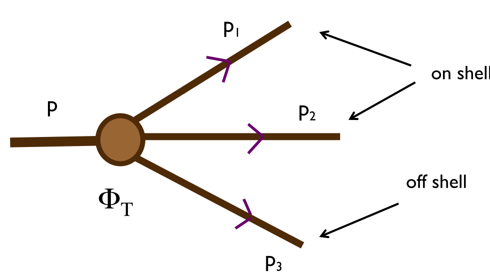


FIG. 12 (color online). $T/^3$He vertex—one nucleon is virtual and two nucleons are on shell.

In the limit $k \to 0$ the zero component must give the total charge of the bound state $\mathcal{J}_0(0) = 2T_o$, or, in other words, the form factor must satisfy $f(0) = 1$. Now we use the relations $q_1 = T - q_2 - q_3$ and then, keeping the prescription of neglecting the antiparticle poles, we set

$$\Delta(q)^2 \approx -\frac{1}{4q_o\omega}\left[\frac{1}{q_o - \omega}\right]^2.$$

The integration over the particle poles in $q_{2_o}$, $q_{3_o}$ is performed in the frame $\mathbf{T} = 0$ with the result

$$\begin{aligned}\mathcal{J}_0(0) &= -(2\pi)^2 \int \prod \frac{d^3q}{2\omega}\Phi(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}) \\ &\quad\times \left[\frac{1}{T_o - \omega_1 - \omega_2 - \omega_3}\right]^2 \\ &\quad\times \Phi(\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3})\delta\left(\sum \mathbf{q}\right) \\ &= 2M_T. \end{aligned} \tag{A8}$$

We compare (A8) with (A6) At first sight, we might conclude that we need an $\mathcal{N}$ that depends on $\omega_i$, but we remember that in the present treatment we neglect $\kappa$ with respect to $m$, but not with respect to $E_B$. Finally, we have

$$\mathcal{N} = 2m\sqrt{mM_T}/\pi. \tag{A9}$$

The function $\varphi$ goes as $q^3$ [see (A6)]; then the function $\Phi$ has power *zero* in $q$ as expected. In fact, $\Phi$ finally plays the role of an effective coupling in a four-boson relativistic vertex. Summing up the procedure, we start from $\varphi$, as it is known in nuclear physics; we then construct $\Phi$ [Eqs. (A5) and (A9)], and from it we obtain [Eq. (6)] the expression of $\Psi$ that, after Fourier transformation, enters in the expression of the cross sections.

The functions $\varphi$ are given in terms of the three-dimensional momenta in the c.m. of the nucleus $\mathbf{q}_i$; they need to be expressed in terms of the light-cone fractional momenta to evaluate the cross sections in the main text. In the case of interest, two nucleons are on shell (see Fig. 12). Introducing the invariants

$$\begin{aligned} s_{12} &= (p_1 + p_2)^2 = (T - p_3)^2, \\ s_{23} &= (p_2 + p_3)^2 = (T - p_1)^2, \\ s_{13} &= (p_1 + p_3)^2 = (T - p_2)^2 \end{aligned} \tag{A10}$$

one has

$$\begin{aligned} &p_1^2 = p_2^2 = m^2, \qquad T^2 = M_T^2, \qquad p_3^2 \neq m^2, \\ &s_{12} + s_{23} + s_{13} = M_T^2 + 2m^2 + p_3^2. \end{aligned} \tag{A11}$$

The light-cone four-momenta components are given by

$$\begin{aligned} T &\equiv \left(\frac{M_T^2}{T_-}, T_-, 0\right), \\ p_1 &\equiv \left(\frac{m_{1\perp}^2}{Z_1(T_-/3)}, Z_1T_-/3, \mathbf{p}_{1\perp}\right), \\ p_2 &\equiv \left(\frac{m_{2\perp}^2}{Z_2(T_-/3)}, Z_2T_-/3, \mathbf{p}_{2\perp}\right), \\ p_3 &\equiv \left(\frac{p_3^2 + (\mathbf{p}_{1\perp} + \mathbf{p}_{2\perp})^2}{(3 - Z_1 - Z_2)(T_-/3)}, (3 - Z_1 - Z_2)\right. \\ &\qquad \left. \times T_-/3, -(\mathbf{p}_{1\perp} + \mathbf{p}_{2\perp})\right) \end{aligned} \tag{A12}$$

where $m_{i\perp}^2 = m^2 + \mathbf{p}_{i\perp}^2$ are the transverse masses. The four-momentum conservation

$$\begin{aligned} M_T^2 &= \frac{3}{Z_1}m_{1\perp}^2 + \frac{3}{Z_2}m_{2\perp}^2 \\ &\quad + \frac{3}{3 - Z_1 - Z_2}[p_3^2 + (\mathbf{p}_{1\perp} + \mathbf{p}_{2\perp})]^2 \end{aligned} \tag{A13}$$

implies

$$\begin{aligned} p_3^2 - m^2 &= (3 - Z_1 - Z_2)\left[\frac{M_T^2}{3} - \frac{m_{1\perp}^2}{Z_1} - \frac{m_{2\perp}^2}{Z_2}\right. \\ &\qquad \left. - \frac{m_{3\perp}^2}{3 - Z_1 - Z_2}\right]. \end{aligned} \tag{A14}$$

In the $T/^3$He center-of-mass frame, the nucleon's energies $E_i$, $i = 1, 2, 3$ are expressed in terms of the invariants $t_i$ as follows:

$$\begin{aligned} E_1 &= \frac{M_T^2 + m^2 - s_{23}}{2M_T}, \\ E_2 &= \frac{M_T^2 + m^2 - s_{31}}{2M_T}, \\ E_3 &= \frac{M_T^2 + p_3^2 - s_{12}}{2M_T}. \end{aligned} \tag{A15}$$

The relations

$$\begin{aligned} s_{23} &= M_T^2 + m^2 - M_T^2\frac{Z_1}{3} - m_{1\perp}^2\frac{3}{Z_1}, \\ s_{31} &= M_T^2 + m^2 - M_T^2\frac{Z_2}{3} - m_{2\perp}^2\frac{3}{Z_2} \end{aligned} \tag{A16}$$

allow us to express $E_1$ and $E_2$ in terms of fractional momenta and transverse masses,

$$\begin{aligned} E_1 &= \frac{1}{2M_T}\left(M_T^2\frac{Z_1}{3} + m_{1\perp}^2\frac{3}{Z_1}\right), \\ E_2 &= \frac{1}{2M_T}\left(M_T^2\frac{Z_2}{3} + m_{2\perp}^2\frac{3}{Z_2}\right). \end{aligned} \tag{A17}$$

By using $E_i^2 = \mathbf{q}_i^2 + m^2$ one obtains similar expressions for center-of-mass three-momenta $q_{1z}$ and $q_{2z}$,

$$q_{1z} = \frac{1}{2M_T}\left(M_T^2 \frac{Z_1}{3} - m_{1\perp}^2 \frac{3}{Z_1}\right),$$
$$q_{2z} = \frac{1}{2M_T}\left(M_T^2 \frac{Z_2}{3} - m_{2\perp}^2 \frac{3}{Z_2}\right). \tag{A18}$$

Taking into account

$$M_T = E_1 + E_2 + E_3, \qquad \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0 \tag{A19}$$

one obtains the analogous relations for $E_3$ and $q_{3z}$,

$$E_3 = \frac{1}{2M_T}\left[M_T^2\left(2 - \frac{Z_1}{3} - \frac{Z_2}{3}\right) - 3\frac{m_{1\perp}^2}{Z_1} - 3\frac{m_{2\perp}^2}{Z_2}\right],$$
$$q_{3z} = \frac{-1}{2M_T}\left[M_T^2\left(\frac{Z_1}{3} + \frac{Z_2}{3}\right) - 3\frac{m_{1\perp}^2}{Z_1} - 3\frac{m_{2\perp}^2}{Z_2}\right], \tag{A20}$$

which allow us to express explicitly the nonrelativistic nuclear wave functions as a function of the fractional momenta $Z_i$ and of the transverse momenta $\mathbf{p}_{i\perp}$, i.e. in terms of variables invariant under longitudinal boost.

## APPENDIX B: MODELS WORKED OUT COMPLETELY

We present here two models where more explicit calculations are carried out after having introduced more or less strong simplifications of the real dynamics.

### 1. A model with the Hulthén wave function

Since the Hulthén potential, with its relative wave function, is one of the simplest potentials used in preliminary analyses of the deuteron properties, we present here a short derivation of some properties which are relevant for our investigations, in particular, for an estimate of the relevance of the interference terms.

The relative two-body wave function for the ground state ($S$ wave) in $r$ and $p$ representations is

$$h(\mathbf{r}) = \frac{\sqrt{\kappa\tau(\tau+\kappa)/2\pi}}{\tau - \kappa}\frac{1}{r}\left[e^{-\kappa r} - e^{-\tau r}\right],$$
$$\tilde{h}(\mathbf{p}) = \frac{\sqrt{\kappa\tau(\tau+\kappa)}}{\pi(\tau-\kappa)}\left[\frac{1}{\mathbf{p}^2 + \kappa^2} - \frac{1}{\mathbf{p}^2 + \tau^2}\right] \tag{B1}$$

where $\kappa = \sqrt{mE}$, $\tau = \kappa + \mu$, the binding energy is $E = 2m - M_D$, $1/\mu$ should represent the range of the potential (actually it has been fitted phenomenologically to $\mu \approx 5\kappa$), and $m/2$ is the reduced mass, assuming equal masses for the nucleons. The normalization is

$$4\pi\int |h(\mathbf{r})|^2 r^2 dr = 1, \qquad 4\pi\int |\tilde{h}(\mathbf{p})|^2 p^2 dp = 1.$$

Now we may calculate the mean value and the dispersion of the radial coordinate, with the results

$$\langle r\rangle = \frac{\tau^2 + 4\tau\kappa + \kappa}{2\kappa\tau(\kappa+\tau)},$$
$$\langle r^2\rangle - \langle r\rangle^2 = \frac{\tau^4 + 2\tau^3\kappa + 6\tau^2\kappa^2 + 2\tau\kappa^3 + \kappa^4}{[2\tau\kappa(\kappa+\tau)]^2}. \tag{B2}$$

In the actual case the parameters satisfy the condition $\kappa \ll \tau$ and there is a strong simplification:

$$\langle r\rangle \approx \sqrt{\langle r^2\rangle - \langle r\rangle^2} \approx 1/2\kappa.$$

The longitudinal variable $Z$ is studied in an analogous way, and the results are

$$\langle Z^2\rangle = \frac{4}{M_D^2}\left[\frac{4}{3}\langle\mathbf{p}^2\rangle + m^2\right],$$
$$\langle Z^4\rangle = \frac{16}{M_D^4}\left[\frac{16}{5}\langle\mathbf{p}^4\rangle + 4m^2\langle\mathbf{p}^2\rangle + m^4\right] \tag{B3}$$

with the mean values

$$\langle\mathbf{p}^2\rangle = \kappa\tau, \qquad \langle\mathbf{p}^4\rangle = \kappa\tau(\kappa^2 + 3\kappa\tau + \tau^2).$$

The qualitative behaviors of the parameters $B$, $Z$ are that the transverse extension is as large as one expected and that the relative dispersion is quite large; the longitudinal variable $Z$ is centered around 1, with a relatively small dispersion at least for $\kappa < \tau < m$, where we find

$$\sqrt{\langle Z^4\rangle - \langle Z^2\rangle^2}/\langle Z^2\rangle = \sqrt{3\kappa\tau}/6m.$$

The fact that the fractional momentum is slightly larger than 1 is due to the fact the we are, really, considering an unsymmetrical situation where one of the bound nucleons is put on mass shell; the configuration is symmetrical in the space variables $\mathbf{p}$, but it is not symmetrical in the relative energies. It seems that the more interesting result is the dispersion in $Z$; in fact, this is the parameter which says how much the contribution of the interference term, where $Z \neq Z'$, differs from the diagonal term.

We are also interested in a mixed representation where the transverse degrees of freedom are expressed in space variables $B$ while the longitudinal degree is given in light-cone variables $p_+$, $p_-$. We recall that we are interested in a particular kinematical situation where one of the bound nucleons is treated as real on mass shell, but we are still in the center-of-momentum frame so that the two three-momenta are opposite; then we shall consider a longitudinal boost. In this situation we obtain $p_z = m_\perp^2/2p_- - p_-/2$, $m_\perp^2 = m^2 + p_\perp^2$. With these definitions we obtain

$$\frac{1}{\mathbf{p}^2 + \alpha^2} = \frac{p_-}{\sqrt{m^2 - \alpha^2}}\left[\frac{1}{p_\perp^2 + w^2} - \frac{1}{p_\perp^2 + v^2}\right]$$

where

$$v_\alpha^2 = p_-^2 + 2p_-\sqrt{m^2-\alpha^2} + m^2,$$
$$w_\alpha^2 = p_-^2 - 2p_-\sqrt{m^2-\alpha^2} + m^2$$

and $\alpha$ is either $\kappa$ or $\tau$. The definition $Z = 2p_-/D_-$ gives, in the center-of-momentum frame, $p_- = M_D Z/2$, and so we get

$$\hat{h}(B,Z) = \frac{M_D}{\mu}\sqrt{\kappa\tau(\kappa+\tau)}\bigg[\frac{Z}{\sqrt{m^2-\kappa^2}}[K_o(w_\kappa B) - K_o(v_\kappa B)] - \frac{Z}{\sqrt{m^2-\tau^2}}[K_o(w_\tau B) - K_o(v_\tau B)]\bigg]. \quad \text{(B4)}$$

In order to apply these expressions to the deuteron case we take into account that $\kappa < \mu < m$; we have already noted that we are interested in small values of $B$ compared with the nuclear scale $1/2\kappa$, so we look for the limit $B \to 0$ which gives $K_o(w_\kappa B) - K_o(v_\kappa B) \to \ln(v_\kappa/w_\kappa)$.

We can study numerically the above limiting form of $\hat{h}$. The results for $E/m = 0.0021$ are presented in the graph as a function of $u = p_-/m$, which is slightly different from $Z = 2mu/M_D$. Since $\mu = 5\kappa$ is a phenomenological fit without a direct interpretation, the numerical study has been performed also for $\mu = 3\kappa$, which would better describe a binding potential generated by pion exchange; the qualitative conclusions are, however, the same for both choices of the parameters. In Fig. 13 we plot $\hat{h}(0, Z)$ as a function of $Z$ in the cases $\mu = 3\kappa$ and $\mu = 5\kappa$. A numerical study shows that the shape of $\hat{h}(B, Z)$ has little variation with increasing $B$. In accordance with the previous results the spread in $Z$ is sizable; in order to go from the maximum of $\hat{h}$ to half of this maximum, $Z$ must vary at least by 0.1. When the difference in $Z$ of the two functions in the interference term is only a few percent, the corresponding overlap integral is thus not very depressed with respect to the diagonal term. In practice, this means that the interference terms are smaller than the diagonal ones, but not by orders of magnitude.

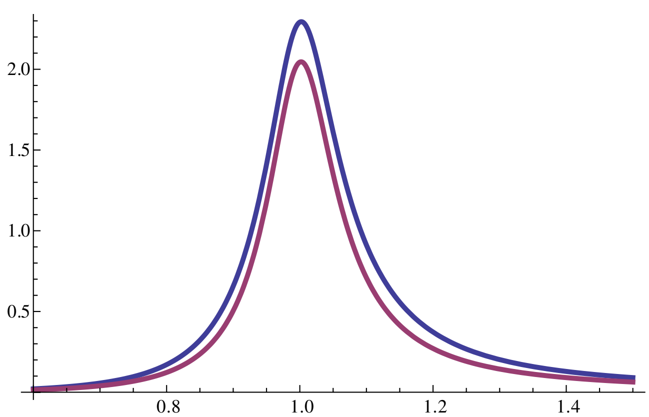


FIG. 13 (color online). Graph of $\hat{h}(B = 0)$ as a function of $p_-/m$. Arbitrary vertical normalization: the upper curve corresponds to $\mu = 3\kappa$, and the lower curve corresponds to $\mu = 5\kappa$.

### 2. A Gaussian model for the transverse dynamics

#### *1. General features of the model*

Here the simplification is heavier, with the aim of dealing explicitly with the transverse degrees of freedom for all the graphs we considered. No correlations among transverse and longitudinal degrees of freedom are taken into account; moreover, the transverse distributions are Gaussian. One of the results, which are likely to be less model dependent, is that the transverse dynamics is less sensitive to the difference between diagonal terms and interference terms.

The one-parton inclusive distribution is written as

$$\Gamma_1 = G(x)f_1(b), \qquad f_1(b) = \frac{1}{\pi R^2}\exp[-b^2/R^2]. \quad \text{(B5)}$$

The two-parton distribution is

$$\Gamma_2 = KG(x_1)G(x_2)f_2(b_1, b_2),$$
$$f_2(b_1,b_2) = \frac{1}{(\pi R^2)^2(1-\lambda^2)} \times \exp[-(b_1^2 + b_2^2 + 2\lambda b_1 b_2)/R^2(1-\lambda^2)],$$
$$\int f_2(b_1, b_2)db_2 = f_1(b_1). \quad \text{(B6)}$$

$K$ controls the parton multiplicity, and $\lambda$ controls the spatial correlation; both can still depend on the fractional momenta $x_i$.

The three-parton distribution, with a minimal number of new parameters, is written as

$$\Gamma_3 = G(x_1)G(x_2)G(x_3)K_3 f_3(b_1, b_2, b_3),$$
$$f_3(b_1,b_2,b_3) = \frac{1}{(\pi R^2)^3(1-2\lambda)(1+\lambda)^2} \times \exp\bigg[-\frac{(1-\lambda)\sum_i b_i^2 + 2\lambda\sum_{i<j}(b_i\cdot b_j)}{R^2(1+\lambda)(1-2\lambda)}\bigg] \quad \text{(B7)}$$

where we have implemented the requirement that $f_3$ be symmetric in the parton variables and that

$$\int f_3(b_1, b_2, b_3)db_3 = f_2(b_1, b_2).$$

The information embodied in the parameters $K_N$ and $\lambda$ can be seen in this way. The distributions $f_N$ are inclusive; if the corresponding exclusive distributions, integrated on the longitudinal variables in a given interval, were Poissonian, then $K_N$ would be 1. Of course, even a Poissonian distribution of the partons' multiplicities can imply spatial correlations if $\lambda \neq 0$.

The nuclear distribution, i.e. the square of the wave function, is expressed for the deuteron as

$$g(z;b) = W(Z)F(B), \qquad F(B) = \frac{1}{\pi S^2}\exp[-B^2/S^2], \qquad 0 < Z < 2, \qquad B = B_1 - B_2. \tag{B8}$$

Since the distribution will be integrated at the end, for tritium we write

$$g(Z_i, B_i) = W(Z_1, Z_2, Z_3)\delta(Z_1 + Z_2 + Z_3 - 3)F(B_i), \quad 0 < Z < 3,$$
$$F(B_i) = \frac{4}{3(\pi S^2)^2}\exp[-2[(B_1 - B_2)^2 + (B_2 - B_3)^2 + (B_2 - B_3)^2]/3S^2]. \tag{B9}$$

The normalization is

$$\int W(Z)dZ = 1,$$
$$\int W(Z_1, Z_2, Z_3)\delta(Z_1 + Z_2 + Z_3 - 3)dZ_1 dZ_2 dZ_3 = 1.$$

In the nonrelativistic case $Z \approx 1$. The transverse three-body distribution satisfies

$$\int F(B_i)dB_3 = \frac{1}{\pi S^2}\exp[-B^2/S^2], \qquad B = B_1 - B_2,$$

which defines the normalization. It has already been noted that the sizes $S$ in the deuteron and in tritium, although similar, are in fact different.

### 2. Free nucleons

As reference quantities we consider, within the model, the simple double and triple hard scattering among free nucleons at fixed fractional momenta. The simple hard scattering is described by

$$\sigma_1(x, x') = \hat{\sigma}_{xx'}G(x)G(x')\int f_1(b)f_1(b - \beta)dbd\beta = \hat{\sigma}_{xx'}G(x)G(x')$$

due to the normalization of the transverse distributions. The double hard scattering is described by

$$\sigma_2(x_1, x_2; x_1', x_2') = K_2^2\hat{\sigma}_{x_1x_1'}G(x_1)G(x_1')\hat{\sigma}_{x_2x_2'}G(x_2)G(x_2')I_o,$$
$$I_o = \int f_2(b_1, b_2)f_2(b_1 - \beta, b_2 - \beta)db_1db_2d\beta. \tag{B10}$$

The triple hard scattering is described by

$$\sigma_3(x_1, x_2, x_3; x_1', x_2', x_3') = K_3^2\hat{\sigma}_{x_1x_1'}G(x_1)G(x_1')\hat{\sigma}_{x_2x_2'}G(x_2)G(x_2')\hat{\sigma}_{x_3x_3'}G(x_3)G(x_3')J_o,$$
$$J_o = \int f_3(b_1, b_2, b_3)f_3(b_1 - \beta, b_2 - \beta, b_3 - \beta)db_1db_2db_3d\beta. \tag{B11}$$

All the transverse variables "$b$" are two dimensional, and the cylindrical symmetry is always preserved. Thus, all the calculations regarding the transverse variables have the standard form

$$C\Big[\int d^N y \exp[-y\cdot \mathsf{M}\cdot y]\Big]^2 = C\frac{\pi^N}{\det\mathsf{M}}. \tag{B12}$$

Here $\mathsf{M}$ is an $N \times N$ matrix, and $C$ embodies the normalizing factors as given in Eqs. (B5)–(B7).

In the cases considered above, after a rescaling of the variables, $b \to bR\sqrt{1-\lambda^2}$ in the first case and $b \to bR\sqrt{1+\lambda}\sqrt{1-2\lambda}$ in the second case, the matrices take the form

$$\mathsf{M}_I = \begin{vmatrix} 2 & 2\lambda & -1-\lambda \\ 2\lambda & 2 & -1-\lambda \\ -1-\lambda & -1-\lambda & 2+2\lambda \end{vmatrix}, \tag{B13}$$

$$\mathsf{M}_J = \begin{vmatrix} 2(1-\lambda) & 2\lambda & 2\lambda & -1 \\ 2\lambda & 2(1-\lambda) & 2\lambda & -1 \\ 2\lambda & 2\lambda & 2(1-\lambda) & -1 \\ -1 & -1 & -1 & 3(1+\lambda) \end{vmatrix}. \tag{B14}$$

So, finally, we have

$$I_o = \frac{1}{4\pi R^2}\frac{1}{1+\lambda}, \qquad J_o = \frac{1}{12(\pi R^2)^2}\frac{1}{1+4\lambda+2\lambda^2}. \tag{B15}$$

### 3. Bound nucleons

We start by considering the scattering process when one of the nucleons is bound in a deuteron,

$$\sigma_{D,1}(x, x') = \hat{\sigma}_{xx'}G(x)G(x'/Z)W(Z)dZdxdx'\int f_1(b)f_1(b - B_1)F(B_1 - B_2)dbdB_i.$$

By integrating in $B_2$ the factor $F$, one simply gets 1. According to the discussion in Sec. III, in the distribution $W$ the variable $Z$ is shrunk around the value $Z = 1$ so we take $\int G(x'/Z)dZW(Z) \approx G(x')\int dZW(Z) = G(x')$; we obtain the same expression as for the free case. From the simple procedure described it is seen that the same result holds for the case of tritium, with a suitable redefinition of the size $S$ also when we consider double or triple hard scatterings.

When we look at hard scatterings where more than one bound nucleon participates, new features appear. In the simplified model described here where the longitudinal degrees of freedom are factorized, we have seen that the $Z \approx 1$ approximation is inconsistent with the conservation of longitudinal momentum in the nondiagonal cases, but we do not have anything new to add to this point. We investigate the transverse degrees of freedom of the partons (inside the nucleon) and of the nucleon (inside the nucleus), which are dynamically connected.

Now we study the double scattering when both nucleons of the deuteron are involved (here and below the $x$, $x'$ arguments of $\hat{\sigma}$ will usually be omitted).

It has been shown that in this case there are two possibilities: the direct term and an interference term. In the latter case we cannot simply work with the density distribution of partons; we need the "wave function," whose absolute square gives the distribution of the partons inside the hadron. In our case, where the distributions are Gaussian, we take as wave function the square root of the distribution; i.e. we ignore the possible phases. Then the relevant quantities are

$$\begin{aligned} I_2^d &= \int f_2(b_1, b_2) f_1(b_1 - B_1) f_1(b_2 - B_2) F(B_1 - B_2) db dB, \\ I_2^i &= \int f_2(b_1, b_2) \sqrt{f_1(b_1 - B_1)} \sqrt{f_1(b_1 - B_2)} \\ &\quad \times \sqrt{f_1(b_2 - B_2)} \sqrt{f_1(b_2 - B_1)} F(B_1 - B_2) db dB. \end{aligned} \tag{B16}$$

The subsequent calculations are very similar to the previous one. There is only one new feature, the new dimensional parameter $S$. We have an integration over the transverse variables $b$. The Gaussian integration involves the calculation of $4 \times 4$ determinants, and finally, taking into account the normalizations, for the diagonal and for the interference terms we obtain, respectively,

$$\begin{aligned} I_2^d &= \frac{1}{\pi[S^2 + 2(2 + \lambda)R^2]}, \\ I_2^i &= \frac{1}{\pi(2 + \lambda)[S^2 + 2R^2]}. \end{aligned} \tag{B17}$$

In the double collision involving tritium there is necessarily a spectator. Since by integrating the three-body distribution (B5) over the spectator's coordinates we obtain the two-body distribution (B4), the final expression is the same, provided we rescale the size in (B11) according to the experimental values.

Now we consider the triple hard interaction where one of the bound nucleons interacts twice, another only once, and there is, in tritium case, a spectator. It has already been shown that there are two kinds of processes in this case, with a diagonal term and some interference terms which must be treated separately.

We do not repeat the consideration about the longitudinal variables; as far as the transverse variables are concerned, the integration implies the calculation of $5 \times 5$ determinants, and the final result is, for the diagonal term,

$$J_2^d = \frac{1}{1 + \lambda} \frac{1}{4\pi^2 R^2 [S^2 + (3 + \lambda)R^2]}. \tag{B18}$$

For the interference terms we get

$$\begin{aligned} J_2^{i,2} &= \frac{1 - \lambda}{(1 + \lambda)(6 - \lambda)} \\ &\quad \times \frac{4}{\pi^2 R^2 [(4 - \lambda)S^2 + 2(4 - 2\lambda - \lambda^2)R^2]}, \\ J_2^{i,3} &= \frac{1 - \lambda}{(1 + \lambda)(3 + \lambda)} \frac{1}{\pi^2 R^2 [(3 - \lambda)S^2 + 4(1 - \lambda)R^2]}, \\ J_2^{i,4} &= \frac{1 - \lambda}{(1 + \lambda)(4 - 2\lambda - \lambda^2)} \frac{4}{\pi^2 R^2 [7S^2 + 2(6 - \lambda)R^2]}. \end{aligned} \tag{B19}$$

Finally, we consider the process where we have three bound nucleons, all interacting once; evidently, now we must consider tritium. We find a diagonal term and two different interference terms; the integration implies the calculation of $6 \times 6$ determinants, and the final result is, for the diagonal term,

$$J_3^d = \frac{4}{3} \frac{1}{\pi^2 [S^2 + 2(2 + \lambda)R^2]^2}. \tag{B20}$$

For the two interference terms we get

$$\begin{aligned} J_3^{i,2} &= \frac{4}{3} \frac{1}{\pi^2 (2 + \lambda)[(S^2 + 2R^2)(S^2 + 2(2 + \lambda)R^2]}, \\ J_3^{i,3} &= \frac{4}{3} \frac{16}{\pi^2 [(7 + 3\lambda)S^2 + 8(2 + \lambda)R^2]^2}. \end{aligned} \tag{B21}$$

A feature that appears very clearly from the model but that reflects a more general property is the dependence on the geometrical parameters: since the cross section has dimension $\ell^2$ and there are three factors $\hat{\sigma}$, each with dimension $\ell^2$, the terms $J$ must have dimensions $\ell^{-4}$. When only one nucleon interacts, this factor is necessarily $1/R^4$; when two nucleons interact, the factor is $1/R^2(\mu S^2 + \nu R^2)$; when three nucleons interact, the factor has the form $1/(\mu S^2 + \nu R^2)^2$. The meaning is clear considering the hypothetical situation $S \gg R$. In this case the first cross section remains unaltered, the second vanishes as $1/S^2$, and the third one vanishes as $1/S^4$.

## APPENDIX C: INFRARED BEHAVIOR

We take a brief look at the infrared properties of the amplitudes and densities we have used.

Using Eqs. (5) and (9) we can conclude that the one-particle density has the behavior $\Gamma(z;b) \propto 1/z$, but we know that this is not supported by experimental data, so we conclude that the vertex $\phi$ must also show an infrared singularity in such a way that $\Gamma(z;b) \propto 1/z^{1+\nu}$. Since the integration (with an infrared cutoff) of the two-body distribution must give the one-body distribution, the same behavior must be found in the two-body vertex $\hat{\phi}$.

Thus, we find a relevant simplification of the two-parton amplitude when one of the two partons has a very soft four-momentum; this is the particular case of the general features found in the emission of soft particles. For definiteness, we consider the free proton case: when the parton four-momentum goes to zero linearly in all its components, we have $l_{1\perp} \propto x_1$. Then, in the expression for $\psi_2$ we neglect the term in $l_{1\perp}^2$ and also $x_1$ with respect to $x_2$. The result is

$$\psi_{\text{i.r.}} = \frac{1}{\sqrt{2}L_+ x_1} \frac{\hat{\phi}}{x_2[m^2 - M_\perp^2/(1-x_2)] - l_{2\perp}{}^2}$$

but we have seen that $\hat{\phi}$ must have a singularity; we extract it by writing $\hat{\phi} \propto \frac{\phi}{x^{\nu/2}}$. In the same kinematical configurations the other factor used in defining the densities $\Gamma$ can also be decomposed as

$$\frac{x_1, x_2}{1 - x_1 - x_2} \simeq x_1 \frac{x_2}{1-x_2}, \qquad x_1 \ll x_2.$$

So the original density is decomposed into two factors: the second factor is precisely the probability of finding one parton with finite fractional momentum, as seen in Eq. (5), and the first one can be thought of as the usual infrared term of QED corrected phenomenologically, thus enhancing the singularity. The well-known term of QED is $P \cdot \varepsilon / P \cdot q$. It could also represent a gluon emission, but since we have not taken into account the spin, we do not have the numerator $P \cdot \varepsilon$; the denominator is dominated by the "large" component $x_1 L_+$.

We use the mixed representation of $\psi$ with longitudinal momenta and transverse coordinates, so we should perform the Fourier transformation in $l_\perp$. The operation on $l_{2\perp}$ yields precisely $\psi(x_2; b_2)$; on $l_{1\perp}$ the integration must run only on the infrared domain of $l_{1\perp}$, limited by a cutoff $|l_{1\perp}| < \ell_{\text{i.r.}}$ for dimensional reasons. The result is proportional to a function $\ell_{\text{i.r.}}^2 G(b_1 \ell_{\text{i.r.}})$, and we get for the density $\Gamma(x_1, x_2; b_1, b_2) \simeq \ell_{\text{i.r.}}^2 G(b_1 \ell_{\text{i.r.}}) \Gamma(x_2; b_2)/2x_1$. Integrating over the longitudinal infrared momentum $x_1$, we obtain the usual phenomenological divergence $dx/x^{1+\nu}$.

It is in fact a very general property that the soft emission is relatively independent of the rest of the dynamics, so it holds also for the more complicated expressions, like the terms $W_j$, and also for three-body amplitudes, which become completely factorized in the limit $x_1 \ll x_3$, $x_2 \ll x_3$. In detail, the treatment applies to a soft emission originating directly from a free nucleon. When the parton emission comes from a nucleus, one finds the usual complication due to the binding, but when the treatment is applied to a configuration with $z \ll 1$, the previous discussion still holds [see Eq. (21)]. The factorization which is produced in this particular case can be useful in order to make a simpler relation, at least in this kinematical configuration, between the diagonal and the interference term.


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