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0^{++} glueball wave function for improved SU(2) lattice gauge theory in (2 + 1) dimensions

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Using the improved lattice gauge field Hamiltonian and the truncated eigenvalue equation method, we compute the 0^{++} glueball wave function of (2 + 1)-dimensional SU(2) gauge field theory. The result shows a good scaling behavior in the weak coupling region $3 \leq 1/g^2 \leq 6$.

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I. INTRODUCTION

Lattice gauge field theory (LGT) is the most reliable and powerful nonperturbative approach to QCD. LGT has two equivalent formulations, namely, the Lagrangian formulation and the Hamiltonian formulation. QCD predicts the existence of new hadrons, named glueballs. They are bound states of gluons, the vector bosons mediating strong interactions. According to Lagrangian lattice QCD Monte Carlo simulation, the lightest glueball 0^{++} mass is about 1.5–1.8 Gev. There have been a lot of candidates in experiments [1]. Investigation of glueball production and decays would give additional important information to determine which is the 0^{++} glueball. In such a study, the calculation of the 0^{++} glueball wave function is necessary. The disadvantage of the conventional Lagrangian formulation is that it is very difficult to compute the wave function. In our opinion the Hamiltonian approach is a viable alternative.

But LGT's progress has been hampered by systematical errors mainly due to the finite value of the lattice spacing a. The standard Wilson gluonic (bosonic) action (or Kogut Susskind Hamiltonian) differs from the continuum Yang-Mills action (or corresponding Hamiltonian) by order of $O(a^2)$, while the error of the standard Wilson quark (fermionic) action (or Hamiltonian) is bigger, being of the order $O(a)$. In the continuum limit $a \to 0$ or equivalently $1/g^2 \to \infty$ in an asymptotically free theory; these differences in principle disappear and the action (or Hamiltonian) becomes the continuum one. If the practical lattice calculations could be carried out up to a weak enough coupling region, the finite lattice errors would be negligible. Unfortunately, both the standard numerical simulation and the standard Hamiltonian method can only be carried out up to the intermediate coupling region. For example, in the standard Kogut Susskind Hamiltonian method, the calculations of the vacuum wave function and glueball mass as well as glueball wave function for (2 + 1)-dimensional SU(N_C) can only be carried out up to $1/g^2 < 2.5$ [2–7]. For such a lattice parameter, violation of scaling is still obvious and extrapolation of the results to the $1/g^2 \to \infty$ limit induces unknown systematic uncertainties when extracting continuum physics.

One possible way to tackle these problems is to improve the lattice action (or Hamiltonian) so that finite errors become higher order in a. There have been several proposals on this subject:

(a) For the fermionic sector, Hamber and Wu, in 1983, proposed an improved lattice action [8] by adding next-nearest-neighbor interaction terms to the Wilson quark action so as to remove the $O(a)$ error. Several numerical simulations [9–11] of hadron spectroscopy have been performed by using the Hamber-Wu action. In 1994, we proposed several improved lattice Hamiltonians for fermions [12] to reduce the errors from $O(a)$ to $O(a^2)$. In 1999, we showed that our improved theory leads to a significant reduction of finite errors through the calculations of the quark condensate and the vector mass of (1 + 1)-dimensional QCD [13].

(b) For the gluonic sector, Luo *et al.*, in 1999, proposed an improved Kogut Susskind lattice Hamiltonian [14] to reduce the errors from $O(a^2)$ to $O(a^4)$. This improvement has been tested in (2 + 1)-dimensional SU(2) LGT through the calculations of the vacuum wave function and 0^{++} glueball mass [15] in 2003. The results showed that this improved theory leads to a significant reduction of violation of scaling.

As mentioned above, the glueball wave functions can give more physical information than the glueball masses. As a further test, in this paper we shall compute the 0^{++} glueball wave function of (2 + 1)-dimensional SU(2) gauge field theory using the improved Kogut Susskind lattice Hamiltonian.

II. IMPROVED LATTICE HAMILTONIAN AND TRUNCATED EIGENVALUE EQUATION OF GLUEBALL STATE

In SU(2) LGT, $\mathrm{Tr}U_p^+ = \mathrm{Tr}U_p$, so that all loops with crossing can be transformed into loops without crossing. According to the improved Kogut Susskind lattice Hamiltonian [14], we have

$$H = \frac{g^2}{2a}\,\mathrm{Tr}\sum_{x,\,i}\left\{\frac{5}{6}E_i^\alpha(x)E_i^\alpha(x) + \frac{1}{6}U_i^+E_i^\alpha(x)U_i(x)E_i^\alpha(x+i)\right\} - \frac{2}{g^2a}\left(\frac{5}{3}\sum_p \mathrm{Tr}U_p - \frac{1}{6}\sum_{x,\,i<j}R_{ij}\right), \quad (1)$$

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where g is the dimensionless coupling constant that is related to the lattice spacing a and the invariant charge e by $g^2 = e^2 a$, $E_i^\alpha(x)$ is the color-electric field, $\sum_p \mathrm{Tr} U_p$ is the square loop $\square$, and $\sum_{x,\ i<j} R_{ij}$ is the rectangular loop $\square\!\square$.

In order to study the glueball state, one needs detailed information on the structure of the vacuum wave function.

The vacuum wave function in exponential form is written as

$$|\Omega\rangle = \mathrm{e}^R |0\rangle, \tag{2}$$

where R contains closed loops and the state $|0\rangle$ is defined as $E_i^\alpha(x)|0\rangle = 0$.

Substituting H and $|\Omega\rangle$ in the eigenvalue equation (namely, Schrödinger equation) $H|\Omega\rangle = E_\Omega|\Omega\rangle$, using the relation

$$\mathrm{e}^{-R} H \mathrm{e}^R = H - [R, H] + \frac{1}{2!}[R, [R, H]] - \frac{1}{3!}[R, [R, [R, H]]] + \cdots, \tag{3}$$

and the commutation relations

$$\begin{aligned} [U_i(x), E_j^\alpha(y)] &= \lambda^\alpha U_i(x) \delta_{x,y} \delta_{i,j}, \\ [U_i^+(x), E_j^\alpha(y)] &= -U_i^+(x) \lambda^\alpha \delta_{x,y} \delta_{i,j}, \end{aligned} \tag{4}$$

we can obtain the eigenvalue equation of the vacuum state for the improved Hamiltonian:

$$\begin{aligned} &\frac{5}{6}\Big\{\sum_{x,i}[E_i^\alpha(x), [E_i^\alpha(x), R]] + \sum_{x,i}[E_i^\alpha(x), R][E_i^\alpha(x), R]\Big\} \\ &+ \frac{1}{3}\Big\{\sum_{x,i}[E_i^\alpha(x), [E_i^\alpha(x+i), R]] \\ &+ \sum_{x,i}[E_i^\alpha(x+i), R][E_i^\alpha(x), R]\Big\} \\ &- \frac{4}{g^4}\Big(\frac{5}{3}\sum_p \mathrm{Tr} U_p - \frac{1}{6}\sum_{x,\ i<j} R_{ij}\Big) = \frac{2a}{g^2} E_\Omega. \end{aligned} \tag{5}$$

Defining the order of a loop graph as the number of plaquettes involved, we expand R in the order of graphs:

$$R = R_1 + R_2 + R_3 + \cdots. \tag{6}$$

The lowest order loop graph (i.e. the first order graph) is $R_1 = C_1 \square$. Given $R = R_1$, from Eq. (5), we can obtain the second-order loop graphs:

$R_2 = C_2$  $+ C_3$  $+ C_4$ $+ C_5$  $+ C_6$  , (7)

where the coefficients C_i will be given by solving the algebraic equations.

Given $R = R_1 + R_2$, from Eq. (5), we can obtain not only the whole of the third-order loop graphs, but also a part of the fourth-order loop graphs, and so on.

Let R contain up to the Nth-order graphs:

$$R = R_1 + R_2 + \cdots + R_N, \tag{8}$$

and we obtain the truncated eigenvalue equation of the vacuum state for the improved gluonic Hamiltonian at the Nth order:

$$\begin{aligned} &\frac{5}{6}\Big\{\sum_{x,i}[E_i^\alpha(x), [E_i^\alpha(x), R]] + \sum_{\substack{x,i \\ n_1+n_2\le N}}[E_i^\alpha(x), R_{n_1}][E_i^\alpha(x), R_{n_2}]\Big\} \\ &+ \frac{1}{3}\Big\{\sum_{x,i}[E_i^\alpha(x), [E_i^\alpha(x+i), R]] \\ &+ \sum_{\substack{x,i \\ n_1+n_2\le N}}[E_i^\alpha(x+i), R_{n_1}][E_i^\alpha(x), R_{n_2}]\Big\} \\ &- \frac{4}{g^4}\Big(\frac{5}{3}\sum_p \mathrm{Tr} U_p - \frac{1}{6}\sum_{x,\ i<j} R_{ij}\Big) = \frac{2a}{g^2} E_\Omega. \end{aligned} \tag{9}$$

The 0^{++} glueball wave function in exponential form is written as

$$|\Psi\rangle = (G - \langle G\rangle)|\Omega\rangle = (G - \langle G\rangle)\mathrm{e}^R|0\rangle, \tag{10}$$

where G consist of closed loops, which is also expanded up to the Nth-order loop graphs

$$G = G_1 + G_2 + \cdots + G_N. \tag{11}$$

Here G_N is the Nth-order loop graphs, according to the same rules as R_N.

In a similar way of educing Eq. (9), we obtain the truncated eigenvalue equation of the glueball state for the improved Hamiltonian at the Nth order:

$$\begin{aligned} &\frac{5}{6}\Big\{\sum_{x,i}[E_i^\alpha(x), [E_i^\alpha(x), G]] + 2\sum_{\substack{x,i \\ n_1+n_2\le N}}[E_i^\alpha(x), R_{n_1}][E_i^\alpha(x), G_{n_2}]\Big\} \\ &+ \frac{1}{3}\Big\{\sum_{x,i}[E_i^\alpha(x+i), [E_i^\alpha(x), G]] \\ &+ \sum_{\substack{x,i \\ n_1+n_2\le N}}[E_i^\alpha(x+i), R_{n_1}][E_i^\alpha(x), G_{n_2}] \\ &+ \sum_{\substack{x,i \\ n_1+n_2\le N}}[E_i^\alpha(x+i), G_{n_1}][E_i^\alpha(x), R_{n_2}]\Big\} = \frac{2a\Delta m}{g^2} G, \end{aligned} \tag{12}$$

where Δm is the mass gap.

III. CALCULATION OF 0^{++} GLUEBALL WAVE FUNCTION

In this paper we compute the wave function up to the third-order loop graphs. From the terms $\sum_{x,i}[E_i^\alpha(x), R_1][E_i^\alpha(x), R_2]$, $\sum_{x,i}[E_i^\alpha(x+i), R_1][E_i^\alpha(x), R_2]$, and $\sum_{x,i}[E_i^\alpha(x+i), R_2][E_i^\alpha(x), R_1]$, we can obtain the third-order loop graphs for the vacuum wave function:

$$R_3 = C_7 + C_8 + C_9 + C_{10} + C_{11} + C_{12} + C_{13} + C_{14} + C_{15} + C_{16} + C_{17} + C_{18} + C_{19} + C_{20} + C_{21} + C_{22} + C_{23} + C_{24} + C_{25} + C_{26} + C_{27} + C_{28}. \quad (13)$$

Substituting R_1, R_2, and R_3 in Eq. (9), we obtain nonlinear equations for the coefficients $C_i (i = 1, 2, 3, \ldots, 28)$. Numerically solving these nonlinear equations, we can obtain C_i as the functions of $1/g^2$.

Equally, we expand G up to the third-order loop graphs:

$$G = B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9 + B_{10} + B_{11} + B_{12} + B_{13} + B_{14} + B_{15} + B_{16} + B_{17} + B_{18} + B_{19} + B_{20} + B_{21} + B_{22} + B_{23} + B_{24} + B_{25} + B_{26} + B_{27} + B_{28}. \quad (14)$$

Substituting R and G in Eq. (12), we can obtain the nonlinear equations for the coefficients $B_i (i = 1, 2, 3, \ldots, 28)$. Numerically solving these equations, we obtain B_i as a function of $1/g^2$.

In the continuum limit, the long wavelength glueball wave function of (2 + 1)-dimensional SU(2) gauge field theory is

$$|\Psi\rangle = \exp\left\{-\kappa_0 \iint \mathrm{Tr}F^2(x, y)\mathrm{d}x\mathrm{d}y - \kappa_2 \iint \mathrm{Tr}[\mathrm{D}_i F(x, y)]^2 \mathrm{d}x\mathrm{d}y + \text{higher order terms}\right\}, \quad (15)$$

where κ_0 and κ_2 are the expansion coefficients of the long wavelength glueball wave function, $F(x, y)$ is the field strength tensor, and D_i is the covariant derivative.

κ_0 and κ_2 are the linear combination of the coefficients B_i, from expanding $\sum_p \mathrm{Tr}U_p$ in order of a. Analyzing the long wavelength limit of all graphs up to the third order, we obtain

$$\kappa_0 = \frac{g^4}{2e^2}(B_1 + 4B_2 + 4B_4 + 4B_5 + 9B_7 + 9B_8 + B_9 + B_{10} + 9B_{11} + 9B_{12} + B_{13} + B_{14} + B_{15} + 9B_{16} + B_{17} + B_{18} + B_{19} + B_{20} + B_{21} + 9B_{22} + B_{23} + B_{24} + 9B_{25} + B_{26} + 9B_{27} + B_{28}), \quad (16)$$

$$\kappa_2 = \frac{g^8}{2e^6}(-B_1/12 - B_2/3 + B_3/2 - 5B_4/6 - 4B_5/3 + B_6 - 11B_7/4 - 15B_8/4 + 11B_9/12 + 23B_{10}/12 - 7B_{11}/4 - 3B_{12}/4 - B_{13}/12 - 13B_{14}/12 + 11B_{15}/12 - 33B_{16}/12 + 23B_{17}/12 - 13B_{18}/12 + 23B_{19}/12 - B_{20}/12 - 25B_{21}/12 - 19B_{22}/4 + 35B_{23}/12 - B_{24}/12 - 27B_{25}/4 + 47B_{26}/12 - 19B_{27}/4 + 23B_{28}/12). \quad (17)$$

Substituting B_i ($i = 1, 2, 3, \ldots, 28$) in Eqs. (16) and (17), we obtain κ_0 and κ_2 as the functions of $1/g^2$.

IV. NUMERICAL RESULTS AND DISCUSSIONS

It is well known that SU(2) LGT in $(2 + 1)$ dimensions is super-renormalizable, and possesses a simple scaling property, that is, when the lattice spacing a goes to zero,

$$e^2\kappa_0 \sim \text{const}, \qquad e^6\kappa_2 \sim \text{const} \quad \text{as } a \to 0. \tag{18}$$

Figures 1(a) and 1(b) show $e^2\kappa_0$ and $e^6\kappa_2$ as functions of $1/g^2$ in the coupling region $1 \le 1/g^2 \le 6$ and $3 \le 1/g^2 \le 6$, respectinely.

In this work we have computed the 0^{++} glueball wave function of $(2 + 1)$-dimensional SU(2) LGT using the improved lattice gauge field Hamiltonian and the truncated eigenvalue equation method. Our calculation can be carried out up to a deep weak coupling region $1/g^2 = 6.0$ (namely, $\beta = 24.0$), with a good scaling behavior, while using the unimproved Hamiltonian the calculations can only be carried out up to the intermediate coupling region $1/g^2 = 2.4$ (namely, $\beta = 9.6$), which would result in large uncertainty. In other words, from the test of the glueball wave function for $(2 + 1)$-dimensional SU(2) LGT, we see that the improved Hamiltonian indeed leads to much better results.

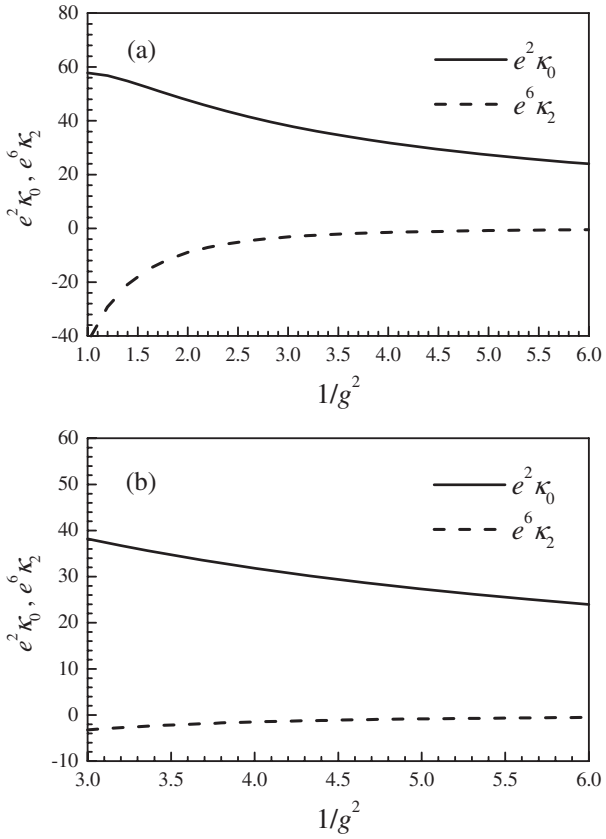

FIG. 1. $e^2\kappa_0$ and $e^6\kappa_2$ versus $1/g^2$.

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