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Leptonic and digamma decay properties of *S*-wave quarkonia states

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Based on Martin-like potential, the masses of quarkonia states and their leptonic decay widths have been reviewed. The hyperfine, spin orbit and tensor interactions are employed to compute the spin splitting of the nS states and the fine splittings of the P and D states. The analysis based on the predicted masses and leptonic decay widths clearly indicates that $\psi(3686)$ is a mixed state with a 50%-50% admixture of $c\bar{c}$ $(2S)$ and the hybrid $c\bar{c}g$ in accordance with the suggestion that resolves the $\rho - \pi$ puzzle related to $\psi(2S)$. And $\Upsilon(10355)$ as similar admixture of $b\bar{b}$ $(3S)$ with $b\bar{b}g$ in accordance with the resolution of Vogel puzzle related to $\Upsilon(3S)$ state. Analyses on the level differences of S-wave excited states of quantum mechanical bound systems show a systematic behavior as n increases. In view of such systematic behavior expected for quarkonia, we observe that $Y(4260)$ and $Z(4430)$ 1^{--} states are closer to the $4S$ and $5S$ states with leptonic decay widths predicted as 0.65 keV and 0.49 keV, respectively. The $c\bar{c}$ $(6S)$ 1^{--} state is predicted to be around 4600 MeV and its leptonic decay width 0.39 keV. The present study also favors other charmonialike states, $Y(4360)$ and $Y(4660)$, as admixtures of charmonia $S - D$ states. Similarly we find $\Upsilon(10865)$ does not fit either the $5S$ state or an admixture of $S - D$ states of a $b\bar{b}$ system. We identify $Y_b(10888)$ observed by Belle as the 6S state of bottonia whose leptonic width is predicted as 0.158 keV. Our predicted leptonic width, 0.242 keV of $\Upsilon(10575, 4S)$, is in good agreement with the experimental value of 0.272 ± 0.029 keV. We predict the pure $\Upsilon(5S)$ state at about 100 MeV lower than 10865 MeV and its leptonic width 0.191 keV. The upsilon state $\Upsilon(11019)$ seems to be the right candidate for the $7S$ state, with 0.134 keV as its predicted leptonic width, which is in very good agreement with the experimental value of 0.13 ± 0.03 keV.

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I. INTRODUCTION

The recent experimental observations particularly in the quarkonia sector have generated renewed interest in the study of hadron spectroscopy [1–4]. The discovery of the $\eta_b(1S)$, $h_b(1P)$ states [1,3] (the *BABAR* and CLEO collaborations) and $\eta_c(2S)$, $\chi_{c2}(2P)$ states and many high-precision experimental observations of various hadronic states [4] have necessitated reconsideration of the parameters involved in the previous studies [5,6].

Until recently, all that was known above the $D\bar{D}$ threshold were the four vector states $\psi(3770)$, $\psi(4040)$, $\psi(4160)$, $\psi(4415)$. The new renaissance in hadron spectroscopy has come from the recent discovery of the large numbers of new states X, Y, Z [7–12]. The challenges paused by these new states include the right identification with the proper J^{PC} values and their decay modes.

Even though the spectroscopy of quarkonium states is well recorded experimentally, the S-wave masses of charmonium states beyond $3S$ and the bottonium states beyond $4S$ are still not very well resolved. There seemed to be mixing of other resonances nearby. For example, the 1^{--} states such as $\psi(3770)$, $Y(4008)$, $Y(4260)$, $Y(4360)$, $X(4630)$, $Y(4660)$, $\Upsilon(10865)$, $\Upsilon(11020)$, $Y_b(10880)$, etc., may be the quarkonia states either with or without mixing with the nearby resonance states. For instance, the $\Upsilon(11020)$ state has recently been reported to be a mixed bottonium $\Upsilon(6S)$ and $\Upsilon(5D)$ states with mixing angle of $\theta = 40^o \pm 5^o$ [13].

Moreover, the decay properties of the higher quarkonia states are interesting with reference to two well-known puzzles. One is the $\rho - \pi$ puzzle [14–16] related to the branching ratio of hadronic and leptonic decays of $\psi(2S)$ states in comparison with the decays of $J/\psi(1S)$ state. The second one is the Vogel $[\Upsilon(\Delta n = 2)]$ puzzle [16,17], where $\Upsilon(2S) \rightarrow \Upsilon(1S) + 2\pi$ has large branching ratio without σ (scalar meson), while $\Upsilon(3S) \rightarrow \Upsilon(1S) + 2\pi$ has large branching ratio to σ. Both of the puzzles are currently being resolved by invoking these higher quarkonia states as admixtures of the respective $Q\bar{Q}$ states with $Q\bar{Q}g$ hybrids [16]. Looking into the experimental energy level differences and leptonic decay rates of the known $c\bar{c}$ (1^{--}) and $b\bar{b}$ (1^{--}) states and their deviations from the expected behavior provide us the clue to consider them as admixture of the nearby S and D states [13]. The disagreement with the experimental dileptonic widths of many of these 1^{--} quarkonia states with the theoretical estimations [18–20] also suggests to treat them as admixtures.

In this context, we compute the spectra of bottonium and charmonium to study their properties. The spectroscopic parameters deduced using a phenomenological approach will be employed to compute the decay properties such as

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the leptonic and digamma decay widths with no additional parameters. We consider the admixture of $\Upsilon(nS, n'D)$ and $\psi(nS, n'D)$ states to understand many of the newly discovered quarkonia 1^{--} states and their leptonic decay widths.

The spectroscopic parameters like the interquark potential parameters that provide the masses of the bound states and the corresponding wave functions obtained from the phenomenology are detrimental in the predictions of their decay widths. Most of the existing theoretical values for the decay rates are based on potential model calculations that employ different types of interquark potentials [21–24].

Although this work is the extension of our earlier study on quarkonia [18], here we employ nonrelativistic non-Coulombic Martin-like Potential and in view of the new experimental observations of various quarkonia like states. We compute masses of higher $P-$ wave and $D-$ wave charmonia and bottonia states.

II. METHODOLOGY

It has been shown that a purely phenomenological approach to the nonrelativistic potential-model study of Υ spectra and ψ spectra can lead to a static non-Coulombic power law potential of the form [25,26]

$$V(r) = \lambda r^{\nu} + V_{\circ} \tag{1}$$

where ν is chosen to be 0.1 for the Martin-like potential with $\lambda > 0$.

Following general quantum mechanical rules applicable to powerlike potentials as discussed in [27], the binding energy of a system with reduced mass μ in a power law potential, λr^{ν} is given by

$$E_{nl} = \lambda^{2/(2+\nu)}(2\mu)^{-\nu/(2+\nu)}\left[A(\nu)\left(n+\frac{l}{2}-\frac{1}{4}\right)\right]^{2\nu/(2+\nu)}. \tag{2}$$

and the corresponding square of the probability amplitude of the S waves at the zero separation of the quark-antiquark system is given by

$$|\psi_{nl}(0)|^2 = \frac{1}{2\pi^2}\left(\frac{2\mu\lambda}{\hbar^2}\right)^{3/(2+\nu)}\frac{\nu}{(2+\nu)}[A(\nu)]^{3\nu/(2+\nu)} \times\left(n+\frac{l}{2}-\frac{1}{4}\right)^{2(\nu-1)/(2+\nu)}, \tag{3}$$

where

$$A(\nu) = \left[2\nu\sqrt{\pi}\Gamma\left(\frac{3}{2}+\frac{1}{\nu}\right)\right]/\Gamma(1/\nu), \qquad \nu > 0. \tag{4}$$

For P, D waves, the radial wave function $R_{nl}(r)$ as $r \to 0$ is given by [27],

$$R_{nl}(r) \sim a_{nl} r^{l}. \tag{5}$$

The lth derivatives of the radial wave function $R_{nl}(r)$ at zero separation of the quark-antiquark system is obtained as

TABLE I. The best fitted model parameters for the $b\bar{b}$ and $c\bar{c}$ systems.

System parameters	$b\bar{b}$	$c\bar{c}$
Quark mass (in GeV)	$m_b = 4.67$ [4]	$m_c = 1.27$ [4]
Potential parameter (λ)	6.5177 GeV$^{\nu+1}$	6.5080 GeV$^{\nu+1}$
$V_{\circ}$	-6.7050 GeV	-6.7050 GeV
Centrifugal parameter (B) (for $l \neq 0$)	0.067 GeV^{-1}	0.194 GeV^{-1}

TABLE II. The radial wave function and corresponding $\langle r^{-2}\rangle$ for $b\bar{b}$ and $c\bar{c}$ systems.

Bottonium							
State	$R(0)$ GeV$^{3/2}$	State	$R'(0)$ GeV$^{5/2}$	$\langle r^{-2}\rangle$ GeV^2	State	$R''(0)$ GeV$^{7/2}$	$\langle r^{-2}\rangle$ GeV^2
1S	2.6919	1P	3.8351	0.8477	1D	8.0992	0.3751
2S	1.8722	2P	3.1743	0.4981	2D	7.1146	0.2492
3S	1.5425	3P	2.7986	0.3571	3D	6.4814	0.1882
4S	1.3505				4D	6.0263	0.1520
5S	1.2204				5D	5.6768	0.1279
6S	1.1245				6D	5.3965	0.1106
7S	1.0498						
Charmonium							
State	$R(0)$ GeV$^{3/2}$	State	$R'(0)$ GeV$^{5/2}$	$\langle r^{-2}\rangle$ GeV^2	State	$R''(0)$ GeV$^{7/2}$	$\langle r^{-2}\rangle$ GeV^2
1S	1.0609	1P	0.8124	0.2449	1D	0.9222	0.1084
2S	0.7378	2P	0.6724	0.1439	2D	0.8101	0.0720
3S	0.6079	3P	0.5928	0.1032	3D	0.7380	0.0544
4S	0.5322				4D	0.6862	0.0439
5S	0.4809				5D	0.6464	0.0369
6S	0.4431				6D	0.6145	0.0320

$$R_{nl}^{(l)}(0) = l! a_{nl}, \tag{6}$$

where

$$a_{nl} = \left[\frac{(2\mu E_{nl})^{(l+1/2)}}{\pi((2l+1)!!)^2} \frac{4\mu\nu}{(2+\nu)} \lambda^{(2/2+\nu)}(2\mu)^{(-\nu/2+\nu)} \times [A(\nu)]^{(2\nu/2+\nu)} \left(n + \frac{l}{2}\right)^{(\nu-2/\nu+2)} \right]^{(1/2)}. \tag{7}$$

In this case, the nonrelativistic Schrodinger bound-state mass (spin average mass) of the $Q\bar{Q}$ ($Q \in b, c$) system is expressed as

$$M_{\rm SA} = 2m_Q + V_0 + E_{nl}. \tag{8}$$

For the hyperfine splitting we have considered the standard one gluon exchange interactions [18]. Accordingly, the hyperfine mass split for the S wave is given by

$$\Delta M = A_{\rm hyp}|\psi_n(0)|^2/m_Q^2. \tag{9}$$

The b and c quark mass parameters m_b and m_c are taken as 4.67 GeV and 1.27 GeV, respectively, as given by the Particle Data Group (PDG) [4]. The vector $\Upsilon(nS)$, $\psi(nS)$ and the pseudoscalar $\eta_b(nS)$, $\eta_c(nS)$ masses are obtained by adding $\Delta M/4$ and $-3\Delta M/4$, respectively, to the corresponding spin average mass of the nS state given by Eq. (8). A fit to this mass formula using the experimental masses of $\Upsilon(1S, 2S)$ and the newly discovered $\eta_b(1S)$ states provides us with the potential parameters λ, V_0 and the hyperfine parameter ($A_{\rm hyp}$) in the case of bottonium system. Similarly, a fit to this mass formula using the experimental masses of $\psi(1S, 2S)$ and $\eta_c(1S)$ states provides us the potential parameters λ, V_0 and the hyperfine parameter ($A_{\rm hyp}$) of the charmonium systems.

TABLE III. S-wave $b\bar{b}$ spectrum (in MeV).

nS	[Our]	[19]	CPP_ν [18] $(0.5 \le \nu \le 1.5)$	Experiment [4]
1^3S_1	9460.43	9460.38	9463 − 9472	$\Upsilon(9460.30 \pm 0.26)$
1^1S_0	9392.38	9392.91	9426 − 9399	$\eta_b(9390.7 \pm 2.9)$ [30,31]
2^3S_1	10023.80	10023.3	9702 − 9951	$\Upsilon(10023.26 \pm 0.31)$
2^1S_0	9990.88	9987.42	9696 − 9924	...
3^3S_1	10345.80	10364.2	9827 − 10334	$\Upsilon(10355.2 \pm 0.5)$
3^1S_0	10323.40	10333.9	9824 − 10317	...
4^3S_1	10575.50	10636.4	...	$\Upsilon(10579.4 \pm 1.2)$
4^1S_0	10558.30	10609.4	...	...
5^3S_1	10755.40	...	...	$\Upsilon(10865 \pm 8.0)$
5^1S_0	10741.40	...	...	...
6^3S_1	10903.90	...	...	$Y_b(10888.4 \pm 3.0)$ [8,9]
6^1S_0	10892.00	...	...	-
7^3S_1	11030.7	...	...	$\Upsilon(11019 \pm 8.0)$
7^1S_0	11020.3	...	...	...

TABLE IV. P-wave $b\bar{b}$ spectrum (in MeV).

nS	M_{cw}	State notation	Tensor contribution	Spin-orbit contribution	[our]	[19]	CPP_ν [18] $(0.5 \le \nu \le 1.5)$	Experiment [4]
$1P$	9896.07	1^3P_2	−0.844	12.667	9907.89	9910.63	9683 − 9866	$\chi_{b2}(9912.21 \pm 0.26)$
		1^3P_1	4.222	−12.667	9887.63	9891.33	9670 − 9842	$\chi_{b1}(9892.78 \pm 0.26)$
		1^3P_0	−8.445	−25.335	9862.29	9861.39	9664 − 9820	$\chi_{b0}(9859.44 \pm 0.42)$
		1^1P_1			9896.07	9899.93	9672 − 9852	$h_b(9900.00)$ [32]
$2P$	10260.70	2^3P_2	−0.496	7.443	10267.65	10271.2	9811 − 10282	$\chi_{b2}(10268.65 \pm 0.22)$
		2^3P_1	2.481	−7.443	10255.74	10254.8	9806 − 10246	$\chi_{b1}(10255.46 \pm 0.22)$
		2^3P_0	−4.962	−14.885	10240.85	10230.5	9803 − 10228	$\chi_{b0}(10232.50 \pm 0.40)$
		2^1P_1			10260.70	10261.8	9808 − 10264	-
$3P$	10511.30	3^3P_2	−0.356	5.336	10516.28	...	...	...
		3^3P_1	1.779	−5.336	10507.74	...	...	...
		3^1P_1	−3.557	−10.672	10497.07	...	...	...
		3^3P_0			10511.30	...	...	$h_b(10551 \pm 14 \pm 17)$ [33]

TABLE V. D-wave $b\bar{b}$ spectrum (in MeV).

nS	M_{cw}	State notation	Tensor contribution	Spin-orbit contribution	[Our]	[19]	Experiment
1D	10166.00	1^3D_3	−0.534	11.211	10176.68	10163.1	...
		1^3D_2	1.869	−5.606	10162.26	10157.3	10163.8 ± 1.4 [34,35]
		1^3D_1	−1.869	−16.817	10147.31	...	...
		1^1D_2			10166.00	10158.6	...
2D	10440.00	2^3D_3	−0.355	7.448	10447.09	10455.7	...
		2^3D_2	1.241	−3.724	10437.52	10450.3	...
		2^3D_1	−1.241	−11.172	10427.59	-	...
		2^1D_2			10440.00	10451.4	...
3D	10646.50	3^3D_3	−0.268	5.626	10651.86	...	...
		3^3D_2	0.938	−2.813	10644.62	...	...
		3^3D_1	−0.938	−8.439	10637.12	...	...
		3^1D_2			10646.50	...	...
4D	10812.60	4^3D_3	−0.216	4.543	10816.93	...	...
		4^3D_2	0.757	−2.271	10811.09	...	...
		4^3D_1	−0.757	−6.814	10805.03	...	...
		4^1D_2			10812.60	...	...
5D	10952.00	5^3D_3	−0.182	3.821	10955.64	...	...
		5^3D_2	0.637	−1.911	10950.73	...	...
		5^3D_1	−0.637	−5.732	10945.63	...	...
		5^1D_2			10952.00	...	...
6D	11072.20	6^3D_3	−0.157	3.305	11075.35	...	...
		6^3D_2	0.551	−1.653	11071.10	...	...
		6^3D_1	−0.551	−4.958	11066.69	...	...
		6^1D_2			11072.20	...	...

In the case of $l \neq 0$ orbitally excited states, we find a small variations in the choice of $V_\circ$ fixed for the $l = 0$ states due to the centrifugal repulsion from the center of mass of the bound system which is proportional to $l(l+1)\langle r^{-2}\rangle$. Accordingly, we find a linear relationship for $V_\circ$ in the present study as

$$V_\circ(l) = V_\circ(l=0) + Bl(l+1)\langle r^{-2}\rangle. \qquad (10)$$

Further, we have considered the tensor and spin orbit components of the Breit interaction terms as given by [28]

$$H_T = \frac{1}{12m_Q^2} S_{12}\left[\frac{1}{r}\frac{d}{dr}V_V(r) - \frac{d^2}{dr^2}V_V(r)\right] \qquad (11)$$

and

$$H_{LS} = \frac{1}{2m_Q^2 r}\vec{L}.\vec{S}\left[3\frac{d}{dr}V_V(r) - \frac{d}{dr}V_S(r)\right], \qquad (12)$$

where

$$S_{12} = \frac{4}{(2l-1)(2l+3)}\left[\vec{S}^2\vec{L}^2 - \frac{3}{2}\vec{L}.\vec{S} - 3(\vec{L}.\vec{S})^2\right], \qquad (13)$$

$$\vec{L}.\vec{S} = \frac{1}{2}[J(J+1) - L(L+1) - S(S+1)]. \qquad (14)$$

For the present case of the potential $V(r)$ given in Eq. (1), we consider

$$V_V(r) = V_S(r) = \frac{1}{2}V(r). \qquad (15)$$

For the choice, $\nu = 0.1$ and approximating $|\nu - 2| = 1.9 \approx 2$ in the resultant expressions of Eqs. (11) and (12),

TABLE VI. S-wave $c\bar{c}$ spectrum (in MeV)

nS	[Our]	[20]	CPP_ν [18] ($0.5 \le \nu \le 1.5$)	Experiment [4]
1^3S_1	3097.14	3096.92	3092 − 3129	$\psi(3096.91 \pm 0.011)$
1^1S_0	2980.40	2981.7	3000 − 2912	$\eta_c(2980.3 \pm 1.2)$
2^3S_1	3689.95	3686.1	3375 − 3739	$\psi'(3686.09 \pm 0.04)$
2^1S_0	3633.49	3619.2	3352 − 3636	$\eta_c'(3637 \pm 4)$
3^3S_1	4030.32	4102.0	3553 − 4285	$\psi(4039 \pm 1)$
3^1S_0	3991.99	4052.5	3541 − 4212	...
4^3S_1	4273.49	4446.8	...	$Y(4263 \pm 5)$ [36,37]
4^1S_0	4244.11	...	...	...
5^3S_1	4464.12	...	...	$\psi(4421 \pm 4)$ $Z(4443^{+24}_{-18})$ [38,39]
5^1S_0	4440.12	...	...	...
6^3S_1	4621.56	...	...	...
6^1S_0	4601.19	...	...	...

we get

$$\langle H_T\rangle = \frac{\lambda\nu}{12m_Q^2} S_{12}\langle r^{-2}\rangle \quad (16)$$

and

$$\langle H_{LS}\rangle = \frac{\lambda\nu}{2m_Q^2} \vec{L}.\vec{S}\langle r^{-2}\rangle, \quad (17)$$

where $\langle r^{-2}\rangle_{nl}$ is obtained as [29],

$$\langle r^{-2}\rangle_{nl} = \frac{2m}{\hbar^2}\frac{1}{2l+1}\left(\frac{\partial E_{nl}}{\partial l}\right). \quad (18)$$

The best fitted model parameters including the quark masses m_c and m_b are listed in Table I and the l^{th} derivatives of the radial wave function at the origin and the corresponding $\langle r^{-2}\rangle$ for bottonium and charmonium states are listed in Table II. The computed S-wave, P-wave and

TABLE VII. P-wave $c\bar{c}$ spectrum (in MeV)

nS	M_{cw}	State notation	Tensor contribution	Spin-orbit contribution	[Our]	[20]	CPP_ν [18] $(0.5 \le \nu \le 1.5)$	Experiment [4]
$1P$	3523.88	1^3P_2	−3.294	49.415	3570.00	3556.2	3323 − 3590	$\chi_{c2}(3556.20 \pm 0.09)$
		1^3P_1	16.472	−49.415	3490.94	3510.6	3302 − 3504	$\chi_{c1}(3510.66 \pm 0.07)$
		1^3P_0	−32.943	−98.829	3392.11	3415.2	3292 − 3461	$\chi_{c0}(3414.75 \pm 0.31)$
		1^1P_1			3523.88	3523.7	3313 − 3547	$h_c(3525.42 \pm 0.29)$ [40]
$2P$	3921.91	2^3P_2	−1.936	29.033	3949.01	3992.3	3531 − 4239	$\chi_{c2}(3927.2 \pm 2.6)$ [41,42]
		2^3P_1	9.678	−29.033	3902.55	3950.0	3507 − 4085	...
		2^3P_0	−19.355	−58.066	3844.49	3864.3	3494 − 4009	...
		2^1P_1			3921.91	3963.2	3519 − 4162	...
$3P$	4192.35	3^3P_2	−1.388	20.816	4211.78	...	...	...
		3^3P_1	6.939	−20.816	4178.47	...	...	...
		3^1P_1	−13.878	−41.633	4136.84	...	...	...
		3^3P_0			4192.35	...	...	...

TABLE VIII. D-wave $c\bar{c}$ spectrum (in MeV)

nS	M_{cw}	State notation	Tensor contribution	Spin-orbit contribution	[Our]	[20]	Experiment [4]
$1D$	3802.30	1^3D_3	−2.083	43.735	3843.95	3844.8	...
		1^3D_2	7.289	−21.867	3787.72	3822.1	...
		1^3D_1	−7.29	−65.602	3729.41	3789.4	$\psi(3772.92 \pm 0.35)$
		1^1D_2			3802.30	3822.2	...
$2D$	4104.86	2^3D_3	−1.380	29.055	4132.53	4218.9	...
		2^3D_2	4.840	−14.528	4095.17	4195.8	...
		2^3D_1	−4.840	−43.583	4056.43	4159.2	...
		2^1D_2			4104.86	4196.9	...
$3D$	4329.76	3^3D_3	−1.045	21.946	4350.66	...	...
		3^3D_2	3.660	−10.970	4322.44	...	...
		3^3D_1	−3.660	−32.920	4293.18	4478.9	...
		3^1D_2			4329.76	...	...
$4D$	4509.54	4^3D_3	−0.844	17.720	4526.41	...	...
		4^3D_2	2.950	−8.860	4503.63	...	...
		4^3D_1	−2.950	−26.580	4480.01	...	...
		4^1D_2			4509.54	...	...
$5D$	4659.77	5^3D_3	−0.710	14.907	4673.96	...	...
		5^3D_2	2.485	−7.454	4654.80	...	...
		5^3D_1	−2.485	−22.361	4634.92	...	$X(4634^{+9}_{-11})$ [43]
		5^1D_2			4659.77	...	...
$6D$	4789.10	6^3D_3	−0.614	12.894	4801.38	...	...
		6^3D_2	2.149	−6.447	4784.80	...	...
		6^3D_1	−2.149	−19.341	4767.61	...	...
		6^1D_2			4789.10	...	...

D-wave masses of the bottonium states are listed in Tables III, IV, and V and corresponding charmonium states are listed in Tables VI, VII, and VIII, respectively.

We compare our present results (listed in the column named "[Our]" of the respective tables) with the experimentally known states ("Experiment") as well as with other potential model predictions [19] including our own previous study [18] based on coulomb plus power potential (CPP_ν) with power index ν ranging from $0.5 \leq \nu \leq 2.0$. The range of values predicted in the CPP_ν model [18] for the choice of $0.5 \leq \nu \leq 1.5$ are compared with the present results.

III. LEPTONIC AND DIGAMMA DECAY WIDTHS OF BOTTONIUM AND CHARMONIUM STATES

Apart from the masses of the low-lying states, the hyperfine splits due to chromomagnetic interaction and the right behavior of the wave function that provides as the correct predictions of the decay rates are important features of any successful model. Accordingly, the radial wave functions of the identified nS states of quarkonia ($c\bar{c}$, $b\bar{b}$) obtained from Eq. (5) are employed to predict the leptonic and digamma widths of the vector 1^{--} and 0^{-+} states, respectively. The leptonic decay widths with the radiative correction are computed using the expression [44–46],

$$\Gamma^{l^+l^-} = \frac{4\alpha_e^2 e_Q^2}{M_V^2}|R_{ns}(0)|^2\left[1 - \frac{16}{3\pi}\alpha_s\right], \tag{19}$$

and the digamma (two photon) decay widths with radiative correction are computed as [44–46]

$$\Gamma^{\gamma\gamma} = \frac{12\alpha_e^2 e_Q^4}{M_P^2}|R_{ns}(0)|^2\left[1 - \frac{\alpha_s}{\pi}\left(\frac{20-\pi^2}{3}\right)\right]. \tag{20}$$

Here, α_e is the electromagnetic coupling constant and α_s is the running strong coupling constant which is computed as

$$\alpha_s = \frac{4\pi}{(11 - \frac{2}{3}n_f)\log(\frac{m_Q^2}{\Lambda^2})}. \tag{21}$$

Here, $\Lambda_{\rm QCD}$ is taken as 0.15 GeV that provides $\alpha_s = 0.118$, at the Z-boson mass (91 GeV) [4]. The flavor number is taken as $n_f = 3$ in the case of $c\bar{c}$ and $n_f = 4$ in the case of $b\bar{b}$ system. The predicted results in the case of bottonia $\Upsilon(nS) \to l^+l^-$ and $\eta_b(nS) \to \gamma\gamma$ and in the case of charmonia $\psi(nS) \to l^+l^-$ and $\eta_c(nS) \to \gamma\gamma$ are tabulated in Tables IX and X respectively, along with the known experimental as well as with other theoretically predicted values.

IV. RESULTS AND DISCUSSION

We have computed the charmonium and bottonium spectral states which are in good agreement with the reported PDG values of known states. Though there are many excited 1^{--} states of quarkonia known experimentally, most of them beyond $3S$ states are still not understood completely. And in the case of P-wave states only 1^3P_J, 1^1P_1, and 2^3P_2 of the charmonia are known experimentally. While in the case of bottonia all the $1P$ states and 2^3P_J states are known. Recently the only $l = 2$,

TABLE IX. The leptonic widths (in keV) of the $\Upsilon(nS)$ and the digamma widths (in keV) of $\eta_b(nS)$ states.

nS	$\Gamma^{l^+l^-}$ [Our]	$\Gamma^{l^+l^-}$ [19]	$\Gamma^{l^+l^-}$ CPP_ν [18] ($0.5 \leq \nu \leq 1.5$)	$\Gamma^{l^+l^-}$ Experiment [4]	$\Gamma^{\gamma\gamma}$ [Our]	$\Gamma^{\gamma\gamma}$ [47]	$\Gamma^{\gamma\gamma}$ CPP_ν [18] ($0.5 \leq \nu \leq 1.5$)	$\Gamma^{\gamma\gamma}$ [48]
1S	1.203	1.33	1.025 − 2.025	1.34 ± 0.018 $\Upsilon(9460)$	0.496	0.460	0.259 − 0.512	0.580
2S	0.519	0.62	⋯	0.612 ± 0.011 $\Upsilon(10023)$	0.212	0.20	⋯	⋯
3S	0.330	0.48	⋯	0.443 ± 0.008 $\Upsilon(10355)$	0.135	⋯	⋯	⋯
4S	0.242	0.40	⋯	0.272 ± 0.029 $\Upsilon(10579)$	0.099	⋯	⋯	⋯
5S	0.191	⋯	⋯	0.310 ± 0.07 $\Upsilon(10865)$	0.078	⋯	⋯	⋯
6S	0.158	⋯	⋯	⋯	0.064	⋯	⋯	⋯
7S	0.134	⋯	⋯	0.130 ± 0.030 $\Upsilon(11019)$	0.055	⋯	⋯	⋯

TABLE X. The leptonic widths (in keV) of the $\psi(nS)$ and the digamma widths (in keV) of $\eta_c(nS)$ states

nS	$\Gamma^{l^+l^-}$ [Our]	$\Gamma^{l^+l^-}$ [20]	$\Gamma^{l^+l^-}$ CPP_ν [18] ($0.5 \leq \nu \leq 1.5$)	$\Gamma^{l^+l^-}$ Experiment [4]	$\Gamma^{\gamma\gamma}$ [Our]	$\Gamma^{\gamma\gamma}$ [47]	$\Gamma^{\gamma\gamma}$ CPP_ν [18] ($0.5 \leq \nu \leq 1.5$)	$\Gamma^{\gamma\gamma}$ [48]	$\Gamma^{\gamma\gamma}$ Experiment [4]
1S	4.947	1.89	5.506 − 13.520	5.55 ± 0.14 $\psi(3097)$	10.380	7.8	5.538 − 13.531	11.8	7.2 ± 0.7 ± 2.0
2S	1.686	1.04	⋯	2.35 ± 0.04 $\psi(3686)$	3.378	3.5	⋯	⋯	7 ± 3.5
3S	0.959	0.77	⋯	0.86 ± 0.07 $\psi(4040)$ 0.83 ± 0.07 [43]	1.900	⋯	⋯	⋯	⋯
4S	0.654	0.65	⋯	-	1.288	⋯	⋯	⋯	⋯
5S	0.489	⋯	⋯	0.58 ± 0.07 $\psi(4421)$ 0.35 ± 0.12 [43]	0.961	⋯	⋯	⋯	⋯
6S	0.388	⋯	⋯	⋯	0.760	⋯	⋯	⋯	⋯

D state known experimentally is the bottonia state $1^3D_2(10163.8 \pm 1.4$ MeV) reported by *BABAR* collaboration [35]. To compare and identify our predicted masses with the respective experimental S-wave states (PDG [4] average values) as well as with many of the recently known 1^{--} states (X, Y, Z) [7], we compute here the energy level differences of the S-wave masses, $\Delta M = M_{(n+1)S} - M_{nS}$ between the successive 1^{--} states. These mass differences are then plotted against $(n+1)S - nS$ in the case of bottonia and charmonia in Figs. 1 and 2 respectively. It is expected that the excited states must follow a specific pattern representing its characteristic spectral property.

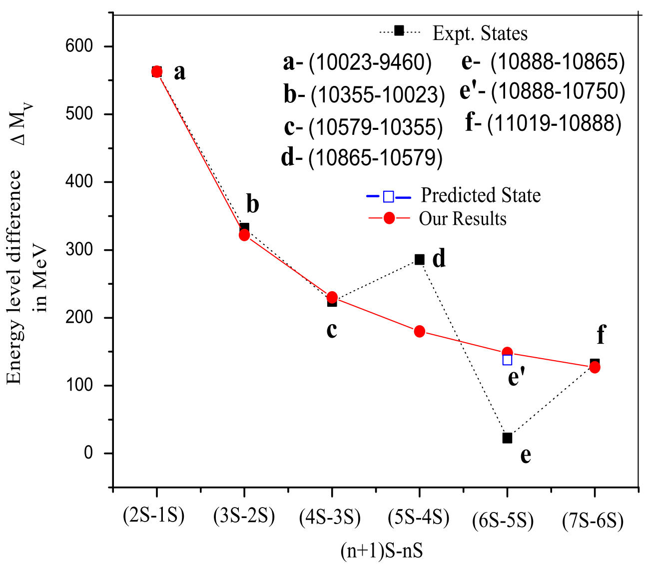

FIG. 1 (color online). Behavior of energy level shift of the $(n+1)S - nS$ bottonium states.

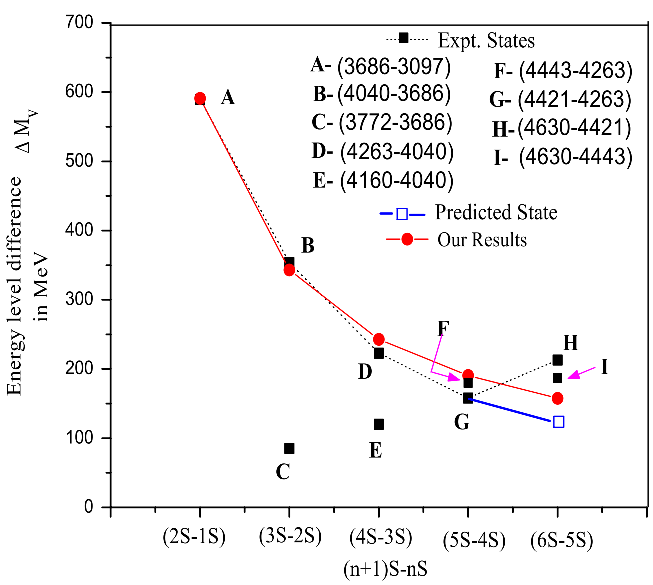

FIG. 2 (color online). Behavior of energy level shift of the $(n+1)S - nS$ charmonium states.

So we compare our predicted nS states with those which are closer to the expected behavior shown by the solid curve in the graph. The states which are widely off from the expected behavior are then identified as either mixed (disturbed) states or exotic states. Even PDG found it difficult to assign $\Upsilon(10860)$ and $\Upsilon(11020)$ as the $5S$ and $6S$ states, respectively, as there seemed to be mixing of two Briet-Wigner resonances [4]. Accordingly, we identify that $\Upsilon(10865)$ does not fit to be the $5S$ state while $\Upsilon(11019)$, $Y_b(10888)$, $\Upsilon(10579)$ and $\Upsilon(10355)$ are closer to the bottonium $7S$, $6S$, $4S$ and $3S$ states, respectively. Similarly in the case of charmonia, the states $\psi(4040)$, $Y(4260)$, $Z(4443)$ are close to our predicted $3S$, $4S$ and $5S$ states, while $\psi(3770)$, $\psi(4160)$, $X(4630)$, etc., do not match with the predicted pattern, particularly the state $X(4630)$ does not fall nearer to the $6S$ state. If the experimental states $Y(4260)$, $\psi(4421)$ represent the charmonia $4S$ and $5S$ states, then the trend curve predicts the $6S$ state at about 40 MeV below our predicted $6S$ state [open square in Fig. 2].

We have also predicted the $\eta_b(2S–6S)$ states within the mass range 9.989 GeV to 10.891 GeV and $\eta_c(2S–6S)$ states within the mass range 3631.97 GeV to 4601.18 GeV. We hope to find future experimental support in favor of these predictions.

The precise experimental measurements of the masses of quarkonia states provide a real test for the choice of the hyperfine and the fine structure interactions adopted in the study of quarkonia spectroscopy. Recently, quarkonium mass splittings in three flavor lattice QCD has been studied by Fermilab Lattice and MILC collaborations [49]. In Table XI, we compare our results on the mass splittings with the lattice results as well as with the respective experimental results. Here, we quoted the CPP_ν model [18] values for $\nu = 1.1$ for $c\bar{c}$ and $\nu = 1.5$ for $b\bar{b}$ as their predicted values for these potential indices are found to be closer to the respective experimental values. The splittings obtained in the present study are in agreement with the experimental values as well as with the lattice results [49] except in the case of charmonium P-wave spin orbit splittings.

In the case of bottonia, our predicted $1P_J$ and $2P_J$ states are in very good agreement with the available experimental states $\chi_{b_J}(1P)$ and $\chi_{b_J}(2P)$ states. We also predicted the $h_b(1P)$ state at 9896.07 MeV which is in close agreement with the recently reported value of $h_b(9900$ MeV) by the *BABAR* collaboration [32]. The present results are in better agreement with the known experimental values compared to similar studies reported by our group earlier [18]. Our predicted mass of $h_b(2P)$ at 10260.70 MeV is in good agreement with other model prediction [19]. Our predicted $b\bar{b}$ (1^3D_2) state at 10162.26 MeV is also in good agreement with the only experimental state of 10163.8 ± 1.4 MeV reported by the *BABAR* [35] and CLEO [34] collaborations. We have predicted $h_b(3P)$ at 10511.3 MeV and other $\chi_b(3^3P_J)$ states to lie between

TABLE XI. Mass splitting in charmonium and bottonium in MeV.

	Charmonium				Bottonium			
Splitting	[Our]	[49]	$CPP_{\nu=1.1}$ [18]	Experiment	[Our]	[49]	$CPP_{\nu=1.5}$ [18]	Experiment
$\overline{1P} - \overline{1S}$	455	473 ± 12	863.5	457.5 ± 0.3	453	446 ± 18	398.25	456.9 ± 0.8
$\overline{2S} - \overline{1S}$	606	792 ± 42	529	606 ± 1	572.57	599 ± 36	490.5	580.3 ± 0.8
$1^3S_1 - 1^1S_0$	116.74	116 ± 7.4	174	116.4 ± 1.2	68.05	54.0 ± 12.4	73	69.4 ± 2.8
$1P$ tensor	13.17	15.0 ± 2.3	–	16.25 ± 0.07	3.38	4.5 ± 2.2	–	5.25 ± 0.13
$1P$ spin orbit	65.88	43.3 ± 6.6	–	46.61 ± 0.09	16.88	16.9 ± 7.0	–	18.2 ± 0.2

10497–10516 MeV. Very recently Fermilab has reported the $\chi_b(3P)$ state at 10551 ± 14(stat) ± 17(syst) MeV [33] and the ATLAS collaboration has observed the state at 10530 ± 5(stat) ± 9(syst) MeV. They are in close agreement with our predicted $\chi_b(3P)$ spin average mass. Other $1D$ and $2D$ wave masses are in accordance with the recently predicted values based on potential model consisting of a linear confining term with scalar and vector relativistic corrections, a relativistic kinetic energy term and the complete perturbative one-loop QCD short distance potential [20].

In the case of charmonia, our predicted mass of the $h_c(1P)$ state at 3523.88 MeV is in very good agreement with the experimental value of 3525.42 ± 0.29 MeV reported by the PDG and BES III collaborations [40]. The known experimental masses of 1^3P_J [4] and 2^3P_2 [41,42] states are about 20 MeV off compared to our predicted values. It is attributed to the large contribution coming from the spin orbit interaction to the P states. The present results are in better agreement with the respective experimental values compared to similar studies reported by our group earlier [18]. Our predicted ($1D$) state is in close agreement with the value reported by other model [20]. We have predicted $h_c(2P)$, $h_c(3P)$ at 3921.91 MeV and 4192.35 MeV, respectively, and other $\chi_c(2^3P_J)$, $\chi_c(3^3P_J)$ states to lie between 3949 MeV to 3844 MeV, and 4211 MeV to 4136 MeV, respectively. The present $1D$ and $2D$ wave masses are comparable with values predicted by [20]. We do not find other model predictions for higher $P-$ wave and $D-$ wave quarkonia masses for comparison.

With no additional parameters we have predicted the leptonic decay widths of $\Upsilon(1S\text{–}7S)$ as well as $\psi(1S\text{–}6S)$ states and are compared with the known experimental values of the quarkonia states which are closer to the respective S-wave masses [4].

In the case of digamma decay widths of bottonia, there is no experimental measurements available even for the $\eta_b(1S)$ case for comparison and hence we compare our results with the available other theoretical predictions [18,47,48]. While in the case of $\eta_c(1S)$, our prediction is over estimated as we compare with the PDG average value but close to the value reported by [48] and lie within the error bar of 7.2 ± 0.7 ± 2.0 (PDG-2010 evaluation). The digamma widths of $\eta_c(2S)$ is in good agreement with the predicted value of [47] and lie closer to the experimental result nearer to the lower error bar. However, more precise experimental measurements of these widths are required to identify the higher 0^{-+} resonances. Further, we have predicted the leptonic decay widths of the pure $\psi(4S, 5S, 6S)$ states as 0.65 keV, 0.49 keV and 0.39 keV respectively. Similarly, the leptonic decay widths of $\Upsilon(5S, 6S)$ states as 0.19 keV and 0.16 keV, respectively.

The variations of 15% and above seen in the case of the leptonic decays with some of the compared experimental states particularly in the case of $\Upsilon(5S)$ and $\psi(2S)$ leptonic decay widths indicate the complex nature of these states. Such observations in the case of charmonium states have been reported by others while looking for the resolution of the well-known 12% rule related to the $\rho - \pi$ puzzle corresponds to the ratio of the branching ratios of the hadronic decays of $\Psi(2S)$ with $J/\Psi(1S)$ [16]. Another issue related to the $\Upsilon(3S)$ state is the Vogel, $\Delta n = 2$ puzzle [16]. Both the puzzles have been resolved by considering admixture of the respective S states with 50% of the lowest hybrid ($Q\bar{Q}g$) state [16]. When we consider similar admixture of $c\bar{c}g$ hybrid state bearing its mass equal to 4.1 GeV given by [50] and 10.5 GeV for $b\bar{b}g$ given by [51] yield the leptonic decay widths of $\Psi(3686)$ as 2.376 keV as against the predicted 1.686 keV and that of $\Upsilon(10355)$ as 0.421 keV as against 0.33 keV obtained for pure $3S$ state which are now in good agreement with the reported experimental values of 2.35 ± 0.04 keV and 0.443 ± 0.008 keV, respectively. We find the admixture of hybrid state excludes the radiative correction to the leptonic decay widths. Thus providing a strong support to treat them as hybrid admixture states [16].

Similar disparities of the predicted higher S-wave masses and leptonic decay widths with the observed quarkonia states are reported to be due to the admixture of the S states with nearby D states [13]. Thus, to identify the different quarkonia 1^{--} states observed recently, it is important to consider such admixtures of S waves and D waves.

Accordingly, the mixed state $R_{nS'}$ is represented in terms of the mixing angle θ as [13]

$$R_{nS'} = \cos\theta R_{nS} - \sin\theta R_{n'D}. \tag{22}$$

As the S-D wave mixed state candidates, we consider $\psi(4040)$, $\psi(4160)$, $Y(4260)$, $\psi(4415)$, $X(4630)$, $Y(4360)$, $Y(4660)$ of the 1^{--} charmonium like states and the

TABLE XII. Mixing of mass spectra (in GeV) and leptonic widths (in keV) of quarkonium with S or D state.

Exp. state	Mixed state configuration	Mixing angle θ	Mixed state decay width $\Gamma^{e^+e^-}$	$\Gamma^{e^+e^-}_{[\mathrm{Exp}]}$ [4]
$\psi(4040)$	3^3S_1 & 3^3D_1	11.07°	**0.896**	0.86 ± 0.07
	3^3S_1 & 2^3D_1	37.53°	0.528	
$\psi(4160)$	4^3S_1 & 2^3D_1	46.31°	0.268	0.48 ± 0.22 [43,52]
	3^3S_1 & 3^3D_1	44.62°	0.398	
$Y(4260)$	4^3S_1 & 2^3D_1	14.44°	0.588	...
	3^3S_1 & 3^3D_1	69.19°	0.074	...
	4^3S_1 & 3^3D_1	Not possible	...	...
$\psi(4415)$	5^3S_1 & 3^3D_1	32.38°	0.320	0.58 ± 0.07
	4^3S_1 & 4^3D_1	55.87°	0.158	
	5^3S_1 & 4^3D_1	Not possible	...	
$X(4630)$	5^3S_1 & 5^3D_1	80.23°	0.005	...
	6^3S_1 & 5^3D_1	53.52°	0.112	...
$Y(4360)$	4^3S_1 & 4^3D_1	40.33°	0.326	...
	5^3S_1 & 3^3D_1	51.28°	0.161	...
$Y(4660)$	6^3S_1 & 5^3D_1	31.05°	0.259	...
$\Upsilon(10860)$	5^3S_1 & 5^3D_1	49.44°	0.072	0.31 ± 0.07
	5^3S_1 & 4^3D_1	Not possible	...	
$Y_b(10888)$	6^3S_1 & 6^3D_1	Not possible	...	...
	6^3S_1 & 5^3D_1	Not possible	...	...
$\Upsilon(10996)$	6^3S_1 & 6^3D_1	48.7°	0.062	...
	6^3S_1 & 5^3D_1	Not possible	...	...
$\Upsilon(11020)$	6^3S_1 & 6^3D_1	57.13°	0.040	0.13 ± 0.03
	6^3S_1 & 5^3D_1	Not possible	...	

1^{--} $\Upsilon(10860)$, $Y_b(10888)$, $\Upsilon(10996)$ and $\Upsilon(11020)$ of bottonium states. The mixing angle θ is determined by expressing the mass of the disturbed S-wave state in terms of the $S-$ wave mass and the near by D-wave mass as

$$M_{(nS')} = |a|^2 M_{(nS)} + (1 - |a|^2) M_{(n'D)}, \quad (23)$$

where $|a|^2 = \cos^2\theta$. Such mixed state configuration and the corresponding mixing angles obtained for the above stated quarkonia like states are presented in Table XII. Using these mixing angles we get the wave function of the mixed (disturbed) state at the origin as

$$R_{nS'}(0) = \cos\theta R_{nS}(0) - \sin\theta R_{n'D}(0), \quad (24)$$

where the wave functions at zero of the D wave, $R_{n'D}(0)$ is defined in terms of the second derivative of the D wave as $R''_{n'D}(0)/M^2_{n'D}$ [13]. These disturbed wave function at the origin given by Eq. (24) are then employed to compute the leptonic decay widths of the mixed states. The results are also tabulated in Table XII. Mixing probabilities lead to greater than one are listed as the "not possible" mixing configurations in the table.

From the results presented in Table XII, it is straightforward to conclude that $\psi(4040)$ is the admixture of 3^3S_1 and 3^3D_1, with mixing angle $\theta = 11.07°$, corresponding to 96.31% of the $3S$ state and 3.69% of the $3D$ state with a leptonic decay width 0.896 keV, which is in close agreement with the experimental value of 0.86 ± 0.07 keV compared to 0.959 keV obtained for the pure $3S$ description of the state. The leptonic decay widths of $\psi(4160)$ obtained here with the mixing configuration of $(3^3S_1, 3^3D_1)$ and $(4^3S_1, 2^3D_1)$ are in agreement with the experimental value 0.48 ± 0.22 and lie within the error bar reported by the Belle and BES collaborations [39,52] but are completely in disagreement with the value of 0.83 ± 0.07 reported by [4]. Though $Y(4260)$ can be obtained by $4S - 2D$ admixture state with mixing angle $\theta = 14.44°$ that predicts its leptonic decay width 0.588 keV, the mixing may not be possible as the $4S$ and $2D$ masses differ by more than 200 MeV. So, we consider $Y(4260)$ close to the $c\bar{c}$ $(4S)$ state with its leptonic decay width of 0.65 keV predicted. However experimental determination of this width is awaited. Though experimentally the J^{PC} for $Z(4443)$ is not known, our predicted $c\bar{c}$ $(5S)$ is very close to this state, while the state $\psi(4421 \pm 4)$ does not qualify to be the pure $5S$ state or S-D admixture. Among the other charmonia like states, the present study strongly favors $Y(4360)$ as the admixture of $(4^3S_1, 4^3D_1)$ with mixing angle $\theta = 40.33°$ whose leptonic decay width is then predicted as 0.326 keV, and $Y(4660)$ as admixture of $(6^3S_1, 5^3D_1)$ with $\theta = 31.05°$ that yields its leptonic decay width 0.259 keV.

We look forward to seeing the experimental leptonic decay widths of $Y(4260)$, $X(4630)$, $Y(4360)$ and $Y(4660)$ states before making further conclusions about their statuses. However, the states $\psi(4160)$, $\psi(4415)$, $\Upsilon(10860)$ and $\Upsilon(10996)$ do not qualify to be either pure S wave or admixtures and they may be treated as exotic states as listed in [7]. The $Y_b(10888)$ is identified to be close to the pure $b\bar{b}$ $(6S)$ 1^{--} state with its leptonic decay width of 0.16 keV and $\Upsilon(11019)$ is identified to be the $b\bar{b}$ $(7S)$ 1^{--} state whose predicted leptonic width 0.134 keV is in very good agreement with the experimental result of 0.13 ± 0.03 keV.

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