Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory

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Recently we have shown by Monte Carlo simulation that an expanding (3 + 1)-dimensional universe appears dynamically from a Lorentzian matrix model for type IIB superstring theory in (9 + 1) dimensions. Here we study the classical equation of motion as a complementary approach. While the Monte Carlo results represent the behavior at earlier times, the classical equation of motion is expected to be valid at later times. In particular, we find a class of SO(3) symmetric solutions, which exhibits the time-dependence compatible with the expanding universe, while having no spatial noncommutativity. Based on this result, we speculate that the spatial noncommutativity, which plays a crucial role in the spontaneous breaking of rotational symmetry, vanishes at later times due to some dynamical mechanism.

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I. INTRODUCTION

It is widely believed that the birth of our universe can be described by superstring theory, which is a natural candidate for a unified theory including quantum gravity. Indeed, a lot of insights into this issue have been obtained by string cosmology over the last decade.[1] These studies are based on perturbative formulations incorporating nonperturbative effects through D-branes. An obvious drawback in such an approach, however, is that one has to choose a particular string vacuum from numerous vacua that are theoretically allowed. On the other hand, there is also a possibility that one can actually determine the true string vacuum uniquely if one uses a nonperturbative formulation.

Along this line of thought, we have studied a SO(9,1) symmetric Lorentzian matrix model, which is considered to be a nonperturbative definition of type IIB superstring theory in $(9+1)$ dimensions [2]. Surprisingly our Monte Carlo results provide clear evidence that three out of nine directions start to expand at some critical time. The observed spontaneous breaking of the SO(9) rotational symmetry down to SO(3) has been understood intuitively by a mechanism, which relies crucially on the noncommutative nature of the space. While this is certainly intriguing, it also poses a crucial question whether the space-time becomes commutative at later times as we observe it now.

In this article we study the classical equations of motion of the model as a complementary approach, which is expected to be valid in describing the behavior at later times. In particular, we find a class of SO(3) symmetric solutions, which turns out to have the time dependence compatible with the expanding universe. For this solution, the space-space noncommutativity is exactly zero, whereas the space-time noncommutativity becomes significant only towards the end of the expansion.

II. LORENTZIAN MATRIX MODEL

The matrix model proposed as a nonperturbative formulation of type IIB superstring theory has the action $S = S_{\mathrm{b}} + S_{\mathrm{f}}$, where [3]

$$\begin{aligned} S_{\mathrm{b}} &= -\frac{1}{4g^2}\,\mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu]), \\ S_{\mathrm{f}} &= -\frac{1}{2g^2}\,\mathrm{tr}(\Psi_\alpha(\mathcal{C}\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta]), \end{aligned} \tag{1}$$

with A_μ $(\mu = 0, \ldots, 9)$ and Ψ_α $(\alpha = 1, \ldots, 16)$ being $N \times N$ traceless Hermitian matrices. The Lorentz indices μ and ν are raised and lowered using the invariant tensor $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. The 16×16 matrices Γ^μ are 10-dimensional gamma matrices after the Weyl projection, and the unitary matrix $\mathcal{C}$ is the charge conjugation matrix. The action has manifest SO(9,1) symmetry, where A_μ and Ψ_α transform as a vector and a Majorana-Weyl spinor, respectively. The space-time is represented dynamically by the ten bosonic matrices A_μ [4].

An important feature of the Lorentzian model is that the bosonic part of the action is proportional to

$$\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -2\,\mathrm{tr}(F_{0i})^2 + \mathrm{tr}(F_{ij})^2, \tag{2}$$

where $F_{\mu\nu} = -i[A_\mu, A_\nu]$ are Hermitian matrices, and hence the two terms in Eq. (2) have opposite signs. A common approach to study the nonperturbative dynamics of this model was to make the Wick rotation $A_0 = iA_{10}$ and to

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study the SO(10) symmetric Euclidean model, which is proved to have finite partition function [5,6]. Recent calculations based on the Gaussian expansion method [7] suggest that the SO(10) symmetry is spontaneously broken down to SO(3) in the Euclidean model.[2] On the other hand, it is known that the Lorentzian signature of the metric can play an important role in the dynamics of quantum gravity [10,11].

In Ref. [2] we studied, for the first time, the nonperturbative dynamics of the Lorentzian model defined by

$$Z = \int dA d\Psi e^{iS} = \int dA e^{iS_b} \mathrm{Pf}\mathcal{M}(A), \tag{3}$$

where the Pfaffian $\mathrm{Pf}\mathcal{M}(A)$ appears from integrating out the fermionic matrices Ψ_α. We made the partition function [Eq. (3)] finite by introducing infrared cutoffs in both the spatial and temporal directions instead of making the Wick rotation. It was shown by Monte Carlo simulation that one can remove these cutoffs in the large-N limit in such a way that physical quantities scale. This implies that the obtained results are universal, and the cutoffs become irrelevant in the large-N limit. In fact the theory thus obtained has no parameters other than one scale parameter.

III. THE CLASSICAL SOLUTIONS

Taking account of the infrared cutoffs introduced in the Lorentzian model, we search for stationary points of the bosonic action S_b for fixed $\frac{1}{N}\mathrm{tr}(A_0)^2$ and $\frac{1}{N}\mathrm{tr}(A_i)^2$. Then the problem reduces to solving the classical equations of motion

$$\begin{aligned} -[A_0,[A_0,A_i]] + [A_j,[A_j,A_i]] - \lambda A_i = 0, \\ -[A_j,[A_j,A_0]] - \tilde{\lambda} A_0 = 0, \end{aligned} \tag{4}$$

where λ and $\tilde{\lambda}$ represent the Lagrange multipliers corresponding to the constraints.

We look for solutions, which are given by a unitary representation of a Lie algebra $[A_\mu, A_\nu] = i f_{\mu\nu\lambda} A_\lambda$, which guarantees automatically that the Jacobi identity is satisfied. (See Ref. [12] for an analogous study in the Euclidean model.) Motivated by the Monte Carlo results mentioned above, we restrict ourselves to solutions with $A_I = 0$ ($4 \le I \le 9$) and with SO(3) symmetry corresponding to rotations in the $i = 1, 2, 3$ directions. From the complete list of real Lie algebras with four generators e_μ ($0 \le \mu \le 3$), we pick up the one with

$$[e_0, e_i] = -i e_i \quad \text{for } i = 1, 2, 3, \tag{5}$$

whereas all the other commutators vanish. This corresponds to the algebra $A_{4,5}^{ab}$ in Table I of Ref. [13] for $a = b = 1$, and it corresponds to the so-called κ-Minkowski space [14] with $\kappa = 1$. Note, however, that our space-time interpretation based on the unitary representation discussed below is very different from those in previous studies on the κ-Minkowski space.

The unitary irreducible representations of the above algebra are classified into two categories. One consists of the trivial one-dimensional representations given by $e_0 = a$ and $e_i = 0$, where a is a real parameter. The other consists of the infinite-dimensional representations given by the operators $e_0 = -i\frac{d}{dx}$ and $e_i = a_i \exp(x)$ on the space of functions of x with L^2 integrability, where the three real parameters a_i specify a representation. As the basis of the functional space, we use the eigenfunctions of the Hamiltonian of a one-dimensional harmonic oscillator, which are given as

$$f_n(x) = c_n H_n(x) e^{-(1/2)x^2}, \qquad c_n = (\pi^{1/4}\sqrt{n!}2^{n/2})^{-1}.$$

The representation matrices of e_0 and e_i/a_i, which we denote as $\hat{P}$ and $\hat{K}$, respectively, have the following elements:

$$\begin{aligned} P_{nm} &= \int dx f_n(x)^* (-i) \frac{d}{dx} f_m(x) \\ &= -i\frac{1}{\sqrt{2}}(\sqrt{m}\delta_{n,m-1} - \sqrt{m+1}\delta_{n,m+1}), \end{aligned}$$

$$\begin{aligned} K_{nm} &= \int dx f_n(x)^* e^x f_m(x) \\ &= c_n c_m e^{1/4} \int dx e^{-x^2} H_n\left(x+\frac{1}{2}\right) H_m\left(x+\frac{1}{2}\right) \\ &= e^{1/4} 2^{-|n-m|/2} \sqrt{n!m!} \sum_{l=0}^{M} [2^l l!(M-l)!(|n-m|+l)!]^{-1}, \end{aligned}$$

where $M = \min(n, m)$. In the last equality, we have used the property $\frac{d}{dx}H_n(x) = 2nH_{n-1}$ of the Hermite polynomials.

Using a direct sum of k infinite-dimensional representations, we find a set of SO(3) symmetric solutions to Eq. (4), which are given by

$$A_0 = \sqrt{\lambda}\hat{P} \otimes \mathbf{1}_k, \tag{6}$$

$$A_i = \hat{K} \otimes \mathrm{diag}(x_{1i}, \ldots, x_{ki}). \tag{7}$$

The $3k$ parameters $x_{ai} \equiv (\mathbf{x}_a)_i$ should be chosen such that the points $\mathbf{x}_a$ ($a = 1, \ldots, k$) have spherically symmetric distribution in the three-dimensional space. One of the Lagrange multipliers is fixed as $\tilde{\lambda} = 0$.

In the following analysis, the $k \times k$ matrices that appear in Eqs. (6) and (7) are omitted since they only give an irrelevant constant factor. Also we consider only one spatial direction $i = 1$ for simplicity since it turns out that the number of spatial directions does not play any role.

[2] The mechanism of the spontaneous symmetry breaking in the Euclidean model [8,9] is quite different from that in the Lorentzian model. In the former, the Pfaffian that appears from the fermion integral is complex, and it becomes real positive for collapsed configurations.

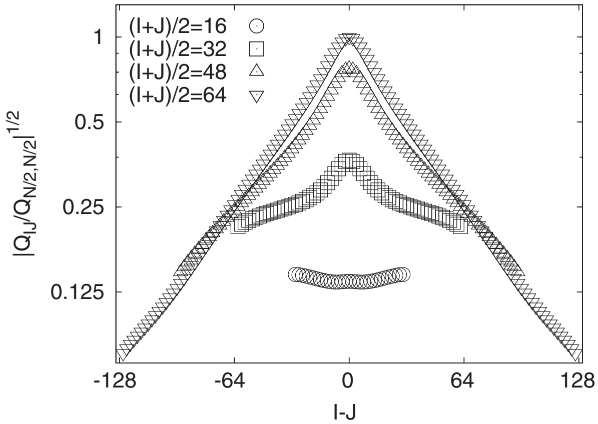

FIG. 1. The plot of $\sqrt{|Q_{IJ}/Q_{N/2,N/2}|}$ against $I - J$ for four values of $(I + J)/2$ with $N = 128$.

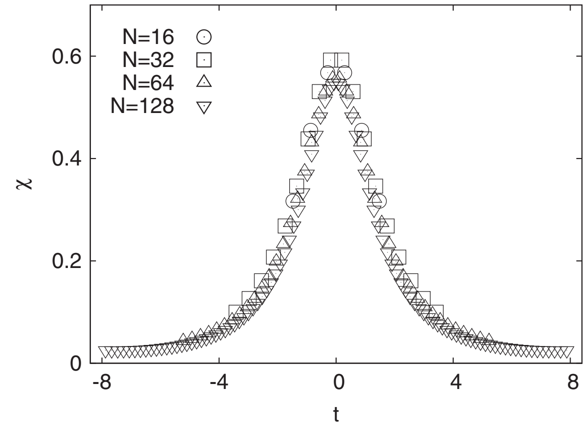

FIG. 3. The dimensionless parameter $\chi(t)$ representing the space-time noncommutativity is plotted against t for $N = 16$, 32, 64, 128. We have used the same set of values of n and λ for each N as in the previous figure. The data show nice scaling in N.

IV. THE SPACE-TIME STRUCTURE

In order to extract the space-time structure from the solution, we first need to diagonalize A_0 and calculate A_1 in that basis. Unfortunately it seems to be impossible to do this analytically. We therefore do it numerically by truncating the functional space to the N-dimensional space spanned by $f_n(x)$ with $0 \leq n \leq N - 1$. Let us define the eigenvectors $|t_I\rangle$ corresponding to the eigenvalues t_I of A_0 ($I = 1, \ldots, N$) with the specific order $t_1 < \cdots < t_N$. The spatial matrix $\langle t_I|A_1|t_J\rangle$ in that basis is not diagonal. However, it turns out that the off-diagonal elements decay exponentially in the direction orthogonal to the diagonal line. To see it explicitly, let us consider the $N \times N$ matrix $Q_{IJ} = \langle t_I|(A_1)^2|t_J\rangle$. In Fig. 1 we plot $\sqrt{|Q_{IJ}/Q_{N/2,N/2}|}$ against $I - J$ for four values of $(I + J)/2$ with $N = 128$, which shows that it decreases exponentially with $|I - J|$ for sufficiently large $(I + J)/2$. The results for smaller N show similar behaviors. The half width for $(I + J)/2 = N/2$, which we denote by n for later convenience, is obtained as $n = 11, 15, 23, 33$ for $N = 16, 32, 64, 128$.

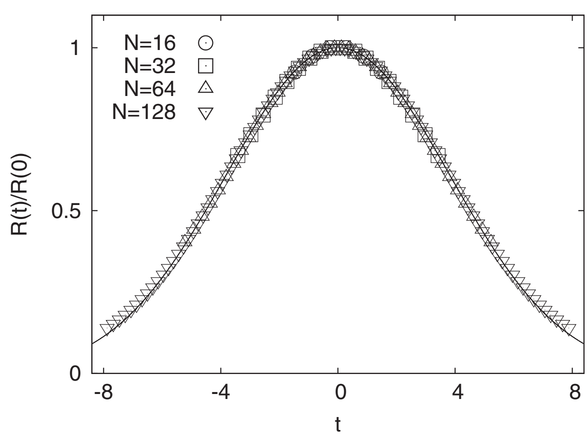

FIG. 2. The extent of space $R(t)/R(0)$ is plotted as a function of t for four values of N. The block size n is determined from the decay rate of the off-diagonal elements of A_1 in the basis which diagonalizes A_0. The value of λ is chosen for each N in such a way that the results scale in N. The solid line represents $y = \exp(-0.034t^2)$, which is obtained by fitting the $N = 128$ data to the Gaussian function.

The above observation motivates us to define $n \times n$ matrices $\bar{A}_1^{(ab)}(t) \equiv \langle t_{\nu+a}|A_1|t_{\nu+b}\rangle$ with $1 \leq a, b \leq n$ and $t = \frac{1}{n}\sum_{a=1}^{n} t_{\nu+a}$ for $\nu = 0, \ldots, (N - n)$. These matrices represent the space structure at fixed time t. Let us define the extent of space at the time t as $R(t)^2 \equiv \frac{1}{n}\,\mathrm{tr}\bar{A}_1(t)^2$. In Fig. 2 we plot $R(t)/R(0)$ for $N = 16, 32, 64, 128$. It is symmetric under the time reflection $t \to -t$ as one can prove analytically even at finite N. For each N, we have chosen the Lagrange multiplier λ, which determines the scale of t, so that $R(t)$ scales around $t = 0$. We have fixed $\lambda = 1$ for $N = 16$ without loss of generality. Then we obtain $\lambda = 0.92, 0.72, 0.59$ for $N = 32, 64, 128$, respectively. As we increase N, the scaling region extends to larger $|t|$. The solid line is a fit to the Gaussian function. Thus we find that the time evolution of the space is compatible with the expanding behavior observed in the Monte Carlo simulation [2].

Let us next turn our attention to the space-time noncommutativity. We define the dimensionless parameter

$$\chi(t) = \frac{-\frac{1}{n}\,\mathrm{tr}[\bar{A}_0(t), \bar{A}_1(t)]^2}{\frac{1}{n}\,\mathrm{tr}\,\bar{A}_0(t)^2 \cdot \frac{1}{n}\,\mathrm{tr}\bar{A}_1(t)^2} \tag{8}$$

as an estimate on the space-time noncommutativity.[3] In Fig. 3 we plot $\chi(t)$ for $N = 16, 32, 64, 128$. We find that it is of O(1) at $t = 0$ and decreases at large $|t|$. Therefore, the space-time noncommutativity is significant only around $t \sim 0$, and it becomes smaller as we go back in time.

[3]This quantity may be related to the space-time uncertainty principle. See, for instance, Ref. [15].

V. SUMMARY AND DISCUSSIONS

We have studied the classical equation of motion in the Lorentzian matrix model for type IIB superstring theory. Restricting ourselves to the class of solutions that are written in terms of Lie algebras with four generators, we find a simple solution with SO(3) symmetry. The space-time structure extracted from the solution exhibits the time dependence, which is compatible with the expanding behavior. Note, however, that the classical equation of motion is expected to be valid at later times, whereas our previous Monte Carlo results represent the behavior at earlier times. The space-time noncommutativity becomes significant only towards the end of the expansion, whereas the space-space noncommutativity is identically zero. The existence of a commutative and expanding classical solution suggests a possibility for the appearance of an expanding $(3 + 1)$-dimensional (almost commutative) space-time from the Lorentzian matrix model at later times.

We speculate that the noncommutativity of space, which plays a crucial role in making three directions expand at earlier times, disappears at some point for some dynamical reason. For instance, let us consider the model obtained after integrating out the scale factor [2]. In that model we have a constraint that requires the quantity (2) to vanish. If the expansion with large space-space noncommutativity in the earlier times continues for too long a period, the second term of (2) will be too large to satisfy the constraint (2)$= 0$. Such an effect may lead to an end of the noncommutative expansion. One might speculate that this corresponds to the end of "inflation."

Our classical solution is symmetric under time reflection, and the size of the space becomes maximum at $t = 0$, after which it has a contracting behavior. At $t = 0$, the dimensionless space-time noncommutativity becomes maximum, too, and it is of the order 1. Hence the physics there will be quite exotic. This may be taken as a prediction on the fate of our universe from the Lorentzian matrix model given that our classical solution is valid around $t = 0$.

Obviously one can generalize our solution to SO(d) symmetric ones with $1 \le d \le 9$. The time evolution of the size of the space and that of the space-time noncommutativity are essentially the same as in the SO(3) case. Let us recall here that the space-space noncommutativity seems to play a crucial role in the spontaneous breaking of SO(9) rotational symmetry as the mechanism proposed in our previous work [2] suggests. It is therefore not surprising that the dimensionality of space is not fixed by classical solutions without space-space noncommutativity.

Some comments on related works are in order. In Ref. [16] the Einstein equation was derived from the classical equations of motion of the matrix model. In this derivation, however, the matrices A_μ were interpreted as the covariant derivative on a curved space. It would be interesting to clarify the relationship to our work.

Reference [17] reports on interesting solutions to the classical equations of motion [Eq. (4)] with $\lambda = \tilde{\lambda} = 0$. They represent a flat Minkowski space with extra dimensions described by fuzzy spheres. An interesting feature of these solutions is that there exists noncommutativity between the four extended directions and the extra dimensions. This is crucial for realizing a nontrivial structure in the extra dimensions even without the Myers-like term. Here we emphasize that the nonzero λ, which is introduced in our work, is crucial for the expanding behavior. Let us recall that λ is the Lagrange multiplier corresponding to the infrared cutoff, which turns out to be needed according to our previous Monte Carlo studies.

In Ref. [18] the Matrix theory [19] has been applied to cosmology. A classical solution with three expanding (commutative) directions and six oscillating (noncommutative) directions was discussed. (The number of expanding directions does not have to be three.) In order to have such a solution, the authors introduced a SO(9) symmetric tachyonic mass term, which was interpreted as the cosmological term. The relationship to our solution is not clear, though, since the time is treated in a different way. The idea to use the matrices to avoid the big-bang singularity is also pursued in Ref. [20].

It is tempting to imagine that the rapid growth of $R(t)$ observed in the present solution has something to do with the accelerating expansion confirmed by recent cosmological observations. The power-law expansion at earlier times may be understood by considering the quantum corrections around the classical solution. It is also expected that the gauge interactions and the matter content in the $(3 + 1)$-dimensional space-time are determined by the structure in the extra dimensions [21,22] analogously to the case of intersecting D-brane models. From this point of view, it would be interesting to search for other SO(3) symmetric solutions with more nontrivial structure.

To conclude, we consider that the result of this work provides a prototype of what can happen at later times in the Lorentzian matrix model, and in that sense it is complementary to our previous Monte Carlo result [2], which seems to describe the birth of the universe. We hope that this model opens up new perspectives on particle physics beyond the standard model as well as on microscopic descriptions of cosmological models for inflation and modified gravity.

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