PHYSICAL REVIEW D **86,** 026004 (2012)

Holographic representation of bulk fields with spin in AdS/CFT

Daniel Kabat,[1,*] Gilad Lifschytz,[2,3,†] Shubho Roy,[1,4,‡] and Debajyoti Sarkar[1,5,§]

[1]*Department of Physics and Astronomy, Lehman College of the CUNY, Bronx, New York 10468, USA*
[2]*Department of Mathematics and Physics, University of Haifa at Oranim, Kiryat Tivon 36006, Israel*
[3]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030*
[4]*Physics Department, City College of the CUNY, New York, New York 10031, USA*
[5]*Graduate School and University Center, City University of New York, New York, New York 10036, USA*
(Received 27 April 2012; published 9 July 2012; corrected 23 July 2012)

We develop the representation of bulk fields with spin one and spin two in anti-de Sitter space, as nonlocal observables in the dual CFT. Working in holographic gauge in the bulk, at leading order in $1/N$ bulk gauge fields are obtained by smearing boundary currents over a sphere on the complexified boundary, while linearized metric fluctuations are obtained by smearing the boundary stress tensor over a ball. This representation respects AdS covariance up to a compensating gauge transformation. We also consider massive vector fields, where the bulk field is obtained by smearing a nonconserved current. We compute bulk two-point functions and show that bulk locality is respected. We show how to include interactions of massive vectors using $1/N$ perturbation theory, and we comment on the issue of general backgrounds.

DOI: 10.1103/PhysRevD.86.026004 PACS numbers: 11.25.Tq

I. INTRODUCTION

The question of locality and causality in quantum gravity is an old and unresolved issue. AdS/CFT implies that at best locality and causality are approximate notions. However it is vital to understand in what situations and in what way the notion of bulk locality arises. One approach to this issue, pursued since the early days of AdS/CFT, is to construct operators in the CFT which can mimic the local field operators of bulk supergravity.

In [1–4] free scalar fields in the bulk were expressed as CFT operators, and it was shown that bulk locality was obeyed in the leading large-N limit. This approach was refined to obtain CFT expressions that are covariant and convenient in [5–7]. In particular it was shown that one can represent bulk scalar fields as smeared operators in the CFT, where the smearing has support on a ball on the complexified boundary. In [8] it was shown that for scalar fields this construction can be extended to include interactions using $1/N$ perturbation theory. The construction of bulk operators in asymptotically AdS spacetimes has been further extended and clarified in [9].

In this paper we build upon two approaches that have been successfully used to construct scalar fields in the bulk.

(1) Given a bulk Lagrangian one can solve the bulk equations of motion perturbatively, to express the Heisenberg picture field operators in terms of boundary data. This leads to an expression for the bulk field as a sum of smeared CFT operators. The bulk operator constructed in this way of course respects locality, assuming one starts from a local Lagrangian in the bulk, but the construction seems tied to knowing the bulk equations of motion.

(2) Alternatively, one can start in the CFT with a candidate bulk operator, constructed by solving free equations of motion, then demand that bulk microcausality holds at the level of three-point functions. This can be achieved order-by-order in the $1/N$ expansion, by modifying the definition of the bulk field in the CFT to include a sum of appropriately-smeared higher dimension operators. In this construction, the guiding principle is bulk microcausality.

The latter construction can be carried out fully inside the CFT, without knowing the bulk Lagrangian. Hence, it may enable one to see the limitations of bulk perturbation theory, and understand the way in which microcausality breaks down at the nonperturbative level. A difficulty of extending the second approach to gauge fields is that the correct statement of bulk microcausality is necessarily somewhat subtle [9].

An outline of this paper is as follows. In the first part of this paper we extend the program of [5–7] to free fields with spin one and spin two. A closely related construction has been carried out by Heemskerk [10]. In Sec. II we derive the smearing function for a bulk gauge field and show that it is covariant under conformal transformations. We compute the bulk-to-boundary two point function and show that, although the gauge field does not obey microcausality, the corresponding field strength does. In Sec. III we obtain analogous results for gravity: we work out the smearing function for a graviton, and show that the graviton has nonlocal correlators. In the context of gravity, it is the Weyl tensor that obeys bulk microcausality. In Sec. IV

*daniel.kabat@lehman.cuny.edu
†giladl@research.haifa.ac.il
‡sroy@ccny.cuny.edu
§dsarkar@gc.cuny.edu

we derive the smearing function for a massive vector field, and show that a massive vector directly obeys microcausality. This helps to clarify the relation between gauge symmetry and locality.

In the second part of this paper we discuss interactions and general backgrounds. In Sec. V we show how to extend the definition of a massive vector field in the bulk to include interactions, using perturbation theory in $1/N$, and we discuss the difficulty with gauge fields resulting from the existence of conserved charges. In Sec. VI we provide a framework for extending the construction to general backgrounds and for going beyond the approximation of having a fixed background. We also explain the necessary conditions for the existence of approximately local operators in the bulk.

II. GAUGE SMEARING FUNCTIONS

In this section we develop the representation of an Abelian bulk gauge field as a nonlocal observable in the dual CFT. Our basic result is given in Eq. (4) below: the bulk gauge field at a point (x, z) in the bulk is obtained by integrating the boundary current over a sphere of radius z on the complexified boundary.

Our conventions are as follows. We work in Poincaré coordinates in AdS_{d+1} with metric

$$ds^2 = G_{MN}dX^M dX^N = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2),$$
$$\mu, \nu = 0, \ldots, d-1.$$

The boundary at $z = 0$ carries a flat Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +)$. Boundary indices μ, ν are raised and lowered with $\eta_{\mu\nu}$.

Our goal is to solve the source-free Maxwell equations in the bulk, $\nabla_M F^{MN} = 0$, with the boundary conditions

$$F_{z\mu}(x, z) \sim (d-2)z^{d-3} j_\mu(x) \quad \text{as } z \to 0. \qquad (1)$$

The factor $d - 2$ is inserted for later convenience.[1] From the bulk perspective this defines $j_\mu(x)$ as the coefficient of the leading small-z behavior of the bulk field. But in the dual CFT, $j_\mu(x)$ is interpreted as a conserved current. So if we can solve for the bulk field in terms of its near-boundary behavior, via a kernel of the form

$$A_M(x, z) = \int d^d x' K_M{}^\mu(x, z|x') j_\mu(x'), \qquad (2)$$

then we will have succeeded in representing the bulk gauge field as a nonlocal observable in the dual CFT. We'll refer to $K_M{}^\mu$ as a smearing function, although as we'll see below, smearing distribution might be more appropriate.

A few comments are in order.

(i) The smearing function we are after should not be confused with Witten's bulk-to-boundary propagator, which relates a non-normalizable field in the bulk to a source in the dual CFT [11]. Rather, we wish to express a *normalizable* field in the bulk in terms of an *operator* in the CFT.

(ii) The AdS boundary is timelike, so this is not a standard Cauchy problem. Nonetheless, in all cases of interest, it seems an explicit solution is possible. There is some discussion of this fact in [9]. Also note that we will construct smearing functions with compact support on the complexified boundary, along the lines of [7]. For a construction with support on a real section of the boundary, see [10].

Of course the CFT does not know about bulk gauge symmetries—it only sees global conservation laws—so in order to reconstruct a bulk gauge field we will need to make some choice of gauge in the bulk. It's convenient to work in "holographic gauge" and set

$$A_z(x, z) = 0.$$

This allows a residual gauge freedom

$$A_\mu(x, z) \to A_\mu(x, z) + \partial_\mu \lambda(x),$$

where the gauge parameter λ is independent of z. The equation of motion from varying A_z is

$$\partial_z(\eta^{\mu\nu}\partial_\mu A_\nu) = 0.$$

Thus $\partial_\mu A^\mu$ is independent of z, and we can use a residual gauge transformation to set $\partial_\mu A^\mu = 0$ everywhere.[2] The remaining Maxwell equations then simplify to

$$\partial_\mu \partial^\mu A_\nu + z^{d-3}\partial_z \frac{1}{z^{d-3}}\partial_z A_\nu = 0.$$

Defining $\phi_\mu(x, z) = zA_\mu(x, z)$, one finds that[3]

$$\partial_\mu \partial^\mu \phi_\nu + z^{d-1}\partial_z \frac{1}{z^{d-1}}\partial_z \phi_\nu + \frac{d-1}{z^2}\phi_\nu = 0. \qquad (3)$$

This shows that each component of ϕ obeys the usual scalar wave equation,[4] and from the mass term we can read off $m^2R^2 = 1 - d$.

Although tachyonic, the scalar satisfies the BF bound [12]. It is dual to an operator of conformal dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2} = d - 1.$$

[1] The special case $d = 2$ will be discussed in Sec. II B 1.

[2] From the CFT point of view this is guaranteed by the boundary conditions at $z = 0$, where the bulk gauge field approaches a conserved current in the CFT.

[3] This amounts to expressing the gauge field in a vielbein basis, setting $A_a = e_a{}^\mu A_\mu$ where $e_a{}^\mu = \frac{z}{R}\delta_a{}^\mu$.

[4] The mass term actually represents a nonminimal coupling to curvature, $(\Box + \xi R)\phi = 0$ where $\xi = -\frac{d-1}{d(d+1)}$.

The normalizable near-boundary behavior for such a scalar field is

$$\phi_\mu(x, z) \sim z^{d-1} j_\mu(x) \quad \text{as } z \to 0.$$

In Appendix A we show how to construct a smearing function for such a scalar field. The result, given in Eq. (A9), can be used to represent a bulk gauge field in terms of the boundary current:

$$zA_\mu(t, \mathbf{x}, z) = \frac{1}{\text{vol}(S^{d-1})} \int_{t'^2+|\mathbf{y}'|^2=z^2} dt' d^{d-1}y' j_\mu(t+t', \mathbf{x}+i\mathbf{y}'),$$
$$\text{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{4}$$

Here we are splitting the boundary coordinates $x^\mu = (t; \mathbf{x})$ into a time coordinate t and $d-1$ spatial coordinates $\mathbf{x}$. Note that the boundary current is evaluated at complex values of the spatial coordinates. The integral is over a sphere of radius z on the complexified boundary, with the center of the sphere located at $(t; \mathbf{x})$.

The basic claim is that Eq. (4) gives a gauge field that satisfies Maxwell's equations and has the boundary behavior

$$A_\mu(x, z) \sim z^{d-2} j_\mu(x) \quad \text{as } z \to 0. \tag{5}$$

The fact that A_μ satisfies Maxwell's equations follows from Appendix A, while the boundary conditions are easy to check. As $z \to 0$ the integration region shrinks to a point, so we can bring the current outside the integral: the factors of $\text{vol}(S^{d-1})$ cancel and we are left with Eq. (5). The corresponding field strength then satisfies Eq. (1). This is one nice feature of working on the complexified boundary: it's manifest that local fields in the bulk go over to local operators in the CFT, in the limit that the bulk point approaches the boundary.

Finally, note that Eq. (4) can be written in a covariant form. The invariant distance between two points in AdS is

$$\sigma(x, z|x', z') = \frac{z^2 + z'^2 + (x-x')_\mu (x-x')^\mu}{2zz'}.$$

The invariant distance diverges as $z' \to 0$. However, we can define a regulated bulk-boundary distance

$$(\sigma z')_{z'\to 0} = \frac{z^2 + (x-x')_\mu (x-x')^\mu}{2z}. \tag{6}$$

In terms of $\sigma z'$, the smearing integral Eq. (4) can be written as

$$zA_\mu(t, \mathbf{x}, z) = \frac{1}{\text{vol}(S^{d-1})} \int dt' d^{d-1}y' \delta(\sigma z') j_\mu(t+t', \mathbf{x}+i\mathbf{y}'). \tag{7}$$

A. AdS covariance for gauge fields

It is instructive to check that the smearing function Eq. (7) behaves covariantly under conformal transformations. First note that it is manifestly covariant under Poincaré transformations of the x^μ coordinates. Under a dilation, which corresponds to the bulk isometry

$$x^\mu \to x'^\mu = \lambda x^\mu, \qquad z \to z' = \lambda z,$$

we have

$$A_\mu \to A'_\mu = \frac{1}{\lambda} A_\mu, \qquad A_z \to A'_z = \frac{1}{\lambda} A_z.$$

Thus, holographic gauge is preserved, $A'_z = 0$, and the quantity zA_μ appearing on the left-hand side of Eq. (7) transforms like a scalar. This is consistent with the right-hand side of Eq. (7), since under a dilation d^dx has dimension $-d$, $\delta(\sigma z')$ has dimension 1, and j_μ has dimension $d-1$.

Special conformal transformations are a little more subtle. These correspond to the bulk isometry

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu (x^2 + z^2)}{1 - 2b \cdot x + b^2(x^2+z^2)}, \tag{8}$$

$$z \to z' = \frac{z}{1 - 2b \cdot x + b^2(x^2+z^2)}. \tag{9}$$

Starting from holographic gauge $A_z = 0$ and working to first order in b^μ, we find

$$A'_z = 2zb \cdot A, \tag{10}$$

$$A'_\mu = A_\mu + 2x_\mu b \cdot A - 2b_\mu x \cdot A - 2b \cdot x A_\mu. \tag{11}$$

So holographic gauge isn't preserved. To restore it we make a compensating gauge transformation $A \to A + d\lambda$, where

$$\lambda = -\frac{1}{\text{vol}(S^{d-1})} \int d^dx' \theta(\sigma z') 2b \cdot j.$$

The gauge parameter λ has been chosen so that

$$\partial_z \lambda = -\frac{1}{\text{vol}(S^{d-1})} \int d^dx' \delta(\sigma z') 2b \cdot j = -2zb \cdot A, \tag{12}$$

and

$$\partial_\mu \lambda = -\frac{1}{\text{vol}(S^{d-1})} \int d^dx' \delta(\sigma z') \frac{1}{z}(x-x')_\mu 2b \cdot j \tag{13}$$

$$= -2x_\mu b \cdot A + \frac{1}{\text{vol}(S^{d-1})} \int d^dx' \delta(\sigma z') \frac{1}{z} x'_\mu 2b \cdot j, \tag{14}$$

The gauge transformation restores holographic gauge, $A'_z = 0$, while combining Eqs. (11) and (13) we find

$$(zA_\mu)' = zA_\mu - 2zb_\mu x \cdot A + \frac{1}{\mathrm{vol}(S^{d-1})} \times \int d^d x' \delta(\sigma z') x'_\mu 2b \cdot j \quad (15)$$

$$= zA_\mu + \frac{1}{\mathrm{vol}(S^{d-1})} \int d^d x' \delta(\sigma z') 2(x'_\mu b \cdot j - b_\mu x \cdot j). \quad (16)$$

Current conservation implies $\int d^d x' \theta(\sigma z') \partial_\mu j^\mu = 0$, which after integrating by parts means

$$\int d^d x' \delta(\sigma z')(x - x')_\mu j^\mu = 0. \quad (17)$$

So we can replace x with x' in the last term of Eq. (16) to obtain

$$(zA_\mu)' = zA_\mu + \frac{1}{\mathrm{vol}(S^{d-1})} \int d^d x' \delta(\sigma z') 2(x'_\mu b \cdot j - b_\mu x' \cdot j). \quad (18)$$

This establishes how the left hand side of Eq. (7) behaves under a special conformal transformation. Now let's look at the right-hand side. Under a special conformal transformation

$$x^\mu \to x'^\mu = x^\mu + 2b \cdot x x^\mu - b^\mu x^2, \quad (19)$$

a vector of dimension Δ transforms according to

$$j_\mu \to j'_\mu = j_\mu + 2x_\mu b \cdot j - 2b_\mu x \cdot j - 2\Delta b \cdot x j_\mu. \quad (20)$$

The measure $d^d x' \delta(\sigma z')$ has dimension $1 - d$ and transforms according to

$$d^d x' \delta(\sigma z') \to d^d x' \delta(\sigma z')[1 - 2(1 - d) b \cdot x]. \quad (21)$$

Combining Eqs. (20) and (21) for $\Delta = d - 1$ reproduces the transformation seen in Eq. (18).

This shows explicitly that the smearing function we have defined behaves covariantly under conformal transformations. Indeed, it seems that, aside from the freedom to choose a different gauge in the bulk, the smearing function is uniquely fixed by the requirement of AdS covariance, at least if one works on the complexified boundary. This means that, even though we derived the smearing function by solving Maxwell's equations, it actually has a more general scope of validity. It can be used whenever one seeks a linear map from a conserved current on the boundary to a gauge field in the bulk.

B. Two-point functions and bulk causality for gauge fields

In this section we use the smearing functions we have constructed to study bulk locality and causality for gauge fields. Since we are working at leading order in the $1/N$ expansion of the CFT, we are restricted to studying bulk physics at the level of two-point functions. We consider two basic cases: in Sec. II B 1 we consider Chern-Simons theory in AdS_3, and in Sec. II B 2 we consider Maxwell theory in AdS_4 and higher.

1. Chern-Simons fields in AdS_3

AdS_3 is something of a special case, since a conserved current in the CFT is dual to a Chern-Simons gauge field in the bulk [13]. Fortunately we can still use our smearing functions in this context, since they're essentially fixed by AdS covariance.

From the smearing function Eq. (4) we have

$$zA_\mu(t, x, z) = \frac{1}{2\pi} \int_0^{2\pi} z d\theta j_\mu(t + z \sin\theta, x + iz \cos\theta). \quad (22)$$

It's convenient to introduce light-front coordinates $x^\pm = t \pm x$ and write the AdS_3 metric as

$$ds^2 = \frac{R^2}{z^2}(-dx^+ dx^- + dz^2).$$

For concreteness, consider a CFT with a right-moving Abelian current $j_- = j_-(x^-)$. We assume the left-moving current vanishes, $j_+ = 0$. Then the only nontrivial smearing integral is

$$A_-(x^+, x^-, z) = \int_0^{2\pi} \frac{d\theta}{2\pi} j_-(x^- - ize^{i\theta}).$$

Defining $\xi = e^{i\theta}$ the contour integral picks up the pole at $\xi = 0$ and gives $A_-(x^+, x^-, z) = j_-(x^-)$. So a right-moving current in the CFT is dual to a bulk gauge field

$$A_+ = 0, \qquad A_-(x^+, x^-, z) = j_-(x^-), \qquad A_z = 0. \quad (23)$$

This is the world's simplest example of holography: the boundary current is lifted to be z-independent, and declared to be a gauge field in the bulk.

Although "reading the hologram" in this case is almost trivial, there are a few things to check. First of all, Eq. (23) defines a flat gauge field in AdS, which satisfies the Chern-Simons equations of motion.[5] Working backwards, the boundary conditions on the gauge field are a bit different from Eq. (1), since we have

[5]The smearing functions were constructed by solving Maxwell's equations, but they are essentially fixed by AdS covariance and therefore hold more generally. In AdS_3 the smearing functions seem to know that a current in the CFT is dual to a Chern-Simons gauge field in the bulk.

$$A_\mu(x, z) \sim j_\mu(x) \quad \text{as } z \to 0.$$

We can use this framework to compute 2-point functions in the bulk. The boundary correlator is fixed by conformal invariance. With a Wightman $i\epsilon$ prescription

$$\langle j_-(x^-) j_-(x^{-\prime})\rangle = -\frac{k}{8\pi^2}\frac{1}{(x^- - x^{-\prime} - i\epsilon)^2}, \tag{24}$$

where k is the level of the current algebra. This lifts to a bulk correlator

$$\begin{aligned}&\langle A_-(x^+, x^-, z) A_-(x^{+\prime}, x^{-\prime}, z')\rangle \\ &\quad = -\frac{k}{8\pi^2}\frac{1}{(x^- - x^{-\prime} - i\epsilon)^2}.\end{aligned}$$

Note that the bulk 2-point function is independent of x^+ and z, which is perhaps not so surprising in a topological theory.

We can also study bulk locality and causality in this framework. The correlator Eq. (24) implies that the CFT currents obey the standard current algebra

$$i[j_-(x^-), j_-(x^{-\prime})] = -\frac{k}{4\pi}\delta'(x^- - x^{-\prime}).$$

This lifts to a bulk commutator

$$i[A_-(x^+, x^-, z), A_-(x^{+\prime}, x^{-\prime}, z')] = -\frac{k}{4\pi}\delta'(x^- - x^{-\prime}). \tag{25}$$

This bulk commutator is clearly nonlocal, being independent of both x^+ and z. But causality is respected: the field strength vanishes, so all local gauge-invariant quantities obey causal (in fact trivial) commutation relations.

We obtained these results by applying our smearing functions to the current algebra on the boundary. In Appendix B we show that they can also be obtained from the bulk point of view, by quantizing Chern-Simons theory in holographic gauge.

2. Maxwell fields in $\mathbf{AdS_4}$ *and higher*

We now consider Maxwell fields in AdS_4 and higher, where a bulk gauge field obeying Maxwell's equations is dual to a conserved current on the boundary.[6]

Our starting point is the current-current correlator in a d-dimensional CFT,

$$\langle j_\mu(x) j_\nu(0)\rangle = \left(\frac{1}{x^2}\right)^{d-1}\left(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right). \tag{26}$$

Up to an overall normalization, this correlator is fixed by current conservation and conformal invariance. We will be interested in Wightman correlators, defined by the $i\epsilon$ prescription

$$x^2 \equiv -(t - i\epsilon)^2 + |\mathbf{x}|^2.$$

Our goal is to apply the smearing function Eq. (4) to the first operator in Eq. (26), to obtain a bulk-boundary correlator

$$\langle A_\mu(t, \mathbf{x}, z) j_\nu(0)\rangle.$$

To deal with the vector indices it is useful to write the current-current correlator in the form

$$\begin{aligned}\langle j_\mu(x) j_\nu(0)\rangle = &\frac{d-2}{d-1}\eta_{\mu\nu}\left(\frac{1}{x^2}\right)^{d-1} \\ &- \frac{1}{2(d-1)(d-2)}\partial_\mu\partial_\nu\left(\frac{1}{x^2}\right)^{d-2}.\end{aligned}$$

Applying the smearing function Eq. (4) gives the bulk-boundary correlator in terms of two scalar integrals,

$$\begin{aligned}\langle zA_\mu(t, \mathbf{x}, z) j_\nu(0)\rangle = &\frac{\Gamma(d/2)}{2\pi^{d/2}}\left(\frac{d-2}{d-1}\eta_{\mu\nu} I_1\right. \\ &\left. - \frac{1}{2(d-1)(d-2)}\partial_\mu\partial_\nu I_2\right),\end{aligned} \tag{27}$$

where

$$I_n = \int_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \frac{1}{(-(t+t')^2 + |\mathbf{x} + i\mathbf{y}'|^2)^{d-n}}. \tag{28}$$

The integral is over a $(d-1)$-sphere of radius z on the boundary. We write the metric on this sphere as

$$ds^2 = \frac{z^2}{z^2 - y^2}dy^2 + (z^2 - y^2)d\Omega^2_{d-2}.$$

Here $-z < y < z$ and $d\Omega^2_{d-2}$ is the metric on a unit S^{d-2}. To take advantage of spherical symmetry on S^{d-2} we work at spacelike separation in the x_1 direction, setting

$$x_1 = x \qquad t = x_2 = \cdots = x_{d-1} = 0.$$

Then I_n reduces to a one-dimensional integral,

$$I_n = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}\int_{-z}^{z} dy \frac{z(z^2 - y^2)^{(d-3)/2}}{(x^2 - z^2 + 2ixy)^{d-n}}.$$

The prescription for defining this integral is to begin at large spacelike separation, $x \gg 0$, where the operators are well-separated on the boundary and the integral is well-defined. It can be extended to smaller values of x by analytic continuation, as described in Fig. 1. This prescription gives I_n in terms of a hypergeometric function,

$$I_n = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{z^{d-1}}{(x^2)^{d-n}}F\left(d-n, \frac{d}{2} - n + 1, \frac{d}{2}, -\frac{z^2}{x^2}\right). \tag{29}$$

When $n = 1$ this reduces to

[6] Low dimensions are special, for example, in AdS_3 a bulk Maxwell field is dual to a gauge field in the CFT [13,14]. Strictly speaking, AdS_4 Maxwell is also special since the boundary currents only capture the "electric" sector of the bulk theory [15].

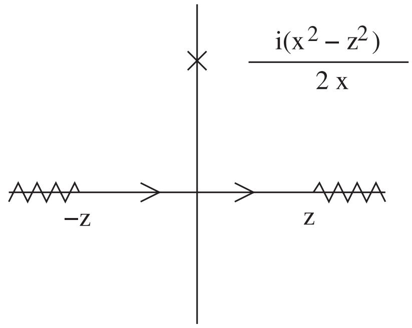

FIG. 1. Integration contour for I_n. At large spacelike separation the pole is far up the imaginary axis. The pole moves down and crosses the integration contour when $x = z$; one can continue to smaller values of x by deforming the contour. The integral may be singular when $x \to 0^+$ and the pole moves to $-i\infty$. There are singularities when $x \to \pm iz$ and the pole hits an endpoint of the integration contour.

$$I_1 = \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{z^{d-1}}{(x^2 + z^2)^{d-1}}. \tag{30}$$

Note that I_1 is only singular on the bulk lightcone, at $x^2 + z^2 = 0$. It has an AdS-covariant form, with $I_1 \sim 1/(\sigma z')^{d-1}$. These properties could have been anticipated since, up to an overall coefficient, I_1 is the bulk-boundary correlator for a scalar field with dimension $\Delta = d - 1$.

We are also interested in $n = 2$. In any given dimension I_2 can be reduced to elementary functions, see for example Table I; however, the expressions become unwieldy as d increases. For our purposes a key observation is that I_2 is singular on the boundary lightcone, with

$$I_2 \sim \frac{\pi^{(d+1)/2}}{2^{d-4}\Gamma((d-1)/2)} \frac{z}{x^{d-2}} \quad \text{as } x \to 0.$$

I_2 is also singular on the bulk lightcone, at $x^2 + z^2 = 0$.

Bulk-boundary correlators follow from Eqs. (27) and (29). For example, in AdS_4 we find

$$\langle A_\mu(t, \mathbf{x}, z) j_\nu(0)\rangle = \eta_{\mu\nu}\left[\frac{z(3x^2+z^2)}{4x^2(x^2+z^2)^2} - \frac{i}{8x^3}\log\frac{x+iz}{x-iz}\right] - x_\mu x_\nu\left[\frac{z(5x^2+3z^2)}{4x^4(x^2+z^2)^2} - \frac{3i}{8x^5}\log\frac{x+iz}{x-iz}\right],$$

while in AdS_5 we have

TABLE I. I_2 in various dimensions.

d	I_2
3	$-\frac{2\pi iz}{x}\log\frac{x+iz}{x-iz}$
4	$\frac{2\pi^2 z^3}{x^2(x^2+z^2)}$
5	$-\frac{i\pi^2 z}{2x^3}\log\frac{x+iz}{x-iz} - \frac{\pi^2 z^2(x^2-z^2)}{x^2(x^2+z^2)^2}$
6	$\frac{\pi^3 z^5(z^2+3x^2)}{3x^4(x^2+z^2)^3}$

$$\langle A_\mu(t, \mathbf{x}, z) j_\nu(0)\rangle = \eta_{\mu\nu}\frac{z^2(6x^4 + 3x^2z^2 + z^4)}{6x^4(x^2+z^2)^3} - x_\mu x_\nu \frac{2z^2(3x^4 + 3x^2z^2 + z^4)}{3x^6(x^2+z^2)^3}.$$

Explicit expressions in higher dimensions become rather unwieldy. In general, the A-j correlators inherit the singularity structure of I_2: they are singular on the boundary lightcone $x^2 = 0$, as well as on the bulk lightcone $x^2 + z^2 = 0$. Correlators involving field strengths are both simpler and better behaved. In any dimension we find

$$\langle F_{\lambda\mu}(t, \mathbf{x}, z) j_\nu(0)\rangle = -\frac{2(d-2)z^{d-2}}{(x^2+z^2)^d}(x_\lambda \eta_{\mu\nu} - x_\mu \eta_{\lambda\nu}),$$
$$\langle F_{z\mu}(t, \mathbf{x}, z) j_\nu(0)\rangle = \frac{(d-2)z^{d-3}}{(x^2+z^2)^d}(\eta_{\mu\nu}(x^2 - z^2) - 2x_\mu x_\nu). \tag{31}$$

Note that F-j correlators are only singular on the bulk lightcone.

Finally we can use these results to discuss bulk locality and causality. The expectation value of a commutator $\langle [A_\mu(t, \mathbf{x}, z), j_\nu(0)]\rangle$ is given by the difference in the prescriptions $t \to t - i\epsilon$ and $t \to t + i\epsilon$. It follows that the commutator of a bulk gauge field with a boundary current is nonzero at lightlike separation on the boundary. Lightlike separation on the boundary implies spacelike separation in the bulk, so we appear to have nonlocal or acausal correlators. Of course there is no real violation of causality here, since A-j correlators are gauge-dependent. For Maxwell fields we can test causality by looking at gauge-invariant quantities, and indeed field strengths have causal correlators: they commute with the boundary currents at bulk spacelike separation.

III. GRAVITON SMEARING FUNCTIONS

We now turn our attention to constructing a smearing function that describes a fluctuation of the bulk metric. To this end we consider a linearized perturbation of the AdS metric,

$$ds^2 = \frac{R^2}{z^2}(dz^2 + g_{\mu\nu}dx^\mu dx^\nu), \qquad g_{\mu\nu} = \eta_{\mu\nu} + \frac{z^2}{R^2}h_{\mu\nu}. \tag{32}$$

Here we are working in "holographic gauge" (or Fefferman-Graham coordinates [16]), in which

$$g_{zz} = g_{z\mu} = 0.$$

The source-free Einstein equations in this coordinate system can be found in [17].[7] Working to linear order in $h_{\mu\nu}$

[7]Ref. [17] uses $\rho = z^2/R^2$ as a radial coordinate.

the zz, the $z\nu$, and the trace of the $\mu\nu$ components of the Einstein equations read

$$zz\colon \left(\partial_z^2 + \frac{3}{z}\partial_z\right)h = 0, \tag{33}$$

$$z\nu\colon \left(\partial_z + \frac{2}{z}\right)(\partial_\mu h^{\mu\nu} - \partial^\nu h) = 0, \tag{34}$$

$$\text{trace:} \left(\partial_z^2 - \frac{2d-5}{z}\partial_z - \frac{4(d-1)}{z^2}\right)h + 2(\partial_\mu\partial^\mu h - \partial_\mu\partial_\nu h^{\mu\nu}) = 0. \tag{35}$$

Here $h \equiv h^\mu{}_\mu$. The only solution to this system of equations compatible with normalizable behavior as $z \to 0$ is to set[8]

$$h = 0, \qquad \partial_\mu h^{\mu\nu} = 0. \tag{36}$$

Thus $h_{\mu\nu}$ is traceless and conserved, which enables us to consistently identify its boundary behavior with the stress tensor of the CFT.

It only remains to solve the $\mu\nu$ components of the Einstein equations, which given Eq. (36) can be simplified to

$$\left(\partial_\alpha\partial^\alpha + \partial_z^2 + \frac{5-d}{z}\partial_z - \frac{2(d-2)}{z^2}\right)h_{\mu\nu} = 0.$$

Following the procedure that worked for Maxwell fields, we define $\phi_{\mu\nu} = z^2 h_{\mu\nu}$ and find that[9]

$$\left(\partial_\alpha\partial^\alpha + z^{d-1}\partial_z\frac{1}{z^{d-1}}\partial_z\right)\phi_{\mu\nu} = 0.$$

That is, each component of $\phi_{\mu\nu}$ obeys the massless scalar wave equation. A massless scalar is dual to an operator of dimension $\Delta = d$ in the CFT, and has the asymptotic falloff

$$\phi_{\mu\nu}(x, z) \sim z^d T_{\mu\nu}(x) \quad \text{as } z \to 0.$$

We identify $T_{\mu\nu}$ with the stress tensor of the CFT. To reconstruct the bulk metric perturbation from the stress tensor we use the scalar smearing function Eq. (A2) given in Appendix A. Setting $\Delta = d$, this gives

$$z^2 h_{\mu\nu}(t, \mathbf{x}, z) = \frac{1}{\mathrm{vol}(B^d)}\int_{t'^2+|\mathbf{y}'|^2<z^2} dt' d^{d-1}y' T_{\mu\nu}(t+t', \mathbf{x}+i\mathbf{y}'),$$
$$\text{volume of a unit } d\text{-ball} = \mathrm{vol}(B^d) = \frac{2\pi^{d/2}}{d\Gamma(d/2)}. \tag{37}$$

Thus the bulk metric perturbation is obtained by smearing the stress tensor over a ball of radius z on the complexified boundary.

A. AdS covariance

It is instructive to check that the smearing function Eq. (37) respects AdS covariance. We will be somewhat brief, since the steps are very similar to those in Sec. II A. Covariance under Poincaré transformations of x^μ is manifest. A dilation corresponds to the bulk isometry

$$x^\mu \to x'^\mu = \lambda x^\mu, \qquad z \to z' = \lambda z.$$

Holographic gauge is preserved since $h'_{zz} = h'_{z\mu} = 0$, while the combination $z^2 h_{\mu\nu}$ which appears on the left-hand side of Eq. (37) transforms like a scalar. This matches the behavior of the right-hand side: the stress tensor has dimension d, while the measure $d^d x'$ has dimension $-d$.

Special conformal transformations are a little more involved. A special conformal transformation corresponds to an infinitesimal bulk isometry

$$x^\mu \to x'^\mu = x^\mu + 2b\cdot x x^\mu - b^\mu(x^2+z^2),$$
$$z \to z' = z + 2b\cdot x z.$$

Under this isometry,

$$h'_{zz} = 0, \qquad h'_{z\mu} = 2zb^\alpha h_{\alpha\mu},$$
$$h'_{\mu\nu} = h_{\mu\nu} + 2b^\alpha(x_\mu h_{\alpha\nu} + x_\nu h_{\alpha\mu}) - 2x^\alpha(b_\mu h_{\alpha\nu} + b_\nu h_{\alpha\mu}) - 4b\cdot x h_{\mu\nu}. \tag{38}$$

Holographic gauge is not preserved, so to restore it we make a compensating diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu(x, z)$, under which

$$\delta h_{\mu\nu} = \frac{R^2}{z^2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu), \qquad \delta h_{z\mu} = \frac{R^2}{z^2}\partial_z\epsilon_\mu,$$
$$\delta h_{zz} = 0.$$

The appropriate diffeomorphism is

$$\epsilon^\mu = -\frac{1}{R^2\mathrm{vol}(B^d)}\int d^d x'\theta(\sigma z')\sigma z z' 2b_\alpha T^{\alpha\mu}, \tag{39}$$

for which

[8] To see this, note that (35) implies $\partial_\mu h^{\mu\nu} - \partial^\nu h \sim 1/z^2$. To avoid this non-normalizable behavior we must set $\partial_\mu h^{\mu\nu} - \partial^\nu h = 0$. The divergence of this equation means the last term in Eq. (35) drops out. Then the difference of Eqs. (33) and (35) gives $(\partial_z + \frac{2}{z})h = 0$, which requires that we set $h = 0$.

[9] This amounts to working in a vielbein basis, $h_{ab} = e_a{}^\mu e_b{}^\nu h_{\mu\nu}$, where $e_a{}^\mu = \frac{z}{R}\delta_a{}^\mu$.

$$\delta h_{z\mu} = -2zb^\alpha h_{\alpha\mu},$$
$$\delta h_{\mu\nu} = -\frac{1}{z^2 \mathrm{vol}(B^d)} \int d^d x' \theta(\sigma z') 2b^\alpha (x-x')_\mu T_{\alpha\nu} + (\mu \leftrightarrow \nu). \quad (40)$$

This restores holographic gauge. Combining Eqs. (38) and (40) we find

$$(z^2 h_{\mu\nu})' = z^2 h_{\mu\nu} + \frac{1}{\mathrm{vol}(B^d)} \int d^d x' \theta(\sigma z') \times [2b^\alpha x'_\mu T_{\alpha\nu} - 2x^\alpha b_\mu T_{\alpha\nu} + (\mu \leftrightarrow \nu)]. \quad (41)$$

Current conservation in the form $\int d^d x' \theta(\sigma z') \times \sigma z z' \partial_\mu T^{\mu\nu} = 0$ implies

$$\int d^d x' \theta(\sigma z')(x-x')^\mu T_{\mu\nu} = 0.$$

This means we can replace x^α with x'^α in Eq. (41), to obtain the transformation of the left-hand side of Eq. (37). The result exactly matches the transformation of the right-hand side, since under a special conformal transformation

$$T_{\mu\nu} \to T'_{\mu\nu} = T_{\mu\nu} + 2b^\alpha (x_\mu T_{\alpha\nu} + x_\nu T_{\alpha\mu}) - 2x^\alpha (b_\mu T_{\alpha\nu} + b_\nu T_{\alpha\mu}) - 2db \cdot x T_{\mu\nu}.$$

The last term cancels the transformation of the measure $d^d x' \theta(\sigma z')$.

B. Two-point functions and bulk causality for gravity

We now use the smearing functions we have constructed to compute 2-point functions for the graviton. We consider gravity in AdS_3 in Sec. III B 1, and gravity in AdS_4 and higher in Sec. III B 2.

1. Gravity in AdS_3

AdS_3 is special because there is no propagating graviton [18]. Rather, the bulk curvature is completely determined by the vacuum Einstein equations

$$R_{MN} = \frac{\Lambda}{d-1} G_{MN}, \quad (42)$$

where the cosmological constant $\Lambda = -d(d-1)/R^2$. This uniquely fixes the geometry. So in AdS_3 we expect the smearing function to generate a metric perturbation which corresponds to an infinitesimal (but non-normalizable) diffeomorphism of the background AdS metric.

We work in light-front coordinates $x^\pm = t \pm x$ and write the perturbed AdS metric as

$$ds^2 = \frac{R^2}{z^2}(dz^2 - dx^+ dx^-) + h_{\mu\nu} dx^\mu dx^\nu. \quad (43)$$

From the smearing function Eq. (37) we have, for instance,

$$z^2 h_{--} = \frac{1}{\pi} \int_{t'^2 + y'^2 < z^2} dt' dy' T_{--}(t+t', x+iy'). \quad (44)$$

Since T_{--} only depends on x^- this becomes ($t' = r\sin\theta$, $y' = r\cos\theta$)

$$z^2 h_{--} = \frac{1}{\pi} \int_0^z r dr \int_0^{2\pi} d\theta T_{--}(x^- - ire^{i\theta}). \quad (45)$$

Defining $\xi = e^{i\theta}$ the contour integral picks up the pole at $\xi = 0$ and ends up giving $h_{--} = T_{--}$. So at the linearized level a stress tensor in the CFT corresponds to a bulk metric perturbation

$$h_{--} = T_{--}(x^-), \qquad h_{++} = T_{++}(x^+), \qquad h_{+-} = 0. \quad (46)$$

This provides a remarkably simple example of holography: the boundary stress tensor is lifted to be z-independent and reinterpreted as a metric perturbation in the bulk. Not surprisingly, this is very reminiscent of the Chern-Simons correspondence Eq. (23).

We can use this to compute the bulk 2-point function for the graviton. For instance, the CFT 2-point function

$$\langle T_{--}(x^-) T_{--}(x'^-) \rangle = \frac{c}{8\pi^2} \frac{1}{(x^- - x'^- - i\epsilon)^4} \quad (47)$$

lifts to a bulk correlator

$$\langle h_{--}(x^+, x^-, z) h_{--}(x'^+, x'^-, z') \rangle = \frac{c}{8\pi^2} \frac{1}{(x^- - x'^- - i\epsilon)^4}.$$

Here we have used a Wightman $i\epsilon$ prescription and c is the central charge of the CFT.

To study bulk locality and causality in this framework, note that the CFT correlator Eq. (47) corresponds to a Virasoro algebra

$$i[T_{--}(x^-), T_{--}(x'^-)] = \frac{c}{24\pi} \delta'''(x^- - x'^-).$$

This lifts to the bulk commutator

$$i[h_{--}(x^+, x^-, z), h_{--}(x'^+, x'^-, z')] = \frac{c}{24\pi} \delta'''(x^- - x'^-).$$

Metric perturbations in the bulk have nonlocal commutators; this behavior is acceptable since metric perturbations are coordinate-dependent. One might ask if there is a quantity—analogous to the field strength for a gauge field—which obeys causal commutation relations. In the next section we will claim that, for gravity, such a quantity is provided by the Weyl tensor. This claim becomes vacuous in three dimensions since the Weyl tensor vanishes identically.

We began this section by recalling that the source-free Einstein equations fix the bulk geometry to be pure AdS. So, to complete the story, one might ask for a coordinate transformation which brings the perturbed metric Eq. (43) and (46) back to the canonical form $ds^2 = \frac{R^2}{z^2}(dz^2 - dx^+ dx^-)$. The required transformation is

TABLE II. J_1 and J_2 in low dimensions.

d	J_1	J_2
3	$-\frac{3i}{4x}\log\frac{x+iz}{x-iz}-\frac{3z}{2(x^2+z^2)}$	$-\frac{3i(x^2+z^2)}{4x}\log\frac{x+iz}{x-iz}-\frac{3z}{2}$
4	$\frac{z^4}{x^2(x^2+z^2)^2}$	$-2\log\frac{x^2+z^2}{x^2}+\frac{2z^2}{x^2}$
5	$-\frac{5i}{32x^3}\log\frac{x+iz}{x-iz}-\frac{5z(3x^4+8x^2z^2-3z^4)}{48x^2(x^2+z^2)^3}$	$\frac{15i}{32x^3}(3x^2-z^2)\log\frac{x+iz}{x-iz}+\frac{15z(3x^2+z^2)}{16x^2(x^2+z^2)}$
6	$\frac{z^6(4x^2+z^2)}{4x^4(x^2+z^2)^4}$	$\frac{z^6}{x^4(x^2+z^2)^2}$

$$\delta x^+ = -\frac{2}{R^2}\frac{1}{\partial_+^3}T_{++} - \frac{z^2}{R^2}\frac{1}{\partial_-}T_{--},$$
$$\delta x^- = -\frac{2}{R^2}\frac{1}{\partial_-^3}T_{--} - \frac{z^2}{R^2}\frac{1}{\partial_+}T_{++}, \qquad (48)$$
$$\delta z = -\frac{z}{R^2}\left(\frac{1}{\partial_+^2}T_{++} + \frac{1}{\partial_-^2}T_{--}\right).$$

Note that the transformation does not vanish at the boundary, so it does not correspond to a (normalizable) gauge symmetry of the bulk theory.

2. *Gravity in* AdS_4 *and higher*

Our starting point for gravity in AdS_4 and higher is the 2-point function of the stress tensor in a general CFT. Up to an overall coefficient proportional to the central charge, this has the form[10]

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle = X_{\mu\nu\alpha\beta}\frac{1}{(x^2)^d} + Y_{\mu\nu\alpha\beta}\frac{1}{(x^2)^{d-1}} + Z_{\mu\nu\alpha\beta}\frac{1}{(x^2)^{d-2}}, \qquad (49)$$

where we've introduced

$$X_{\mu\nu\alpha\beta} = -2d\eta_{\mu\nu}\eta_{\alpha\beta} + d(d-1)(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}),$$
$$Y_{\mu\nu\alpha\beta} = \frac{1}{d-1}(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu) - \frac{1}{2}(\eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\mu\beta}\partial_\nu\partial_\alpha + \eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha), \qquad (50)$$
$$Z_{\mu\nu\alpha\beta} = \frac{1}{2(d-1)(d-2)}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta.$$

Up to an overall normalization this correlator is uniquely determined by requiring that the stress tensor be traceless and conserved with the correct scaling dimension. Applying the smearing function Eq. (37) gives the bulk-boundary correlator

$$z^2\langle h_{\mu\nu}(t,\mathbf{x},z)T_{\alpha\beta}(0)\rangle = X_{\mu\nu\alpha\beta}J_0 + Y_{\mu\nu\alpha\beta}J_1 + Z_{\mu\nu\alpha\beta}J_2, \qquad (51)$$

where

$$J_n = \frac{1}{\mathrm{vol}(B^d)}\int_{t'^2+|\mathbf{y}'|^2<z^2} dt'd^{d-1}y' \times \frac{1}{(-(t+t')^2+|\mathbf{x}+i\mathbf{y}'|^2)^{d-n}}. \qquad (52)$$

Note that J_n is related to the integral Eq. (28) we encountered for gauge fields,

$$\frac{d}{dz}J_n = \frac{1}{\mathrm{vol}(B^d)}I_n.$$

Integrating Eq. (29) gives

$$J_n = \frac{z^d}{(x^2)^{d-n}}F\left(d-n, \frac{d}{2}-n+1, \frac{d}{2}+1, -\frac{z^2}{x^2}\right). \qquad (53)$$

In general, J_n has singularities on both the boundary lightcone (where $x^2=0$) and the bulk lightcone (where $x^2+z^2=0$). The case $n=0$ is an exception to this general rule, since

$$J_0 = \frac{z^d}{(x^2+z^2)^d}.$$

J_0 is only singular on the bulk lightcone, and in fact has an AdS-covariant form $J_0 \sim 1/(\sigma z')^d$. This was to be expected since, up to an overall normalization, J_0 is the bulk-boundary correlator for a massless scalar field. Some other cases of interest can be found in Table II.

At this stage we have an expression for the h-T correlator in terms of differential operators acting on J_n's. We will stop here, since explicitly evaluating the derivatives in Eq. (51) leads to lengthy expressions. But one important observation we can make is that the h-T correlator inherits the singularity structure of J_1 and J_2: it has singularities on both the bulk and boundary lightcones. This means the commutator $[h_{\mu\nu}(t,\mathbf{x},z), T_{\alpha\beta}(0)]$ will be nonzero at light-like separation on the boundary (where $x^2=0$), even

[10] See for example (2.37) and (A5) in Ref. [19].

though this corresponds to spacelike separation in the bulk (since $x^2 + z^2 > 0$). This shows that in holographic gauge metric perturbations have acausal commutators. This is acceptable because the commutator is gauge-dependent.

This raises an interesting question; whether there is a quantity one can define in linearized gravity which obeys causal commutation relations. That is, whether there is something analogous to the Maxwell field strength $F_{\mu\nu}$, which as we saw in Eq. (31) has correlators that are only singular on the bulk lightcone. At first one might think the gravitational analog is provided by the Riemann tensor. However, this can't be right: perturbing the source-free Einstein equations Eq. (42) shows that $\delta R_{\mu\nu} = -\frac{d}{R^2} h_{\mu\nu}$. Since we've already shown that the metric perturbation has acausal commutators, the same must be true for the Ricci tensor.

This suggests that we split off the Ricci part of the curvature and work with the Weyl tensor. In fact the Weyl tensor commutes with the boundary stress tensor at bulk spacelike separation. We will show this in two ways: first by an intuitive argument, then by an explicit calculation in holographic gauge.

The intuitive argument runs as follows. Imagine quantizing the bulk theory perturbatively using a covariant gauge condition. Then locality would be manifest, and all fields (including the metric perturbation) would obey canonical local commutation relations. It follows that in covariant gauge the Weyl tensor commutes with the boundary stress tensor at spacelike separation. But since the Weyl tensor transforms homogeneously under changes of coordinates, if the commutator vanishes in covariant gauge it should also vanish in holographic gauge.[11]

The explicit calculation proceeds as follows. Linearizing around an AdS background, the nontrivial components of the Weyl tensor are

$$
\begin{aligned}
z^2 C_{\alpha\beta\gamma\delta} &= \frac{1}{2}(\partial_\alpha\partial_\gamma\phi_{\beta\delta} - \partial_\alpha\partial_\delta\phi_{\beta\gamma} - \partial_\beta\partial_\gamma\phi_{\alpha\delta} + \partial_\beta\partial_\delta\phi_{\alpha\gamma}) - \frac{1}{2z}\partial_z(\eta_{\alpha\gamma}\phi_{\beta\delta} - \eta_{\alpha\delta}\phi_{\beta\gamma} - \eta_{\beta\gamma}\phi_{\alpha\delta} + \eta_{\beta\delta}\phi_{\alpha\gamma}),\\
z^2 C_{z\beta\gamma\delta} &= \frac{1}{2}\partial_z(\partial_\gamma\phi_{\beta\delta} - \partial_\delta\phi_{\beta\gamma}).
\end{aligned} \tag{54}
$$

Here $\phi_{\alpha\beta} = z^2 h_{\alpha\beta}$, and we have used the fact that $\phi_{\alpha\beta}$ obeys the massless scalar wave equation $(\partial_\alpha\partial^\alpha + \partial_z^2)\phi_{\mu\nu} = \frac{d-1}{z}\partial_z\phi_{\mu\nu}$. The remaining components of the Weyl tensor $C_{z\beta z\delta}$ are not independent by the trace-free condition.

In principle it is straightforward to compute C-T correlators. Consider for example $z^2\langle C_{z\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle$. Using the ϕ-T correlator Eq. (51) and the operators Eq. (50), one obtains a rather long expression. However many terms drop out when antisymmetrized on γ and δ. What survives has the form ("stuff" meaning metrics and derivatives tangent to the boundary)

$$
\begin{aligned}
z^2\langle C_{z\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle &= \partial_z \int_{x'^2<z^2} d^d x' \left\{(\text{stuff})\cdot\frac{1}{(x^2)^d} + (\text{stuff})\cdot\frac{1}{(x^2)^{d-1}}\right\}\\
&= \int_{x'^2=z^2} d^d x' \left\{(\text{stuff})\cdot\frac{1}{(x^2)^d} + (\text{stuff})\cdot\frac{1}{(x^2)^{d-1}}\right\}\\
&= (\text{stuff})\cdot I_0 + (\text{stuff})\cdot I_1.
\end{aligned} \tag{55}
$$

As we saw in Eq. (30) I_1 is analytic on the boundary lightcone. It turns out that I_0 is also analytic at $x^2 = 0$:

$$
I_0 = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{z^{d-1}(x^2 - z^2)}{(x^2 + z^2)^{d+1}}. \tag{56}
$$

So the correlator Eq. (55) is analytic at $x^2 = 0$, and $C_{z\beta\gamma\delta}$ obeys causal commutation relations with the boundary stress tensor.

Now consider $z^2\langle C_{\alpha\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle$. Again one obtains a rather long expression. However, many terms drop out when antisymmetrized on α and β, or on γ and δ. Also, many terms involve either J_0, I_0 or I_1, which we know are analytic at $x^2 = 0$. Dropping all such contributions up to an overall coefficient, we find that only two terms survive:

$$
z^2\langle C_{\alpha\beta\gamma\delta}(x)T_{\rho\sigma}(0)\rangle \sim \partial_{[\alpha}\eta_{\beta][\gamma}\partial_{\delta]}\partial_\rho\partial_\sigma\left(\int_{x'^2<z^2} d^d x' \frac{1}{(x^2)^{d-1}} - \frac{1}{2(d-2)z}\int_{x'^2=z^2} d^d x' \frac{1}{(x^2)^{d-2}}\right). \tag{57}
$$

With the help of one of Gauss' recursion relations for hypergeometric functions one can show that the quantity in parenthesis is

[11]This argument breaks down for the Riemann tensor. In an AdS background the Riemann tensor acquires a vev, and a perturbation $\delta R_{\alpha\beta\gamma\delta}$ transforms inhomogeneously under changes of coordinates. By contrast, the Weyl tensor has a vanishing vev and transforms homogeneously. It follows that, at the linearized level, the Weyl tensor is gauge-invariant around an AdS background.

$$\mathrm{vol}(B^d)J_1 - \frac{1}{2(d-2)z}I_2 = -\frac{\pi^{d/2}}{(d-2)\Gamma(d/2)}\frac{z^{d-2}}{(x^2+z^2)^{d-2}}.$$

This is analytic on the boundary lightcone, so $C_{\alpha\beta\gamma\delta}$ obeys causal commutation relations with the boundary stress tensor.

IV. MASSIVE VECTOR FIELDS

In this section we derive the smearing function for a massive vector. Our starting point is the Lagrangian for a massive vector field in Lorentzian AdS_{d+1}:

$$S = \int dz d^d x\sqrt{-G}\Big(-\frac{1}{4}F^{MN}F_{MN} - \frac{1}{2}m^2 A_M A^M\Big). \tag{58}$$

The equations of motion $\nabla_M F^{MN} - m^2 A^N = 0$ imply

$$\nabla_M A^M = 0. \tag{59}$$

Decomposing $A_M = (A_z, A_\mu)$, the equations of motion for A_z are

$$\Big(\partial_z^2 + \partial_\mu\partial^\mu - \frac{1}{z}(d-1)\partial_z - \frac{m^2-d+1}{z^2}\Big)A_z = 0. \tag{60}$$

This is identical to the equation of motion for a scalar field with $(\mathrm{mass})^2 = m^2 - d + 1$. For the other components one has (defining $\phi_\mu = zA_\mu$)

$$\Big(\partial_z^2 + \partial_\nu\partial^\nu - \frac{1}{z}(d-1)\partial_z - \frac{m^2-d+1}{z^2}\Big)\phi_\mu = 2\partial_\mu A_z. \tag{61}$$

Let

$$\Delta = \frac{d}{2} + \sqrt{\frac{(d-2)^2}{4} + m^2}, \tag{62}$$

and define the boundary value of A_z by

$$A_z \sim z^\Delta A_z^0 \quad \text{as } z \to 0.$$

The equation of motion for A_z can be solved in the same way as for a scalar field (see Appendix A):

$$A_z(t, \mathbf{x}, z) = \int_{t'^2+\mathbf{y}'^2<z^2} dt' d\mathbf{y}'\Big(\frac{z^2 - t'^2 - \mathbf{y}'^2}{z}\Big)^{\Delta-d} \times A_z^0(t+t', \mathbf{x}+i\mathbf{y}'). \tag{63}$$

What is the boundary value of A_z^0 in terms of CFT data? Since $\phi_\mu(z\to 0) \sim z^\Delta$ then $A_\mu \sim z^{\Delta-1}j_\mu$, and inserting this in Eq. (59) gives

$$A_z^0 = \frac{1}{d-\Delta-1}\partial_\mu j^\mu. \tag{64}$$

So A_z^0 is sourced by the divergence of the boundary current.

Now let us solve Eq. (61). First note that a solution to the homogeneous Eq. (60) can be expanded in modes as

$$A_z = \int_{|\omega|>|k|} d\omega d^{d-1}k a^z_{\omega k} e^{-i\omega t + i\mathbf{k}\mathbf{x}} z^{d/2} J_\nu(z\sqrt{\omega^2 - \mathbf{k}^2}), \tag{65}$$

where $\nu = \Delta - d/2$ and $J_\nu(y)$ is a Bessel function. A similar solution would hold for Eq. (61) if the right-hand side was zero. The complete solution to Eq. (61) can then be written in the form [20]

$$\phi_\mu(t, \mathbf{x}, z) = \int_{|\omega|>|k|} d\omega d^{d-1}k z^{d/2} e^{-i\omega t + i\mathbf{k}\mathbf{x}}\Big(a^\mu_{\omega k} J_\nu(z\sqrt{\omega^2-\mathbf{k}^2}) + a^z_{\omega k}\frac{izk_\mu}{\sqrt{\omega^2-\mathbf{k}^2}} J_{\nu+1}(z\sqrt{\omega^2-\mathbf{k}^2})\Big). \tag{66}$$

Now from the boundary behavior of A_z one has

$$a^z_{\omega k} = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-\mathbf{k}^2)^{\nu/2}}\int dt' d^{d-1}x' e^{i\omega t' - i\mathbf{k}\mathbf{x}'} A_z^0(t', \mathbf{x}'), \tag{67}$$

and since the term proportional to $a^z_{\omega k}$ in Eq. (66) is subleading as $z \to 0$, one also has

$$a^\mu_{\omega k} = \frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-\mathbf{k}^2)^{\nu/2}}\int dt' d^{d-1}x' e^{i\omega t' - i\mathbf{k}\mathbf{x}'} z j_\mu(t', \mathbf{x}'). \tag{68}$$

By inserting the expressions for $a^\mu_{\omega k}$ and $a^z_{\omega k}$ into Eq. (66) one gets an expression for the bulk field in terms of boundary data. The first term looks just like the smearing function for a scalar field of dimension Δ, while the second term (aside from a factor $\frac{izk_\mu}{2(\nu+1)}$) is just the smearing function for a scalar field of dimension $\Delta + 1$ [7]. As a result, we get the following expression:

$$\phi_\mu(t, \mathbf{x}, z) = \int K_\Delta(x, x') j_\mu(x') + \frac{z}{2(\nu+1)}\int K_{\Delta+1}(x, x')\partial_\mu A_z^0(x'). \tag{69}$$

More explicitly,

$$zA_\mu(t,\mathbf{x},z) = \frac{\Gamma(\Delta - d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)}\int_{t'^2+\mathbf{y}'^2<z^2} dt' d^{d-1}y' \left(\frac{z^2-t'^2-\mathbf{y}'^2}{z}\right)^{\Delta-d} A^0_\mu(t+t', x+i\mathbf{y}') + \frac{z\Gamma(\Delta-d/2+1)}{2\pi^{d/2}\Gamma(\Delta-d+2)}\int_{t'^2+\mathbf{y}'^2<z^2} dt' d^{d-1}y' \left(\frac{z^2-t'^2-\mathbf{y}'^2}{z}\right)^{\Delta-d+1}\partial_\mu A^0_z(t+t', x+i\mathbf{y}').$$

A. Two-point functions and bulk causality

In this section we compute the two-point function of a massive vector. The CFT two-point function for a spin-1 field is

$$\langle j_\mu(x) j_\nu(0)\rangle = \left(\eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right)\frac{1}{(x^2)^\Delta}. \tag{70}$$

It can also be written in the form

$$\langle j_\mu(x) j_\nu(0)\rangle = \frac{\Delta-1}{\Delta}\eta_{\mu\nu}\frac{1}{(x^2)^\Delta} - \frac{1}{2\Delta(\Delta-1)}\partial_\mu\partial_\nu\frac{1}{(x^2)^{\Delta-1}}. \tag{71}$$

Since our expression for the bulk operator involves the divergence of the current, we will also need

$$\langle \partial_\mu j^\mu(x) j_\nu(0)\rangle = \frac{d-\Delta-1}{\Delta}\partial_\nu\frac{1}{(x^2)^\Delta}. \tag{72}$$

The correlator of a bulk field A_z with a boundary current j_ν is easy to read off from the smearing function for A_z, which as we showed is just the smearing function of a scalar field of dimension Δ. Since $A_z(x) = \frac{1}{d-\Delta-1}\partial_\mu j^\mu(x)$ we have

$$\langle A_z(z,x) j_\nu(0)\rangle = \frac{1}{\Delta}\partial_\nu\left(\frac{z}{x^2+z^2}\right)^\Delta. \tag{73}$$

This two-point function respects bulk causality.

For the other components of the bulk field we have

$$\langle zA_\mu(t,\mathbf{x},z) j_\nu(0)\rangle = \int_{t'^2+|\mathbf{y}'|^2<z^2} dt' d^{d-1}y' \left[\frac{\Gamma(\Delta-d/2+1)}{\pi^{d/2}\Gamma(\Delta-d+1)}\left(\frac{z^2-t'^2-|\mathbf{y}'|^2}{z}\right)^{\Delta-d}\langle j_\mu(t+t',\mathbf{x}+i\mathbf{y}') j_\nu(0)\rangle + \frac{z\Gamma(\Delta-d/2+1)}{2\pi^{d/2}\Gamma(\Delta-d+2)}\left(\frac{z^2-t'^2-|\mathbf{y}'|^2}{z}\right)^{\Delta-d+1}\partial_\mu\langle A_z(t+t',\mathbf{x}+i\mathbf{y}') j_\nu(0)\rangle\right]. \tag{74}$$

Using Eqs. (71) and (73) we write this as

$$\langle zA_\mu(x,z) j_\nu(0)\rangle = \frac{\Delta-1}{\Delta}\eta_{\mu\nu}\left(\frac{z}{z^2+x^2}\right)^\Delta - \frac{\Gamma(\Delta-d/2+1)}{2\Delta\pi^{d/2}\Gamma(\Delta-d+1)}\partial_\mu\partial_\nu\left(\frac{1}{\Delta-1}f_\Delta(z,x) - \frac{z}{\Delta-d+1}f_{\Delta+1}(z,x)\right),$$

where

$$f_\Delta(z,x) = \int_{t'^2+|y'|^2<z^2} dt' d^{d-1}y' \left(\frac{z^2-t'^2-|y'|^2}{z}\right)^{\Delta-d} \times \left(\frac{1}{-(t+t')^2+(x_1+iy_1)^2+\cdots+(x_{d-1}+iy_{d-1})^2}\right)^{\Delta-1}.$$

We set $t=0$, $x_1 = x$, $x_2 = \cdots = x_{d-1} = 0$. We will compute f_Δ for this case, then restore the dependence on the other coordinates using Lorentz invariance. Switching from (t', y') to spherical coordinates we get

$$f_\Delta = \mathrm{vol}(S^{d-2})\int_0^z dr r^{d-1}\left(\frac{z^2-r^2}{z}\right)^{\Delta-d}\int_0^\pi \frac{\sin^{d-2}\theta}{(x^2+2ixr\cos\theta - r^2)^{\Delta-1}}. \tag{75}$$

We use the integrals

$$\int_0^\pi \frac{\sin^{2\mu-1}\theta}{(1+2a\cos\theta+a^2)^\nu} = \frac{\Gamma(\mu)\Gamma(\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}F\left(\nu,\nu-\mu+\frac{1}{2},\mu+\frac{1}{2},a^2\right),$$
$$\int_0^1 (1-x)^{\mu-1}x^{\gamma-1}F(\alpha,\beta,\gamma,ax) = \frac{\Gamma(\mu)\Gamma(\gamma)}{\Gamma(\mu+\gamma)}F(\alpha,\beta,\gamma+\mu,a), \tag{76}$$

to find

$$f_\Delta = \frac{\pi^{d/2}\Gamma(\Delta-d+1)}{\Gamma(\Delta-\frac{d}{2}+1)}\frac{z^\Delta}{x^{2\Delta-2}}F\left(\Delta-1,\Delta-\frac{d}{2},\Delta-\frac{d}{2}+1,-\frac{z^2}{x^2}\right). \tag{77}$$

Then we use the identity

$$\gamma F(\alpha,\beta,\gamma,x)-\gamma F(\alpha,\beta+1,\gamma,x)+x\alpha F(\alpha+1,\beta+1,\gamma+1,x)=0, \tag{78}$$

and restore Lorentz invariance to find

$$\langle zA_\mu(x,z)j_\nu(0)\rangle=\frac{\Delta-1}{\Delta}\eta_{\mu\nu}\left(\frac{z}{x^2+z^2}\right)^{\Delta}-\frac{z^\Delta}{2\Delta(\Delta-1)}\partial_\mu\partial_\nu\left(\frac{1}{x^2+z^2}\right)^{\Delta-1}. \tag{79}$$

Note that the final answer is only nonanalytic on the bulk lightcone. This however was achieved by a cancellation of terms that are nonanalytic on the boundary lightcone between f_Δ and $f_{\Delta+1}$. So the locality of a massive vector field in the bulk is made possible by the fact that the dual boundary current is not conserved, which allowed us to cancel nonanalytic terms in the correlator. This mechanism is not available for a gauge field since it is dual to a conserved current.

V. INTERACTIONS

In this section we make some remarks on constructing bulk operators at higher orders in $1/N$. For scalar fields it was shown in [8] that one can construct interacting local bulk fields without any knowledge of the bulk Lagrangian. Rather, by adopting bulk microcausality as a guiding principle, one can construct the appropriate bulk operators just from knowing CFT correlators. Here we show that something similar can be done for a massive vector field in AdS_3: a local bulk operator can be constructed, even in the presence of interactions. However, for a gauge field in AdS_3 we show that the analogous procedure breaks down. In this section, to avoid notational complexity, we denote

$$w=x^+=t+x,\qquad \bar{w}=x^-=t-x.$$

Up to an overall coefficient, the three-point function of three primary operators in a two dimensional CFT is

$$\langle\mathcal{O}_{1,h_1,\bar{h}_1}(w_1,\bar{w}_1)\mathcal{O}_{2,h_2,\bar{h}_2}(w_2,\bar{w}_2)\mathcal{O}_{3,h_3,\bar{h}_3}(w_3,\bar{w}_3)\rangle=\frac{1}{w_{12}^{h_1+h_2-h_3}w_{23}^{h_2+h_3-h_1}w_{13}^{h_3+h_1-h_2}}\frac{1}{\bar{w}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{w}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\bar{w}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}. \tag{80}$$

Here $w_{ij}=w_i-w_j$. Let us for simplicity assume that $\mathcal{O}_2$ and $\mathcal{O}_3$ are scalar operators, so $h_2=\bar{h}_2$ and $h_3=\bar{h}_3$, but $\mathcal{O}_1$ has spin 1 with $h_1=\bar{h}_1+1$. To explore bulk locality we smear $\mathcal{O}_2$ into a bulk operator using the free field smearing function

$$\mathcal{O}_2(z,w_2,\bar{w}_2)=\int_0^z rdr\left(\frac{z^2-r^2}{z}\right)^{2h-2}\times\int_{|\alpha|=1}\frac{d\alpha}{i\alpha}\mathcal{O}(w_2+r\alpha,\bar{w}_2-r\alpha^{-1}). \tag{81}$$

We can get the CFT three-point function with $h_1\to h_1+1$ (as long as $h_1\neq 0$) by acting on a three-point correlator with the operator

$$\frac{1}{h_3-h_2-h_1}\frac{\partial}{\partial w_{12}}-\frac{1}{h_2-h_3-h_1}\frac{\partial}{\partial w_{13}}. \tag{82}$$

So the result for $h_1=\bar{h}_1+1$ can be gotten from the result for $h_1=\bar{h}_1$ by acting with the operator Eq. (82). The situation with $h_1=\bar{h}_1$ was analyzed in [8]. It was found that for scalar operators one can add a series of appropriately smeared higher dimension scalar operators that will cancel the causality-violating terms in the three-point function. Here we see that this is still true if one of the boundary operators has spin. Note however that for the special case of conserved current (meaning $h=0$, $\bar{h}=1$ or $h=1$, $\bar{h}=0$) this argument does not apply. This is not only because acting with the operator Eq. (82) is not possible, but also because if $\mathcal{O}_1$ is a conserved current then Ward identities restrict its three-point function. For instance, for a conserved current the three-point function will vanish unless the two-point function $\langle\mathcal{O}_2\mathcal{O}_3\rangle$ is nonzero. So for a conserved current, adding smeared higher dimension primaries is not in general possible.

We now consider the case where $\mathcal{O}_1$ is smeared into the bulk. We'll work in terms of the operator product expansion, similarly to what was done in [8]. For simplicity we denote $h_1=n$, $\bar{h}_1=n-1$ and assume that $h_2=\bar{h}_2=1$. We look at terms in the OPE proportional to the scalar operator

$$\begin{aligned} j^{n,n-1}(w,\bar{w})\mathcal{O}^{1,1}(0)&=\frac{\mathcal{O}^{1,1}(0)}{w^n\bar{w}^{n-1}}+\cdots,\\ j^{n-1,n}(w,\bar{w})\mathcal{O}^{1,1}(0)&=\frac{\mathcal{O}^{1,1}(0)}{w^{n-1}\bar{w}^{n}}+\cdots. \end{aligned} \tag{83}$$

When $n=1$ the smearing function Eq. (23) for a massless gauge field in AdS_3 gives

$$A^{1,0}(z,w,\bar{w})\mathcal{O}^{1,1}(0)=\frac{1}{w}\mathcal{O}^{1,1}(0)+\cdots. \tag{84}$$

On the other hand, for a massive vector the smearing function Eq. (69) leads to

$$A^{n,n-1}(z,w,\bar{w})\mathcal{O}^{1,1}(0)=\left(-\frac{2}{\pi}\frac{d}{dw}I_1^{(n-1)}+\frac{z}{\pi}\frac{d}{dw}I_2^{(n)}\right)\times\mathcal{O}^{1,1}(0)+\cdots, \tag{85}$$

where

$$I_1^{(n-1)} = \int_0^z rdr\left(\frac{z^2 - r^2}{z}\right)^{2n-3} \times \int_{|\alpha|=1} \frac{d\alpha}{\alpha(w + r\alpha)^{n-1}(\bar{w} - r/\alpha)^{n-1}},$$
$$I_2^{(n)} = \int_0^z rdr\left(\frac{z^2 - r^2}{z}\right)^{2n-2} \times \int_{|\alpha|=1} \frac{d\alpha}{\alpha(w + r\alpha)^{n}(\bar{w} - r/\alpha)^{n}}. \tag{86}$$

Using Eq. (76) one gets

$$I_1^{(n-1)} = \frac{\pi z^{2n-1}}{(2n-2)(w\bar{w})^{n-1}} F\left(n-1, n-1, 2n-1, -\frac{z^2}{w\bar{w}}\right),$$
$$I_1^{(n-1)} = \frac{\pi z^{2n}}{(2n-1)(w\bar{w})^{n}} F\left(n, n, 2n, -\frac{z^2}{w\bar{w}}\right), \tag{87}$$

and finally, using Eq. (78) one gets

$$A^{n,n-1}(z, w, \bar{w})\mathcal{O}^{1,1}(0) = -\mathcal{O}^{1,1}(0)\frac{d}{dw}\left(\frac{z^{2n-1}}{(n-1)(w\bar{w})^{n-1}} \times F(n-1, n, 2n-1, -\frac{z^2}{w\bar{w}})\right). \tag{88}$$

A similar result holds for $A^{n-1,n}$ by replacing $w \to \bar{w}$. The quantity in parentheses in Eq. (88) is nonanalytic due to terms of the form

$$\left(\frac{w\bar{w}}{z^2}\right)^m \ln\frac{z^2 + w\bar{w}}{w\bar{w}}, \tag{89}$$

with $n \geq m \geq 1$.

Suppose we have a massless gauge field in the bulk. The singular term in Eq. (84) leads to a nonvanishing commutator at bulk spacelike separation, and must be canceled if the gauge field is to commute at spacelike separation. But given the structure Eq. (89) there is no massive vector we can add to our definition of a bulk gauge field that will cancel the divergent term in Eq. (84). This means that it is not possible to promote a boundary conserved current to a local bulk field.[12]

On the other hand, starting from a nonconserved current in the CFT, there is no obstacle to restoring bulk locality. One can cancel nonanalytic terms of the form Eq. (89) by adding a tower of higher dimension spin-1 fields with appropriately chosen masses and coefficients to our definition of a bulk vector field. This will leave a nonanalytic term of the form

$$\left(\frac{w\bar{w}}{z^2}\right)^{n_{\max}} \ln(w\bar{w}), \tag{90}$$

where $n_{\max}$ is the largest n used in the sum over higher dimension primaries. So, just as in the scalar case [8], we can make a massive vector field in the bulk as local as we wish.

A. A comment on gauge fields

If there is a gauge symmetry in the bulk, i.e., a conserved current on the boundary, the issue of constructing bulk operators become a bit more involved. Of course one could start from the bulk equations of motion and solve them perturbatively, to express bulk fields in terms of boundary data. If one starts from a local bulk Lagrangian, this procedure is guaranteed to describe a local theory in the bulk (at least perturbatively). But if one wants to construct bulk operators purely in terms of the CFT, without making reference to bulk equations of motion, then having bulk gauge symmetries complicates matters. If there is a gauge symmetry in the bulk then the corresponding charge can be expressed as a surface term and identified with a conserved quantity in the CFT. The charge generates global gauge transformations, so as discussed in [9,10], charged fields in the bulk must have nonlocal commutators in order to properly implement the Gauss constraint. In the context of gravity this discussion applies to time evolution, since the CFT Hamilton should generate time translation everywhere in the bulk. While these nonlocal commutators do not actually violate causality, they do complicate the CFT construction, in the sense that the guiding principle of bulk causality must be stated more carefully. It is tempting to speculate that the good causal properties we found for the field strength and Weyl tensor at the linearized level can provide a basis for constructing the interacting theory, at least in perturbation theory.

VI. GENERAL BACKGROUNDS

In a given fixed background one can solve the bulk equations of motion perturbatively, to write an expression for the Heisenberg picture fields in the bulk in terms of the boundary values of those same fields, now interpreted as operators in the dual CFT. Correlation function of these CFT operators then reproduce bulk correlation functions. The computations are done from the bulk point of view in a particular gauge $G_{z\mu} = 0$, $G_{zz} = R^2/z^2$. With gauge fields one also sets $A_z = 0$. These conditions completely fix the gauge. The resulting computations are thus physical since all redundant degrees of freedom have been eliminated. In a fixed gauge one can reproduce bulk calculations using boundary data, and since the boundary data comes from a unitary field theory, this constitutes holography. From the CFT point of view, one corrects the naive smeared operator

[12]The lesson here is not that causality is violated. For example, in AdS_3 the field strength associated with Eq. (23) vanishes identically, and in this sense microcausality is trivially satisfied even in the presence of interactions. Rather, the lesson is that there is an obstacle to constructing bulk gauge fields which have local commutators. This is a feature, not a bug, since as we—discuss in Sec. VA—gauge fields are expected to have nonlocal commutators.

(constructed to represent a free field in the bulk) by adding higher dimension smeared operators to get a local bulk operator. However, these calculations as presented are done in a fixed background metric with a fixed causal structure. This causal structure cannot be circumvented or changed in perturbation theory since it is built in to the hardware of the approach. The approach based on microcausality and CFT correlators has the same difficulty. One must define a smearing function which is determined by the background metric, and this smearing function cannot be changed in perturbation theory, aside from corrections to incorporate anomalous dimensions.

Besides the question, How local can bulk operators be in this formalism?, one can ask how this formalism could work without an *a priori* notion of a background. Here we make a few comments on these issues.

In a fixed background the equations of motion for the bulk fields come from a radial Hamiltonian H_r. (By radial Hamiltonian we mean the operator which generates radial evolution of fluctuations about this particular background.) Schematically (ϕ stands for any perturbative field including gravitons on this background),

$$\frac{\partial \phi}{\partial z} = -[H_r, \phi]. \tag{91}$$

We also need to impose an initial condition, given by normalizable falloff as $z \to 0$ for each field. The radial Hamiltonian can be explicitly written down in the supergravity approximation. If we had a different background metric, then the radial Hamiltonian would be some different operator, but for each background we can think of the radial Hamiltonian as some operator in the CFT, generating the transformation from boundary operators to bulk operators via the map

$$\mathcal{O}(x, t) \to e^{-\int_0^z H_r} \mathcal{O}(x, t) e^{\int_0^z H_r}. \tag{92}$$

However the idea that we will just get a different smearing function for each background is still problematic. The construction of smearing functions relies on having a classical spacetime (perhaps with a few perturbative quantum fluctuations). This clearly does not have to be the case for a generic state in the CFT.

The approximation of getting a fixed background with a few supergravity excitations on it involves two steps. First, one needs to integrate out all the bulk stringy modes, which in the CFT means integrating out all high dimension operators. Second, one must do a semiclassical approximation to get a well-defined background metric. We won't have much to say about the first step, other than that one has to be careful later on when discussing high dimension operators. For instance, in the promotion of a boundary operator to a field in the bulk, one needs to include from the CFT perspective a tower of high dimension operators. If one includes high dimension operators only up to some $\Delta_{\max}$ then, according to [8], a good estimate of the commutator of a bulk operator with a boundary operator (taken to be scalars in AdS_3), which are spacelike separated in the bulk but not on the boundary, is

$$[\phi(t, \mathbf{x}, z), \mathcal{O}(0)] \sim \left(\frac{t^2 - |\mathbf{x}|^2}{z^2}\right)^{\Delta_{\max}}. \tag{93}$$

Although nonzero, the commutator is exponentially suppressed away from the bulk lightcone provided $\Delta_{\max}$ is large. A nice way to characterize the bulk nonlocality associated with a finite value of $\Delta_{\max}$ is to ask how far from the bulk lightcone one can go before the commutator becomes exponentially small. This is given by

$$\delta S \sim R/\Delta_{\max}, \tag{94}$$

where R is the AdS radius and S is proper length in the bulk. For $\Delta_{\max} \sim (g_{\rm YM}^2 N)^{1/4}$—appropriate for stringy modes—one gets $\delta S \sim l_s$.

Even if the approximation of integrating out the stringy modes is good it does not mean that the CFT state describes a semiclassical spacetime. In the supergravity approximation we can write down the equations of motion for the metric and matter fields in holographic gauge without choosing a particular background. This is done by replacing the radial Hamiltonian in Eq. (91) with the appropriate Hamiltonian for the supergravity system, namely $H_g = \int d^d x \frac{1}{z^2} H_{\rm WD}$, where $H_{\rm WD}$ is the Wheeler-de Witt operator. The radial evolution equations are then

$$\frac{\partial \mathcal{O}}{\partial z} = -[H_g, \mathcal{O}] \frac{\partial g_{\mu\nu}}{\partial z} = -[H_g, g_{\mu\nu}], \tag{95}$$

and similarly for the conjugate momenta. Once the constraints are satisfied on the initial slice ($z = 0$), the equations of motion guarantee that they are obeyed at any z. We assume here that

$$g_{\mu\nu}(z \to 0) = \eta_{\mu\nu}. \tag{96}$$

So corrections to the bulk metric come from normalizable modes, with the leading correction for small z being proportional to $T_{\mu\nu}$. This together with $\partial^\mu T_{\mu\nu} = 0$ and $T^\mu_\mu = 0$ gives enough initial data to solve the equations.[13]

The equations of motion can formally be solved to give the bulk fields as functionals of the boundary data:

$$\begin{aligned} \phi(x, z) &= \phi(x, z)[T_{\mu\nu}(x'), \mathcal{O}(x'')], \\ g_{\mu\nu}(x, z) &= g_{\mu\nu}(x, z)[T_{\mu\nu}(x'), \mathcal{O}(x'')]. \end{aligned} \tag{97}$$

So far this is independent of the state of the CFT. But now, given some state of the CFT, we would like to obtain a set of bulk operators which look like fields propagating

[13]We are ignoring the question of whether holographic gauge can be extended all the way to $z = \infty$. Also, since we are working in a Poincaré patch, we are ignoring any anomalous trace of the stress tensor.

on some semiclassical spacetime. To do this, to a good approximation one needs to be able to substitute

$$T_{\mu\nu} = \langle T_{\mu\nu} \rangle + \delta T_{\mu\nu}. \tag{98}$$

If this approximation is valid then we are guaranteed that correlators of our bulk operators, calculated in the CFT, will look like correlation function of supergravity fields on a background which solves the Einstein equations with asymptotics set by $\langle T_{\mu\nu} \rangle$.

Clearly, such an approximation is valid in a CFT state if connected correlation functions of CFT operators obey large N factorization. Thus, CFT states with large N factorization will be dual to semiclassical spacetimes, while those which do not obey large N factorization will not have a local spacetime interpretation.

Finally we want to speculate about a method for constructing bulk operators purely within the CFT. It seems possible from the above considerations that one can define a master set of "bulk operators" in the CFT, regardless of the state of the CFT or any low-energy approximation. These operators would not have a bulk interpretation, except on a restricted set of states where large N factorization holds. What are these master bulk operators? We propose to extrapolate from the supergravity situation Eq. (95). A natural guess is that they are defined by replacing the radial Hamiltonian in Eq. (92) with a more fundamental operator in the CFT, such as the exact RG Hamiltonian or Fokker-Planck Hamiltonian (see for instance [21,22]).

VII. CONCLUSIONS

In this paper we worked out the smearing functions which describe linearized spin-1 and spin-2 excitations in AdS. We showed that bulk locality is respected: although gauge fields and metric perturbations have nonlocal commutators when one works in holographic gauge, the corresponding curvatures—the field strength for A_μ, or the Weyl tensor in the case of gravity—are causal. We also studied massive vector fields, where the vector field itself is causal due to the nonconserved nature of the dual boundary current.

These results could be extended in several directions. For example, we computed the smearing function for a Chern-Simons gauge field in AdS3. It would be interesting to work out the smearing function for a Maxwell field in AdS_3, dual to a CFT with a dynamical gauge field [13,14] (see however [23]). Our results could be used to study the Maxwell-Chern-Simons theory recently analyzed in [24]. Since the smearing functions are basically fixed by AdS covariance, our results should also apply if there is a duality between AdS_2 and CFT_1, although the physical interpretation in this context is not so clear.

Perhaps a more interesting direction is to extend our results to include interactions. For massive vector fields we showed how this works in Sec. V: in a $1/N$ expansion one adds appropriately smeared higher dimension vector operators, with coefficients that are fixed by the requirement of bulk causality. It would be very interesting to extend this to gauge fields and metric perturbations, perhaps using the good causal properties of the field strength and Weyl tensor as a guiding principle. Ultimately one might hope to make contact between the 'bottom-up' approach of constructing bulk observables in $1/N$ perturbation theory, and the 'top-down' approach of Sec. VI where bulk operators are constructed from a fundamental operator of the boundary theory.

ACKNOWLEDGMENTS

We are grateful to Idse Heemskerk, Don Marolf, Joe Polchinski and James Sully for valuable discussions. The work of D. K., S. R. and D. S. was supported by U.S. National Science Foundation Grant No. PHY-0855582 and by PSC-CUNY grants. The work of G. L. was supported in part by the Israel Science Foundation under Grant No. 392/09. This research was supported in part by the National Science Foundation under Grant No. PHY11-25915.

APPENDIX A: SCALAR SMEARING FUNCTIONS

Consider a scalar field of mass m in AdS_{d+1}. It is dual to an operator of dimension Δ in the CFT, where $m^2R^2 = \Delta(\Delta - d)$. The mode expansion is

$$\phi(t, \mathbf{x}, z) = \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k a_{\omega\mathbf{k}} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} z^{d/2} J_\nu \times (z\sqrt{\omega^2 - |\mathbf{k}|^2}), \tag{A1}$$

where $\nu = \Delta - d/2$. As $z \to 0$ we have $\phi(t, \mathbf{x}, z) \sim z^\Delta \phi_0(t, \mathbf{x})$, where the boundary field

$$\phi_0(t, \mathbf{x}) = \frac{1}{2^\nu \Gamma(\nu + 1)} \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k a_{\omega\mathbf{k}} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} \times (\omega^2 - |\mathbf{k}|^2)^{\nu/2}.$$

Our basic goal is to express the bulk field in terms of the boundary field. A straightforward way to do this is to express the coefficients $a_{\omega\mathbf{k}}$ as a Fourier transform of ϕ_0,

$$a_{\omega\mathbf{k}} = \frac{2^\nu \Gamma(\nu + 1)}{(2\pi)^d (\omega^2 - |\mathbf{k}|^2)^{\nu/2}} \int dt d^{d-1}x e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{x}} \phi_0(t, \mathbf{x}).$$

Substituting this back in Eq. (A1) leads to an integral representation of the smearing function. Generically one obtains a smearing function with support on the entire boundary of the Poincaré patch; however, by complexifying the boundary spatial coordinates one can obtain a smearing function with compact support. As shown in [7] this leads to

$$\phi(t,\mathbf{x},z) = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\pi^{d/2}\Gamma(\Delta - d + 1)} \int_{t'^2 + |\mathbf{y}'|^2 < z^2} dt' d^{d-1} y' \times \left(\frac{z^2 - t'^2 - |\mathbf{y}'|^2}{z}\right)^{\Delta - d} \phi_0(t + t', \mathbf{x} + i\mathbf{y}'). \tag{A2}$$

This expression is fine for $\Delta > d - 1$. However when $\Delta = d - 1$ it is ill-defined: the integral diverges, and the coefficient in front goes to zero.

To construct a smearing function for $\Delta = d - 1$ we return to the mode expansion Eq. (A1). As a warm-up example take a massless field in AdS_2 with $\Delta = 0$. The mode expansion is $\phi(t,z) = \int d\omega a_\omega e^{-i\omega t}\cos(\omega z)$. Then $a_\omega = \frac{1}{2\pi}\int dt e^{i\omega t}\phi_0(t)$ and

$$\begin{aligned}\phi(t,z) &= \int dt' \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\cos(\omega z)\phi_0(t') \\ &= \frac{1}{2}(\phi_0(t+z) + \phi_0(t-z)).\end{aligned} \tag{A3}$$

This clearly satisfies the wave equation $(\partial_t^2 - \partial_x^2)\phi = 0$ and obeys the boundary condition $\phi(t,z) \to \phi_0(t)$ as $z \to 0$. It can be written in the covariant form

$$\phi(t,z) = \frac{1}{2}\int dt' \delta(\sigma z')\phi_0(t'),$$

where $\sigma z' = \frac{z^2 - (t-t')^2}{2z}$.

We now consider the general case of a field with $\Delta = d - 1$. In any dimension solving for $a_{\omega\mathbf{k}}$ in terms of ϕ_0 and plugging back into the mode expansion gives

$$\phi(t,\mathbf{x},z) = \int_{|\omega|>|\mathbf{k}|} d\omega d^{d-1}k \frac{2^\nu \Gamma(d/2) z^{d/2}}{(2\pi)^d(\omega^2 - |\mathbf{k}|^2)^{\nu/2}} J_\nu \times \left(z\sqrt{\omega^2 - |\mathbf{k}|^2}\right) e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_0(\omega,\mathbf{k}). \tag{A4}$$

Here $\nu = \frac{d}{2} - 1$ and $\phi_0(\omega,\mathbf{k})$ is the Fourier transform of the boundary field. The Bessel function has an integral representation

$$J_\nu(a) = \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}\left(\frac{a}{2}\right)^\nu \int_0^\pi d\theta e^{-ia\cos\theta}\sin^{2\nu}\theta, \tag{A5}$$

or equivalently

$$J_\nu(a) = \frac{1}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}\left(\frac{a}{2}\right)^\nu \frac{1}{\mathrm{vol}(S^{d-2})} \int_{|\mathbf{n}|=1} d\mathbf{n} e^{-i\mathbf{a}\cdot\mathbf{n}}. \tag{A6}$$

Here $\mathbf{a}$ is a d-component vector with Euclidean norm a and $\mathbf{n} \in S^{d-1}$ is a unit vector. Setting $\mathbf{a} = z(\omega, -ik_1, \ldots, -ik_{d-1})$ and using

$$\mathrm{vol}\,(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})\mathrm{vol}(S^{d-2})}{\Gamma(d/2)}, \tag{A7}$$

this becomes

$$\frac{2^\nu\Gamma(d/2)z^{d/2}}{(\omega^2 - |\mathbf{k}|^2)^{\nu/2}} J_\nu(z\sqrt{\omega^2 - |\mathbf{k}|^2}) = \frac{1}{\mathrm{vol}(S^{d-1})}\int_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' e^{-i\omega t'} e^{-\mathbf{k}\cdot\mathbf{y}'}.$$

Using this representation in Eq. (A4) leads to[14]

$$\phi(t,\mathbf{x},z) = \frac{1}{\mathrm{vol}(S^{d-1})}\int_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \times \int \frac{d\omega d^{d-1}k}{(2\pi)^d} e^{-i\omega(t+t')} e^{i\mathbf{k}\cdot(\mathbf{x}+i\mathbf{y}')}\phi_0(\omega,\mathbf{k}). \tag{A8}$$

We interpret the Fourier transforms in Eq. (A8) as defining the analytic continuation of $\phi_0(t,\mathbf{x})$ to complex $\mathbf{x}$. Thus the smearing function for a scalar field with $\Delta = d - 1$ is

$$\phi(t,\mathbf{x},z) = \frac{1}{\mathrm{vol}(S^{d-1})}\int_{t'^2 + |\mathbf{y}'|^2 = z^2} dt' d^{d-1}y' \phi_0(t + t', \mathbf{x} + i\mathbf{y}'). \tag{A9}$$

This can be written in a covariant form

$$\phi(t,\mathbf{x},z) = \frac{1}{\mathrm{vol}(S^{d-1})}\int dt' d^{d-1}y' \delta(\sigma z')\phi_0(t + t', \mathbf{x} + i\mathbf{y}') \tag{A10}$$

in terms of the bulk-boundary distance Eq. (6).

It is clear that Eqs. (A9) and (A10) satisfy the correct boundary conditions. As $z \to 0$ the integration region on the boundary shrinks to a point, so we can bring the boundary field outside the integral and recover

$$\phi(t,\mathbf{x},z) \sim z^{d-1}\phi_0(t,\mathbf{x}) \quad \text{as } z \to 0.$$

One can also check that Eq. (A10) satisfies the wave equation. Acting on a function of the AdS-invariant distance σ, the wave equation $(\Box - m^2)\phi = 0$ reduces to

$$(\sigma^2 - 1)\phi'' + (d+1)\sigma\phi' - \Delta(\Delta - d)\phi = 0.$$

With a small fixed cutoff z', the smearing kernel appearing in Eq. (A10) is $\frac{1}{z'}\delta(\sigma)$. We want to check that this is annihilated by the wave operator in the limit $z' \to 0$. To do this we act with the wave operator and integrate against a test function $f(\sigma z')$ (the test function can be thought of as the boundary field). For $\Delta = d - 1$ this gives

[14]The boundary field ϕ_0 only has Fourier components with $|\omega| > |\mathbf{k}|$, so we can integrate over ω and $\mathbf{k}$ without restriction.

$$\int d(\sigma z')f(\sigma z')\left[(\sigma^2-1)\frac{d^2}{d\sigma^2}+(d+1)\sigma\frac{d}{d\sigma}+(d-1)\right]\frac{1}{z'}\delta(\sigma)$$
$$=\int d(\sigma z')\frac{1}{z'}\delta(\sigma)\left[\frac{d^2}{d\sigma^2}(\sigma^2-1)-(d+1)\frac{d}{d\sigma}\sigma+(d-1)\right]f(\sigma z')$$
$$=-z'^2 f''(0).$$

This vanishes as $z'\to 0$, which shows that the wave equation is satisfied when the regulator is removed.

APPENDIX B: CHERN-SIMONS IN HOLOGRAPHIC GAUGE

Our goal in this appendix is to quantize Chern-Simons theory in holographic gauge. We want to show that we recover the bulk commutator Eq. (25) obtained in Sec. II B 1 by applying our smearing functions to the current algebra on the boundary.

We begin from the Abelian Chern-Simons action[15]

$$S_{\rm bulk}=\int d^3x\frac{1}{2}\kappa\epsilon^{ABC}A_A\partial_B A_C.$$

To obtain a right-moving current algebra on the boundary we supplement this with a surface term [25]

$$S_{\rm bdy}=\int d^2x\kappa A_+A_-.$$

The surface term leads to a well-defined variational principle provided we impose the boundary condition that A_- is fixed (that is, $\delta A_-=0$) on the boundary.

In light-front coordinates one can integrate by parts to find (the surface terms cancel against $S_{\rm bdy}$)

$$S_{\rm bulk+bdy}=\int dx^+dx^-dz\kappa A_z\partial_+A_-+\kappa A_+(\partial_-A_z-\partial_zA_-).$$

We adopt x^+ as light-front time [26] and read off the Poisson bracket [27]

$$\{A_z(x^-,z),A_-(x^{-\prime},z')\}=\frac{1}{\kappa}\delta(x^--x^{-\prime})\delta(z-z').$$

A_+ is a Lagrange multiplier that enforces the Chern-Simons Gauss law. Thus we have a (primary, first-class) constraint,

$$\chi_1=\partial_zA_--\partial_-A_z\approx 0.$$

The constraint generates the expected gauge transformation:

$$\delta A_z=\left\{\int dx^{-\prime}dz'\lambda_1\chi_1,A_z(x^-,z)\right\}=\frac{1}{\kappa}\partial_z\lambda_1,$$
$$\delta A_-=\left\{\int dx^{-\prime}dz'\lambda_1\chi_1,A_-(x^-,z)\right\}=\frac{1}{\kappa}\partial_-\lambda_1.$$

To preserve the boundary condition $\delta A_-|_{z=0}=0$, we require that the gauge parameter satisfy $\lambda_1|_{z=0}=0$. We wish to work in holographic gauge, so we impose an additional constraint (a gauge-fixing condition)

$$\chi_2=A_z\approx 0.$$

The constraints obey

$$\Delta_{ij}\equiv\{\chi_i,\chi_j\}$$
$$=\begin{pmatrix}0 & -\frac{1}{\kappa}\delta(x^--x^{-\prime})\delta'(z-z')\\ -\frac{1}{\kappa}\delta(x^--x^{-\prime})\delta'(z-z') & 0\end{pmatrix}.$$

Acting on functions

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix},$$

this operator has zero modes, but as we will see the zero modes can be eliminated by requiring

$$\lambda_1(x^-,z=0)=0\qquad \lambda_2(x^-,z=\infty)=0.$$

Then Δ has a well-defined inverse,

$$\Delta^{-1}=\begin{pmatrix}0 & -\kappa\delta(x^--x^{-\prime})\theta(z-z')\\ \kappa\delta(x^--x^{-\prime})\theta(z'-z) & 0\end{pmatrix}.$$

Note that Δ^{-1} is antisymmetric. One can easily check the basic property

$$\Delta^{-1}\Delta\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}=\begin{pmatrix}\lambda_1(x^-,z)-\lambda_1(x^-,0)\\ \lambda_2(x^-,z)-\lambda_2(x^-,\infty)\end{pmatrix},$$

which shows that Δ is invertible given our boundary conditions. The constraints can be eliminated by defining Dirac brackets. The Dirac bracket of A_z with anything will vanish, while the Dirac bracket of A_- with itself is

$$\{A_-(x^-,z),A_-(x^{-\prime},z')\}=0-\{A_-,\chi_i\}\Delta^{-1}_{ij}\{\chi_j,A_-\}$$
$$=-\frac{1}{\kappa}\delta'(x^--x^{-\prime}).$$

Quantizing via $\{\cdot,\cdot\}\to i[\cdot,\cdot]$ reproduces the bulk commutator Eq. (25) and fixes the normalization $\kappa=4\pi/k$.

[15]Conventions: light-front coordinates are $x^\pm=t\pm x$. We take $\epsilon_{012}=+1$ and relate the bulk and boundary orientations by $\int d^3x\partial_zf=-\int d^2xf|_{z=0}$.

[1] V. Balasubramanian, P. Kraus, and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999).
[2] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, arXiv:hep-th/9808016.
[3] V. K. Dobrev, Nucl. Phys. **B553**, 559 (1999).
[4] I. Bena, Phys. Rev. D **62**, 066007 (2000).
[5] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **73**, 086003 (2006).
[6] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **74**, 066009 (2006).
[7] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **75**, 106001 (2007).
[8] D. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **83**, 106009 (2011).
[9] I. Heemskerk, D. Marolf, and J. Polchinski, arXiv:1201.3664.
[10] I. Heemskerk, arXiv:1201.3666.
[11] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[12] P. Breitenlohner and D. Z. Freedman, Ann. Phys. (N.Y.) **144**, 249 (1982).
[13] K. Jensen, J. High Energy Phys. 01 (2011) 109.
[14] D. Marolf and S. F. Ross, J. High Energy Phys. 11 (2006) 085.
[15] E. Witten, *From Fields to Strings*, edited by M. Shifman *et al.* (World Scientific, Singapore, 2004), Vol. 2, pp. 1173–1200.
[16] C. Fefferman and C. Robin Graham, *Conformal Invariants, The Mathematical Heritage of Élie Cartan* (Société Mathématique de France, Lyon, 1984) [Astérisque Numero Hors Serie, 95 (1985)].
[17] M. Henningson and K. Skenderis, J. High Energy Phys. 07 (1998) 023.
[18] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).
[19] J. Erdmenger and H. Osborn, Nucl. Phys. **B483**, 431 (1997).
[20] W. Mueck and K. Viswanathan, Phys. Rev. D **58**, 106006 (1998).
[21] G. Lifschytz and V. Periwal, J. High Energy Phys. 04 (2000) 026.
[22] I. Heemskerk and J. Polchinski, J. High Energy Phys. 06 (2011) 031.
[23] T. Andrade and D. Marolf, J. High Energy Phys. 01 (2012) 049.
[24] T. Andrade, J. I. Jottar, and R. G. Leigh, J. High Energy Phys. 05 (2012) 071.
[25] P. Kraus, Lect. Notes Phys. **755**, 1 (2008).
[26] T. Heinzl, Lect. Notes Phys. **572**, 55 (2001).
[27] L. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988).