Fermions on a Lifshitz background

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We study a nonrelativistic fermionic retarded Green's function by making use of a fermion on the Lifshitz geometry with critical exponent $z = 2$. With a natural boundary condition, respecting the symmetries of the model, the resultant retarded Green's function exhibits a number of interesting features including a flat band. We also study the finite temperature and finite chemical potential cases where the geometry is replaced by Lifshitz black hole solutions.

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I. INTRODUCTION

At critical points, physics is usually described by a scale-invariant model. Typically, the scale invariance arises in the relativistic conformal group where we have

$$t \longrightarrow \lambda t, \qquad x_i \longrightarrow \lambda x_i. \tag{1.1}$$

Here t is time and x_i's are spatial directions of the space-time.

We note, however, that in many physical systems the critical points are governed by dynamical scalings in which space and time scale differently. In fact spatially isotropic scale invariance is characterized by the dynamical exponent, z, as follows [1]

$$t \longrightarrow \lambda^z t, \qquad x_i \longrightarrow \lambda x_i. \tag{1.2}$$

The corresponding critical points are known as Lifshitz fixed points.

In light of AdS/CFT correspondence, [2] it is natural to seek for gravity duals of Lifshitz fixed points. Indeed gravity descriptions of Lifshitz fixed points have been considered in [3] (see also [4] for an earlier work on a geometry with the Lifshitz scaling.), where a metric invariant under the scaling (1.2) was introduced. The corresponding metric is[1]

$$ds^2 = L^2\left(-\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2}\right), \tag{1.3}$$

where L is the radius of curvature.[2] The action of the scale transformation (1.2) on the metric is given by

$$t \longrightarrow \lambda^z t, \qquad x_i \longrightarrow \lambda x_i, \qquad r \longrightarrow \lambda^{-1} r. \tag{1.4}$$

As a physical application, the Lifshitz geometry has been used to provide a possible holographic description for strange metals [5].[3] In this setup the Lifshitz background is probed by D-branes with nonzero gauge fields in their world volume. By appropriately choosing the dynamical critical exponent, z, the authors of [5] have been able to match the non-Fermi liquid scalings, such as linear resistivity, observed in strange metal regimes. Having found the non-Fermi liquid scalings, it is natural to study the fermionic properties of the system to explore, for example, a possibility of having a Fermi surface in the model. To do so, one needs to consider a fermion on the Lifshitz geometry to find the retarded Green's function of the corresponding dual fermionic operator via AdS/CFT correspondence.

Indeed, utilizing fermions on asymptotically AdS geometries, it was shown that the AdS/CFT correspondence can holographically describe Fermi surfaces [7–13]. Actually to see a Fermi surface, one should look for a sharp behavior in the fermionic retarded Green's function at finite momentum and small frequencies (for a review see, e.g., [14]). Moreover the spectrum of quasiparticle excitations near the Fermi surface is governed by an emergent CFT corresponding to the AdS_2 near-horizon geometry of the black hole [10].

The aim of this article is to study fermions on the Lifshitz geometry, which in turn can be used to study the fermionic retarded Green's function of the corresponding nonrelativistic dual theory.[4] Fermions on the Lifshitz geometry and also on a geometry with Lifshitz IR fixed point have been studied in [17,18], respectively. In these papers, with a Lorentz-symmetry-preserving boundary term, the Green's function of the fermion has been obtained. It was shown that the resultant Green's function has no imaginary part.

On the other hand, in order to find the retarded Green's function we consider the Lorentz-symmetry-breaking boundary condition introduced in [19].[5] Since the corresponding

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[1]As has been mentioned in [3], although the metric is nonsingular, it is not geodesically complete and, in particular, an infalling object into $r = \infty$ feels a large tidal force.

[2]In what follows we set $L = 1$.

[3]See also [6] for drag force computations in Lifshitz geometries.

[4]Fermions on Schrodinger space-time has also been studied in [15,16].

[5]Throughout this paper we refer to this boundary condition as *nonstandard* boundary condition, while we refer to that introduced in [20,21] as *standard* boundary condition.

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boundary condition preserves rotational and scale invariances, but breaks the boost, it is more natural to impose such a boundary condition on the geometries with Lifshitz isometry. Of course it also breaks the parity which is preserved by the Lifshitz symmetry.[6] Note that in this paper we consider the fermion as a probe. It would be interesting to extend this work to the case where the back reaction of the fermions is taken into account.

The paper is organized as follows. In the next section, we study fermions on the Lifshitz geometry, where we find a solution for the equation of motion with a proper boundary condition. Then, using the solution, we calculate the corresponding retarded Green's function, where we see that the model exhibits a flat band. In Sec. III we extend our study to the finite-temperature case, where we show that, although the system has excited zero-energy fermionic modes at low momenta, at high momenta still it has a flat band. In Sec. IV, we consider charged fermions probing a charged Lifshitz black hole, where we show that, while with the standard boundary condition the system exhibits a Fermi surface, in the nonstandard case it still has flat band. The last section is devoted to discussions.

II. ZERO TEMPERATURE

The aim of this section is to study fermions on the Lifshitz background which will be used to find the retarded Green's function for the corresponding fermionic dual operator in the dual nonrelativistic field theory. Before going into computations, it is worthwhile to note that the Lifshitz geometry is not a solution of the pure Einstein gravity with or without cosmological constant.

In general, to get the Lifshitz geometry one needs to couple the Einstein gravity to other fields. In particular, the Lifshitz geometry may be obtained from gravity coupled to massive gauge fields. In the minimal case where we have only one massive gauge field, the corresponding action is given as follows

$$I=\frac{1}{16\pi G_{d+1}}\int d^{d+1}x\sqrt{g}\left(R+\Lambda-\frac{1}{4}F^2-\frac{1}{4}m^2A^2\right), \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is easy to see that, with a suitable choice of the parameters m and Λ, the model admits the Lifshitz solution (1.3) with a nonzero gauge field given by [24]

$$A_t = \sqrt{\frac{2(z-1)}{z}}\frac{1}{r^z}. \quad (2.2)$$

For this solution the parameters m and Λ are $m^2 = 4z$, $\Lambda = z^2 + (d-2)z + (d-1)^2$.

[6]The Lorentz-symmetry-breaking boundary condition has also been imposed for fermions with dipole coupling in [22] and, on the charged dilatonic black hole in [23].

Alternatively, the Lifshitz metric may also be obtained as a solution of the pure gravity modified by curvature squared terms [25]. As the simplest case consider a $d+1$-dimensional gravitational action as follows

$$I = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}(R-\Lambda+\beta R^2). \quad (2.3)$$

Using the equations of motion derived from the above action one can show that the Lifshitz geometry (1.3) is a solution of the equations of motion for a suitable choice of the cosmological constant Λ and the coupling constant β that are given by

$$\Lambda = -\frac{2z^2+(d-1)(2z+d)}{2}, \qquad \beta = -\frac{1}{4\Lambda}. \quad (2.4)$$

Although we could have Lifshitz metric in arbitrary dimensions, in what follows we will consider the four-dimensional Lifshitz geometry which could provide a holographic description for a three-dimensional nonrelativistic field theory.

A. Fermions on Lifshitz geometry

Let us consider a four-dimensional Dirac fermion on the Lifshitz background whose action is

$$S_{\rm bulk} = \int d^4x\sqrt{-g}i\bar{\Psi}\left[\frac{1}{2}(\Gamma^a\vec{D}_a - \overleftarrow{D}_a\Gamma^a) - m\right]\Psi, \quad (2.5)$$

where $\not{D} = (e_\mu)^a\Gamma^\mu[\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a\Gamma^{\rho\sigma}]$, with $\Gamma^{\mu\nu} = \frac{1}{2}\times[\Gamma^\mu, \Gamma^\nu]$. In our notation the space-time indices are denoted by $a, b\ldots$, though the tangent space indices are labeled by $\mu, \nu\ldots.$

Since the Lifshitz metric may be obtained from a gravity coupled to a massive gauge field, in general, the solution may also support a nonzero gauge field. Therefore, one could consider a fermion that is charged under the background gauge field. Nevertheless, in what follows we will consider a neutral fermion. We will come back to charged fermions, later, in the discussion section.

To write the equation of motion one should use the variational principle, which typically comes with a proper boundary condition. It is important to note that the boundary term is not unique and indeed there are several ways to make the variational principle well-defined using different boundary terms [19]. For the moment, we assume that there is a suitable boundary condition such that the variational principle will be well-defined. With this assumption. the equation of motion is

$$\left((e_\mu)^a\Gamma^\mu\left[\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a\Gamma^{\rho\sigma}\right] - m\right)\Psi = 0, \quad (2.6)$$

where the nonzero components of vierbeins and spin connections for the Lifshitz metric (1.3) are

$$(e_t)^a = r^z\delta^{ta}, \qquad (e_i)^a = r\delta^{ia}, \qquad (e_r)^a = r\delta^{ra},$$

and

$$(\omega_{tr})_a = -(\omega_{rt})_a = \frac{z}{r^z}\delta_{ta}, \quad (\omega_{ir})_a = -(\omega_{ri})_a = -\frac{1}{r}\delta_{ia}.$$

Using these expressions, the equation of motion reduces to

$$\left[\Gamma^t r^z \partial_t - \left(\frac{z}{2}+1\right)\Gamma^r + r\Gamma^i\partial_i + r\Gamma^r\partial_r - m\right]\Psi = 0. \quad (2.7)$$

To proceed, it is useful to work in the momentum space where we may set $\Psi = e^{i\omega t + ik.x}\psi(r)$. In this notation the equation of motion reads

$$ir(k.\Gamma)\psi = \left[-i\omega r^z \Gamma^t + \left(\frac{z}{2}+1\right)\Gamma^r - r\Gamma^r\partial_r + m\right]\psi. \quad (2.8)$$

It is also useful to act by $(\not{D} + m)$ on the first-order equation of motion to find a second-order differential equation which typically is easier to solve. Doing so, and using Eq. (2.8), one arrives at

$$(\not{D}\not{D} - m^2)\psi = \Big[r^2\partial_r^2 - (z+2)r\partial_r + \omega^2 r^{2z} + \left(\frac{z}{2}+1\right)\left(\frac{z}{2}+2\right) - r^2\vec{k}^2 + i(z-1)\omega r^z\Gamma^r\Gamma^t + m\Gamma^r - m^2\Big]\psi = 0. \quad (2.9)$$

In general, this equation may not have analytic solutions. We note, however, that for a particular case of $m = 0$ and $z = 2$ the equation has, indeed, an analytic solution. This is the case we will consider in this paper. In this case defining $\psi_\pm = \frac{1}{2}(1 \pm \Gamma^r\Gamma^t)\psi$, one gets

$$[r^2\partial_r^2 - 4r\partial_r + \omega^2 r^4 - r^2(\vec{k}^2 \mp i\omega) + 6]\psi_\pm = 0. \quad (2.10)$$

To solve the above equation, we make the following change of variable

$$\psi_\pm(r) = r^{3/2} e^{(i\omega/2)r^2} f_\pm(i\omega r^2) \quad (2.11)$$

by which Eq. (2.10) reduces to a well-known differential equation for $f_\pm(\xi)$

$$\frac{d^2 f_\pm(\xi)}{d\xi^2} + \frac{df_\pm(\xi)}{d\xi} + \left(\frac{\lambda_\pm}{\xi} + \frac{\frac{1}{4} - \mu_\pm^2}{\xi^2}\right)f_\pm(\xi) = 0, \quad (2.12)$$

where

$$\lambda_\pm = -\frac{k^2}{4i\omega} \pm \frac{1}{4}, \qquad \mu_\pm = \frac{1}{4}. \quad (2.13)$$

We recognize the above equation as the hypergeometric differential equation whose solution is

$$f_\pm(\xi) = c_1^\pm \xi^{(1/2)-\mu_\pm} e^{-\xi} F(\alpha_\pm, -2\mu_\pm + 1, \xi) + c_2^\pm \xi^{(1/2)+\mu_\pm} e^{-\xi} F(\beta_\pm, 2\mu_\pm + 1, \xi), \quad (2.14)$$

where $F(a, b, \xi)$ is the confluent hypergeometric function, $c_{1,2}^\pm$ are two constant spinors and

$$\alpha_\pm = \frac{1}{2} - \mu_\pm - \lambda_\pm, \qquad \beta_\pm = \frac{1}{2} + \mu_\pm - \lambda_\pm. \quad (2.15)$$

Therefore, altogether we find

$$\psi_\pm(r) = e^{-(i\omega/2)r^2} r^2\left[D_1^\pm F\left(\alpha_\pm, \frac{1}{2}, i\omega r^2\right) + D_2^\pm rF\left(\beta_\pm, \frac{3}{2}, i\omega r^2\right)\right], \quad (2.16)$$

with $D_1^\pm = (i\omega)^{1/4} c_1^\pm$, $D_2^\pm = (i\omega)^{3/4} c_2^\pm$.

It is important to note that so far we have solved the second-order differential equation and thus the constant spinors $c_{1,2}^\pm$ are not independent and, indeed, restricting the above solution to be a solution of the first-order equation of motion (2.8) leads to certain relations among them. More precisely, one finds

$$c_2^+ = \frac{-i}{\sqrt{i\omega}}\Gamma^r(k.\Gamma)c_1^-, \qquad c_2^- = \frac{-i}{\sqrt{i\omega}}\Gamma^r(k.\Gamma)c_1^+. \quad (2.17)$$

We note, also, that the solution has not been uniquely fixed yet. In fact in the context of AdS/CFT correspondence one usually imposes a boundary condition at IR. In the Euclidean case the proper boundary condition is to assume that the wave function is finite at IR. When we are dealing with the real-time AdS/CFT correspondence, the proper boundary condition is to impose an ingoing boundary condition on the wave function at the horizon [26]. In our case using the asymptotic behavior of the hypergeometric function,

$$F(a, b, \xi) \sim \frac{\Gamma(b)}{\Gamma(b-a)}(-\xi)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^\xi \xi^{a-b}, \quad \text{for large}|\xi|, \quad (2.18)$$

the wave function is ingoing at "Lifshitz horizon," $r = \infty$, if the parameters c_1^+ and c_2^+ satisfy the following relation

$$c_2^+ = -2\frac{\Gamma(\alpha_+ + \frac{1}{2})}{\Gamma(\alpha_+)} c_1^+, \quad (2.19)$$

by which the ingoing wave function near the Lifshitz horizon behaves as follows[7]

$$\psi \sim r^2 e^{i\omega(t-(r^2/2)+(k^2/2\omega^2)\ln r)}. \quad (2.20)$$

One may wonder what "Lifshitz horizon" means! Actually, the situation could be compared with that of fermions on the pure AdS case studied in, e.g., [27], where the ingoing boundary condition has been imposed at the AdS horizon at $r = \infty$. In order to understand it better, one may think of this boundary condition as a limiting procedure starting from a black hole solution and then approaching the zero-temperature limit, as we will do in the next section.

Alternatively, to obtain the relation (2.19) and then the corresponding retarded Green's function[8] one may use the

[7] Since in what follows we are interested in the lowenergy limit of the retarded Green's function, we will consider the case where the momentum is spacelike, i.e., $k^2 \geq \omega$.

[8] The prescription for calculating retarded Green's function in the context of AdS/CFT correspondence has been first considered in [26] and further studied in the literature in, e.g., [28–33].

prescription explored in [27] where the authors presented a derivation of the real-time AdS/CFT prescription as an analytic continuation of the corresponding problem in the Euclidean signature. Indeed, in our case, we have checked that, using this prescription, we will arrive at the same results as those in this and the next subsections.[9]

B. Retarded Green's function

In this subsection, we compute the retarded Green's function of a fermionic operator in the dual nonrelativistic three-dimensional field theory by making use of the solution we obtained in the previous section. One should note that, in the context of the AdS/CFT correspondence, in order to find the corresponding retarded Green's function it is crucial to appropriately identify the source and response of the dual operator.

On the other hand, the identification of the source and response depends on the boundary conditions which one imposes to get a well-defined variational principle. Thus it is important to study the possible boundary terms one may add to the action to make the variational principle well-defined. Therefore, in what follows we will first find a proper boundary action for our model. To do so it is useful to explicitly fix our notation.

Since we have been working in a basis in which $\Gamma^r\Gamma^t$ is diagonal, we use the following representation for four-dimensional gamma matrices

$$\Gamma^r = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \qquad \Gamma^t = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix},$$
$$\Gamma^1 = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}. \quad (2.21)$$

In this notation, one has

$$\Psi_+ = \frac{1}{2}(1 + \Gamma^r\Gamma^t)\Psi = \mathrm{diag}(0,1,1,0)\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_2 \\ \Psi_3 \\ 0 \end{pmatrix}, \quad (2.22)$$

$$\Psi_- = \frac{1}{2}(1 - \Gamma^r\Gamma^t)\Psi = \mathrm{diag}(1,0,0,1)\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \\ \Psi_4 \end{pmatrix}. \quad (2.23)$$

Therefore the boundary terms coming from the variation of the bulk action,

$$\delta S_{\mathrm{bulk}} = \frac{i}{2}\int d^3x\sqrt{-h}(\bar{\Psi}\Gamma^r\delta\Psi - \delta\bar{\Psi}\Gamma^r\Psi), \quad (2.24)$$

reads

$$\delta S_{\mathrm{bulk}} = \frac{i}{2}\int d^3x\sqrt{-h}[\Psi_1^\dagger\delta\Psi_1 - \Psi_2^\dagger\delta\Psi_2 - \Psi_3^\dagger\delta\Psi_3 + \Psi_4^\dagger\delta\Psi_4 - \delta\Psi_1^\dagger\Psi_1 + \delta\Psi_2^\dagger\Psi_2 + \delta\Psi_3^\dagger\Psi_3 - \delta\Psi_4^\dagger\Psi_4]. \quad (2.25)$$

Since the Dirac equation is a first-order differential equation, we are not allowed to impose the boundary condition on all components of the spinors. Thus the aim is to add a proper boundary term such that half of the degrees of freedom do not appear on the boundary. So we will have to fix only half of the spinors.

We note, however, that the boundary terms may not be unique [19]. Of course, different boundary terms lead to different physics. In our case, since the dual theory is a nonrelativistic field theory, one may relax the condition to have Lorentz-symmetric boundary terms.

Following the suggestion of [19], it is natural to consider the following boundary term[10]

$$S_{\mathrm{bdy}} = \frac{1}{2}\int d^3x\sqrt{-h}\bar{\Psi}\Gamma^1\Gamma^2\Psi, \quad (2.26)$$

which in our notation reads

$$S_{\mathrm{bdy}} = \frac{i}{2}\int d^3x\sqrt{-h}(\Psi_1^\dagger\Psi_3 + \Psi_2^\dagger\Psi_4 - \Psi_3^\dagger\Psi_1 - \Psi_4^\dagger\Psi_2). \quad (2.27)$$

This boundary term is invariant under rotation and scaling but breaks the boost symmetry. Of course in our model, being Lifshitz geometry, the boost symmetry has already been broken by the geometry at first place.

Adding this boundary term to the bulk action and varying the total action, we arrive at

[9]Euclidean Green's function for fermions on the Lifshitz geometry has recently been studied in [17].

[10]Note that this boundary term is different from that considered in [17].

$$\begin{aligned}\delta S_{\text{bulk}} + \delta S_{\text{bdy}} &= \frac{i}{2}\int\sqrt{-h}[\delta(\Psi_1^\dagger + \Psi_3^\dagger)(\Psi_3 - \Psi_1) + \delta(\Psi_2^\dagger - \Psi_4^\dagger)(\Psi_2 + \Psi_4) \\ &\quad + (\Psi_1^\dagger - \Psi_3^\dagger)\delta(\Psi_1 + \Psi_3) + (\Psi_2^\dagger + \Psi_4^\dagger)\delta(\Psi_4 - \Psi_2)] \\ &= i\int\sqrt{-h}[-\delta\chi_1^\dagger\chi_2 + \delta\zeta_2^\dagger\zeta_1 + \chi_2^\dagger\delta\chi_1 - \zeta_1^\dagger\delta\zeta_2],\end{aligned}$$

where

$$(\chi_1, \chi_2) = \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3, \Psi_1 - \Psi_3) \qquad (\zeta_1, \zeta_2) = \frac{1}{\sqrt{2}}(\Psi_2 + \Psi_4, \Psi_2 - \Psi_4). \tag{2.28}$$

Therefore we get a well-defined variational principle by setting a Dirichlet boundary condition on χ_1 and ζ_2.[11] As a result the source and response are given by (χ_1, ζ_2) and (χ_2, ζ_1), respectively. The retarded Green's function is essentially a matrix which maps the source to the response. To compute the corresponding retarded Green's function, it is illustrative to explicitly write the solution we have found in the previous section in components

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = r^2 e^{-(i\omega/2)r^2}\begin{pmatrix} D_{1\uparrow}^- F\Big(\alpha_-;\frac{1}{2};i\omega r^2\Big) + D_{2\uparrow}^- rF\Big(\beta_-;\frac{3}{2};i\omega r^2\Big) \\ D_{1\downarrow}^+ F\Big(\alpha_+;\frac{1}{2};i\omega r^2\Big) + D_{2\downarrow}^+ rF\Big(\beta_+;\frac{3}{2};i\omega r^2\Big) \\ D_{1\uparrow}^+ F\Big(\alpha_+;\frac{1}{2};i\omega r^2\Big) + D_{2\uparrow}^+ rF\Big(\beta_+;\frac{3}{2};i\omega r^2\Big) \\ D_{1\downarrow}^- F\Big(\alpha_-;\frac{1}{2};i\omega r^2\Big) + D_{2\downarrow}^- rF\Big(\beta_-;\frac{3}{2};i\omega r^2\Big)\end{pmatrix}. \tag{2.29}$$

On the other hand, in our basis one has

$$\Gamma^r(k.\Gamma) = i\begin{pmatrix} k_1\sigma^1 & k_2 \\ k_2 & -k_1\sigma^1\end{pmatrix}. \tag{2.30}$$

Therefore the Eq. (2.17) reads

$$\begin{pmatrix}0\\ c_{2\downarrow}^+\\ c_{2\uparrow}^+\\ 0\end{pmatrix} = \frac{1}{\sqrt{i\omega}}\begin{pmatrix}0\\ k_1c_{1\uparrow}^- + k_2c_{1\downarrow}^-\\ k_2c_{1\uparrow}^- - k_1c_{1\downarrow}^-\\ 0\end{pmatrix} = \frac{1}{\sqrt{i\omega}}\begin{pmatrix}0\\ Ac_{1\downarrow}^+\\ Ac_{1\uparrow}^+\\ 0\end{pmatrix}, \tag{2.31}$$

where the last equality is the ingoing condition (2.19) with

$$A = -2\sqrt{i\omega}\frac{\Gamma(\alpha_+ + \frac{1}{2})}{\Gamma(\alpha_+)}. \tag{2.32}$$

By making use of this relation and utilizing the asymptotic behaviors of the solution, the retarded Green's function can be read as follows

$$G_R(k) = -\frac{1}{A^2 - k^2} \times\begin{pmatrix} A^2 - 2Ak_2 + k^2 & -2k_1A \\ -2k_1A & A^2 + 2Ak_2 + k^2\end{pmatrix}. \tag{2.33}$$

[11] If we had considered $\delta S_{\text{bulk}} - \delta S_{\text{bdy}}$, the boundary condition should have been imposed on χ_2 and ζ_1.

As is evident from the above expression for the retarded Green's function $G_{11}(\omega, k_2) = G_{22}(\omega, -k_2)$ and $\det(G_R) = -1$. The spectral function is also given by

$$\mathcal{A}(k) = -\frac{1}{\pi}\,\text{Im}(\text{Tr}(G_R)) = \frac{2}{\pi}\,\text{Im}\Big(\frac{A^2 + k^2}{A^2 - k^2}\Big). \tag{2.34}$$

To study different features of the retarded Green's function one may use the rotational symmetry to set $k_1 = 0$.[12] In this case, the retarded Green's function becomes diagonal. In fact, taking into account that $k = k_2$, one gets

$$G_R(k_2) = -\begin{pmatrix}\frac{A-k_2}{A+k_2} & 0 \\ 0 & \frac{A+k_2}{A-k_2}\end{pmatrix}. \tag{2.35}$$

Moreover, one finds

$$\mathcal{A}(k_2) = \frac{2}{\pi}\text{Im}\Big(\frac{\Gamma^2(\eta+\frac{1}{2}) + \Gamma(\eta)\Gamma(\eta+1)}{\Gamma^2(\eta+\frac{1}{2}) - \Gamma(\eta)\Gamma(\eta+1)}\Big), \quad \text{with } \eta = \frac{k_2^2}{4i\omega}. \tag{2.36}$$

The behavior of the spectral function as a function of k_2 and ω is shown in Fig. 1. Since the spectral function is symmetric under $k \to -k$, it is sufficient to draw the figure

[12] Although the model has rotational symmetry, the resultant retarded Green's function (2.33) seems asymmetric with respect to exchanging k_1 and k_2. We note that it is the artificial of our asymmetric representation of the Gamma matrices. We will come back to this point latter.

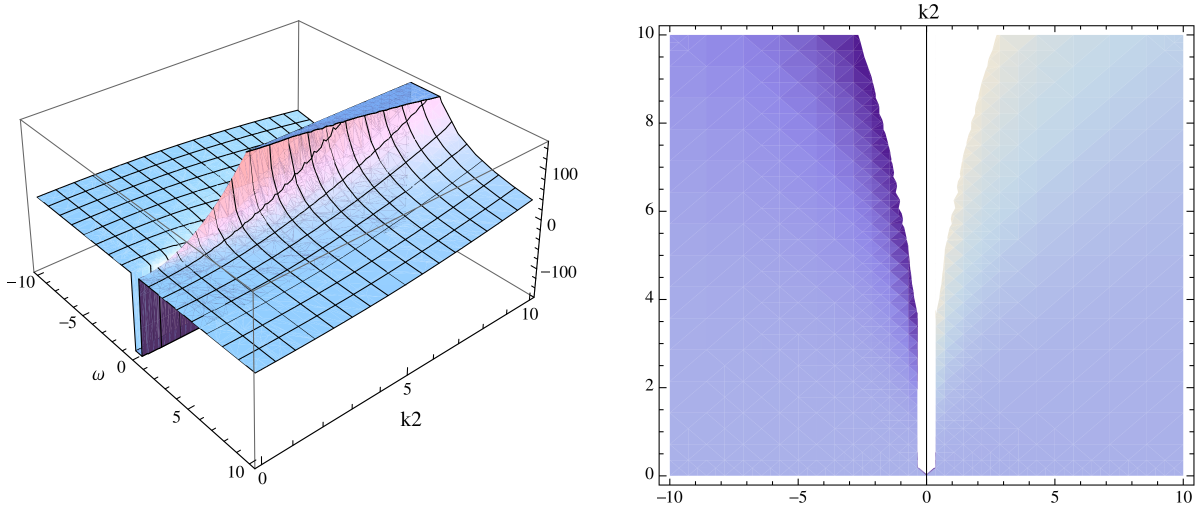

FIG. 1 (color online). Three-dimensional and density plots of the spectral function as a function of ω and k_2. It is positive for $\mathrm{sign}(\omega) > 0$ and has a pole at $\omega = 0$ for fixed k_2.

for positive k. As we observe, the spectral function is positive for $\mathrm{sign}(\omega) > 0$ and diverges at $\omega \to 0$. Actually, by making use of the asymptotic behavior of the Gamma functions one can read the asymptotic behavior of the spectral function near $\omega = 0$. Indeed, for finite k_2 one finds $\mathcal{A} \sim \frac{k_2^2}{\omega}$ showing that it has a simple pole.

To further explore the physical content of the model, it is useful to study the behavior of the eigenvalues of the retarded Green's function as we approach $\omega = 0$ for fixed and finite k_2. Indeed, using the asymptotic behavior of the Gamma functions, for $k_2^2 \gg \omega$, one finds

$$\lambda_1 = \frac{k_2 - A}{k_2 + A} \approx i\frac{k_2^2}{\omega}, \qquad \lambda_2 = \frac{k_2 + A}{k_2 - A} \approx -i\frac{\omega}{k_2^2}. \tag{2.37}$$

This shows that for finite values of k_2, one of the eigenvalues, λ_1, has a pole at $\omega = 0$. More generally, one can see that the eigenvalue λ_1 has a pole at $\omega = 0$ for all values of spatial momenta. This can be seen, for example, from the behavior of the real part of the eigenvalue λ_1 where there is a delta function at $\omega = 0$ as shown in Fig. 2. As a result, one may conclude that there are localized nonpropagating excitations in the model, showing that the theory exhibits a flat band.

Note that, as we have already mentioned in the introduction, the behavior of the retarded Green's function is different from that considered in [18] where, by making use of a semiholographic method, the authors have shown that the corresponding Green's function has no imaginary part (This is also the case for that in [17]). We note, however, that since in our case we are using the Lorentz-symmetry-breaking boundary term, the resultant Green's function has, indeed, an imaginary part. Actually, the situation is similar to the pure AdS case with Lorentz-symmetry-breaking boundary term (2.26) studied in [19]. Although in this case the bulk AdS geometry respects the Lorentz symmetry, the boundary term breaks this symmetry, leading to a nonrelativistic boundary theory. On the other hand since the boundary term (2.26), up to parity, is invariant under the Lifshitz symmetry, the nonrelativistic theory one gets from AdS bulk geometry has the same symmetry as if we had started with Lifshitz geometry in the bulk. Therefore one may conclude that the appearance of flat band is, indeed, the consequence of the nonrelativistic feature of the dual theory.

III. FINITE TEMPERATURE

In this section, we would like to redo the computations of the previous section for a nonrelativistic theory at finite temperature. Following the general idea of gauge/gravity

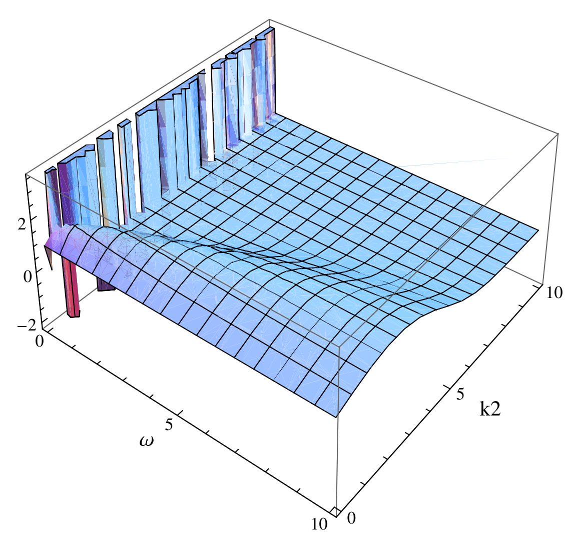

FIG. 2 (color online). The real part of the flat band eigenvalue, λ_1 which shows a delta function behavior at $\omega = 0$, indicating that the imaginary part of λ_1 has a pole at $\omega = 0$.

duality, placing the dual theory at finite temperature corresponds to having a black hole in the bulk gravity. Therefore, in our case we should look for a black hole solution in the asymptotic Lifshitz geometry. Actually, black hole solutions in the asymptotic Lifshitz geometry have been studied in [34–37]. In particular, the authors of [37] have analytically constructed a black hole which asymptotes to a vacuum Lifshitz solution with $z = 2$. The solution may be supported by different actions with different field contents, though the metric has the same form as follows

$$ds^2 = -\left(1 - \frac{r^2}{r_H^2}\right)\frac{dt^2}{r^4} + \frac{dr^2}{r^2(1 - \frac{r^2}{r_H^2})} + \frac{d\vec{x}^2}{r^2}, \quad (3.1)$$

where r_H is the radius of horizon. The Hawking temperature is [37]

$$T = \frac{1}{2\pi r_H^2}. \quad (3.2)$$

A. Fermions on Lifshitz black hole

Following our study in the previous section, we will consider a neutral massless fermion on the Lifshitz black hole given by Eq. (3.1). For this geometry the nonzero components of vierbeins and spin connections are

$$(e_t)^a = \frac{r^2}{\sqrt{1 - \frac{r^2}{r_H^2}}}\delta^{ta}, \quad (e_i)^a = r\delta^{ia}, \quad (e_r)^a = r\sqrt{1 - \frac{r^2}{r_H^2}}\delta^{ra},$$

and

$$(\omega_{tr})_a = -(\omega_{rt})_a = \left(\frac{2}{r^2} - \frac{1}{r_H^2}\right)\delta_{ta},$$
$$(\omega_{ir})_a = -(\omega_{ri})_a = -\sqrt{\frac{1}{r^2} - \frac{1}{r_H^2}}\delta_{ia}.$$

Therefore the equation of motion for a massless fermion in this background, setting

$$\Psi = e^{i\omega t + ik\cdot x}\psi(r) = e^{i\omega t + ik\cdot x}r^2\phi(r), \quad (3.3)$$

reduces to

$$\left[\sqrt{1 - \frac{r^2}{r_H^2}}\partial_r - \frac{r}{r_H^2\sqrt{1 - \frac{r^2}{r_H^2}}}\left(\frac{1}{2} - i\omega r_H^2\Gamma^r\Gamma^t\right) + i\Gamma^r k\cdot\Gamma\right]\phi(r) = 0. \quad (3.4)$$

To solve this equation, it is useful to act by $\not{D}$ on the first-order equation to get a second-order differential equation. In fact, defining a new variable $x = \frac{r}{r_H}$, one finds

$$(1-x^2)\frac{d^2\phi_\pm}{dx^2} - 2x\frac{d\phi_\pm}{dx} + \left[\nu(\nu+1) - \frac{\mu_\pm^2}{1-x^2}\right]\phi_\pm = 0, \quad (3.5)$$

where $\phi_\pm = \frac{1}{2}(1 \pm \Gamma^r\Gamma^t)\phi$, and

$$\nu = -\frac{1}{2} + ir_H\sqrt{k^2 + \omega^2 r_H^2}, \quad \mu_\pm = \pm\frac{1}{2} - i\omega r_H^2. \quad (3.6)$$

The resultant differential equation has a well-known form whose solutions are the associated Legendre functions P and Q. Therefore the most general solution of the above equation is

$$\phi_\pm(r) = c_1^\pm P(\nu, \mu_\pm, x) + c_2^\pm Q(\nu, \mu_\pm, x), \quad (3.7)$$

where $c_{1,2}^\pm$ are constant spinors.

Of course, so far we have solved the second-order differential equation, but its solution is not necessarily a solution of the equation of motion, which is a first-order differential equation. In other words, the constant spinors $c_{1,2}^\pm$ are not independent. In fact, in order to find a solution of the equation of motion, one needs to plug the solution (3.7) into the equation of motion, which in general leads to certain relations between the constant spinors $c_{1,2}^\pm$. Indeed, using the recursion relations between the associated Legendre functions [38,39],

$$\begin{aligned}(1-x^2)\frac{dP(\nu,\mu_\pm,x)}{dx} &= (\nu - \mu_\pm + 1)(\nu + \mu_\pm) \\ &\quad \times\sqrt{1-x^2}P(\nu,\mu_\pm - 1,x) \\ &\quad + \mu_\pm xP(\nu,\mu_\pm,x) \\ &= -\sqrt{1-x^2}P(\nu,\mu_\pm + 1,x) \\ &\quad - \mu_\pm xP(\nu,\mu_\pm,x),\end{aligned} \quad (3.8)$$

one finds

$$c_{1,2}^- = ir_H\Gamma^r(k\cdot\Gamma)c_{1,2}^+. \quad (3.9)$$

In order to impose the ingoing boundary condition on the wave function at the horizon, we note that at near horizon the oscillating part of the solution has the following form

$$\left(1 - \frac{r}{r_H}\right)^{\pm i\omega r_H^2}, \quad (3.10)$$

which, in our notation, the ingoing and outgoing waves correspond to plus and minus signs, respectively. On the other hand, using the asymptotic behaviors of the associated Legendre functions near $x = 1$, one observes that the function Q has both the ingoing and the outgoing components, though the function P has only the ingoing part. Therefore, in order to have a physical solution one needs to set $c_2^\pm = 0$. As a result the solution of the equation of motion of the massless fermions on the Lifshitz black hole satisfying the ingoing boundary condition is

$$\Psi^{\pm} = c^{\pm} r^2 e^{i\omega t + ik\cdot x} P\left(\nu, \mu_{\pm}, \frac{r}{r_H}\right), \tag{3.11}$$

with $c^{\pm}$ being constant two-component spinors satisfying

$$c^{-} = i r_H \Gamma^r (k\cdot\Gamma) c^{+}. \tag{3.12}$$

B. Retarded Green's function

In this subsection, using the solution we just found, we will compute the retarded Green's function of a fermionic operator in the dual nonrelativistic theory at finite temperature. As we mentioned in the previous section, in order to compute the corresponding retarded Green's function, one needs to properly identify the source and response of the dual operator, which in turn depends on the boundary condition. In this section, we will follow our notation in the previous section and will consider the same boundary action as that given by the Eq. (2.26). In this notation, the source and the response of the dual operator are given by (χ_1, ζ_2) and (χ_2, ζ_1), respectively.

In order to read the proper source and response, one needs to find the asymptotic behavior of the solution as we approach the boundary. In fact, by making use of the asymptotic behaviors of the associated Legendre functions (see for example [38]) one gets

$$\begin{aligned}\Psi_{\pm} &\sim \frac{2^{\mu_{\pm}}\sqrt{\pi}}{\Gamma(\frac{1-\nu-\mu_{\pm}}{2})\Gamma(1+\frac{\nu-\mu_{\pm}}{2})} r^2 c^{\pm} e^{i\omega t+ik\cdot x} \\ &\equiv A_{\pm} r^2 c^{\pm} e^{i\omega t + ik\cdot x}.\end{aligned} \tag{3.13}$$

In other words, one may write

$$\Psi \sim r^2 \begin{pmatrix} A_- c_{\uparrow}^- \\ A_+ c_{\downarrow}^+ \\ A_+ c_{\uparrow}^+ \\ A_- c_{\downarrow}^- \end{pmatrix} e^{i\omega t + ik\cdot x}. \tag{3.14}$$

Note also that in our notation, Eq. (3.12) reads

$$\begin{pmatrix} c_{\uparrow}^- \\ 0 \\ 0 \\ c_{\downarrow}^- \end{pmatrix} = -r_H \begin{pmatrix} k_1 c_{\downarrow}^+ + k_2 c_{\uparrow}^+ \\ 0 \\ 0 \\ k_2 c_{\downarrow}^+ - k_1 c_{\uparrow}^+ \end{pmatrix}. \tag{3.15}$$

Altogether with this information the retarded Green's function of the dual fermionic operator in the finite-temperature nonrelativistic theory is

$$G_R(k) = -\begin{pmatrix} \frac{A_+^2 + r_H^2 k^2 A_-^2 + 2 r_H k_2 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} & \frac{2 r_H k_1 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} \\ \frac{2 r_H k_1 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} & \frac{A_+^2 + r_H^2 k^2 A_-^2 - 2 r_H k_2 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} \end{pmatrix}. \tag{3.16}$$

It follows from this expression that $G_{11}(\omega, k_2) = G_{22}(\omega, -k_2)$ and $\det(G_R) = -1$. The spectral function is also given by

$$\mathcal{A}(k) = -\frac{1}{\pi}\,\mathrm{Im}(\mathrm{Tr} G_R) = \frac{2}{\pi}\,\mathrm{Im}\left(\frac{r_H^2 k^2 A_-^2 + A_+^2}{r_H^2 k^2 A_-^2 - A_+^2}\right). \tag{3.17}$$

To explore different features of the retarded Green's function, it is useful to use the rotational symmetry to set $k_1 = 0$. In this case, the retarded Green's function reads

$$G_R(k_2) = -\begin{pmatrix} \frac{r_H k_2 A_- + A_+}{r_H k_2 A_- - A_+} & 0 \\ 0 & \frac{r_H k_2 A_- - A_+}{A_+ + r_H k_2 A_-} \end{pmatrix}. \tag{3.18}$$

Moreover for the spectral function one also finds

$$\mathcal{A}(k_2) = \frac{2}{\pi}\,\mathrm{Im}\left(\frac{\frac{r_H^2 k_2^2}{4}\Gamma^2(\frac{1}{2}+X^+)\Gamma^2(\frac{1}{2}+X^-) + \Gamma^2(1+X^+)\Gamma^2(1+X^-)}{\frac{r_H^2 k_2^2}{4}\Gamma^2(\frac{1}{2}+X^+)\Gamma^2(\frac{1}{2}+X^-) - \Gamma^2(1+X^+)\Gamma^2(1+X^-)}\right), \tag{3.19}$$

with $X^{\pm} = \frac{i}{2}(\omega r_H^2 \pm r_H\sqrt{k_2^2 + \omega^2 r_H^2})$.

It is instructive to study the behavior of the spectral function in the small-temperature limit. Physically small temperature means that we should look for the energies much higher than the temperature, i.e., $\frac{T}{\omega} \ll 1$. Practically, one may expand the above expression for $\omega r_H^2 \gg 1$. Indeed by making use of the asymptotic behaviors of the Gamma function, up to order of $\mathcal{O}(\frac{T^2}{\omega^2})$, one arrives at

$$\mathcal{A}(k_2) = \frac{2}{\pi}\,\mathrm{Im}\left[\frac{\Gamma^2(\eta+\frac{1}{2}) + \Gamma(\eta)\Gamma(\eta+1)}{\Gamma^2(\eta+\frac{1}{2}) - \Gamma(\eta)\Gamma(\eta+1)}\left(1 + \frac{i\pi T}{4\omega}\,\frac{(4\eta-1)\Gamma^2(\eta+\frac{1}{2})\Gamma(\eta)\Gamma(1+\eta))}{\Gamma^4(\eta+\frac{1}{2}) - \Gamma^2(\eta)\Gamma^2(\eta+1)}\right)\right], \tag{3.20}$$

with $\eta = \frac{k_2^2}{4i\omega}$. As we see, at leading order it is exactly the same expression we have found for the zero-temperature case [see the Eq. (2.36)].

The spectral function as a function of ω and k_2 is depicted in Fig. 3. The plot is drawn for $r_H = \frac{2}{\sqrt{2\pi}}$ where the temperature is $T = 1/4$. To further explore the physical content of the model, it is also illustrative to examine the behavior

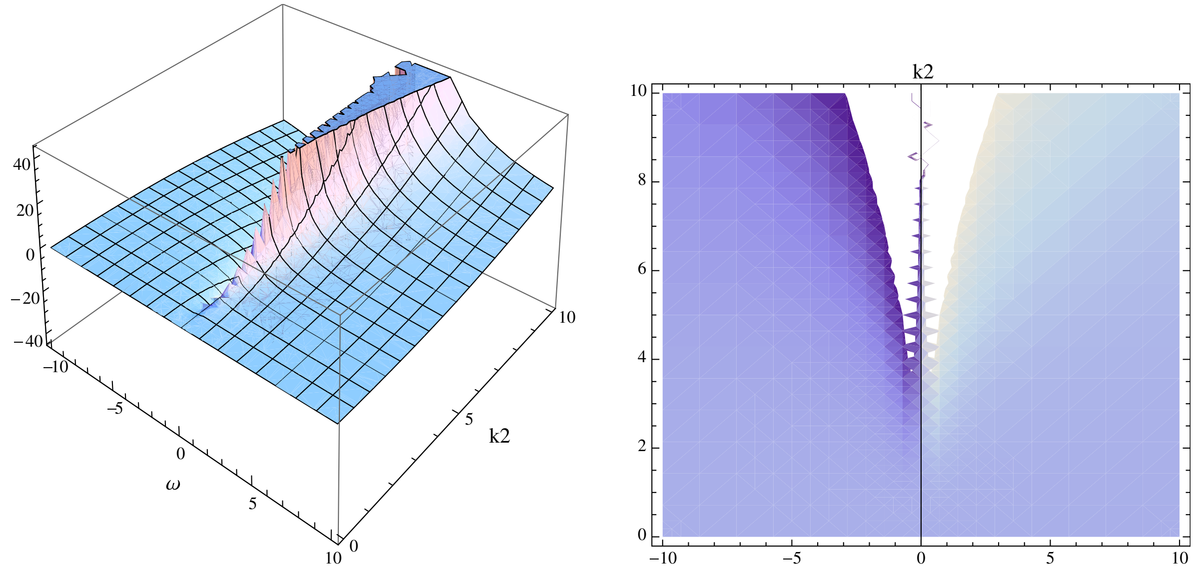

FIG. 3 (color online). Three-dimensional and density plots of the spectral function as a function of ω and k_2 at $r_H = \frac{2}{\sqrt{2\pi}}$. It is positive for sign$(\omega) > 0$. Note there is no pole at $\omega = 0$ for finite k_2.

of the eigenvalues of the retarded Green's function as functions of ω and k_2. In fact, the real and imaginary parts of the first eigenvalue of the retarded Green's function have been plotted in Fig. 4. As it is shown, the imaginary part of the first eigenvalue has a pole at $\omega = 0$. This can also be seen from the delta function behavior of its real part. On the other hand, it can be seen that the second eigenvalue has no pole at $\omega = 0$.

In comparison with the zero-temperature case we see that the spectral function has qualitatively the same shape, though there is a small deviation in the low momentum modes. Nevertheless, for high momenta it remains unchanged. Therefore the system has a flat band for high momenta.

It is worth to note that at low momenta and for low energies there are several nontrivial peaks. In fact, the presence of these peaks at low energies suggest that heating up the system has excited zero-energy fermionic modes at low momenta.

IV. NON-ZERO CHEMICAL POTENTIAL

It is important to mention that when one studies fermionic features of a system in condensed matter physics, usually one looks for a possibility of having a Fermi surface.

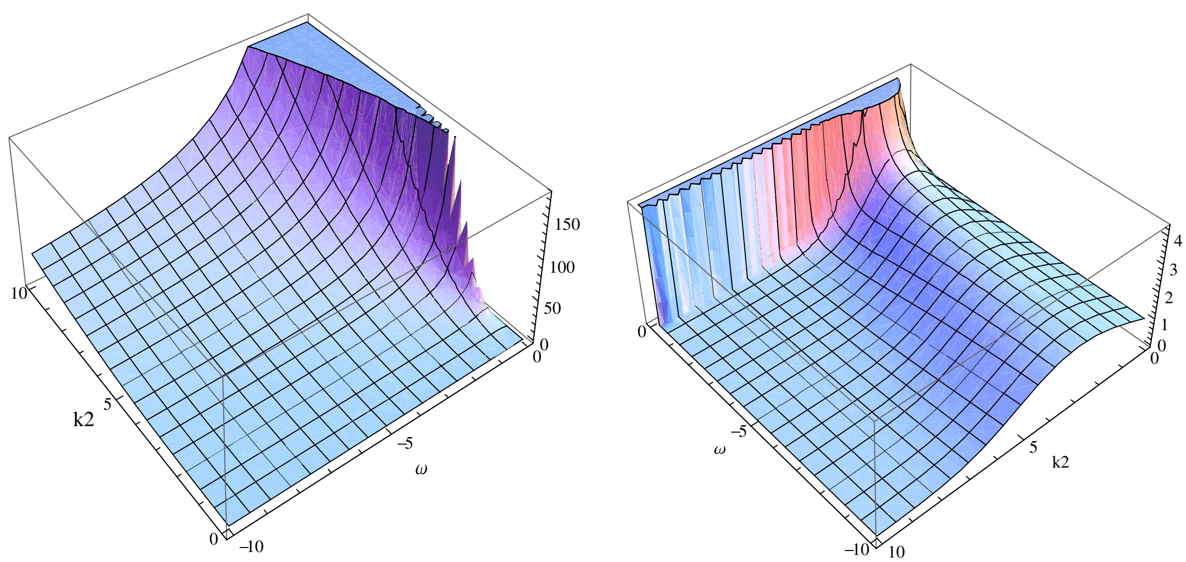

FIG. 4 (color online). The imaginary (left) and real (right) parts of the flat band eigenvalue. As we see, there is a pole at $\omega = 0$ in the imaginary part of the eigenvalue, which is also evident from the delta function behavior of its real part.

Actually, in order to rigorously address this question one needs to consider a charged fermion propagating on a charged Lifshitz black hole[13] where we could have a nonzero chemical potential. In fact, as we have already mentioned in Sec. II, the Lifshitz geometry is not a solution of pure Einstein gravity. In order to find the Lifshitz solution, one may couple gravity to a massive background gauge field. In this case, the background supports a nonzero gauge field.

We note, however, that this gauge field diverges as we approach the boundary and thus cannot play the role of chemical potential. In fact, in order to have a chemical potential, another gauge field is needed [41]. Indeed, the second gauge field has the proper near-boundary behavior to define chemical potential. More precisely, one may start with the following action [41]

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big[R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\lambda_1\phi}(F^{(1)})^2 - \frac{1}{4}e^{\lambda_2\phi}(F^{(2)})^2\Big]. \quad (4.1)$$

This model admits a charged Lifshitz black hole solution with critical exponent z for the particular values of Λ, λ_1 and λ_2 as follows

$$\Lambda = -\frac{(z+1)(z+2)}{2L^2}, \quad \lambda_1 = -\frac{2}{\sqrt{z-1}}, \quad \lambda_2 = \sqrt{z-1}. \quad (4.2)$$

The corresponding black hole solution is [41][14]

$$\begin{aligned} ds^2 &= -r^{2z} f dr^2 + \frac{dr^2}{r^2 f} + r^2 d\vec{x}^2,\\ f &= 1 - \frac{1 + r_0^{2(z+1)}}{r^{z+2}} + \frac{r_0^{2(z+1)}}{r^{2(z+1)}},\\ e^{\sqrt{z-1}\phi} &= \frac{\kappa^2}{4zr_0^{2(z+1)}} r^{2(z-1)},\\ A_t^{(1)} &= -\mu^{(1)}(1 - r^{1+z}),\\ A_t^{(2)} &= \mu^{(2)}\Big(1 - \frac{1}{r^{1+z}}\Big), \end{aligned} \quad (4.3)$$

where r_0 and κ are the only remaining free parameters which determine mass and charges of the solution, and

$$\begin{aligned} \mu^{(1)} &= \sqrt{2(z-1)(z+2)}\Big(\frac{\kappa^2}{4zr_0^{2(z+1)}}\Big)^{1/z-1},\\ \mu^{(2)} &= \frac{4zr_0^{2(z+1)}}{\kappa}. \end{aligned} \quad (4.4)$$

It is, indeed, an asymptotically Lifshitz charged black hole whose Hawking temperature is

$$T = \frac{z+2}{4\pi}\Big(1 - \frac{z}{z+2} r_0^{2(z+1)}\Big). \quad (4.5)$$

At low energy, using the general idea of AdS/CFT correspondence, the physics is governed by near-horizon modes. In this case, for example, one should study fermions on the near-horizon background. At zero temperature where $r_0^{2(z+1)} = \frac{z+2}{2}$ setting

$$r - 1 = \frac{\epsilon}{(2+3z+z^2)\xi}, \qquad t = \frac{1}{\epsilon}\tau, \quad (4.6)$$

the near-horizon background can be obtained in the limit of $\epsilon \to 0$, where one finds

$$\begin{aligned} ds^2 &= \frac{-d\tau^2 + d\xi^2}{(2+3z+z^2)\xi^2} + dx_2^2, \qquad e^{\phi} = \frac{\kappa^2}{4(z+2)},\\ A_t^{(i)} &= \frac{(z+1)\mu^{(i)}}{(2+3z+z^2)\xi}. \end{aligned} \quad (4.7)$$

As we observe, the metric is in the form of $\mathrm{AdS}_2 \times R^2$. Therefore, the low-energy physics is described by an emergent IR conformal field theory (CFT). Actually, the situation is very similar to the relativistic case [10]. As a result, we would expect that the model exhibits a Fermi surface whose physics is governed by an IR fixed point.

A. Charged fermions

Let us first consider a four-dimensional charged Dirac fermion on the Lifshitz background whose action is

$$S_{\mathrm{bulk}} = \int d^4x\sqrt{-g}\, i\bar{\Psi}\Big[\frac{1}{2}(\Gamma^a\vec{D}_a - \overleftarrow{D}_a\Gamma^a) - m\Big]\Psi. \quad (4.8)$$

Here $\not{D} = (e_\mu)^a\Gamma^\mu[\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a\Gamma^{\rho\sigma} - iqA_a^{(2)}]$, with $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$.

As we discussed in the previous sections, one needs to impose a proper boundary condition to get a well-defined variational principle. With a suitable boundary condition the equation of motion is

$$\Big((e_\mu)^a\Gamma^\mu\Big[\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a\Gamma^{\rho\sigma} - iqA_a^{(2)}\Big] - m\Big)\Psi = 0. \quad (4.9)$$

The above equation of motion by the choice of $\Psi = (-h)^{-1/4} e^{-i\omega t + ik.x}\psi(r)$ reduces to

[13], A few days after submitting our paper, another paper [40] appeared on arXiv where charged fermions on the Lifshitz geometry were studied. Of course, their background is different from what we are considering in this paper for charged fermions.

[14]Note that, in comparison with the solution in , we have shifted the gauge field by constants to make sure that $g^{\mu\nu}A_\mu^{(i)}A_\nu^{(i)}$ remains finite. Moreover, by a proper rescaling we have set $L = 1$ and also with a proper choice of the parameters the radius of horizon has also been set to 1.

$$\left[rf^{1/2}\Gamma^r\partial_r - \frac{i}{r^z f^{1/2}}\left(\omega + q\mu^{(2)}\left(1-\frac{1}{r^{z+1}}\right)\right)\Gamma^t + \frac{i}{r}\Gamma\cdot k - m\right]\psi(r) = 0. \tag{4.10}$$

As we already mentioned, the low energy is governed by an emergent IR fixed point. To examine the low-energy limit of the fermions, one should consider the limit of $\omega \ll \mu^{(2)}$. At zero temperature, using the scaling (4.6) and in the limit of $\epsilon \to 0$ keeping ω/ϵ fixed, the above equation reads

$$\left[-\Gamma^\xi \xi\partial_\xi - i\xi\Gamma^t\left(\tilde\omega + \frac{qe}{\xi}\right) + i\Gamma\cdot k - \frac{m}{\sqrt{2+3z+z^2}}\right]\psi(\xi) = 0, \tag{4.11}$$

where

$$e = \frac{(1+z)\mu^{(2)}}{2+3z+z^2}, \qquad \tilde\omega = \frac{\omega}{\epsilon} = \text{finite}. \tag{4.12}$$

We recognize the above equation as a charged fermion probing the $AdS_2 \times R^2$ background (4.7), as expected. As a result, one may go through the construction of [10] to express the retarded Green's function of the fermion on the Lifshitz-charged black hole in terms of the retarded Green's function of AdS_2 model in small ω limit.

Here, instead of doing so, one utilizes the numerical method to solve the equation of motion numerically. To proceed, it is useful to consider the following representation for four-dimensional gamma matrices

$$\Gamma^r = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \qquad \Gamma^t = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix},$$
$$\Gamma^1 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}. \tag{4.13}$$

Because of rotational symmetry in the spatial directions, we may set $k_2 = 0$. Then using the notation

$$\psi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{4.14}$$

the equation of motion (4.10) reduces to the following decoupled equations

$$\left[rf^{1/2}\partial_r - \frac{1}{r^z f^{1/2}}\left(\omega + q\mu^{(2)}\left(1-\frac{1}{r^{z+1}}\right)\right)i\sigma^2 + m\sigma^3 - (-1)^\alpha \frac{k_1}{r}\sigma^1\right]\Phi_\alpha = 0 \tag{4.15}$$

for $\alpha = 1, 2$. It is easy to see that

$$\Phi_\alpha \sim a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } r \to \infty. \tag{4.16}$$

To find the retarded Green's function, following [10], it is useful to defined $\zeta_1 = \psi_1/\psi_2$ and $\zeta_2 = \psi_3/\psi_4$ where ψ_i's are defined via $\Phi_1 = (\psi_1, \psi_2), \Phi_2 = (\psi_3, \psi_4)$. These parameters satisfy the following equations

$$rf^{1/2}\partial_r\zeta_1 + 2m\zeta_1 - \left(\frac{\Omega}{r^z f^{1/2}} + \frac{k_1}{r}\right)\zeta_1^2 = \frac{\Omega}{r^z f^{1/2}} - \frac{k_1}{r},$$
$$rf^{1/2}\partial_r\zeta_2 + 2m\zeta_2 - \left(\frac{\Omega}{r^z f^{1/2}} - \frac{k_1}{r}\right)\zeta_2^2 = \frac{\Omega}{r^z f^{1/2}} + \frac{k_1}{r}, \tag{4.17}$$

where

$$\Omega = \omega + q\mu^{(2)}\left(1 - \frac{1}{r^{z+1}}\right). \tag{4.18}$$

Using these equations, the retarded Green's function is essentially given in terms of functions $G_1(k,\omega)$ and $G_2(k,\omega)$, where[15]

$$G_\alpha(k,\omega) = \lim_{r\to\infty} r^{2m}\zeta_\alpha, \quad \text{for } \alpha = 1, 2 \tag{4.19}$$

with the ingoing boundary condition at the horizon, which in our notation is [10]

$$\zeta_\alpha|_{\text{horizon}} = i. \tag{4.20}$$

The precise expression of the retarded Green's function in terms of G_α depends on the boundary condition one imposes to get a well-defined variational principle. For example, if we impose the standard boundary condition, in our notation, the corresponding retarded Green's function is

$$G(k,\omega) = -\begin{pmatrix} G_1(k,\omega) & 0 \\ 0 & G_2(k,\omega) \end{pmatrix}. \tag{4.21}$$

Therefore the spectral function reads

$$\mathcal{A}(k,\omega) = \frac{1}{\pi}\,\mathrm{Im}(G_1(k,\omega) + G_2(k,\omega)). \tag{4.22}$$

On the other hand, for the boundary condition obtained by adding the boundary term (2.26) the corresponding retarded Green's function as a function of G_α is given by (see also [22])

$$G(k,\omega) = -\begin{pmatrix} \frac{2G_1(k,\omega)G_2(k,\omega)}{G_1(k,\omega)+G_2(k,\omega)} & \frac{G_1(k,\omega)-G_2(k,\omega)}{G_1(k,\omega)+G_2(k,\omega)} \\ \frac{G_1(k,\omega)-G_2(k,\omega)}{G_1(k,\omega)+G_2(k,\omega)} & \frac{-2}{G_1(k,\omega)+G_2(k,\omega)} \end{pmatrix}. \tag{4.23}$$

Thus the corresponding spectral function reads

$$\mathcal{A}(k,\omega) = 2\,\mathrm{Im}\left(\frac{G_1(k,\omega)G_2(k,\omega) - 1}{G_1(k,\omega) + G_2(k,\omega)}\right). \tag{4.24}$$

Note that in the notation we are using in this section [see (4.13)] the boundary term (2.26) reads

[15] Here we set $k_1 = k$.

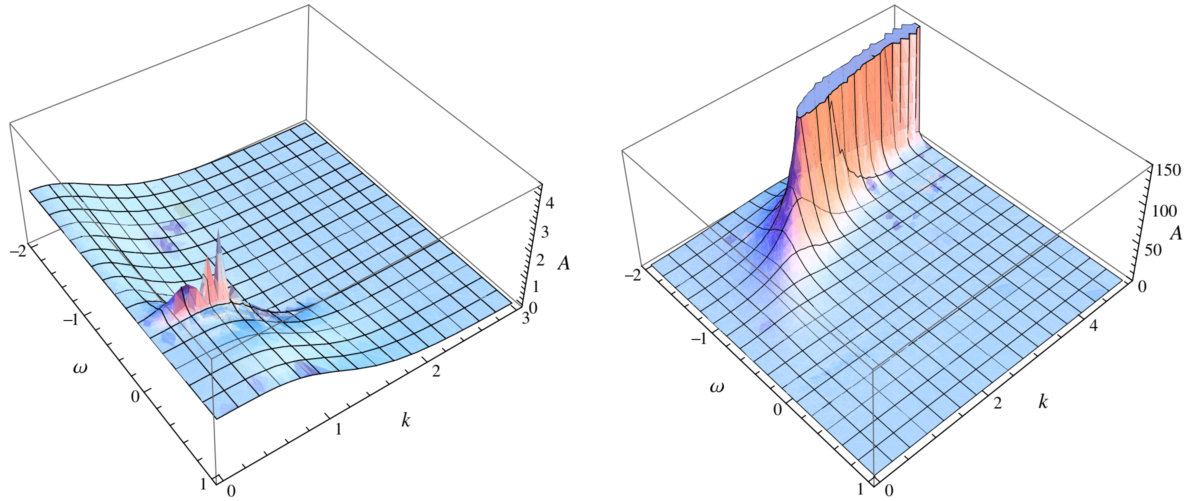

FIG. 5 (color online). The behavior of spectral functions with the standard (left) and nonstandard (right) boundary conditions for $z = 2$ and $T = 0$ as functions of ω and k. For the standard boundary condition the system exhibits a Fermi surface at $k_f = 0.8902$, while for the nonstandard case it has a flat band.

$$S_{\rm bdy} = \frac{i}{2}\int d^3x\sqrt{-h}\bar{\Psi}\Gamma^1\Gamma^2\Psi = \frac{-i}{2}\int d^3x\sqrt{-h}(\psi_1^\dagger\psi_4 + \psi_2^\dagger\psi_3 + \psi_3^\dagger\psi_2 + \psi_4^\dagger\psi_1). \tag{4.25}$$

Therefore, adding this boundary term to the action results in imposing the boundary condition on a combination of the different components of the fermions as follows

$$\delta S = i\int d^3x\sqrt{-h}(\delta\chi_2^\dagger\eta_2 - \delta\eta_1^\dagger\chi_1 - \chi_1^\dagger\delta\eta_1 + \eta_2^\dagger\delta\chi_2), \tag{4.26}$$

where

$$(\chi_1, \chi_2) = \frac{1}{\sqrt{2}}(\psi_2 + \psi_4, \psi_2 - \psi_4),$$
$$(\eta_1, \eta_2) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_3, \psi_1 - \psi_3). \tag{4.27}$$

B. Numerical results

Having found expressions for the retarded Green's function and the spectral function for the cases of standard and nonstandard boundary conditions, it is an easy task to find their behaviors as functions of k and ω. Here, to explore the physical content of the model, we have plotted the spectral function of the model for both standard and nonstandard boundary conditions. At zero temperature, where $r_0^{2(z+1)} = (z+2)/z$, we set $m = 0$, $q\mu^{(2)} = \sqrt{3}$. For $z = 2$, the spectral function is shown in Fig. 5. While in the standard case the model has a Fermi surface at $k_f = 0.8902$, in the nonstandard case it exhibits a flat band. Note that since the retarded Green's function is an even function of k [see Eqs. (4.15)] we only considered $k > 0$.

It is worthwhile to note that the model contains four free parameters which are mass m, critical exponent z, temperature T, and chemical potential $\mu^{(2)}$, which always appears in the combination $q\mu^{(2)}$, where q is the charge of the fermion. Therefore it is natural to explore the physical content of the model when we vary these parameters. Actually, changing the parameters, we find the following behaviors.

For fixed m, T, and $q\mu^{(2)}$ as we increase the critical exponent z for the standard boundary condition, the sharp peak representing the Fermi surface becomes smaller and occurs at smaller k_f and eventually, for large enough z, it distorts the Fermi surface completely. In other words the model does not have a Fermi surface. On the other hand, for the nonstandard boundary condition, the model still exhibits a flat band, though there is a depletion in the low momentum modes (see, for example, Fig. 6 for $z = 3$).

For fixed z, the dependence of the spectral function on the parameters m, T and $q\mu^{(2)}$ for both standard and nonstandard cases is the same as that for $z = 1$, where we have AdS black hole solutions (see for example [10,19] for standard and nonstandard boundary conditions, respectively).

V. DISCUSSIONS

In this paper, following the general idea of AdS/CFT correspondence, we have studied retarded Green's function of a fermionic operator in a three-dimensional nonrelativistic field theory by making use of a massless fermion on the asymptotically Lifshitz geometry. In this paper, we have mainly considered the asymptotically Lifshitz geometry with critical exponent of $z = 2$. We have considered both neutral and charged fermions.

Taking into account that the gravity on asymptotically Lifshitz backgrounds may provide holographic descriptions

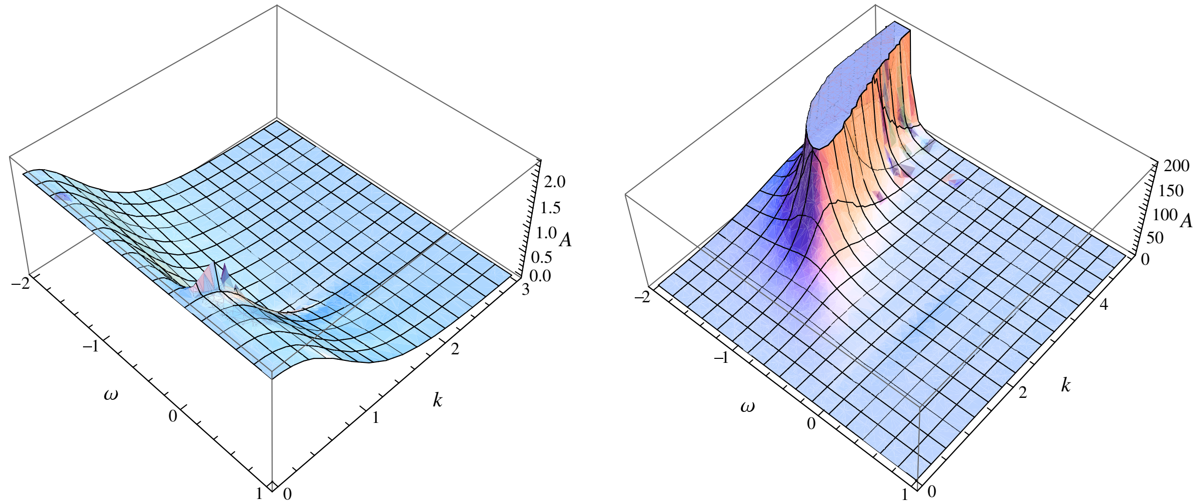

FIG. 6 (color online). The behavior of spectral functions with the standard (left) and nonstandard (right) boundary conditions for $z = 3$ and $T = 0$ as functions of ω and k. For the standard boundary condition the system exhibits a Fermi surface, while for the nonstandard case it has a flat band.

for strange metals [5], our studies might be useful to explore certain features of strange metals.

For neutral fermions at zero temperature, where the bulk fermions propagate on the Lifshitz background, the resultant retarded Green's function of the dual fermionic operator exhibits interesting behaviors. In particular we observe that the spectral function has a pole at $\omega = 0$ for all values of spatial momenta. The appearance of the pole may also be seen from the behavior of the real and imaginary parts of the eigenvalues of the corresponding retarded Green's function.

Having seen a pole at $\omega = 0$ with all values of spatial momenta shows that there are localized nonpropagating excitations, which in turn indicates an infinite flat band. Actually, the situation is similar to that in pure AdS geometry with the Lorentz-breaking boundary condition [19].

We have also considered three-dimensional nonrelativistic theories at finite temperature. To study the three-dimensional model at finite temperature, we have utilized the asymptotically Lifshitz black hole obtained in [37]. We have shown that by heating up the dual theory, although the nonzero temperature can excite low-momenta zero-energy modes, at high momenta there is still an infinite flat band!

An interesting feature we have seen in our model is that the spectral function is positive for $\mathrm{sign}(\omega) > 0$. In fact, unitarity requires that the spectral function be always positive. Since, in our case, the retarded Green's function changes its sign and indeed is negative for $\mathrm{sign}(\omega) < 0$, it is tempting to propose that retarded Green's function we have found for the nonrelativistic model contains information for both particles and antiparticles!

We have also considered charged fermions probing a charged Lifshitz black hole. While for the standard boundary condition the model exhibits a Fermi surface, for nonstandard boundary condition the model still has a flat band. We have also observed that, as one increases the critical exponent z for the standard boundary condition, the sharp peak representing the Fermi surface becomes smaller and occurs at smaller k_f and eventually, for large enough z, it distorts the Fermi surface completely. In other words, the model does not have a Fermi surface. On the other hand for the nonstandard boundary condition the model still exhibits a flat band, though there is a depletion in the low-momentum modes.

It is worthwhile to note that, in order to make the variational principle well-defined, one could also use another boundary action as follows

$$\begin{aligned} S_{\mathrm{bdy}} &= \frac{1}{2}\int d^3x\sqrt{-h}(\bar{\Psi}_+\Psi_- - \bar{\Psi}_-\Psi_+) \\ &= \frac{i}{2}\int d^3x\sqrt{-h}(\Psi_2^\dagger\Psi_1 + \Psi_3^\dagger\Psi_4 - \Psi_1^\dagger\Psi_2 - \Psi_4^\dagger\Psi_3). \end{aligned} \tag{5.1}$$

With this boundary term, the variation of the whole action leads to the following boundary terms

$$i\int\sqrt{-h}[-\delta\chi_2^\dagger\chi_1 + \delta\zeta_2^\dagger\zeta_1 + \chi_1^\dagger\delta\chi_2 - \zeta_1^\dagger\delta\zeta_2], \tag{5.2}$$

where in this case the newly defined fields are given by

$$\begin{aligned} (\chi_1, \chi_2) &= \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_2, \Psi_1 - \Psi_2) \\ (\zeta_1, \zeta_2) &= \frac{1}{\sqrt{2}}(\Psi_3 + \Psi_4, \Psi_3 - \Psi_4). \end{aligned} \tag{5.3}$$

If we follow the steps we went through in the previous sections, one may compute the corresponding retarded Green's function in this case. Doing so, one finds that the resultant retarded Green's functions have the same form as those in the previous sections, except for the fact that the roles of k_1 and k_2 have been changed. Now, in this case we could use the rotational symmetry to set $k_2 = 0$. Of course, the physics remains unchanged after all.

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After submitting our paper we were informed by U. Gursoy that fermionic correlation functions on Lifshitz background has recently been studied in [42]. We note, however, that the authors of this paper have used different UV boundary condition than ours.

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