

Higher-order corrections in a four-fermion Lifshitz modelJ. Alexandre,^{*} J. Brister,[†] and N. Houston[‡]*King's College London, Department of Physics, London WC2R 2LS, United Kingdom*

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We study a flavor-violating four-fermion interaction in the Lifshitz context, in $3 + 1$ dimensions and with a critical exponent $z = 3$. This model is renormalizable, and features dynamical mass generation, as well as asymptotic freedom. At one-loop, it is only logarithmically divergent, but the superficial degree of divergence of the two-point functions is 3. We calculate the two-loop corrections to the propagators, and show that, at this order, the Lorentz-violating corrections to the IR dispersion relation are quadratic in the cut off. Furthermore, these corrections are too significant to represent a physical effect. As a consequence, the predictive power of the model in terms of Lorentz-violating effects in the propagation of particles is limited.

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I. INTRODUCTION

Lifshitz-type theories, where time and space have different mass dimensions and therefore violate Lorentz invariance, have attracted attention in recent years, motivated essentially by the possibility of defining new renormalizable interactions. This is because of an improvement in the convergence of loop integrals, due to higher order space derivatives, which is achieved without the introduction of ghost degrees of freedom, since the order of time derivatives remains minimal. A review of Lifshitz theories can be found in [1] for quantum field theories in particle physics and in [2] for the Horava-Lifshitz alternative approach to gravity. An example of a new renormalizable interaction in the Lifshitz context is the Liouville interaction in $3 + 1$ dimensions [3], where the exponential potential for the scalar field is a relevant interaction because the field is dimensionless if the critical exponent is $z = 3$.

We consider a Lifshitz-type four-fermion interaction model, which, in $d = 3$ space dimensions and for an anisotropic scaling $z = 3$, is renormalizable [4]. Such models have been studied in [5], where two fundamental properties were shown: dynamical mass generation and asymptotic freedom. One motivation for these theories, besides renormalizability, is the apparent improvement of quantum corrections, since the one-loop graphs are only logarithmically divergent, instead of quadratically in the Lorentz case. But the overall superficial degree of divergence of the graphs of the theory is actually $\omega = 6 - 3E/2$, where E is the number of external lines. If one considers the propagator ($E = 2$), the corresponding corrections have a superficial degree of divergence equal to 3. The coefficient of the cubic divergence may cancel for some graphs, but we calculate here a two-loop graph which shows that the divergence in the model is at least quadratic. Therefore, although renormalizable, this Lifshitz model still contains “large” divergences.

Our model features two massless fermion flavors, coupled with four-fermion interactions which do not respect flavor symmetry. After showing the occurrence of dynamical flavor oscillations in this model, we calculate the modified dispersion relations for these two fermions, arising from quantum fluctuations. Classically, all fermions have the same dispersion relations, with higher order powers of the space momentum \vec{p} , rescaled by a large mass M , which represents the crossover scale between the Lifshitz and Lorentz regimes. These dispersion relations coincide with the expected Lorentz-invariant one in the infrared (IR) regime $|\vec{p}| \ll M$. Taking into account quantum corrections, though, modifies this IR limit: it is known in Lifshitz-type studies that different species of particles see different effective light cones [6]. Since our model breaks flavor symmetry, the dressed IR dispersion relations are different from the Lorentz-invariant one, and we show that the corresponding corrections are quadratically divergent. Furthermore, as we will see, these corrections are too significant to represent any physical effect. As a consequence, a proper treatment of the model would consist in defining counterterms to absorb these divergences, and no prediction can be made as far as Lorentz-violating propagation is concerned (a logarithmic divergence could lead to a “realistic” energy dependent effective maximum speed).

The next section derives the dynamical masses for the model, including the mass mixing terms necessary for flavor oscillations. From our study of dynamically induced flavor oscillations, and taking into account experimental data on neutrino oscillations, we derive values for the coupling constants of our model, which, as expected, are perturbative. Although neutrinos are not Dirac fermions, the corresponding experimental constraints give a good order of magnitude for the parameters in our model. Section III shows the asymptotic freedom of the interaction, based on a one-loop calculation. The four-point function has a vanishing superficial degree of divergence, such that higher order corrections cannot change the sign of these beta functions. The effective IR dispersion relations, dressed by quantum fluctuations, are derived in Sec. IV. For

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this, we need to go to two loops, since the one-loop correction to the fermion propagators is momentum-independent. Detailed calculations are given in the Appendix, where we perform part of the integration analytically and then integrate the rest numerically.

II. FLAVOR OSCILLATIONS

We describe here, in the Lifshitz context, how flavor oscillations can arise dynamically from flavor-mixing interactions between two massless bare fermions, as suggested in [7]. From our expressions for the dynamical masses, together with experimental data, we find phenomenologically realistic values for the coupling constants of our model.

Oscillations of massless neutrinos are studied in [8], where neutrinos are considered open systems, interacting with an environment. Such oscillations have also been studied in [9], in the framework of Lorentz-violating models, involving nonvanishing vacuum expectation values for vectors and tensors. Whilst these studies have been questioned by phenomenological constraints [10], our present model, based on anisotropic space time and higher order space derivatives, is not excluded.

Flavor oscillations were also related to superluminality in [11], where it is shown that, if superluminality is due to a tachyonic mode, the latter can be stabilized by flavor mixing. Finally, in [12], superluminal effects are related to the extension of a single neutrino wave function, where the oscillation mechanism plays a role in the uncertainty of the neutrino position.

A. Flavor symmetry violating 4-fermion interactions

We work in the $z = 3$ Lifshitz context, in $d = 3$ space dimensions. We consider two flavors of massless Dirac fermions ψ_1, ψ_2 , and the free action

$$S_{\text{free}} = \int dt d\vec{x} (\bar{\psi}_a i \gamma^0 \dot{\psi}_a - \bar{\psi}_a (M^2 - \Delta) (i\vec{\partial} \cdot \vec{\gamma}) \psi_a) \quad (1)$$

$a = 1, 2,$

where $[M] = 1$ and $[\psi_a] = 3/2$, and a dot over a field represents a time derivative. For the dispersion relations to be consistent in the IR [see Eq. (4) below], one can consider M typically of the order of a grand unified theory scale (GUT), although we will show that our results only slightly depend on the actual value of M . We introduce the following renormalizable, flavor-violating and attractive 4-fermion interactions

$$S_{\text{int}} = \int dt d\vec{x} (g_1 \bar{\psi}_1 \psi_1 + g_2 \bar{\psi}_2 \psi_2 + h (\bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1))^2, \quad (2)$$

where the coupling constants g_1, g_2, h are dimensionless. As shown in [5], this kind of model exhibits dynamical

mass generation, which can be seen only with a nonperturbative approach, as will be shown in the next section. Taking into account the dynamical masses, but ignoring quantum corrections to the kinetic terms, the dispersion relations are of the form

$$\omega^2 = m_{\text{dyn}}^6 + (M^2 + p^2)^2 p^2, \quad a = 1, 2, \quad (3)$$

which, after the rescaling $\omega = M^2 \tilde{\omega}$, leads to

$$\tilde{\omega}^2 = \tilde{m}_{\text{dyn}}^2 + p^2 + \frac{2p^4}{M^2} + \frac{p^6}{M^4}, \quad (4)$$

where $\tilde{m}_{\text{dyn}} = m_{\text{dyn}}^3/M^2$. One can see then that Lorentz-like kinematics are recovered in the IR regime $p^2 \ll M^2$, as expected in the framework of Lifshitz models. After the rescaling $t = \tilde{t}/M^2$, the action reads

$$S = \int d\tilde{t} d\vec{x} \left(\bar{\psi}_a i \not{\partial} \psi_a + \bar{\psi}_a \frac{\Delta}{M^2} (i\vec{\partial} \cdot \vec{\gamma}) \psi_a + \left[\frac{g_1}{M} \bar{\psi}_1 \psi_1 + \frac{g_2}{M} \bar{\psi}_2 \psi_2 + \frac{h}{M} (\bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1) \right]^2 \right), \quad (5)$$

where we can see that the four fermion couplings $(g_a/M)^2$, $g_a h/M^2$, and $(h/M)^2$ are very small compared to the Fermi coupling $\simeq 10^{-5} \text{ GeV}^{-2}$, if M is of the order of a GUT scale, or even several orders of magnitude smaller, and g_a, h are perturbative. Finally, note that, for Large Hadron Collider energies up to few TeVs, the classical Lifshitz corrections p^4/M^2 and p^6/M^4 in the dispersion relation (4) are not detectable, if M is of the order of a GUT scale. For this reason, if one wishes to describe measurable non-relativistic effects in the Lifshitz context, these should be sought in quantum corrections to the IR dispersion relation.

B. Superficial degree of divergence

It is interesting to note that, although this Lifshitz model has only logarithmic divergences at one-loop, quantum corrections actually do not “behave better” than those in the Lorentz-invariant ϕ^4 theory, since the superficial degree of divergence of the propagator is 3.

To show this, we calculate via the usual approach the degree of divergence ω of a graph with E external lines. Each loop gives an integration measure $dp_0 d^3 p$, which has mass dimension 6, and each propagator has mass dimension-3. For a graph with I internal lines and L loops, the superficial degree of divergence is therefore $\omega = 6L - 3I$. As usual, because of momentum conservation, we also have $L = I - n + 1$, where n is the number of vertices of the graph. Finally, since we have 4-leg vertices, we also have the relation $4n = E + 2I$. Taking into account these constraints, we find $\omega = 6 - 3E/2$.

From this result, we see that the four-point function is at most logarithmically divergent, but the propagator has a superficial degree of divergence equal to 3, although the one-loop mass corrections are logarithmically divergent only, as we show in the next subsection.

C. Dynamical generation of masses

We now calculate the dynamical masses generated by the interaction (2). For this, we introduce the auxiliary scalar field ϕ to express the interaction as

$$\exp(iS_{\text{int}}) = \int \mathcal{D}[\phi] \exp(iS_\phi), \quad \text{with}$$

$$S_\phi = \int dt d\vec{x} (-\phi^2 + 2\phi(g_1 \bar{\psi}_1 \psi_1 + g_2 \bar{\psi}_2 \psi_2 + h(\bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1))), \quad (6)$$

and then calculate the effective potential for $\phi = \text{constant}$ as

$$\exp(i\mathcal{V}V_{\text{eff}}(\phi)) = \int \mathcal{D}[\psi_1, \bar{\psi}_1, \psi_2, \bar{\psi}_2] \exp(iS_{\text{free}} + iS_\phi), \quad (7)$$

where \mathcal{V} is the space time volume. This integration can be done exactly, since $S_{\text{free}} + S_\phi$ is quadratic in fermion fields, and leads to an effective potential for ϕ . From the dispersion relation (4), one can see that a nontrivial minimum ϕ_{min} for this effective potential will give the flavor mixing mass matrix

$$\mathcal{O} = \begin{pmatrix} i\gamma^0 \partial_0 - (M^2 - \Delta)(i\vec{\partial} \cdot \vec{\gamma}) + 2g_1 \phi & 2h\phi \\ 2h\phi & i\gamma^0 \partial_0 - (M^2 - \Delta)(i\vec{\partial} \cdot \vec{\gamma}) + 2g_2 \phi \end{pmatrix}. \quad (13)$$

Integration over the fermions then gives the following effective potential for ϕ (where the Euclidean metric is used for the loop momentum)

$$V_{\text{eff}}(\phi) = \phi^2 - \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{d\vec{p}}{(2\pi)^3} \ln[\omega^2 + (M^2 + p^2)^2 p^2] + 4\phi^2[\omega^2 + (M^2 + p^2)^2 p^2](g_1^2 + g_2^2 + 2h^2) + 16(g_1 g_2 - h^2)^2 \phi^4. \quad (14)$$

A derivative with respect to ϕ gives

$$\frac{dV_{\text{eff}}}{d\phi} = 2\phi - \phi \int \frac{d\omega}{2\pi} \frac{d\vec{p}}{(2\pi)^3} \frac{A\omega^2 + B}{(\omega^2 + C_+)(\omega^2 + C_-)}, \quad (15)$$

where

$$A = 4(g_1^2 + g_2^2 + 2h^2)$$

$$B = 4(M^2 + p^2)^2 p^2 (g_1^2 + g_2^2 + 2h^2) + 32\phi^2 (g_1 g_2 - h^2)^2 \quad (16)$$

$$C_\pm = (M^2 + p^2)^2 p^2 + 2\phi^2 [(g_1^2 + g_2^2 + 2h^2) \pm \sqrt{(g_1^2 - g_2^2)^2 + 4h^2 (g_1 + g_2)^2}]. \quad (17)$$

The integration over frequencies ω leads to

$$\begin{pmatrix} m_1^3 & \mu^3 \\ \mu^3 & m_2^3 \end{pmatrix} = 2\phi_{\text{min}} \begin{pmatrix} g_1 & h \\ h & g_2 \end{pmatrix}, \quad (8)$$

which leads to the rescaled masses

$$\begin{pmatrix} \tilde{m}_1 & \tilde{\mu} \\ \tilde{\mu} & \tilde{m}_2 \end{pmatrix} = 2 \frac{\phi_{\text{min}}}{M^2} \begin{pmatrix} g_1 & h \\ h & g_2 \end{pmatrix}. \quad (9)$$

As a consequence, the mass eigenstates are

$$m_\pm = \frac{\phi_{\text{min}}}{M^2} (g_1 + g_2 \pm (g_1 - g_2) \sqrt{1 + \tan^2(2\theta)}), \quad (10)$$

where the mixing angle θ is defined by

$$\tan(2\theta) \equiv \frac{2h}{g_1 - g_2}. \quad (11)$$

With the auxiliary field, the Lagrangian can then be written in the form $\bar{\Psi} \mathcal{O} \Psi$, where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (12)$$

and the operator \mathcal{O} is

$$\frac{dV_{\text{eff}}}{d\phi} = 2\phi - \phi \frac{1}{(2\pi)^2} \int_0^\Lambda p^2 dp \left(\frac{B}{C_+ \sqrt{C_-} + C_- \sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}} \right) \quad (18)$$

where Λ is the UV cutoff, assumed to be large compared to M . A nontrivial minimum $\phi_{\text{min}} \neq 0$ for this effective potential is solution of the equation

$$8\pi^2 = \int_0^\Lambda p^2 dp \left(\frac{B}{C_+ \sqrt{C_-} + C_- \sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}} \right). \quad (19)$$

The dominant contribution of these logarithmically divergent integrals comes from $p \rightarrow \Lambda$, so we can therefore approximate

$$C_- \simeq C_+ \simeq p^6 + \frac{A}{2} \phi^2 \quad \text{and} \quad B \simeq A p^6, \quad (20)$$

such that

$$\int_0^\Lambda p^2 dp \left(\frac{B}{C_+ \sqrt{C_-} + C_- \sqrt{C_+}} + \frac{A}{\sqrt{C_+} + \sqrt{C_-}} \right) \simeq \frac{A}{3} \ln \left(\frac{2\sqrt{2}\Lambda^3}{\phi\sqrt{A}} \right). \quad (21)$$

The gap equation (19) then gives

$$\phi_{\min} \simeq \frac{\Lambda^3}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2}\right), \quad (22)$$

and the rescaled masses (9) are

$$\tilde{m}_a \simeq \frac{2g_a}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \frac{\Lambda^3}{M^2} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2}\right) \quad (23)$$

$$\tilde{\mu} \simeq \frac{2h}{\sqrt{g_1^2 + g_2^2 + 2h^2}} \frac{\Lambda^3}{M^2} \exp\left(\frac{-6\pi^2}{g_1^2 + g_2^2 + 2h^2}\right). \quad (24)$$

As expected, these masses are not analytical in the coupling constants and could not have been obtained with a perturbative expansion. Similar results have been obtained in the context of magnetic catalysis [13], based on the Schwinger-Dyson approach, and also for Lorentz-violating extensions of *QED* [14]. Neither of these studies however feature anisotropic space time studied herein.

Finally, we note that the approach adopted here, based on the effective potential for the auxiliary field ϕ , is in principle valid for a large number of flavors. Indeed, this auxiliary field depends on space and time, and its fluctuations around the minimum ϕ_{\min} induce new fermion interactions. For N fermion flavors though, these fluctuations are suppressed by $1/N$, which justifies the approach. In our case, $N = 2$ is not “large,” but the corresponding order of magnitude for the dynamical masses is sufficient for a suitably accurate determination of the coupling constants g_1, g_2 , as explained in the next subsection.

D. Experimental constraints

From the expressions (10), we obtain the following difference of mass eigenstates squared

$$\Delta m^2 = \frac{4}{\cos(2\theta)} \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2 + \tan^2(2\theta)(g_1 - g_2)^2/2} \frac{\Lambda^6}{M^4} \times \exp\left(\frac{-12\pi^2}{g_1^2 + g_2^2 + \tan^2(2\theta)(g_1 - g_2)^2/2}\right). \quad (25)$$

Experimental constraints are [15]

$$\begin{aligned} \Delta m_{12}^2 &= 7.59(7.22-8.03) \times 10^{-5} \text{ (eV)}^2 \\ \sin^2\theta_{12} &= 0.318(0.29-0.36), \end{aligned} \quad (26)$$

and we plot in Fig. 1, from the expression (25), the set of points in the plane g_1, g_2 which are allowed, given the experimental constraints (26). We consider $\Lambda \simeq 10^{19}$ GeV, corresponding to the Planck mass. An important property is that the result is hardly sensitive to the value of the mass

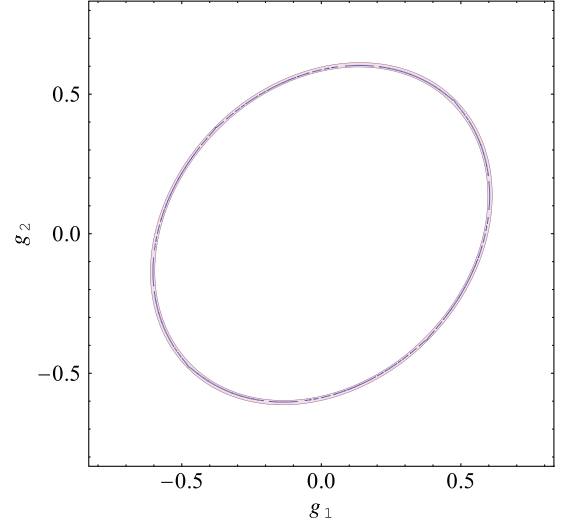


FIG. 1 (color online). Values of g_1 (x-axis) and g_2 (y-axis) allowed by experimental constraints, for $M/\Lambda = 10^{-11}$. Negative values are allowed, since the physical quantities depend on the square of the coupling constants only. Points where $g_1 = g_2$ are strictly speaking not allowed, since at these points $\Delta m^2 = 0$. However, the resulting logarithmic singularity is very localized in the parameter space, such that we can safely choose g_1 and g_2 perturbatively close to each other.

scales M : because of the exponential dependence in Eq. (25), an increase of several orders of magnitude in M leads to an increase of a few percent only for the couplings g_a , as shown in the following table. Considering the situation where $h \ll 1$, such that $g_1 \simeq g_2$, according to Eq. (11), the approximate common value for the coupling constants as a function of the ratio M/Λ is then:

On Fig. 1, the thin line represents the set of points satisfying the constraint

$$\left| \ln\left(\frac{\Delta m_{\text{experimental}}^2}{\Delta m_{\text{calculated}}^2}\right) \right| \leq 1, \quad (27)$$

and the thick line represents the set of points such that the largest mass eigenvalue is between 10^{-3} and 1 eV. We see that the coupling constants appearing in the theory are then of the order $g_a^2 \simeq 0.25$, and can be considered perturbative.

III. ASYMPTOTIC FREEDOM

We now calculate the one-loop coupling constants, for $h \ll 1$, and we show that the theory is asymptotically free. For simplicity, we set $h = 0$ but still keep $g_1 \neq g_2$. The bare interaction can be expressed as

$$g_1^2(\bar{\psi}_1 \psi_1)^2 + G \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 + g_2^2(\bar{\psi}_2 \psi_2)^2, \quad G = 2g_1 g_2, \quad (28)$$

and the dressed interaction is of the form

$$(g_1^2 + \delta g_1^2)(\bar{\psi}_1 \psi_1)^2 + (G + \delta G)\bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2 + (g_2^2 + \delta g_2^2)(\bar{\psi}_2 \psi_2)^2. \quad (29)$$

Note that no symmetry imposes any relation between δG and $\delta g_1^2, \delta g_2^2$: the interaction $\bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$ is dressed independently of the interactions $(\bar{\psi}_1 \psi_1)^2$ and $(\bar{\psi}_2 \psi_2)^2$.

A. One-loop Fermi coupling

The one-loop corrections to the coupling constants correspond to the momentum-independent part of the

$$\begin{aligned} I_{ab} &= - \int \frac{d\omega d\vec{p}}{(2\pi)^4} \left(\frac{m_a^3/(m_a^3 - m_b^3)}{\omega^2 - (M^2 + p^2)^2 p^2 - m_a^6} - \frac{m_b^3/(m_a^3 - m_b^3)}{\omega^2 - (M^2 + p^2)^2 p^2 - m_b^6} \right) \\ &= \frac{i}{4\pi^2(m_a^3 - m_b^3)} \int_0^\Lambda p^2 dp \left(\frac{m_a^3}{\sqrt{(M^2 + p^2)^2 p^2 + m_a^6}} - \frac{m_b^3}{\sqrt{(M^2 + p^2)^2 p^2 + m_b^6}} \right) \\ &\simeq \frac{i}{12\pi^2(m_a^3 - m_b^3)} \int_0^{\Lambda^3} dx \left(\frac{m_a^3}{\sqrt{x^2 + m_a^6}} - \frac{m_b^3}{\sqrt{x^2 + m_b^6}} \right) \simeq \frac{i}{4\pi^2(m_a^3 - m_b^3)} \left[m_a^3 \ln\left(\frac{\Lambda}{m_a}\right) - m_b^3 \ln\left(\frac{\Lambda}{m_b}\right) \right]. \end{aligned} \quad (32)$$

The integral (32) diverges logarithmically, unlike the Lorentz symmetric case where it diverges quadratically. Note that, when $m_b \rightarrow m_a$, the previous result is regular and leads to

$$I_{aa} \simeq \frac{i}{4\pi^2} \ln\left(\frac{\Lambda}{m_a}\right). \quad (33)$$

In order to calculate the number of graphs (32) contributing to the coupling corrections, we introduce the auxiliary field σ and write the four-fermion interactions in the form

$$-\frac{1}{2}\sigma^2 + \sigma\sqrt{2}(g_1\bar{\psi}_1\psi_1 + g_2\bar{\psi}_2\psi_2). \quad (34)$$

The scalar σ does not propagate, but is described by a fictitious propagator, which carries a factor i . This propagator has to be understood in the limit where it shrinks to a

one-loop four-point function, and are therefore obtained for vanishing external momentum. If one denotes

$$\begin{aligned} N_a(\omega, \vec{p}) &= \omega\gamma^0 - (M^2 + p^2)(\vec{p} \cdot \vec{\gamma}) + m_a^3 \\ D_a(\omega, \vec{p}) &= \omega^2 - (M^2 + p^2)^2 p^2 - m_a^6, \end{aligned} \quad (30)$$

the generic graph for the one-loop corrections is thus

$$I_{ab} = \int \frac{d\omega d\vec{p}}{(2\pi)^4} \frac{iN_a(\omega, \vec{p})iN_b(\omega, \vec{p})}{D_a(\omega, \vec{p})D_b(\omega, \vec{p})}. \quad (31)$$

When $\Lambda \gg M$, we obtain

point, leading to the fermion loops given by the expressions (32). The two vertices corresponding to the effective Yukawa interactions are $i\sqrt{2}g_1$ and $i\sqrt{2}g_2$.

The graphs corresponding to the four-point function are represented in Fig. 2, in terms of the equivalent Yukawa interaction (34), where the last two graphs do not contribute to the four-fermion beta functions. Indeed, the general structure of the four point function is

$$\langle 0 | \psi^\dagger \psi \psi^\dagger \psi | 0 \rangle = A \mathbf{1} \otimes \mathbf{1} + B_i \mathbf{1} \otimes \sigma^i + C_{ij} \sigma^i \otimes \sigma^j, \quad (35)$$

where the Dirac indices are omitted and the tensorial product allows for the two in and two out states. A is the only quantity contributing to the coupling constant, since the corresponding term has no Dirac structure. The four-point

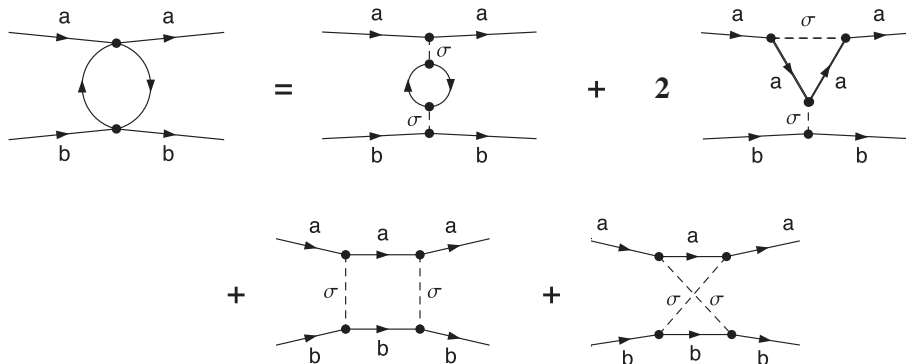


FIG. 2. One-loop graphs involving the auxiliary scalar field, which contribute to the four-point function. Solid lines represent fermions and dashed lines represent the scalar. Only the first two diagrams, where the fermion lines cross an odd number of vertices, contribute to the four-fermion beta functions. The first graph corresponds to the insertion of a scalar self-energy, it involves a factor -4 for the trace over Dirac indices and a summation over both flavors. The second graph has two contributions: one for each insertion of a vertex correction.

function contains one divergence only, which is logarithmic, such that the divergent graphs are obtained only from the highest power of momentum in the numerator of propagators, i.e., from $\vec{p} \cdot \vec{\gamma}$ and not the mass term. The last two graphs of Fig. 2 contain continuous lines of fermions with one internal propagator, such that the divergent part is contained in C_{ij} only. A is finite for these two graphs, and thus does not contribute to the beta function. More generally [16], to any order of the perturbation theory, any graph containing a open fermion line, which meets an even number of vertices, does not contribute to the beta functions of the model. One needs an even number of internal lines for the product of gamma matrices (appearing in $\vec{p} \cdot \vec{\gamma}$) to give a diverging term with a non-vanishing trace.

B. Beta-functions

The divergent one-loop correction to the four-fermion interactions are then given by the first two graphs of Fig. 2, which are as follows.

- (1) For the flavor preserving four-fermion interaction:
- (i) Graphs with the insertion of the one-loop scalar self-energy: both flavors contribute to the fermion loop, which induces a factor -4 for the trace over Dirac indices. The contribution is then,

$$\begin{aligned} & -4i^2(i\sqrt{2}g_a)^4 I_{aa} - 4i^2(i\sqrt{2}g_a)^2(i\sqrt{2}g_b)^2 I_{bb} \\ & = 16g_a^2(g_a^2 I_{aa} + g_b^2 I_{bb}). \end{aligned} \quad (36)$$

- (ii) Graphs with the insertion of the one-loop Yukawa interaction: only the flavor a plays a role, and the contribution is

$$2i^2(i\sqrt{2}g_a)^4 I_{aa} = -8g_a^4 I_{aa}. \quad (37)$$

The total contribution must be identified with the correction to the bare graph $i(i\sqrt{2}g_a)^2$, such that

$$i\delta g_a^2 = -4g_a^2(g_a^2 I_{aa} + 2g_b^2 I_{bb}), \quad (38)$$

and the corresponding beta function is therefore

$$\beta_a \equiv \Lambda \frac{\partial(\delta g_a^2)}{\delta \Lambda} = -\frac{g_a^2}{\pi^2}(g_a^2 + 2g_b^2). \quad (39)$$

- (2) For the flavor-changing interaction:
- (i) Graphs with the insertion of the one-loop scalar self-energy:

$$16g_a g_b (g_a^2 I_{aa} + g_b^2 I_{bb}); \quad (40)$$

- (ii) Graphs with the insertion of the one-loop Yukawa interaction:

$$-4g_b g_a^3 I_{aa} - 4g_a g_b^3 I_{bb}; \quad (41)$$

The total contribution must be identified with the correction to the bare graph iG

$$i\delta G = -12g_a g_b (g_a^2 I_{aa} + g_b^2 I_{bb}), \quad (42)$$

and the corresponding beta function is therefore

$$\beta_G = -3 \frac{g_a g_b}{\pi^2} (g_a^2 + g_b^2). \quad (43)$$

One can infer from this one-loop analysis that the theory is asymptotically free, since higher orders also diverge at most logarithmically, and cannot change the sign of the one-loop beta functions. Note that, when $g_1 = g_2$, then $\beta_G = 2\beta_a$, as expected from the $O(2)$ symmetry. To conclude this section, we add a comment related to the situation where $h \neq 0$. In this case, the fermion propagator is not diagonal in flavor space, and the calculations are therefore more involved. However, one can predict that asymptotic freedom will still hold: the additional graphs to take into account will feature the flavor changing Yukawa interactions $\sigma\sqrt{2}h(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1)$, and the corresponding fermion loop in Fig. 2 will contain the integral I_{ab} for $a \neq b$. But this integral remains positive (along the imaginary axis) when $a \neq b$, as can be seen by setting $m_a^3 = m_b^3(1 + \epsilon)$, which leads to

$$I_{ab} = \frac{i}{4\pi^2} \left[\ln\left(\frac{\Lambda}{m_b}\right) - \frac{1}{3} - \frac{\epsilon}{6} + \dots \right], \quad (44)$$

where dots represent higher orders in ϵ . One can see that, whatever the sign of ϵ is, this integral keeps the same sign as in the situation $a = b$. As a consequence, the additional terms to be added to the beta functions will not change their global sign.

IV. TWO-LOOP PROPAGATOR

Since Lifshitz theories explicitly break Lorentz symmetry, space and time derivatives are dressed differently by quantum corrections. If one considers only one particle, or several particles in a given flavor multiplet, frequency and space momentum can always be rescaled in such a way that the particles have the usual Lorentz-like IR dispersion relation (after neglecting the higher order powers of the space momentum, suppressed by M). However if one considers several particles without flavor symmetry, then it becomes necessary to perform a flavor-independent rescaling of frequency and space momentum, such that different particles see different effective light cones. This is the case we consider here.

The only one-loop correction to the fermion propagator is a tadpole diagram, where the loop is made of one internal propagator only. As a consequence, the external momentum does not flow in this loop, which therefore provides a momentum-independent correction and contributes only to the mass dressing. For this reason, one needs to go at least to two loops, represented in Fig. 3 in terms of the equivalent Yukawa model (34), in order to get a

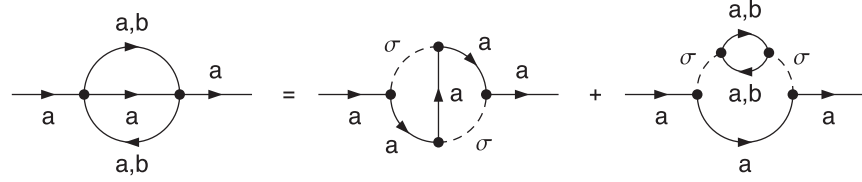


FIG. 3. Two-loop contributions to the propagator. The fermion loop in the second graph involves a contribution from each flavor, and a factor -4 for the trace over Dirac indices.

momentum-dependent correction, which leads to a modification of classical dispersion relations.

We note here that the two-loop propagator is evaluated in [17] for a scalar ϕ^4 theory, in 6 spatial dimensions and for $z = 2$. This calculation is done in the massless case and in the absence of quadratic space derivatives. Dimensional regularization is used there, such that the power of the cutoff does not appear explicitly in the results. The authors conclude that the Lorentz-symmetry breaking terms flow to 0 in the deep IR.

A. Self-energy

We still consider the case where $h \ll 1$ and the corresponding interaction is disregarded. The perturbative graphs on Fig. 3 can be calculated with massless bare propagators, since the two-loop graphs contain no IR divergence. As a consequence, these graphs are flavor independent (besides an overall factor depending on the coupling constants), and they involve the integrals

$$\begin{aligned}
 I(k_0, \vec{k}) &= i^2 \int \frac{dp_0 d\vec{p}}{(2\pi)^4} \int \frac{dq_0 d\vec{q}}{(2\pi)^4} \\
 &\quad \times \frac{iN(-p)iN(-q)iN(p+q+k)}{D(-p)D(-q)D(p+q+k)} \\
 J(k_0, \vec{k}) &= i^2 \int \frac{dp_0 d\vec{p}}{(2\pi)^4} \int \frac{dq_0 d\vec{q}}{(2\pi)^4} \\
 &\quad \times \frac{\text{Tr}[iN(-p)iN(-q)]iN(p+q+k)}{D(-p)D(-q)D(p+q+k)},
 \end{aligned} \tag{45}$$

where the trace in J arises from the fermion loop, and the factors i^2 are for the scalar propagators. Taking into account the different possibilities for the self-energy of flavor a , we obtain

- (i) $(i\sqrt{2}g_a)^4 I$ for the graph without fermion loop;
- (ii) $[(i\sqrt{2}g_a)^4 + (i\sqrt{2}g_a)^2(i\sqrt{2}g_b)^2]J$ for the graph with a fermion loop: one contribution for each flavor in the loop.

We calculate these integrals in the Appendix, where we see that the only role of the fermion loop is to give a factor -4 from the trace over Dirac indices. We therefore have $J = -4I$, and the total contribution to the momentum-dependent two-loop self-energy $\Sigma_a(k_0, \vec{k})$ is given by

$$\begin{aligned}
 -i\Sigma_a(k_0, \vec{k}) &= -4g_a^2(3g_a^2 + 4g_b^2)I \\
 &= -i4g_a^2(3g_a^2 + 4g_b^2) \int \frac{dp_0 d\vec{p}}{(2\pi)^4} \int \frac{dq_0 d\vec{q}}{(2\pi)^4} \\
 &\quad \times \frac{N(-p)N(-q)N(p+q+k)}{D(-p)D(-q)D(p+q+k)}.
 \end{aligned} \tag{46}$$

The bare inverse fermion propagator is

$$S_{\text{bare}}^{-1} = k_0 \gamma^0 - M^2 \vec{k} \cdot \vec{\gamma} + \dots, \tag{47}$$

where dots represent higher orders in \vec{k} . We parametrize the dressed inverse propagator as

$$\begin{aligned}
 S_{\text{dressed}}^{-1} &= -m_a^3 + (1 - Y_a)k_0 \gamma^0 - (1 - Z_a)M^2 \vec{k} \cdot \vec{\gamma} \\
 &\quad + \dots,
 \end{aligned} \tag{48}$$

such that the self-energy is

$$\begin{aligned}
 \Sigma_a(k_0, \vec{k}) &= S_{\text{bare}}^{-1} - S_{\text{dressed}}^{-1} \\
 &= m_a^3 + Y_a k_0 \gamma^0 - Z_a M^2 \vec{k} \cdot \vec{\gamma} + \dots
 \end{aligned} \tag{49}$$

The integrals (45) should then be expanded in the external frequency k_0 and momentum \vec{k} in order to find the corrections Y_a, Z_a . The k -independent mass correction m_a^3 will be disregarded, since the dynamical masses have already been calculated.

B. Dressed dispersion relations

From the self-energy (49), the IR dispersion relation for the flavor a is

$$(1 - Y_a)^2 k_0^2 = m_a^6 + M^4(1 - Z_a)^2 k^2 + \dots, \tag{50}$$

where $k = |\vec{k}|$. If we assume that the two fermion flavors are to be coupled to other particles, then one needs a flavor-independent rescaling of the dispersion relation. $k_0 \rightarrow M^2 \tilde{k}_0$ leads then to the following product of the phase and the group velocities, v_p and v_g , respectively

$$v_a^2 \equiv v_p v_g = \frac{\tilde{k}_0}{k} \frac{\partial \tilde{k}_0}{\partial k} = 1 + 2(Y_a - Z_a) + \mathcal{O}(k/M)^2. \tag{51}$$

We calculate Y_a and Z_a in the Appendix, by expanding analytically the integral I to first order in k_0 and $\vec{k} \cdot \vec{\gamma}$, and we find a quadratic divergence of the form ($\Lambda \gg M$)

TABLE I. Coupling constants for different mass scales M , when $h \ll 1$ and $\Lambda = 10^{19}$ GeV.

M/Λ	10^{-16}	10^{-15}	10^{-14}	10^{-13}	10^{-12}	10^{-11}
$g_1 \approx g_2$	0.46	0.47	0.48	0.48	0.49	0.50
M/Λ	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}	10^{-5}
$g_1 \approx g_2$	0.51	0.53	0.54	0.55	0.56	0.58

$$Y_a - Z_a \approx 4\kappa g_a^2(3g_a^2 + 4g_b^2) \frac{\Lambda^2}{M^2}, \quad \kappa \approx -3.49 \times 10^{-5} \quad (52)$$

where $a \neq b$. This result shows that the present model is of limited use for the prediction of Lorentz-violating propagation. Indeed, with the values of Λ/M , g_1 , g_2 shown in Table I, the result (52) is not perturbative: one needs to absorb the quadratic divergence with counterterms, such that the renormalized value of $Y_a - Z_a$ needs to be fixed by experimental data. Therefore the model cannot predict quantitative deviations from special relativity at low energies.

If these corrections were logarithmic, one could infer from our result a cutoff-independent beta function for the effective maximum speed seen by the fermions, which could lead to “realistic” predictions on potential sub/super-luminal propagation.

Note that the rescaling of frequency which leads to the speed squared (51) does not make apparent the fact that, if flavor symmetry is exactly satisfied, then the IR dispersion relations are relativistic. If one ignores possible interactions with other particles, one can further rescale

$$k^2 = \tilde{k}^2 \frac{1 - Y_1}{1 - Z_1} \frac{1 - Y_2}{1 - Z_2}, \quad (53)$$

which leads to the following IR dispersion relations

$$\begin{aligned} \tilde{k}_0^2 &\approx \tilde{m}_1^2 + (1 + 2\delta\nu)\tilde{k}^2 \\ \tilde{k}_0^2 &\approx \tilde{m}_2^2 + (1 - 2\delta\nu)\tilde{k}^2 \quad \text{where } \delta\nu = 6\kappa(g_1^4 - g_2^4) \frac{\Lambda^2}{M^2}. \end{aligned} \quad (54)$$

One can see here that the Lorentz-invariant IR dispersion relations are recovered when $g_1 = g_2$. But for $g_1 \neq g_2$, one needs the difference $|g_1^2 - g_2^2|$ to be proportional to M^2/Λ^2 in order to deal with realistic phenomenology. One can take the example of the largest value $M/\Lambda \approx 10^{-5}$ with $g_a \approx 0.58$ from Table I. The upper bound $\delta\nu \leq 2 \times 10^{-9}$ given by the supernovae SN1987a data [18] gives then

$$|g_1^2 - g_2^2| \leq \frac{\delta\nu M^2/\Lambda^2}{6\kappa(g_1^2 + g_2^2)} \approx 10^{-15}, \quad (55)$$

such that flavor symmetry can be considered exact, and the corresponding fine tuning is not natural.

V. CONCLUSION

The main aim of this article was to show how quantum fluctuations modify classical dispersion relations, obtained from a four-fermion interaction Lifshitz model, where flavor symmetry is broken at the classical level. The first steps to study the model consisted in showing results which were expected, from similar models previously studied by other authors: (i) Dynamical mass generation, and consequently dynamical arising of flavor oscillations. Using experimental constraints on neutrino oscillations, we showed that the parameters of the model are consistent with a perturbative regime, justifying the next steps; (ii) Asymptotic freedom, derived using an equivalent Yukawa model, which makes use of an auxiliary scalar field. The modification of dispersion relations by quantum corrections has then been calculated at two loops, making use of the same Yukawa model. We find, after a rescaling of frequency and space momentum in the effective dispersion relations, that the mismatch with the speed of light is proportional to the difference of the coupling constants, and scales as the square of the cutoff.

Classically, Lorentz-violating effects in this model are suppressed by inverse powers of M , and therefore might not be measurable at low energies $k \ll M$. Quantum corrections, though, can provide measurable contributions, as the quadratically diverging corrections which were found here, for the fermion self-energies. We believe that these corrections are too significant to avoid an extremely precise fine tuning of the coupling constants, for the model to be consistent with upper bounds on Lorentz violation.

We therefore suggest that a realistic Lifshitz model should have logarithmic divergences at most, in order to have phenomenological relevance. This is the case, for example, of Lifshitz-type Yukawa models [19], where one-loop corrections to the fermion dispersion relations are finite. Also, Lifshitz-type extensions of gauge theories, which are super-renormalizable in $3 + 1$ dimensions and for $z = 3$, feature interesting properties [20]. An essential point is, since the gauge coupling constant has a (positive) mass dimension, fermion dynamical mass naturally appears in the dressed theory. If, in addition, fermion condensates break flavor symmetry, then vectors automatically become massive [21], with a mechanism similar to the one initially derived by Schwinger in $1 + 1$ dimensional quantum electrodynamics, and used in Technicolor studies. Such models are to be looked at in future publications, in order to determine their phenomenological relevance.

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APPENDIX: A TWO-LOOP PROPAGATOR

An expansion in the frequency k_0 of the integrand appearing on the right-hand side of Eq. (46) gives

$$\begin{aligned} \frac{N(-p_0, -\vec{p})N(-q_0, -\vec{q})N(p_0 + q_0 + k_0, \vec{p} + \vec{q})}{D(-p_0, -\vec{p})D(-q_0, -\vec{q})D(p_0 + q_0 + k_0, \vec{p} + \vec{q})} &= \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D(p+q)} + k_0\gamma^0 \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} \\ &\quad - 2k_0(p_0 + q_0) \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} + \mathcal{O}(k_0^2), \end{aligned} \quad (56)$$

where $(p) \equiv (p_0, \vec{p})$, and an expansion in the spatial momentum \vec{k} gives

$$\begin{aligned} &\frac{N(-p_0, -\vec{p})N(-q_0, -\vec{q})N(p_0 + q_0, \vec{p} + \vec{q} + \vec{k})}{D(-p_0, -\vec{p})D(-q_0, -\vec{q})D(p_0 + q_0, \vec{p} + \vec{q} + \vec{k})} \\ &= \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D(p+q)} - \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} [2((\vec{p} + \vec{q}) \cdot \vec{k})(\vec{p} + \vec{q}) \cdot \vec{\gamma}) + (M^2 + (\vec{p} + \vec{q})^2)(\vec{k} \cdot \vec{\gamma})] \\ &\quad + \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} 2((\vec{p} + \vec{q}) \cdot \vec{k})[M^2 + (\vec{p} + \vec{q})^2][M^2 + 3(\vec{p} + \vec{q})^2] + \mathcal{O}(k_0^2). \end{aligned} \quad (57)$$

The first term in the k_0 -expansion leads to the integral

$$\begin{aligned} (k_0\gamma^0) \int_{p,q} \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} &= (k_0\gamma^0) \int_{p,q} \frac{p_0q_0 - (M^2 + \vec{p}^2)(M^2 + \vec{q}^2)\vec{p} \cdot \vec{q}}{D(-p)D(-q)D(p+q)} \\ &\quad + (k_0\gamma^0) \int_{p,q} \frac{p_0(M^2 + \vec{q}^2)\vec{q} \cdot \vec{\gamma} - q_0(M^2 + \vec{p}^2)\vec{p} \cdot \vec{\gamma}}{D(-p)D(-q)D(p+q)}, \end{aligned} \quad (58)$$

and, because of the symmetry $p \leftrightarrow q$, the second integral vanishes. The following rescaling

$$p_0 = M^3 u_0, \quad q_0 = M^3 v_0, \quad \vec{p} = M\vec{u}, \quad \vec{q} = M\vec{v}, \quad (59)$$

together with a Wick rotation on u_0, v_0 finally leads to

$$(k_0\gamma^0) \int_{p,q} \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} \quad (60)$$

$$= -(k_0\gamma^0) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E + v_E)}, \quad (61)$$

where $D_E(u_E) = u_4^2 + (1 + \vec{u}^2)^2 \vec{u}^2$. The second term in the k_0 -expansion gives

$$\begin{aligned} &- 2k_0 \int_{u,v} (p_0 + q_0) \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} \\ &= -2(k_0\gamma^0) \int_{p,q} (p_0 + q_0)^2 \frac{p_0q_0 - (M^2 + \vec{p}^2)(M^2 + \vec{q}^2)\vec{p} \cdot \vec{q}}{D(-p)D(-q)D^2(p+q)} \\ &\quad - 2k_0 \int_{p,q} (p_0 + q_0) \frac{p_0\gamma^0 \vec{q} \cdot \vec{\gamma}(\vec{p} + \vec{q}) \cdot \vec{\gamma}(M^2 + q^2)[M^2 + (\vec{p} + \vec{q})^2] - (p \leftrightarrow q)}{D(-p)D(-q)D^2(p+q)}, \end{aligned} \quad (62)$$

where, by symmetry, the terms proportional to $\vec{\gamma}$ lead to a vanishing integral. After the rescaling (59) and a Wick rotation, we then obtain

$$- 2k_0 \int_{u,v} (p_0 + q_0) \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} = 2(k_0\gamma^0) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} (u_4 + v_4)^2 \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)}. \quad (63)$$

The term proportional to $k_0\gamma^0$ is then

$$(k_0\gamma^0) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E + v_E)} \left(-1 + \frac{2(u_4 + v_4)^2}{D_E(u_E + v_E)} \right). \quad (64)$$

For the first term in the \vec{k} -expansion, we use the identity

$$\int_{p,q} f(p, q) \vec{p} \cdot \vec{k}(\vec{p} + \vec{q}) \cdot \vec{\gamma} = \frac{\vec{k} \cdot \vec{\gamma}}{3} \int_{p,q} f(p, q) \vec{p} \cdot (\vec{p} + \vec{q}), \quad (65)$$

where $f(p, q)$ depends on $(\vec{p})^2$, $(\vec{q})^2$ and $\vec{p} \cdot \vec{q}$ only. The rescaling (59) and a Wick rotation then lead to the integral

$$\begin{aligned} & - \int_{p,q} \frac{N(-p)N(-q)}{D(-p)D(-q)D(p+q)} [2((\vec{p} + \vec{q}) \cdot \vec{k})(\vec{p} + \vec{q}) \cdot \vec{\gamma} + (M^2 + (\vec{p} + \vec{q})^2)(\vec{k} \cdot \vec{\gamma})] \\ & = M^2(\vec{k} \cdot \vec{\gamma}) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \left(1 + \frac{5}{3}(\vec{u} + \vec{v})^2\right) \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E + v_E)}. \end{aligned} \quad (66)$$

The second term in the \vec{k} -expansion leads to the integral

$$\begin{aligned} & 2 \int_{p,q} \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} ((\vec{p} + \vec{q}) \cdot \vec{k}) [M^2 + (\vec{p} + \vec{q})^2] [M^2 + 3(\vec{p} + \vec{q})^2] \\ & = -2M^2 \int_{u,v} \frac{u_0 v_0 - (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)} (1 + (\vec{u} + \vec{v})^2)(\vec{u} + \vec{v}) \cdot \vec{\gamma}(\vec{u} + \vec{v}) \cdot \vec{k} [1 + (\vec{u} + \vec{v})^2] [1 + 3(\vec{u} + \vec{v})^2], \end{aligned} \quad (67)$$

where, by symmetry, the term not proportional to $\vec{\gamma}$ vanishes. Using the identity (65), a Wick rotation then leads to

$$\begin{aligned} & 2 \int_{p,q} \frac{N(-p)N(-q)N(p+q)}{D(-p)D(-q)D^2(p+q)} ((\vec{p} + \vec{q}) \cdot \vec{k}) [M^2 + (\vec{p} + \vec{q})^2] [M^2 + 3(\vec{p} + \vec{q})^2] \\ & = -\frac{2}{3} M^2(\vec{k} \cdot \vec{\gamma}) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)} (\vec{u} + \vec{v})^2 [1 + (\vec{u} + \vec{v})^2]^2 [1 + 3(\vec{u} + \vec{v})^2]. \end{aligned} \quad (68)$$

The term proportional to $(\vec{k} \cdot \vec{\gamma})$ is then

$$M^2(\vec{k} \cdot \vec{\gamma}) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E(u_E + v_E)} \left(1 + \frac{5}{3}(\vec{u} + \vec{v})^2 - \frac{2}{3}(\vec{u} + \vec{v})^2 \frac{[1 + (\vec{u} + \vec{v})^2]^2 [1 + 3(\vec{u} + \vec{v})^2]}{D_E(u_E + v_E)}\right). \quad (69)$$

Finally, from Eqs. (64) and (69), the quantum corrections to the IR dispersion relation are determined by

$$Y_a - Z_a = 4g_a^2(3g_a^2 + 4g_b^2) \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \text{Int}, \quad (70)$$

where the integrand is

$$\text{Int} = \frac{1}{3} \frac{u_4 v_4 + (1 + \vec{u}^2)(1 + \vec{v}^2)\vec{u} \cdot \vec{v}}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)} [(u_4 + v_4)^2(6 + 5(\vec{u} + \vec{v})^2) - (\vec{u} + \vec{v})^2(1 + (\vec{u} + \vec{v})^2)^2(2 + (\vec{u} + \vec{v})^2)]. \quad (71)$$

Note that in the Lorentz-symmetric case, higher orders in \vec{u} , \vec{v} are absent and $Y_a = Z_a$. The integral (70) is evaluated as follows.

We can first perform the exact integration over u_4 , v_4 , using the Feynman parametrization. This introduces two new variables of integration, but which lie in a compact domain of integration:

$$\frac{1}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)} = 6 \int_0^1 dx \int_0^{1-x} dy \frac{1 - x - y}{[xD_E(u_E) + yD_E(v_E) + (1 - x - y)D_E(u_E + v_E)]^4},$$

We then introduce the variables a , b , such that

$$u_4 = s(a + b) \quad \text{and} \quad v_4 = t(a - b), \quad \text{with} \quad s = \sqrt{1 - x} \quad \text{and} \quad t = \sqrt{1 - y}, \quad (72)$$

to obtain

$$\frac{du_4 dv_4}{D_E(u_E)D_E(v_E)D_E^2(u_E + v_E)} = \int_0^1 dx \int_0^{1-x} dy \frac{12dadb\sigma}{[2st(st + \sigma)a^2 + 2st(st - \sigma)b^2 + D]^4}, \quad (73)$$

where

$$D = x(1 + u^2)^2 u^2 + y(1 + v^2)^2 v^2 + \sigma(1 + \Sigma)^2 \Sigma \quad \Sigma = (\vec{u} + \vec{v})^2, \quad \sigma = 1 - x - y. \quad (74)$$

We then write, with $0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$

$$\sqrt{2st(st + \sigma)}a = \rho \cos\phi \quad \text{and} \quad \sqrt{2st(st - \sigma)}b = \rho \sin\phi \quad (75)$$

to obtain

$$\begin{aligned} & \int du_4 \int dv_4 \text{Int} \quad (76) \\ &= 2 \int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2 t^2 - \sigma^2}} \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi \frac{1}{[\rho^2 + D]^4} \left[\frac{\rho^2}{2} \left(\frac{\cos^2 \phi}{st + \sigma} - \frac{\sin^2 \phi}{st - \sigma} \right) + (1 + u^2)(1 + v^2) \vec{u} \cdot \vec{v} \right] \\ & \quad \times \left[\frac{\rho^2}{2st} \left(\frac{(s+t)^2 \cos^2 \phi}{st + \sigma} + \frac{(s-t)^2 \sin^2 \phi}{st - \sigma} \right) (6 + 5\Sigma) - \Sigma(1 + \Sigma)^2(2 + \Sigma) \right] \\ &= 2\pi \int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2 t^2 - \sigma^2}} \int_0^\infty \rho d\rho \frac{A\rho^4 + B\rho^2 + C}{[\rho^2 + D]^4} \\ &= \frac{\pi}{6} \int_0^1 dx \int_0^{1-x} dy \frac{\sigma}{\sqrt{s^2 t^2 - \sigma^2}} \left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3} \right), \quad (77) \end{aligned}$$

where

$$\begin{aligned} A &= \frac{6 + 5\Sigma}{4} \frac{2s^2 t^2 + 4\sigma^2 - 3\sigma(s^2 + t^2)}{(s^2 t^2 - \sigma^2)^2} \\ B &= (1 + u^2)(1 + v^2) \vec{u} \cdot \vec{v} (6 + 5\Sigma) \frac{s^2 + t^2 - 2\sigma}{s^2 t^2 - \sigma^2} + \Sigma(1 + \Sigma)^2(2 + \Sigma) \frac{\sigma}{s^2 t^2 - \sigma^2} \\ C &= -2(1 + u^2)(1 + v^2) \vec{u} \cdot \vec{v} \Sigma(1 + \Sigma)^2(2 + \Sigma). \end{aligned} \quad (78)$$

We then define $\vec{u} \cdot \vec{v} = uv \cos\theta$ and

$$u = r \cos\alpha, \quad v = r \sin\alpha, \quad \text{with } 0 \leq r < \infty \quad \text{and} \quad 0 \leq \alpha \leq \pi/2, \quad (79)$$

and the final integral is

$$\begin{aligned} F(\Lambda/M) &= \int \frac{d^4 u_E}{(2\pi)^4} \frac{d^4 v_E}{(2\pi)^4} \text{Int} \\ &= \frac{1}{3 \times 2^8 \pi^5} \int_0^1 dx \int_0^{1-x} dy \int_0^{\Lambda/M} dr \int_0^\pi d\theta \int_0^{\pi/2} d\alpha \frac{\sigma r^5 \sin^2(2\alpha) \sin\theta}{\sqrt{s^2 t^2 - \sigma^2}} \left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3} \right), \quad (80) \end{aligned}$$

which is quadratically divergent, as $F(z) \sim \kappa z^2$ when $z \rightarrow \infty$. We then find via numerical integration

$$\kappa = \lim_{z \rightarrow \infty} \left\{ \frac{1}{2z} \frac{dF}{dz} \right\} \quad (81)$$

$$\begin{aligned} &= \lim_{r \rightarrow \infty} \left\{ \frac{r^4}{3 \times 2^9 \pi^5} \int_0^1 dx \int_0^{1-x} dy \int_0^\pi d\theta \int_0^{\pi/2} d\alpha \frac{\sigma \sin^2(2\alpha) \sin\theta}{\sqrt{s^2 t^2 - \sigma^2}} \left(\frac{2A}{D} + \frac{B}{D^2} + \frac{2C}{D^3} \right) \right\} \\ &\simeq -3.49 \times 10^{-5}, \quad (\text{to a 1\% accuracy}). \quad (82) \end{aligned}$$

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