Multiparticle form factors of the principal chiral model at large N

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We study the sigma model with $SU(N)\times SU(N)$ symmetry in $1+1$ dimensions. The two- and four-particle form factors of the Noether current operators are found, by combining the integrable-bootstrap method with the large-N expansion.

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I. INTRODUCTION

The quantum principal chiral sigma model is completely integrable in one space and one time dimension [1,2]. Its action is

$$S=\frac{N}{2g_0^2}\int d^2x\eta^{\mu\nu}\mathrm{Tr}\,\partial_\mu U(x)^\dagger\partial_\nu U(x), \tag{1.1}$$

where $U(x)\in SU(N)$, μ, $\nu=0$, 1, and where $\eta^{\mu\nu}$ is the Minkowski metric, $\eta^{00}=1$, $\eta^{11}=-1$, $\eta^{01}=\eta^{10}=0$. The action is invariant under the global transformation $U(x)\rightarrow V_LU(x)V_R$, for V_L, $V_R\in SU(N)$. The model is asymptotically free and has a mass gap m. There are two Noether currents,

$$\begin{aligned} j_\mu^L(x)_a^c&=\frac{-iN}{2g_0^2}\partial_\mu U_{ab}(x)U^{*bc}(x),\\ j_\mu^R(x)_b^d&=\frac{-iN}{2g_0^2}U^{*da}(x)\partial_\mu U_{ab}(x),\end{aligned} \tag{1.2}$$

where a, $b=1,\ldots,N$, associated with the symmetries $U\rightarrow V_LU$ and $U\rightarrow UV_R$, respectively.

In this paper, we calculate the two- and four-excitation form factors of the current operators using a large-N expansion and the form-factor bootstrap method [3]. This approach has been used in Ref. [4], to find the form factors of the renormalized field operator. We also find the two-particle form factor for all $N>2$.

In the next section, we review the exact S matrix for the chiral model. We calculate the two-particle form factors in the planar limit in Sec. III, and for general N in Sec. IV. In Sec. V, we calculate the four-particle form factor, and we discuss our results in the final section.

II. THE EXACT S MATRIX AND MULTIPARTICLE STATES

The sigma model has elementary particles of mass m, which carry both left and right colors. These elementary particles form bound states which obey a sine formula [5]

$$m_r=m\frac{\sin(\frac{\pi r}{N})}{\sin(\frac{\pi}{N})},\qquad r=1,\ldots,N-1, \tag{2.1}$$

where m_r is the mass of a r-particle bound state. In the large-N limit, the mass of a r-particle bound state is $m_r=mr$, for finite r. This means that there are no bound states of a finite number of elementary particles in the planar limit, since the binding energy vanishes.

We introduce particle and antiparticle creation operators $\mathfrak{A}_P^\dagger(\theta)_{ab}$ and $\mathfrak{A}_A^\dagger(\theta)_{ba}$, respectively, where θ is the particle rapidity, defined in terms of the momentum vector by $p_0=m\cosh\theta$, $p_1=m\sinh\theta$, and a, $b=1,\ldots,N$ are left and right color indices, respectively. A product of creation operators acting on the vacuum in order of increasing rapidity, from left to right, gives the multiparticle state

$$\begin{aligned}&|P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;\ldots\rangle_{\rm in}\\ &\quad=\mathfrak{A}_P^\dagger(\theta_1)_{a_1b_2}\mathfrak{A}_A^\dagger(\theta_2)_{b_2a_2}\ldots|0\rangle,\quad \text{where }\theta_1>\theta_2>\ldots.\end{aligned} \tag{2.2}$$

The S matrix of two particles, with incoming rapidities θ_1 and θ_2, outgoing rapidities θ_1' and θ_2', is

$$\begin{aligned}&{}_{\rm out}\langle P,\theta_1',c_1,d_1;P,\theta_2',c_2,d_2|P,\theta_1,a_1,b_1;P,\theta_2,a_2,b_2\rangle_{\rm in}\\ &\quad=S_{PP}(\theta)_{a_1b_1;a_2b_2}^{c_2d_2;c_1d_1}4\pi\delta(\theta_1'-\theta_1)4\pi\delta(\theta_2'-\theta_2),\end{aligned}$$

where $\theta=\theta_1-\theta_2$. We follow convention and call the function $S_{PP}(\theta)_{a_1b_1;a_2b_2}^{c_2d_2;c_1d_1}$ the S matrix. It is given by

$$S_{PP}(\theta)_{a_1b_1;a_2b_2}^{c_2d_2;c_1d_1}=\chi(\theta)S_{\rm CGN}(\theta)_{a_1;a_2}^{c_2;c_1}S_{\rm CGN}(\theta)_{b_1;b_2}^{d_2;d_1}, \tag{2.3}$$

where $S_{\rm CGN}(\theta)$ is the S matrix of two elementary excitations of the $SU(N)$ chiral Gross-Neveu model [6,7]:

$$\begin{aligned}S_{\rm CGN}(\theta)_{a_1;a_2}^{c_2;c_1}&=\frac{\Gamma(i\theta/2\pi+1)\Gamma(-i\theta/2\pi-1/N)}{\Gamma(i\theta/2\pi+1-1/N)\Gamma(-i\theta/2\pi)}\\ &\quad\times\left(\delta_{a_1}^{c_1}\delta_{a_2}^{c_2}-\frac{2\pi i}{N\theta}\delta_{a_2}^{c_1}\delta_{a_1}^{c_2}\right),\end{aligned}$$

and $\chi(\theta)$ is the Castillejo-Dalitz-Dyson (CDD) factor [8,9]:

$$\chi(\theta)=\frac{\sinh(\frac{\theta}{2}-\frac{\pi i}{N})}{\sinh(\frac{\theta}{2}+\frac{\pi i}{N})}. \tag{2.4}$$

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The CDD factor is chosen such that the S matrix has the minimum number of singularities and reproduces the particle mass spectrum of the theory [9]. The particle-antiparticle S matrix is related to the particle-particle S matrix by crossing, i.e. $\theta \to \hat{\theta} = \pi i - \theta$. The S matrix for a particle with incoming rapidity θ_1 and outgoing rapidity θ_1', and an antiparticle with incoming rapidity θ_2 and outgoing rapidity θ_2', is

$$S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_2;b_2a_2} = S(\hat{\theta}, N)\Big[\delta^{d_2}_{b_2}\delta^{c_2}_{a_2}\delta^{c_1}_{a_1}\delta^{d_1}_{b_1} - \frac{2\pi i}{N\hat{\theta}} \times (\delta_{a_1a_2}\delta^{c_1c_2}\delta^{d_2}_{b_2}\delta^{d_1}_{b_1} + \delta^{c_2}_{a_2}\delta^{c_1}_{a_1}\delta_{b_1b_2}\delta^{d_1d_2}) - \frac{4\pi^2}{N^2\hat{\theta}^2}\delta_{a_1a_2}\delta^{c_1c_2}\delta_{b_1b_2}\delta^{d_1d_2}\Big], \qquad (2.5)$$

where

$$S(\theta, N) = \frac{\sinh(\frac{\theta}{2} - \frac{\pi i}{N})}{\sinh(\frac{\theta}{2} + \frac{\pi i}{N})}\Big[\frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - 1/N)}{\Gamma(i\theta/2\pi + 1 - 1/N)\Gamma(-i\theta/2\pi)}\Big]^2 = 1 + \mathcal{O}\Big(\frac{1}{N^2}\Big). \qquad (2.6)$$

The creation operators satisfy the Zamolodchikov algebra:

$$\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2} = S_{PP}(\theta)^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}\mathfrak{A}^\dagger_P(\theta_2)_{c_2d_2}\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_1},$$
$$\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} = S_{AA}(\theta)^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1},$$
$$\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} = S_{AP}(\theta)^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_1}. \qquad (2.7)$$

The r-excitation form factor of an operator $\mathfrak{B}(x)$ is defined as

$$\langle 0|\mathfrak{B}(x)|I_1, \theta_1, C_1; \ldots; I_r, \theta_r, C_r\rangle = e^{-i\sum_{k=1}^r x\cdot p_k}\mathcal{F}^{\mathfrak{B}}_{C_1,\ldots C_r}(\theta_1, \ldots, \theta_r),$$

where $I_k = P$ if the kth excitation is a particle, and $I_k = A$ if the kth excitation is an antiparticle, and C_k is the set of indices a_k, b_k for $I_k = P$ or b_k, a_k for $I_k = A$. The x dependence of the form factor is trivial, due to Lorentz invariance.

The vacuum expectation value of two operators $\mathfrak{B}(x)$ and $\mathfrak{C}(y)$ can be expressed in terms of form factors, using completeness of in states

$$\langle 0|\mathfrak{B}(x)\mathfrak{C}(y)|0\rangle = \langle 0|\mathfrak{B}(x)|0\rangle\langle 0|\mathfrak{C}(y)|0\rangle + \sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\int\frac{d\theta_1\ldots d\theta_r\, d\phi_1\ldots d\phi_t}{(2\pi)^{r+t}(r+t)!} \times \langle 0|\mathfrak{B}(x)|P, \theta_1, a_1, b_1; \ldots; P, \theta_r, a_r, b_r; A, \phi_1, d_1, c_1; \ldots; A, \phi_t, d_t, c_t\rangle \times \langle P, \theta_1, a_1, b_1; \ldots; P, \theta_r, a_r, b_r; A, \phi_1, d_1, c_1; \ldots; A, \phi_t, d_t, c_t|\mathfrak{C}(y)|0\rangle. \qquad (2.8)$$

III. SMIRNOV'S AXIOMS AND THE TWO-PARTICLE FORM FACTORS

In this section, we calculate the first nonvanishing form factor of the current operators at large N. We will discuss only the left-handed current $j^L_\mu(x)^c_a$ in detail, since the same method yields the right-handed-current form factor.

Under a global $SU(N)\times SU(N)$ transformation, the current and the particle and antiparticle creation operators transform as

$$j^L_\mu(x) \to V_L j^L_\mu(x)V^\dagger_L,$$
$$\mathfrak{A}^\dagger_P(\theta) \to V^\dagger_R\mathfrak{A}^\dagger_P(\theta)V^\dagger_L,$$
$$\mathfrak{A}^\dagger_A(\theta) \to V_L\mathfrak{A}^\dagger_A(\theta)V_R.$$

Only form factors with an equal number of particles and antiparticles are invariant under such global transformations. The first nontrivial form factor is

$$\langle 0|j^L_\mu(x)_{a_0c_0}|A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2\rangle = \langle 0|j^L_\mu(x)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}|0\rangle = (p_1 - p_2)_\mu e^{-ix\cdot(p_1+p_2)}F(\theta)\Big(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\Big), \qquad (3.1)$$

for $\theta_1 > \theta_2$, and

$$\langle 0|j^L_\mu(x)_{a_0c_0}|P_1, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2\rangle = \langle 0|j^L_\mu(x)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle = (p_1 - p_2)_\mu e^{-ix\cdot(p_1+p_2)}F'(\theta)\Big(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\Big), \qquad (3.2)$$

for $\theta_2 > \theta_1$, where, as before, $\theta = \theta_1 - \theta_2$. The $\mathcal{O}(\frac{1}{N})$ term in Eq. (3.1) ensures the tracelessness of the current operator. Lorentz invariance requires that the function $F(\theta)$ depend only on the rapidity difference θ [3].

We next apply the scattering axiom, also known as Watson's theorem [3]. This axiom follows from the Zamolodchikov algebra (2.7) on the creation operators of the in-state. This gives a relation between $F(\theta)$ and $F'(\theta)$:

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle = S_{AP}(\theta)^{d_1c_1;c_2d_2}_{a_2b_2;b_1a_1}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_2)_{c_2d_2}|0\rangle, \quad (3.3)$$

or

$$F'(\theta) = S(\hat{\theta}, N)\left(1 - \frac{2\pi i}{\hat{\theta}}\right)F(\theta). \quad (3.4)$$

In obtaining Eq. (3.4), some factors of $1/N$ in the S matrix were canceled by summing over group indices in Eq. (3.3).

We next consider the Smirnov periodicity axiom [3], which follows from crossing symmetry. For the M-excitation form factor of an operator $\mathfrak{B}(0)$, the periodicity axiom is

$$\langle 0|\mathfrak{B}(0)\mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1}\mathfrak{A}^\dagger_{I_1}(\theta_2)_{C_2}\ldots\mathfrak{A}^\dagger_{I_M}(\theta_M)_{C_M}|0\rangle = \langle 0|\mathfrak{B}(0)\mathfrak{A}^\dagger_{I_M}(\theta_M - 2\pi i)_{C_M}\mathfrak{A}^\dagger_{I_1}(\theta_1)_{C_1}\ldots\mathfrak{A}^\dagger_{I_{M-1}} \times(\theta_{M-1})_{C_{M-1}}|0\rangle. \quad (3.5)$$

For more discussion of this axiom, see Refs. [4,10].

Applying the periodicity axiom to our form factors (3.1), we find the two equivalent conditions:

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1 - 2\pi i)_{b_1c_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}|0\rangle \Rightarrow F'(\theta) = F(\theta - 2\pi i), \quad (3.6)$$

and

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2}|0\rangle = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_2 - 2\pi i)_{a_2b_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle \Rightarrow F(\theta) = F'(\theta + 2\pi i). \quad (3.7)$$

Combining Eq. (3.4) with Eq. (3.6) gives

$$F(\theta - 2\pi i) = \hat{S}(\theta, N)\left(\frac{\theta + \pi i}{\theta - \pi i}\right)F(\theta), \quad (3.8)$$

where we have defined the function $\hat{S}(\theta, N) \equiv S(\hat{\theta}, N)$.

For large N, we expand $\hat{S}(\theta, N) = 1 + \mathcal{O}(\frac{1}{N^2})$ and $F(\theta) = F^0(\theta) + \frac{1}{N}F^1(\theta) + \frac{1}{N^2}F^2(\theta) + \ldots$, so that

$$F^0(\theta - 2\pi i) = \left(\frac{\theta + \pi i}{\theta - \pi i}\right)F^0(\theta). \quad (3.9)$$

The general solution to Eq. (3.9) is

$$F^0(\theta) = \frac{g(\theta)}{\theta + \pi i}, \quad (3.10)$$

where $g(\theta)$ satisfies the periodicity condition $g(\theta - 2\pi i) = g(\theta)$. The minimal choice is to take $g(\theta) = g$, a constant. We do not present a proof that this is the right choice for the function $g(\theta)$, but it is the simplest solution and is thus likely to be the correct physical solution.

Next, we determine the value of g. There is a conserved charge $Q^L_{a_0c_0}$, associated with the current operator. This charge is

$$Q^L_{a_0c_0} = \int dx^1 j^L_0(x)_{a_0c_0}.$$

We fix the value of g by requiring that the charge generates the $SU(N)$ Lie algebra:

$$Q^{L\,a}_a = 0, \qquad [Q^{L\ c_1}_{a_1}, Q^{L\ c_2}_{a_2}] = if^{c_1c_2a_3}_{a_1a_2c_3}Q^{L\ c_3}_{a_3}, \quad (3.11)$$

where the structure coefficients are

$$f^{c_1c_2a_3}_{a_1a_2c_3} = i(\delta^{c_2}_{a_1}\delta^{a_3}_{a_2}\delta^{c_1}_{c_3} - \delta^{c_1}_{a_2}\delta^{a_3}_{a_1}\delta^{c_2}_{c_3}).$$

We cross the incoming particle from Eq. (3.1) to an outgoing antiparticle, *via* $\theta_2 \to \theta_2 - \pi i$, to find

$$\langle A, \theta_2, b_2, a_2|j^L_0(x)_{a_0c_0}|A, \theta_1, b_1, a_1\rangle = m(\cosh\theta_1 + \cosh\theta_2)\exp\{-im[x^0(\cosh\theta_1 - \cosh\theta_2) - x^1(\sinh\theta_1 - \sinh\theta_2)]\}F_1(\theta + \pi i) \times\left(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\right).$$

The integral over x^1 gives the matrix element of the charge operator:

$$\langle A, \theta_1, b_2, a_2|Q^L_{a_0c_0}|A, \theta_1, b_1, a_1\rangle = (2\pi)^2 2(p_1)_0\delta(\theta_1 - \theta_2) \times\left(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\right)F_1(\pi i).$$

The matrix element of the commutator of two charges is found by inserting a complete set of one-antiparticle intermediate states:

$$\langle A, \theta_2, b_2, a_2|[Q^L_{a_0c_0}, Q^L_{a_4c_4}]|A, \theta_1, b_1, a_1\rangle = \int\frac{d\theta_3}{4\pi}\langle A, \theta_2, b_2, a_2|Q^L_{a_0c_0}|A, \theta_3, b_3, a_3\rangle \times\langle A, \theta_3, b_3, a_3|Q^L_{a_4c_4}|A, \theta_1, b_1, a_1\rangle - \int\frac{d\theta_3}{4\pi}\langle A, \theta_2, b_2, a_2|Q^L_{a_4c_4}|A, \theta_3, b_3, a_3\rangle \times\langle A, \theta_3, b_3, a_3|Q^L_{a_0c_0}|A, \theta_1, b_1, a_1\rangle. \quad (3.12)$$

With the choice $F(\pi i) = 1$, Eq. (3.12) becomes

$$\langle A, \theta_2, b_2, a_2|[Q^{L\ c_0}_{a_0}, Q^{L\ c_4}_{a_4}]|A, \theta_1, b_1, a_1\rangle = if^{c_0c_4a_5}_{a_0a_4c_5}\langle A, \theta_2, b_2, a_2|Q^{L\ c_5}_{a_5}|A, \theta_1, b_1, a_1\rangle,$$

which is equivalent to Eq. (3.11). This fixes the constant $g = 2\pi i$.

We have not yet discussed the annihilation-pole axiom [3]. This axiom relates the form factors of M particles to the form factors of $M - 2$ particles. The general multiparticle form factor of the current operator is

$$\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1};P,\theta_n,a_n,b_n\rangle$$
$$= [p_1+\ldots+p_l-(p_{l+1}+\ldots+p_{2l})+p_{n-1}-p_n]_\mu \mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}. \quad (3.13)$$

Here, we have factored out the vector-valued prefactor in square brackets, consisting of a linear combination of the particle momenta, chosen to make $\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$ a Lorentz scalar. We define a Lorentz-scalar-valued operator $\mathcal{O}_{a_0c_0}$ by

$$\langle 0|\mathcal{O}_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1};P,\theta_n,a_n,b_n\rangle$$
$$\equiv \mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}.$$

The form factor has a pole at $\theta_{n-1n}\equiv\theta_{n-1}-\theta_n=-\pi i$, corresponding to annihilation of the $(n-1)$st and nth excitations. We cross the nth particle to an outgoing antiparticle, yielding

$$\langle A,\theta_n,b_n,a_n|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1}\rangle$$
$$=[p_1+\ldots+p_l-(p_{l+1}+\ldots+p_{2l})+p_{n-1}+p_n]_\mu$$
$$\times\langle A,\theta_n,b_n,a_n|\mathcal{O}_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1}\rangle.$$

By the generalized crossing formula [10],

$$\langle A,\theta_n,b_n,a_n|\mathcal{O}_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1}\rangle$$
$$=\langle A,\theta_n,b_n,a_n|A,\theta_1,b_1,a_1\rangle\mathcal{F}^{\mathcal{O}}(\theta_2,\ldots,\theta_{n-1})_{a_0c_0a_2\ldots a_{n-1};b_2\ldots b_{n-1}}$$
$$+\mathcal{F}^{\mathcal{O}}(\theta_n-i\pi_-,\theta_1,\ldots,\theta_{n-1})_{a_0c_0a_na_1\ldots,a_{n-1};b_nb_1\ldots b_{n-1}}\quad \text{for } \theta_n\geq\theta_1>\ldots>\theta_{n-1}\quad\text{or}$$
$$=\langle A,\theta_n,b_n,a_n|A,\theta_{n-1},b_{n-1},a_{n-1}\rangle\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{2l})_{a_0c_0a_1\ldots a_{2l};b_1\ldots b_{2l}}$$
$$+\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},\theta_n+i\pi_-)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}\quad\text{for } \theta_1>\ldots>\theta_{n-1}\geq\theta_n, \quad (3.14)$$

where the right-hand side contains the n- and the $n-2$-particle form factors, and $\pi_-=\pi-\epsilon$. Near the annihilation pole at $\theta_{n-1n}=-\pi i$, the form factors are of the form

$$\mathcal{F}^{\mathcal{O}}(\theta_n-i\pi_-,\theta_1,\ldots,\theta_{n-1})_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}=\frac{1}{\theta_{n-1}-\theta_n+i\epsilon}h(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n},\quad\text{and}$$
$$\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{n-1},\theta_n+i\pi_-)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}=\frac{1}{\theta_{n-1}-\theta_n-i\epsilon}h(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n},$$

where $h(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}$ is an analytic function in θ_{n-1n}. We use the identity

$$\frac{1}{\theta_{n-1}-\theta_n\pm i\epsilon}=\mathbf{P}\bigg\{\frac{1}{\theta_{n-1}-\theta_n}\bigg\}\mp i\pi\delta(\theta_{n-1}-\theta_n),$$

where $\mathbf{P}\{f(\theta_{n-1},\theta_n)\}$ is the principal value of $f(\theta_{n-1},\theta_n)$. We apply Watson's theorem to Eq. (3.14), and find

$$\langle A,\theta_n,b_n,a_n|\mathcal{O}_{a_0c_0}|A,\theta_1,b_1,a_1;\ldots;A,\theta_l,b_l,a_l;P,\theta_{l+1},a_{l+1},b_{l+1};\ldots;P,\theta_{2l},a_{2l},b_{2l};A,\theta_{n-1},b_{n-1},a_{n-1}\rangle$$
$$=\langle A,\theta_n,b_n,a_n|A,\theta_{n-1},b'_{n-1},a'_{n-1}\rangle S_{AA}(\theta_{1n-1})^{b'_{n-1}a'_{n-1};b'_1a'_1}_{d_1c_1;b_1a_1}\times\ldots\times S_{AA}(\theta_{ln-1})^{d_{l-1}c_{l-1};b'_la'_l}_{d_lc_l;b_lc_l}$$
$$\times S_{AP}(\theta_{n-1l+1})^{d_lc_l;a'_{l+1}b'_{l+1}}_{c_{l+1}d_{l+1};a_{l+1}b_{l+1}}\times\ldots\times S_{AP}(\theta_{n-12l})^{c_{2l-1}d_{2l-1};a'_{2l}b'_{2l}}_{c_{2l}d_{2l};a_{2l}b_{2l}}\times\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{2l})_{a_0c_0a'_1\ldots a'_{2l};b'_1\ldots b'_{2l}}$$
$$+\bigg(\mathbf{P}\bigg\{\frac{1}{\theta_{n-1}-\theta_n}\bigg\}-i\pi\delta(\theta_{n-1}-\theta_n)\bigg)h(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}$$
$$=\langle A,\theta_n,b_n,a_n|A,\theta_{n-1},b_{n-1},a_{n-1}\rangle\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{2l})_{a_0c_0a_1\ldots a_{2l};b_1\ldots b_{2l}}$$
$$+\bigg(\mathbf{P}\bigg\{\frac{1}{\theta_{n-1}-\theta_n}\bigg\}+i\pi\delta(\theta_{n-1}-\theta_n)\bigg)h(\theta_1,\ldots,\theta_n)_{a_0c_0a_1\ldots a_n;b_1\ldots b_n}. \quad (3.15)$$

We will use the normalization $\langle A,\theta_n,b_n,a_n|A,\theta_{n-1},b_{n-1},a_{n-1}\rangle=4\pi\delta_{a_{n-1}a_n}\delta_{b_{n-1}b_n}\delta(\theta_{n-1}-\theta_n)$. Comparing the terms proportional to $\delta(\theta_{n-1}-\theta_n)$ in Eq. (3.15), we recover the annihilation-pole axiom [10]:

$$\begin{aligned} &h(\theta_1,\ldots,\theta_{n-1},\theta_{n-1})_{a_0c_0a_1\ldots a_n;b_1\ldots b_n} \\ &\quad = \mathrm{Res}|_{\theta_{n-1n}=-\pi i}\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{2l},\theta_{n-1},\theta_n)_{a_0c_0a_1\ldots a_{2l}a_{n-1}a_n;b_1\ldots b_{2l}b_{n-1}b_n} \\ &\quad = 2i\mathcal{F}^{\mathcal{O}}(\theta_1,\ldots,\theta_{2l})_{a_0c_0a'_1\ldots a'_{2l};b'_1\ldots b'_{2l}}\delta_{a'_{n-1}a_n}\delta_{b'_{n-1}b_n} \\ &\qquad \times(\delta_{a'_1a_1}\ldots\delta_{a'_{n-1}a_{n-1}}\delta_{b'_1b_1}\ldots\delta_{b'_{n-1}b_{n-1}} - S_{AA}(\theta_{1n-1})^{b'_{n-1}a'_{n-1};b'_1a'_1}_{d_1c_1;b_1a_1}\times\ldots\times S_{AA}(\theta_{ln-1})^{d_{l-1}c_{l-1};b'_la'_l}_{d_lc_l;b_lc_l} \\ &\qquad \times S_{AP}(\theta_{n-1l+1})^{d_lc_l;a'_{l+1}b'_{l+1}}_{c_{l+1}d_{l+1};a_{l+1}b_{l+1}}\times\ldots\times S_{AP}(\theta_{n-12l})^{c_{2l-1}d_{2l-1};a'_{2l}b'_{2l}}_{c_{2l}d_{2l};a_{2l}b_{2l}}). \end{aligned} \tag{3.16}$$

IV. TWO-PARTICLE FORM FACTORS AT FINITE N

In this section, we find the exact two-particle form factor of the current operator, for arbitrary $N \geq 2$. For $N = 2$, the principal chiral model is equivalent to an $O(4)$-symmetric vector model. The form factors of currents of the $O(4)$ model were found in Ref. [11].

Our result for the two-particle form factor, for general N, is

$$\begin{aligned} &\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2\rangle \\ &\quad = (p_1-p_2)_\mu F(\theta)\Big(\delta_{a_0a_2}\delta_{b_1b_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{b_1b_2}\delta_{a_1a_2}\Big), \end{aligned}$$

where $F_1(\theta)$ satisfies Eq. (3.8). We insert

$$F(\theta) = \frac{g(\theta)}{\theta+\pi i},$$

into Eq. (3.8), finding

$$g(\theta-2\pi i) = \hat{S}(\theta,N)g(\theta). \tag{4.1}$$

We solve Eq. (4.1) by a contour-integration method first used in Ref. [11]. We define a contour C to be that from $-\infty$ to ∞ and from $\infty + 2\pi i$ to $-\infty + 2\pi i$, bounding the strip in which the form factor is holomorphic. Then,

$$\begin{aligned} \ln g(\theta) &= \int_C \frac{dz}{4\pi i}\coth\frac{z-\theta}{2}\ln g(z) \\ &= \int_{-\infty}^{\infty}\frac{dz}{4\pi i}\coth\frac{z-\theta}{2}\ln\frac{g(z)}{g(z+2\pi i)}. \end{aligned}$$

We differentiate both sides with respect to θ, and use Eq. (4.1) to write

$$\frac{d}{d\theta}[\ln g(\theta)] = \frac{1}{8\pi i}\int_{-\infty}^{\infty}\frac{dz}{\sinh^2\frac{1}{2}(z-\theta)}\ln\hat{S}(z,N). \tag{4.2}$$

The solution to Eq. (4.2) is

$$g(\theta) = g\exp\int_0^\infty dx A(x,N)\frac{\sin^2[x(\pi i-\theta)/2\pi]}{\sinh x}, \tag{4.3}$$

where the function $A(x,N)$ is defined by

$$\hat{S}(\theta,N) = \exp\int_0^\infty dx A(x,N)\sinh\Big(\frac{x\theta}{\pi i}\Big), \tag{4.4}$$

and g is a constant. Note that expanding the S matrix in powers of $1/N$ yields $A(x,N) = \frac{1}{N^2}B(x) + \mathcal{O}(\frac{1}{N^3})$.

To express the function $\hat{S}(\theta,N)$, presented in Eq. (2.6), in the form (4.4), we use the integral formula of the gamma function [12,13],

$$\Gamma(z) = \exp\int_0^\infty\frac{dx}{x}\Big[\frac{e^{-xz}-e^{-x}}{1-e^{-x}} + (z-1)e^{-x}\Big], \quad \text{for } \mathrm{Re}\, z > 0.$$

Then,

$$\Big[\frac{\Gamma(\frac{i\hat{\theta}}{2\pi}+1)\Gamma(\frac{-i\hat{\theta}}{2\pi}-\frac{1}{N})}{\Gamma(\frac{i\hat{\theta}}{2\pi}+1-\frac{1}{N})\Gamma(\frac{-i\hat{\theta}}{2\pi})}\Big]^2 = \exp\int_0^\infty\frac{dx}{x}\frac{4e^{-x}(e^{2x/N}-1)}{1-e^{-2x}}\sinh\Big(\frac{x\theta}{\pi i}\Big), \tag{4.5}$$

for $N > 2$. We use the formula [10]

$$\frac{\sin\frac{\pi}{2}(z+a)}{\sin\frac{\pi}{2}(z-a)} = \exp 2\int_0^\infty\frac{dx}{x}\frac{\sinh x(1-z)}{\sinh x}\sinh(xa), \quad \text{for } 0 < z < 1,$$

to write the CDD factor as

$$\frac{\sinh(\frac{\hat{\theta}}{2}-\frac{\pi i}{N})}{\sinh(\frac{\hat{\theta}}{2}+\frac{\pi i}{N})} = \frac{\sin\frac{\pi}{2}((1-\frac{2}{N})-\frac{\theta}{\pi i})}{\sin\frac{\pi}{2}((1-\frac{2}{N})+\frac{\theta}{\pi i})} = \exp\int_0^\infty\frac{dx}{x}\frac{-2\sinh(2x/N)}{\sinh x}\sinh\Big(\frac{x\theta}{\pi i}\Big), \tag{4.6}$$

for $N > 2$. Combining Eqs. (4.5) and (4.6) gives

$$\hat{S}(\theta,N) = \exp\int_0^\infty\frac{dx}{x}\Big[\frac{-2\sinh(2x/N)}{\sinh x} + \frac{4e^{-x}(e^{2x/N}-1)}{1-2^{-2x}}\Big]\times\sinh\Big(\frac{x\theta}{\pi i}\Big). \tag{4.7}$$

From Eqs. (4.1) and (4.3), the form factor is

$$F_1(\theta) = \frac{g}{(\theta+\pi i)}\exp\int_0^\infty\frac{dx}{x}\Big[\frac{-2\sinh(\frac{2x}{N})}{\sinh x} + \frac{4e^{-x}(e^{2x/N}-1)}{1-e^{-2x}}\Big]\frac{\sin^2[x(\pi i-\theta)/2\pi]}{\sinh x}. \tag{4.8}$$

The condition $F_1(\pi i) = 1$ implies $g = 2\pi i$.

V. FOUR-PARTICLE FORM FACTORS

Next, we find the four-excitation form factor of the current operator, in the large-N limit. Only the form factor with two particles and two antiparticles is nonzero, because of the global symmetry. The most general Lorentz- and $SU(N)\times SU(N)$-invariant four-particle form factor, respecting the tracelessness of the current operator is

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;A,\theta_2,b_2,a_2;P,\theta_3,a_3,b_3;P,\theta_4,a_4,b_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.1}$$

for $\theta_1>\theta_2>\theta_3>\theta_4$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2;A,\theta_3,b_3,a_3;P,\theta_4,a_4,b_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.2}$$

for $\theta_1>\theta_3>\theta_2>\theta_4$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;P,\theta_2,a_2,b_2;P,\theta_3,a_3,b_3;A,\theta_4,b_4,a_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{H}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.3}$$

for $\theta_1>\theta_3>\theta_4>\theta_2$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;P,\theta_3,a_3,b_3;A,\theta_4,b_4,a_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.4}$$

for $\theta_3>\theta_1>\theta_4>\theta_2$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|P,\theta_1,a_1,b_1;P,\theta_2,a_2,b_2;A,\theta_3,b_3,a_3;A,\theta_4,b_4,a_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{L}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.5}$$

for $\theta_3>\theta_4>\theta_1>\theta_2$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|P,\theta_1,a_1,b_1;A,\theta_2,b_2,a_2;A,\theta_3,b_3,a_3;P,\theta_4,a_4,b_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3a_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{Q}(\theta_1,\theta_2,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.6}$$

for $\theta_3>\theta_1>\theta_2>\theta_4$,

$$\begin{aligned}&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_2,b_2,a_2;A,\theta_1,b_1,a_1;P,\theta_3,a_3,b_3;P,\theta_4,a_4,b_4\rangle\\&\quad=\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\&\quad=\frac{1}{N}[p_1+p_2-p_3-p_4]_\mu\vec{F}(\theta_2,\theta_1,\theta_3,\theta_4)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},\end{aligned}\tag{5.7}$$

for $\theta_2>\theta_1>\theta_3,>\theta_4$, and

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;A,\theta_2,b_2,a_2;P,\theta_4,a_4,b_4;P,\theta_3,a_3,b_3\rangle \\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle \\
&\quad = \frac{1}{N}[p_1+p_2-p_3-p_4]_\mu \vec{F}(\theta_1,\theta_2,\theta_4,\theta_3)\cdot\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4},
\end{aligned} \tag{5.8}$$

for $\theta_1 > \theta_2 > \theta_4 > \theta_3$, where we define the eight-component vectors

$$[\vec{D}_{a_0c_0a_1a_2a_3a_4;b_1b_2b_3b_4}] = \begin{pmatrix}
\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_4}\delta_{b_2b_3} \\
\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_3}\delta_{b_2b_4} \\
\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2a_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_4}\delta_{b_2b_3}
\end{pmatrix}, \tag{5.9}$$

$$[\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)] = \begin{pmatrix}
F_1(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_2(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_3(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_4(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_5(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_6(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_7(\theta_1,\theta_2,\theta_3,\theta_4) \\
F_8(\theta_1,\theta_2,\theta_3,\theta_4)
\end{pmatrix},$$

and similarly for $\vec{G}$, $\vec{H}$, $\vec{K}$, $\vec{L}$ and $\vec{Q}$.

Watson's theorem relates the form factors with different ordering of rapidities, yielding

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = S_{AP}(\theta_{23})^{d_2c_2;c_3d_3}_{a_3b_3;b_2a_2}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad = S_{AP}(\theta_{24})^{d_2c_2;c_4d_4}_{a_4b_4;b_2a_2}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}|0\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle \\
&\quad = S_{AP}(\theta_{13})^{d_1c_1;c_3d_3}_{a_3b_3;b_1a_1}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_1}|0\rangle \\
&\quad = S_{AP}(\theta_{14})^{d_1c_1;c_4d_4}_{a_4b_4;b_1a_1}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle \\
&\quad = S_{AP}(\theta_{13})^{d_1c_1;c_3d_3}_{a_3b_3;b_1a_1}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}$$

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}_P(\theta_4)_{a_4b_4}|0\rangle = S_{AA}(\theta_{12})^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,$$

$$\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}_P(\theta_4)_{a_4b_4}|0\rangle = S_{PP}(\theta_{34})^{c_4d_4;c_3d_3}_{a_3b_3;a_4b_4}\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{c_4d_4}\mathfrak{A}^\dagger_P(\theta_3)_{c_3d_3}|0\rangle.$$

These imply, respectively,

$$\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\hat\theta_{23}} & \left(1-\frac{2\pi i}{\hat\theta_{23}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\hat\theta_{23}} & 0 & \left(1-\frac{2\pi i}{\hat\theta_{23}}\right) & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{23}} & 0 \\ 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{23}}+\frac{4\pi^2}{\hat\theta_{23}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{23}}+\frac{4\pi^2}{\hat\theta_{23}^2}\right) & \left(1-\frac{4\pi i}{\hat\theta_{23}}-\frac{4\pi^2}{\hat\theta_{23}^2}\right) & 0 & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{23}}+\frac{4\pi^2}{\hat\theta_{23}^2}\right) \\ 0 & 0 & 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{23}}\right) & \frac{-2\pi i}{N\hat\theta_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{23}} & \left(1-\frac{2\pi i}{\hat\theta_{23}}\right) \end{pmatrix} \times \vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \vec{M}_1(\theta_2,\theta_3)\vec{F}(\theta_1\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.10}$$

$$\vec{H}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} \left(1-\frac{4\pi i}{\hat\theta_{24}}-\frac{4\pi^2}{\hat\theta_{24}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}+\frac{4\pi^2}{\hat\theta_{24}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}+\frac{4\pi^2}{\hat\theta_{24}^2}\right) & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{24}}-\frac{4\pi^2}{\hat\theta_{24}^2}\right) & 0 & 0 \\ 0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & 0 & \frac{-2\pi i}{N\hat\theta_{24}} & \frac{-2\pi i}{N\hat\theta_{24}^2} & 0 & 0 & 0 \\ 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & \frac{-2\pi i}{N\hat\theta_{24}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{24}} & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{24}}\right) & \frac{-2\pi i}{N\hat\theta_{24}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \vec{G}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \vec{M}_2(\theta_2,\theta_4)\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.11}$$

$$\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} \left(1-\frac{2\pi i}{\hat\theta_{13}}\right) & \frac{-2\pi i}{N\hat\theta_{13}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{13}}\right) & \frac{-2\pi i}{N\hat\theta_{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{13}} & \left(1-\frac{2\pi i}{\hat\theta_{13}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{13}}+\frac{4\pi^2}{\hat\theta_{13}^2}\right) & \left(1-\frac{4\pi i}{\hat\theta_{13}}-\frac{4\pi^2}{\hat\theta_{13}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{13}}+\frac{4\pi^2}{\hat\theta_{13}^2}\right) \\ 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{13}} & \frac{-2\pi i}{N\hat\theta_{13}} & 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{13}}\right) \end{pmatrix}$$

$$\times \vec{H}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \overleftrightarrow{M}_3(\theta_1,\theta_3)\vec{H}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.12}$$

$$\vec{L}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\hat\theta_{14}} & \left(1-\frac{2\pi i}{\hat\theta_{14}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\pi i}{N\hat\theta_{14}} & \left(1-\frac{2\pi i}{\hat\theta_{14}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(1-\frac{4\pi i}{\hat\theta_{14}}-\frac{4\pi^2}{\hat\theta_{14}^2}\right) & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{14}}+\frac{4\pi^2}{\hat\theta_{14}^2}\right) & 0 & \frac{-1}{N}\left(\frac{2\pi i}{\hat\theta_{14}}+\frac{4\pi^2}{\hat\theta_{14}^2}\right) \\ \frac{-2\pi i}{N\hat\theta_{14}} & 0 & 0 & 0 & 0 & \left(1-\frac{2\pi i}{\hat\theta_{14}}\right) & \frac{-2\pi i}{N\hat\theta_{14}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\hat\theta_{14}} & \left(1-\frac{2\pi i}{\hat\theta_{14}}\right) \end{pmatrix}$$

$$\times \vec{K}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \overleftrightarrow{M}_4(\theta_1,\theta_4)\vec{K}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.13}$$

$$\vec{Q}(\theta_1,\theta_2,\theta_3,\theta_4) = \overleftrightarrow{M}_3(\theta_1,\theta_3)\vec{G}(\theta_1,\theta_2,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.14}$$

$$\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} \\ 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} \\ 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 \end{pmatrix} \vec{F}(\theta_2,\theta_1,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \overleftarrow{T}_1(\theta_1,\theta_2)\vec{F}(\theta_2,\theta_1,\theta_3,\theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{5.15}$$

$$\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) = \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ 1 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 \end{pmatrix} \vec{F}(\theta_1,\theta_2,\theta_4,\theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

$$\equiv \overleftrightarrow{T}_2(\theta_3,\theta_4)\vec{F}(\theta_1,\theta_2,\theta_4,\theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{5.16}$$

Next, we apply the Smirnov periodicity axiom (3.5):

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1-2\pi i)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle,\\
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_2-2\pi i)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}|0\rangle\\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle,\\
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_3-2\pi i)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}|0\rangle\\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle,\\
&\langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_P(\theta_4-2\pi i)_{a_4b_4}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\\
&\quad = \langle 0|j^L_\mu(0)_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle,
\end{aligned}$$

which imply, respectively,

$$\vec{F}(\theta_1-2\pi i,\theta_2,\theta_3,\theta_4) = \vec{H}(\theta_2,\theta_1,\theta_3,\theta_4), \tag{5.17}$$

$$\vec{H}(\theta_2-2\pi i,\theta_1,\theta_3,\theta_4) = \vec{L}(\theta_1,\theta_2,\theta_3,\theta_4), \tag{5.18}$$

$$\vec{L}(\theta_1,\theta_2,\theta_3-2\pi i,\theta_4) = \vec{Q}(\theta_1,\theta_2,\theta_4,\theta_3), \tag{5.19}$$

$$\vec{Q}(\theta_1,\theta_2,\theta_4-2\pi i,\theta_3) = \vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{5.20}$$

We combine Watson's theorem with the periodicity axiom, to express Eqs. (5.17), (5.18), (5.19), and (5.20) in terms of only $\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4)$. We combine Eq. (5.17) with Eqs. (5.13), (5.12), and (5.15), and find

$$\vec{F}(\theta_1-2\pi i,\theta_2,\theta_3,\theta_4) = \overleftrightarrow{M}_4(\theta_1,\theta_4)\overleftrightarrow{M}_3(\theta_1,\theta_3)[\overleftrightarrow{T}_1(\theta_1,\theta_2)]^{-1}\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{5.21}$$

Combining Eq. (5.18) with Eqs. (5.11), (5.10), and (5.15) gives

$$[\overleftrightarrow{T}_1(\theta_1,\theta_2-2\pi i)]^{-1}\vec{F}(\theta_1,\theta_2-2\pi i,\theta_3,\theta_4) = \overleftrightarrow{M}_2(\theta_2,\theta_4)\overleftrightarrow{M}_1(\theta_2,\theta_4)\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{5.22}$$

Combining Eq. (5.19) with Eqs. (5.12), (5.10), and (5.16) gives

$$\overleftrightarrow{M}_3(\theta_1,\theta_3-2\pi i)\overleftrightarrow{M}_1(\theta_2,\theta_3-2\pi i)\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4) = [\overleftrightarrow{T}_2(\theta_3,\theta_4)]^{-1}\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{5.23}$$

Finally, we combine Eq. (5.20) with Eqs. (5.13), (5.11), and (5.16) to find

$$\overleftrightarrow{M}_4(\theta_1,\theta_4-2\pi i)\overleftrightarrow{M}_2(\theta_2,\theta_4-2\pi i)[\overleftrightarrow{T}_2(\theta_3,\theta_4-2\pi i)]^{-1}\vec{F}(\theta_1,\theta_2,\theta_3,\theta_4-2\pi i) = \vec{F}(\theta_1,\theta_2,\theta_3,\theta_4). \tag{5.24}$$

The set of equations (5.21), (5.22), (5.23), and (5.24) are difficult to solve, for finite N. In the large-N limit, the matrices $\overleftrightarrow{M}_{1,2,3,4}$ become diagonal and mutually commute, and the matrices $\overleftrightarrow{T}_{1,2}$ become their own inverses. This greatly simplifies the problem, allowing us to find the form factors. We expand the form factors in powers of $1/N$ as $\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) = \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4) + \frac{1}{N}\vec{F}^1(\theta_1, \theta_2, \theta_3, \theta_4) + \dots$, simplifying the periodicity conditions for $\vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4)$. We combine Eqs. (5.21) and (5.22) and Eq. (5.22) to get

$$\vec{F}^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) = \vec{M}_4(\theta_1, \theta_4)\vec{M}_3(\theta_1, \theta_3)\vec{M}_2(\theta_2, \theta_4)\vec{M}_1(\theta_2, \theta_3)\vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4), \tag{5.25}$$

or explicitly, in terms of the components of $\vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4)$,

$$\begin{aligned}
F_1^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right)^2 F_1^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_2^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right) F_2^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_3^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right) F_3^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_4^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right)^2 F_4^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_5^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)^2\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right) F_5^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_6^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right) F_6^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_7^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)^2\left(\frac{\theta_{24}+\pi i}{\theta_{24}-\pi i}\right) F_7^0(\theta_1, \theta_2, \theta_3, \theta_4),\\
F_8^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14}+\pi i}{\theta_{14}-\pi i}\right)\left(\frac{\theta_{13}+\pi i}{\theta_{13}-\pi i}\right)\left(\frac{\theta_{23}+\pi i}{\theta_{23}-\pi i}\right) F_8^0(\theta_1, \theta_2, \theta_3, \theta_4).
\end{aligned}$$

The solution which satisfies (5.25), (5.15), and (5.16) is

$$\begin{aligned}
F_1^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_1, \theta_2, \theta_3, \theta_4)}{(\theta_{13}+\pi i)(\theta_{24}+\pi i)^2},\\
F_2^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_1, \theta_2, \theta_3, \theta_4)}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)},\\
F_3^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_1, \theta_2, \theta_4, \theta_3)}{(\theta_{13}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)},\\
F_4^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_1, \theta_2, \theta_4, \theta_3)}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)^2},\\
F_5^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_2, \theta_1, \theta_3, \theta_4)}{(\theta_{14}+\pi i)^2(\theta_{23}+\pi i)},\\
F_6^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_2, \theta_1, \theta_3, \theta_4)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{24}+\pi i)},\\
F_7^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_2, \theta_1, \theta_4, \theta_3)}{(\theta_{13}+\pi i)^2(\theta_{24}+\pi i)},\\
F_8^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_2, \theta_1, \theta_4, \theta_3)}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{23}+\pi i)},
\end{aligned} \tag{5.26}$$

where the functions $g_1(\theta_1, \theta_2, \theta_3, \theta_4)$ and $g_2(\theta_1, \theta_2, \theta_3, \theta_4)$ are periodic under $\theta_{1,2} \to \theta_{1,2} - 2\pi i$.

Instead of the analysis of the previous paragraph, we could have combined Eqs. (5.23) and (5.24) to obtain

$$\begin{aligned}
&\vec{M}_4(\theta_1, \theta_4 - 2\pi i)\vec{M}_3(\theta_1, \theta_3 - 2\pi i)\vec{M}_2(\theta_2, \theta_4 - 2\pi i)\vec{M}_1(\theta_2, \theta_3 - 2\pi i)\vec{F}^0(\theta_1, \theta_2, \theta_3 - 2\pi i, \theta_4 - 2\pi i)\\
&\quad = \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4).
\end{aligned} \tag{5.27}$$

The condition (5.27) is equivalent to Eq. (5.25). The solution of Eq. (5.27) is Eq. (5.26)

The minimal choice for the functions $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ is to set them equal to constants, $g_1(\theta_1, \theta_2, \theta_3, \theta_4) = g_1$, $g_2(\theta_1, \theta_2, \theta_3, \theta_4) = g_2$. These constants are fixed using the annihilation-pole axiom. There is an annihilation pole at $\theta_{24} = -\pi i$. The annihilation-pole axiom [Eq. (3.16)] implies

$$\begin{aligned}
&\mathrm{Res}|_{\theta_{24}=-\pi i}\langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2}\mathfrak{A}^\dagger_P(\theta_4)_{a_4b_4}|0\rangle\\
&\quad = 2i\{\langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\delta_{a_2a_4}\delta_{b_2b_4}\\
&\qquad - \langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1'a_1'}\mathfrak{A}_P(\theta_3)_{a_3'b_3'}|0\rangle\delta_{a_2'a_4}\delta_{b_2'b_4}S_{AA}(\theta_{12})^{b_2'a_2';b_1'a_1'}_{d_1c_1;b_1a_1}S_{AP}(\theta_{23})^{d_1c_1;a_3'b_3'}_{a_3b_3;b_2a_2}\}.
\end{aligned} \tag{5.28}$$

We substitute Eq. (3.10) into the right-hand side of Eq. (5.28) to find

$$\begin{aligned}
&\langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\mathfrak{A}^\dagger_P(\theta_3)_{a_3b_3}|0\rangle\delta_{a_2a_4}\delta_{b_2b_4} - \langle 0|\mathcal{O}_{a_0c_0}\mathfrak{A}^\dagger_A(\theta_1)_{b_1'a_1'}\mathfrak{A}_P(\theta_3)_{a_3'b_3'}|0\rangle\delta_{a_2'a_4}\delta_{b_2'b_4}S_{AA}(\theta_{12})^{b_2'a_2';b_1'a_1'}_{d_1c_1;b_1a_1}S_{AP}(\theta_{23})^{d_1c_1;a_3'b_3'}_{a_3b_3;b_2a_2}\\
&\quad = \frac{2\pi}{(\theta_{13}+\pi i)}\bigg\{\frac{2\pi i}{N\hat{\theta}_{23}}\bigg(\delta_{a_0a_4}\delta_{a_2a_3}\delta_{c_0a_1}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4}\bigg)\\
&\qquad + \frac{1}{N}\bigg(\frac{-2\pi i}{\hat{\theta}_{23}} + \frac{2\pi i}{\theta_{12}} - \frac{4\pi^2}{\theta_{12}\hat{\theta}_{23}}\bigg)\bigg(\delta_{a_0a_3}\delta_{a_2a_4}\delta_{a_1c_0}\delta_{b_2b_3}\delta_{b_1b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_2b_3}\delta_{b_1b_4}\bigg)\\
&\qquad \times \frac{-2\pi i}{N\theta_{12}}\bigg(\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4}\bigg)\bigg\}.
\end{aligned}$$

Equation (5.28) yields for the constants $g_2 = 8\pi^2 i$, $g_1 = 0$. It is worth mentioning that the constants g_1 and g_2 are determined in terms of the normalization constant g of the two-particle form factor. If we chose $g(\theta)$ from Eq. (3.10) to be a more general periodic function, and not a constant, then it would be inconsistent with the annihilation-pole axiom to make $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ constants. In this way, the choice of the functions $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ (and probably the arbitrary periodic functions which emerge from form factors with more particles) is at least partially fixed by the choice of solution for the two-particle form factor. We notice that the double poles present in Eq. (5.26) vanish, because $g_1 = 0$. The first term on the right-hand side of Eq. (5.28) is of order $1/N$. This is the reason we introduced a factor of $1/N$ in Eqs. (5.1) through (5.8).

The minimal four-particle form factor satisfying all of Smirnov's axioms for large N is

$$\begin{aligned}
&\langle 0|j^L_\mu(0)_{a_0c_0}|A,\theta_1,b_1,a_1;A,\theta_2,b_2,a_2;P,\theta_3,a_3,b_3;P,\theta_4,a_4,b_4\rangle\\
&\quad = [p_1+p_2-p_3-p_4]_\mu\frac{8\pi^2 i}{N}\\
&\qquad\times\bigg\{\frac{1}{(\theta_{14}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)}\bigg(\delta_{a_0a_3}\delta_{a_1c_0}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3}\bigg)\\
&\qquad + \frac{1}{(\theta_{13}+\pi i)(\theta_{23}+\pi i)(\theta_{24}+\pi i)}\bigg(\delta_{a_0a_4}\delta_{a_1c_0}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4}\bigg)\\
&\qquad + \frac{1}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{24}+\pi i)}\bigg(\delta_{a_0a_3}\delta_{a_1a_4}\delta_{a_2c_0}\delta_{b_1b_3}\delta_{b_2b_4} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4}\bigg)\\
&\qquad + \frac{1}{(\theta_{14}+\pi i)(\theta_{13}+\pi i)(\theta_{23}+\pi i)}\bigg(\delta_{a_0a_4}\delta_{a_1a_3}\delta_{a_2c_0}\delta_{b_1b_4}\delta_{b_2b_3} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\delta_{a_1a_3}\delta_{b_1b_4}\delta_{b_2b_3}\bigg)\bigg\},
\end{aligned} \tag{5.29}$$

which is the main result of this section.

VI. CONCLUSIONS

We found the two-particle form factor of the principal-chiral-model current operator, for general N. We were only able to find the four-particle form factor for large N, because the S matrix is much simpler in this limit.

Form factors of more excitations can be calculated at large N, using this method. As we add particles, the number of functions to determine grows very fast. This will be tedious, but perhaps not impossible. We hope it is possible to calculate all the form factors in the planar limit. With knowledge of all the form factors, we can write down an expression for the Wightman function which should be valid at all energy scales. Since the theory is asymptotically free, the high energy (short distances) limit of this Wightman function should correspond to the perturbative, weakly coupled regime. In principle, we should find Wightman and Green functions which contain the results from perturbation theory in the high-energy limit. This problem is under investigation.

We are interested in applying the form factors found here to $(2+1)$-dimensional anisotropic Yang-Mills theory.

This is a theory were the coupling constants are weak, but different in different directions. The form factors of the $O(4)$-symmetric sigma model were used to calculate the string tension [14], and the glueball masses [15] of the $SU(2)$ gauge theory. We can apply our results to extend this treatment beyond the $SU(2)$ gauge group.

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