Upper limits of particle emission from high-energy collision and reaction near a maximally rotating Kerr black hole

Tomohiro Harada,[1,2,*] Hiroya Nemoto,[1] and Umpei Miyamoto[1]

[1]*Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan*
[2]*Astronomy Unit, School of Physics and Astronomy, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom*
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The center-of-mass energy of two particles colliding near the horizon of a maximally rotating black hole can be arbitrarily high if the angular momentum of either of the incident particles is fine-tuned, which we call a critical particle. We study particle emission from such high-energy collision and reaction in the equatorial plane fully analytically. We show that the unconditional upper limit of the energy of the emitted particle is given by 218.6% of that of the injected critical particle, irrespective of the details of the reaction and this upper limit can be realized for massless particle emission. The upper limit of the energy extraction efficiency for this emission as a collisional Penrose process is given by 146.6%, which can be realized in the collision of two massive particles with optimized mass ratio. Moreover, we analyze perfectly elastic collision, Compton scattering, and pair annihilation and show that net positive energy extraction is really possible for these three reactions. The Compton scattering is most efficient among them and the efficiency can reach 134.3%. On the other hand, our result is qualitatively consistent with the earlier claim that the mass and energy of the emitted particle are at most of order the total energy of the injected particles and hence we can observe neither super-heavy nor super-energetic particles. The present paper places the baseline for the study of particle emission from high-energy collision near a rapidly rotating black hole.

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I. INTRODUCTION

Bañados, Silk, and West (2009) [1] have indicated rapidly rotating Kerr black holes as particle accelerators based on the demonstration that the center-of-mass (CM) energy of two colliding particles can be arbitrarily high near the horizon of a maximally rotating Kerr black hole if the angular momentum of either of the particles is finely tuned. Hereafter, we refer to this process as the *Bañados-Silk-West (BSW) process* or *BSW collision*. In fact, the collision with infinite CM energy has already been noticed by Piran, Shaham, and Katz (1975) [2–4] in the study of an energy extraction process by two colliding particles in the ergo region, which is called a *collisional Penrose process*. Recently, the particle acceleration by Kerr black holes has been investigated in different respects [5–12], while it turns out that this phenomenon can be regarded as one of the general properties of extremal and near-extremal black holes [13–22] and other gravitating objects which are near-extremal in some specific sense [23–27].

As for observability, we need to consider the emission from the BSW process. The observed flux and characteristic spectrum from the pair annihilation of dark matter particles through the BSW collision around a Kerr black hole have been demonstrated in Refs. [28,29]. Since the collision with high CM energy can produce very massive particles, one might expect highly energetic particles can escape to infinity and be observed by a distant observer as the black hole is fed with product counterparts with largely negative energy. On the other hand, Jacobson and Sotiriou (2010) [6] have claimed that for the collision of two particles of equal mass m_0, an ejecta particle cannot be more energetic than $2m_0$ and the energy upper limit of the ejecta tends to m_0 in the limit of infinite CM energy. If this were the case, the BSW process would not be applicable to a collisional Penrose process.

In the present paper, we give the general formulation for the BSW collision and subsequent reaction. Based on this, we study the mass and energy of the particle which escapes to infinity and obtain the unconditional upper limits of its mass and energy. We further derive the upper limit of the energy extraction efficiency for this upper limit of energy emission as a collisional Penrose process. We find that net positive energy extraction is really possible, although the efficiency is not very high but modest. We also study the upper limits of the energy of the emitted particle for specific physical processes and find that the energy extraction is really possible. In summary, although the BSW process can be an applicable energy extraction mechanism to a collisional Penrose process, the mass and energy of the particles observable to a distant observer are at most of order the total energy of the injected particles.

We use the units in which $G = c = 1$ and follows the abstract index notation by Wald [30].

*harada@rikkyo.ac.jp

II. GEODESIC ORBIT, COLLISION AND REACTION

A. Preliminaries

The line element in the Kerr spacetime in the Boyer-Lindquist coordinates is written in the following form [30–32]:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2,$$

where a and M are the spin and mass parameters, respectively, $\rho^2(r) = r^2 + a^2\cos^2\theta$ and $\Delta(r) = r^2 - 2Mr + a^2$. If $0 < a^2 \le M^2$, Δ vanishes at $r = r_\pm = M \pm \sqrt{M^2 - a^2}$, where $r = r_+$ and $r = r_-$ correspond to an event and Cauchy horizons, respectively. Here, we denote $r_+ = r_H$. Later, we will focus on the extremal case $a = M$.

In this paper we concentrate on geodesic particles in the equatorial plane $\theta = \pi/2$. For a particle of mass m, energy E, and angular momentum L, the components of the four-momentum are given by

$$p^t = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E - \frac{2Ma}{r}L\right], \tag{2.1}$$

$$p^\phi = \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)L + \frac{2Ma}{r}E\right], \tag{2.2}$$

$$p^\theta = 0, \tag{2.3}$$

and

$$\frac{1}{2}(p^r)^2 + V(r) = 0, \tag{2.4}$$

where $V(r)$ is the effective potential given by

$$V(r) = -\frac{Mm^2}{r} + \frac{L^2 - a^2(E^2 - m^2)}{2r^2} - \frac{M(L - aE)^2}{r^3} - \frac{E^2 - m^2}{2}. \tag{2.5}$$

For a massless particle, we only have to choose $m = 0$ in the above. For a massive particle, the four-velocity u^a, which is normalized as $u^a u_a = -1$, is given by $u^a = p^a/m$. The forward-in-time condition $p^t > 0$ gives

$$\frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E - \frac{2Ma}{r}L\right] > 0. \tag{2.6}$$

In particular, this condition in the vicinity of the horizon $r \to r_H + 0$ reduces to

$$E - \Omega_H L \ge 0, \tag{2.7}$$

where $\Omega_H = a/(r_H^2 + a^2)$ is the angular velocity of the horizon. We call $L_c = E/\Omega_H$ a critical angular momentum and a particle with this value of angular momentum a critical particle.

B. Escape to infinity

Next we discuss the escape of a particle to infinity based on the effective potential. First we consider massless particles. Solving $V(r) = 0$ for the impact parameter $b = L/E$, we obtain

$$b = b_\pm(r) = \frac{-2aM \pm r\sqrt{\Delta(r)}}{r - 2M}. \tag{2.8}$$

This means that a massless particle with impact parameter $b = b_\pm(r)$ has a turning point at r. In particular, for $a = M$, we have

$$b_+(r) = r + M, \qquad b_-(r) = -\left(r + M + \frac{4M^2}{r - 2M}\right). \tag{2.9}$$

$b_+(r)$ begins with $2M$ and monotonically increases to infinity as r increases from M to infinity. $b_-(r)$ begins with $2M$, is larger than $b_+(r)$, and monotonically increases to infinity as r increases from M to $2M$. As r increases beyond $2M$ to infinity, $b_-(r)$ begins with negative infinity, monotonically increases to a local maximum $-7M$ at $r = 4M$, and monotonically decreases to negative infinity. Thus, for $-7M < b < 2M$, the particle escapes to infinity, if it is moving outwardly outside the turning point initially. For $b = 2M$ or $b = -7M$, the particle escapes to infinity, if it is moving outwardly outside the turning point initially.

For massive particles, the situation is similar except for energy dependence. For convenience, we define $e = E/m$ and $\ell = L/(mM)$. For a massive particle, solving $V(r) = 0$ for ℓ, we obtain

$$\ell = \ell_\pm(r) = \frac{-2aMe \pm r\sqrt{\Delta(r)[(e^2 - 1) + 2M/r]}}{M(r - 2M)}. \tag{2.10}$$

This means that a massive particle with angular momentum $\ell = \ell_\pm(r)$ has a turning point at r. For bound particles, i.e. $e < 1$, $V(r)$ becomes positive as r goes sufficiently large, indicating that they cannot reach infinity but bounce back inwardly. Therefore, we concentrate on marginally bound and unbound particles, i.e. $e \ge 1$. For the maximal rotation $a = M$, $\ell_+(r)$ begins with $2e$ and monotonically increases to infinity as r increases from M to infinity. $\ell_-(r)$ begins with $2e$, is larger than $\ell_+(r)$, and monotonically increases to infinity as r increases from M to $2M$. As r increases beyond $2M$ to infinity, $\ell_-(r)$ begins with negative infinity, monotonically increases to a negative local maximum value $\ell_{-,\max}(e)$, and then monotonically decreases to negative infinity. This means that the particle with ℓ satisfying $\ell_L(e) < \ell < \ell_R(e)$, where $\ell_R(e) = 2e$ and $\ell_L(e) = \ell_{-,\max}(e)$, escapes to infinity if it is moving outwardly initially. If the particle with ℓ satisfying $\ell > \ell_R(e)$ or $\ell < \ell_L(e)$ is outside the outer turning point, it eventually escapes to infinity irrespective of the sign of the initial radial velocity. For $\ell = \ell_L(r)$ or $\ell = \ell_R(r)$, the particle escapes to infinity, if it is moving outwardly outside the turning point initially. In other cases, the particle cannot escape to infinity.

C. Particle collision and reaction

Here we consider the reaction of particles 1 and 2 into 3 and 4. We assume geodesic motion of each particle. The local conservation of four-momentum before and after the collision is given by

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu. \tag{2.11}$$

$\mu = t$ and $\mu = \phi$ yield the conservations of energy and angular momentum before and after the collision, i.e.

$$E_1 + E_2 = E_3 + E_4, \tag{2.12}$$

and

$$L_1 + L_2 = L_3 + L_4, \tag{2.13}$$

respectively. $\mu = r$ yields

$$p_1^r + p_2^r = p_3^r + p_4^r. \tag{2.14}$$

Given incident particles 1 and 2, if we specify m_3, E_3 and L_3, we can determine m_4, E_4, and L_4. In fact, m_4 can be expressed in terms of the quantities of other three particles as follows:

$$m_4^2 = -p_{4a}p_4^a = -(p_1^a + p_2^a - p_3^a)(p_{1a} + p_{2a} - p_{3a}). \tag{2.15}$$

On the other hand, the CM energy of particles 1 and 2 is given by

$$E_{\rm cm}^2 = -(p_1^a + p_2^a)(p_{1a} + p_{2a}). \tag{2.16}$$

From the energy conservation, the total rest mass of product particles 3 and 4 must be smaller than or equal to the CM energy, i.e.

$$m_3 + m_4 \le E_{\rm cm}. \tag{2.17}$$

The BSW process is characterized by $\tilde{L}_1 = 2E_1$, $\tilde{L}_2 < 2E_2$, and $r \approx M$ for a maximally rotating black hole $a = M$, where we have put $\tilde{L} = L/M$ for brevity. The CM energy in this special case is derived in Refs. [1,6–9] in an explicit form as follows:

$$E_{\rm cm} \approx \sqrt{\frac{2(2E_1 - \sqrt{3E_1^2 - m_1^2})(2E_2 - \tilde{L}_2)}{\epsilon}}, \tag{2.18}$$

where we denote the radius of the collision point as $r = M/(1-\epsilon)$ and $0 < \epsilon \ll 1$. For a critical particle, $E_1 > m_1/\sqrt{3}$ must be satisfied. As $\epsilon \to 0$, the CM energy is diverging.

III. COLLISION AND REACTION NEAR THE HORIZON

A. Collision and reaction on the horizon

From now on, we assume that the black hole is maximally rotating or $a = M$. We first consider the collision at $r = r_H = M$. We assume that particle 1 is critical, while particle 2 is subcritical, i.e. $\tilde{L}_1 = 2E_1$ and $\tilde{L}_2 < 2E_2$. Note that although the collision we consider here is unphysical because it takes infinite proper time for particle 1 to reach the horizon, it helps us to consider physical processes later. The forward-in-time condition on the horizon for particles 3 and 4 gives

$$2E_3 - (2E_2 - \tilde{L}_2) \le \tilde{L}_3 \le 2E_3. \tag{3.1}$$

On the horizon $r = r_H = M$, from Eqs. (2.4) and (2.5), we obtain

$$p^r = \sigma(2E - \tilde{L}), \tag{3.2}$$

where σ is the sign of p^r and we have taken the forward-in-time condition into account to open the square root. Using Eq. (3.2), we can show that the left-hand side of Eq. (2.14) becomes

$$\sigma_2(2E_2 - \tilde{L}_2), \tag{3.3}$$

where we choose $\sigma_2 = -1$. The right-hand side of Eq. (2.14) is

$$\sigma_3(2E_2 - \tilde{L}_2) \tag{3.4}$$

for $\sigma_3 = \sigma_4$, while it is

$$\sigma_3[4E_3 - 2E_2 - (2\tilde{L}_3 - \tilde{L}_2)] \tag{3.5}$$

for $\sigma_3 = -\sigma_4$. Hence, we can conclude $\sigma_3 = \sigma_4 = -1$ for the former case, while

$$2E_3 - \tilde{L}_3 = 0 \tag{3.6}$$

for $\sigma_3 = 1$, and

$$2E_3 - \tilde{L}_3 = 2E_2 - \tilde{L}_2 \tag{3.7}$$

for $\sigma_3 = -1$ for the latter case.

Note that particle 3 cannot leave the black hole because it is released on the horizon. It is natural to introduce a reaction in the vicinity of the horizon as a small perturbation of the on-horizon reaction. It is clear that we should concentrate on the case $\tilde{L}_3 = 2E_3$, otherwise there is no chance for a distant observer to observe particle 3 even if the collision is slightly perturbed. This fixes $\sigma_4 = -1$.

B. Near-horizon behavior of particles

We consider a collision near the horizon, where $r = M/(1-\epsilon)$ and $\tilde{L}_3 = 2E_3(1+\delta)$, where $0 < \epsilon \ll 1$ and $|\delta| \ll 1$. We assume that δ can be expanded in powers of ϵ as follows:

$$\delta = \delta_{(1)}\epsilon + \delta_{(2)}\epsilon^2 + O(\epsilon^3). \tag{3.8}$$

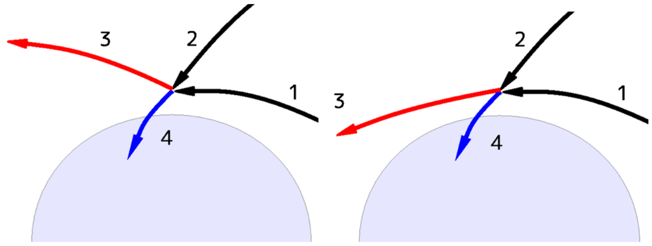

FIG. 1 (color online). The left and right panels are the schematic figures of reactions, where particle 3 has outward ($\sigma_3 = 1$) and inward ($\sigma_3 = -1$) initial velocities, respectively.

This assumption will be justified later because it gives a consistent expansion of the four-momentum conservation.

Here, we require particle 3 to escape to infinity. This is possible in the following two cases: (a) $e_3 \geq 1$, $\ell_L(e_3) < l\ell_3 \leq \ell_R(e_3)$, and $\sigma_3 = 1$ and (b) $e_3 \geq 1$, $\ell_3 > \ell_R(e_3)$, and $r \geq r_{t,+}(e_3)$, where $r_{t,+}(e)$ is the radius of the outer turning point for a particle with e. The left and right panels of Fig. 1 give the schematic figures for $\sigma_3 = 1$ and -1, respectively. The reason why the case $\ell_3 \leq \ell_L(e_3)$ is not considered is that the two turning points are both well outside the horizon in this case.

Under these conditions, we will see the upper limits of the mass m_3 and energy E_3 of particle 3. Since $\ell_R(e) = 2e$ for a maximally rotating Kerr black hole, we need to have $\delta \leq 0$ and $\sigma_3 = 1$ for case (a). For case (b), since the turning points are given by

$$r_{t,\pm}(e) = M\left(1 + \frac{2e}{2e \mp \sqrt{e^2+1}}\delta_{(1)}\epsilon\right) + O(\epsilon^2), \quad (3.9)$$

$r \geq r_{t,+}(e)$ implies

$$0 \leq \delta_{(1)} \leq \frac{2E_3 - \sqrt{E_3^2 + m_3^2}}{2E_3} = \delta_{(1),\max}. \quad (3.10)$$

Note that the forward-in-time condition Eq. (2.6) onto particle 3 in the vicinity of the horizon reduces to

$$\delta < \epsilon + \frac{7}{4}\epsilon^2 + O(\epsilon^3). \quad (3.11)$$

Therefore, $\delta_{(1)} < 1$ gives a sufficient condition and this is already guaranteed for both cases (a) and (b).

We can easily show that the above argument applies for massless particles by the appropriate replacement of ℓ with b and taking the limit $m \to 0$ and $e \to \infty$ in Eqs. (3.9) and (3.10).

C. Local momentum conservation

To look into the local momentum conservation, we use a series of $|p^r|$ in powers of ϵ for each particle as follows:

$$|p_1^r| = \sqrt{3E_1^2 - m_1^2}\epsilon - \frac{E_1^2}{\sqrt{3E_1^2 - m_1^2}}\epsilon^2 + O(\epsilon^3), \quad (3.12)$$

$$|p_2^r| = (2E_2 - \tilde{L}_2) + 2(\tilde{L}_2 - E_2)\epsilon + \frac{\tilde{L}_2^2 - 4\tilde{L}_2E_2 + 3E_2^2 - m_2^2}{2(2E_2 - \tilde{L}_2)}\epsilon^2 + O(\epsilon^3), \quad (3.13)$$

$$|p_3^r| = \sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}\epsilon - \frac{E_3^2[1 - 4(2\delta_{(1)} - \delta_{(2)})(1 - \delta_{(1)})]}{\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}}\epsilon^2 + O(\epsilon^3), \quad (3.14)$$

$$|p_4^r| = (2E_2 - \tilde{L}_2) + [2(\tilde{L}_2 - E_2) + 2E_3(\delta_{(1)} - 1) + 2E_1]\epsilon + \left[\frac{(2E_2 - \tilde{L}_2)}{2} - 2(2\delta_{(1)} - \delta_{(2)})E_3 - \frac{(E_1 + E_2 - E_3)^2 + m_4^2}{2(2E_2 - \tilde{L}_2)}\right]\epsilon^2 + O(\epsilon^3), \quad (3.15)$$

where in the last equation we have used Eqs. (2.12) and (2.13) to eliminate E_4 and $\tilde{L}_4$.

The first and second order terms of ϵ in Eq. (2.14) then give

$$(2E_1 - \sqrt{3E_1^2 - m_1^2}) + 2E_3(\delta_{(1)} - 1) = \sigma_3\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2} \quad (3.16)$$

and

$$\frac{E_1^2}{\sqrt{3E_1^2 - m_1^2}} + \frac{\tilde{L}_2^2 - 4\tilde{L}_2E_2 + 3E_2^2 - m_2^2}{2(\tilde{L}_2 - 2E_2)} = -\sigma_3\frac{E_3^2[1 - 4(2\delta_{(1)} - \delta_{(2)})(1 - \delta_{(1)})]}{\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}} - \left[\frac{(2E_2 - \tilde{L}_2)}{2} - 2(2\delta_{(1)} - \delta_{(2)})E_3 - \frac{(E_1 + E_2 - E_3)^2 + m_4^2}{2(2E_2 - \tilde{L}_2)}\right], \quad (3.17)$$

respectively.

IV. UNCONDITIONAL UPPER LIMITS FOR GENERAL REACTION

A. Mass and energy of the emitted particle

Taking the square of the both sides of Eq. (3.16), we can derive

$$1 - \delta_{(1)} = \frac{A_1^2 + (E_3^2 + m_3^2)}{4A_1E_3}, \quad (4.1)$$

where we put $A_1 = 2E_1 - \sqrt{3E_1^2 - m_1^2} > 0$ for convenience. Note that Eq. (4.1) immediately implies

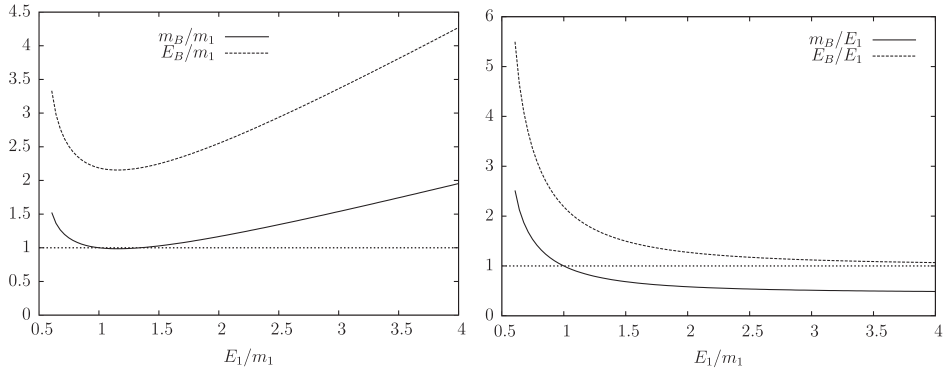

FIG. 2. Upper limits of the mass and energy of the emitted particle as functions of the energy of the incident critical particle. The left and right panels show the ratios to the mass and to the energy of the incident critical particle, respectively. The mass and energy upper limits are denoted by the solid and dashed lines, respectively.

$$\delta_{(1),\max} - \delta_{(1)} = \frac{(A_1 - \sqrt{E_3^2 + m_3^2})^2}{4A_1E_3} \geq 0. \quad (4.2)$$

First we consider case (a). Substituting Eq. (4.1) into the left-hand side of Eq. (3.16), we obtain

$$A_1 - \frac{E_3^2 + m_3^2}{A_1} = 2\sigma_3\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}. \quad (4.3)$$

Since $\sigma_3 = 1$, Eq. (4.3) implies

$$A_1^2 - (E_3^2 + m_3^2) \geq 0. \quad (4.4)$$

This implies $m_3 \leq A_1$ and

$$E_3 \leq \sqrt{A_1^2 - m_3^2} = \lambda_0. \quad (4.5)$$

Since $\lambda_0 \leq (2 - \sqrt{2})E_1$ for $E_1 \geq m_1$, which we assume as the injection of particle 1 from infinity to the system, we cannot extract net positive energy with $\sigma_3 = 1$.

For case (b), only the range given by Eq. (3.10) is permitted. Although both $\sigma_3 = \pm 1$ are possible, we cannot extract net positive energy for $\sigma_3 = 1$ as we have already shown. So, we will concentrate on the case $\sigma_3 = -1$. Equation (4.3) then implies

$$E_3^2 \geq A_1^2 - m_3^2. \quad (4.6)$$

$\delta_{(1)} \geq 0$ in Eq. (4.1) implies

$$A_1^2 + (E_3^2 + m_3^2) - 4A_1E_3 \leq 0. \quad (4.7)$$

The discriminant D and roots $\lambda_\pm$ of the left-hand side of Eq. (4.7) as a quadratic of E_3 are given by

$$D/4 = 3A_1^2 - m_3^2 \quad (4.8)$$

and

$$\lambda_\pm = 2A_1 \pm \sqrt{3A_1^2 - m_3^2}, \quad (4.9)$$

respectively. The solution of Eq. (4.7) is given by

$$\lambda_- \leq E_3 \leq \lambda_+, \quad (4.10)$$

where $D \geq 0$ or $m_3 \leq \sqrt{3}A_1$ must be satisfied. To have $E_3 \geq m_3$, we need $\lambda_+ \geq m_3$, for which $m_3 \leq A_1/(2 - \sqrt{2})$ must be satisfied. The condition (4.6) is satisfied for $E_3 = \lambda_+$ trivially if $m_3 \geq A_1$ and because $\lambda_+ > \lambda_0$ if $0 \leq m_3 < A_1$. $\delta_{(1)} = 0$ holds for $E_3 = \lambda_\pm$. We should note that $\lambda_+ = E_1$ if both particles 1 and 3 are massless, $\lambda_+ < E_1$ if particles 1 and 3 are massless and massive, respectively, but $\lambda_+ > E_1$ if particles 1 and 3 are massive and massless, respectively. Since $\lambda_+ \leq E_1$ in the limit $E_1/m_1 \to \infty$, no net positive energy extraction is possible if the incident critical particle is highly energetic or massless.

From the above argument, the unconditional upper limits of the mass and energy of the emitted particle 3 are given by

$$m_3 \leq (2E_1 - \sqrt{3E_1^2 - m_1^2})/(2 - \sqrt{2}) = m_B \quad (4.11)$$

and

$$E_3 \leq (2E_1 - \sqrt{3E_1^2 - m_1^2})/(2 - \sqrt{3}) = E_B, \quad (4.12)$$

respectively. Note that $\lambda_+ = E_B$ can be realized only if particle 3 is massless. Figure 2 shows the upper limits as functions of E_1/m_1. $m_B/m_1 = 1$ at $E_1/m_1 = 1$ and $7 - 4\sqrt{2}$ and m_B/m_1 takes a minimum $(2 + \sqrt{2})/(2\sqrt{3}) \simeq 0.9856$ at $E_1/m_1 = 2/\sqrt{3}$. $E_B/m_1 = (2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186$ at $E_1/m_1 = 1$ and $7 - 4\sqrt{2}$ and E_B/m_1 takes a minimum $1 + 2/\sqrt{3} \simeq 2.154$ at $E_1/m_1 = 2/\sqrt{3}$. On the other hand, both m_B/E_1 and E_B/E_1 monotonically decrease as E_1/m_1 increases. m_B/E_1 takes a maximum 1 at $E_1/m_1 = 1$ and approaches $(2 - \sqrt{3})/(2 - \sqrt{2}) \simeq 0.4574$ as E_1/m_1 increases from 1 to infinity. E_B/E_1 takes a maximum $(2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186$ at $E_1/m_1 = 1$ and approaches 1 as E_1/m_1 increases from 1 to infinity. The mass and energy of the emitted particle can be at most of order the energy of the incident critical particle. The upper limit m_B of the mass of the emitted particle is approximately equal to m_1 for $E_1 \simeq m_1$ but can be much larger than m_1 for $E_1 \gg m_1$. Since $E_B > E_1$, we might obtain the energy of the ejecta particle more than the total energy of the injected particles. This possibility will be investigated in Sec. IV B.

B. Energy extraction efficiency

Equation (3.17) can be solved for m_4^2 as follows:

$$m_4^2 = (2E_2 - \tilde{L}_2)\Big[\frac{2E_1^2}{\sqrt{3E_1^2 - m_1^2}} - 4(2\delta_{(1)} - \delta_{(2)})E_3 + 2\sigma_3 \frac{E_3^2[1 - 4(2\delta_{(1)} - \delta_{(2)})(1 - \delta_{(1)})]}{\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}}\Big] + (E_2^2 + m_2^2) - (E_1 + E_2 - E_3)^2. \quad (4.13)$$

Since $\delta_{(1)}$ is given by Eq. (4.1), we can obtain $\delta_{(2)}$ using m_3 and E_3 for given m_4. E_4 and $\tilde{L}_4$ are given by Eqs. (2.12) and (2.13). For the collision to occur, $m_4^2 \geq 0$ must be satisfied. We should note that since m_2, E_2, $\tilde{L}_2$, and $\delta_{(2)}$, which do not appear in Eq. (4.1), do appear in Eq. (4.13), the condition $m_4^2 \geq 0$ can be generally satisfied. Equation (4.13) seems to suggest that we can expect very large m_4 as $E_1 \to m_1/\sqrt{3}$, although particle 4 cannot escape to infinity. However, $E_1 \to m_1/\sqrt{3}$ is a singular limit in the series of $|p_1^r|$ given by Eq. (3.12). In Appendix A, we demonstrate that the apparently divergent term in this limit is replaced with a finite term for a particle circularly orbiting near the horizon.

In Sec. IV A, we have seen that the upper limit $E_3 = \lambda_+$ can be realized only for $\delta_{(1)} = 0$. Here we show that this emission can be realized and place the upper limit of the efficiency of the energy extraction for this emission. The expression for m_4 is reduced to a simpler form for $\delta_{(1)} = 0$ and $E_3 = \lambda_+$ as follows:

$$m_4^2 = (2E_2 - \tilde{L}_2)\Big[\frac{2E_1^2}{\sqrt{3E_1^2 - m_1^2}} - \frac{2\lambda_+^2}{\sqrt{3\lambda_+^2 - m_3^2}} - 4\frac{2\lambda_+ - \sqrt{3\lambda_+^2 - m_3^2}}{\sqrt{3\lambda_+^2 - m_3^2}}\lambda_+\delta_{(2)}\Big] + (E_2^2 + m_2^2) - (E_1 + E_2 - \lambda_+)^2. \quad (4.14)$$

This means that even if $\delta_{(1)} = 0$, we can still have different values for m_4 by adjusting $\delta_{(2)}$.

As we have already seen, we can obtain net positive energy gain only for $\delta > 0$. Since the upper limit $E_3 = \lambda_+$ is obtained for $\delta_{(1)} = 0$, we need to assume $\delta_{(2)} \geq 0$. Then, since $r_{t,+} = M + O(\epsilon^2)$, the collision point $r = M/(1 - \epsilon)$ is outside the outer turning point. Since $\lambda_+ = E_B$ only if particle 3 is massless, we concentrate on this case. In this case, we can prove that the first term on the right-hand side of Eq. (4.14) is negative. The condition for E_2 is then given by

$$E_2 \geq \frac{1}{2}\Big[(\lambda_+ - E_1) - \frac{m_2^2}{\lambda_+ - E_1}\Big] = \kappa. \quad (4.15)$$

The proof for this condition will be postponed until Appendix B. This implies that E_2 cannot vanish but greater than or equal to $(\lambda_+ - E_1)/2$ even if particle 2 is massless and that particle 2 must be unbound if $\kappa > m_2$. Conversely, we can always find m_4 and $\tilde{L}_2$ satisfying $m_4^2 \geq 0$ and $\tilde{L}_2 < 2E_2$ if the above inequality is satisfied.

Since Eq. (4.15) potentially gives a lower limit of E_2, this can constrain the efficiency of the energy extraction $\eta = E_3/(E_1 + E_2)$ for $E_3 = \lambda_+$. To estimate η, we here assume $E_2 \geq m_2$ as usual, i.e. we inject the two incident particles from infinity. If $\kappa > m_2$ or $m_2 < (\lambda_+ - E_1)/(\sqrt{2} + 1)$, we find

$$\eta \leq \frac{\lambda_+}{E_1 + \kappa} = 1 + \frac{(\lambda_+ - E_1)^2 + m_2^2}{\lambda_+^2 - E_1^2 - m_2^2}. \quad (4.16)$$

Therefore, the upper limit exceeds unity and hence we can obtain net positive energy extraction. If $\kappa \leq m_2$ or $m_2 \geq (\lambda_+ - E_1)/(\sqrt{2} + 1)$, we find

$$\eta \leq \frac{\lambda_+}{E_1 + m_2}. \quad (4.17)$$

Hence, net positive energy extraction is possible with the upper limit if and only if $m_2 < \lambda_+ - E_1$.

We can here determine the unconditional upper limit of η for $E_3 = \lambda_+$. Since λ_+ does not depend on E_2, to maximize the upper limit of η, we should first find the value for m_2 which minimizes E_2 for fixed E_1 and m_1. This corresponds to the case $\kappa = m_2$ or $m_2 = (\lambda_+ - E_1)/(\sqrt{2} + 1)$. η is then maximized for $E_1 = m_1$. Therefore, the unconditional upper limit is given by

$$\eta_B = \frac{\lambda_+}{m_1 + m_2} = \frac{(\sqrt{2} + 1)\lambda_+}{\sqrt{2}m_1 + \lambda_+}.$$

Since λ_+ takes the upper limit $E_B = (2 + \sqrt{3})(2 - \sqrt{2})m_1$ for $m_3 = 0$, the unconditional upper limit is given by

$$\eta_B = \frac{2 + \sqrt{2} + \sqrt{6}}{4} \simeq 1.466,$$

where $m_2/m_1 = (5\sqrt{2} - 4\sqrt{3} + 3\sqrt{6} - 7) \simeq 0.4913$. Because we impose the condition $E_2 \geq m_2$, the upper limit of the efficiency is realized at the crossing point of the two curves, $E_2 = \kappa$ and $E_2 = m_2$. Figure 3 shows the upper limit of η as a function of the mass ratio m_2/m_1, where we choose particle 1 as marginally bound, i.e. $E_1 = m_1$ for reference. It begins with $2(54 - 10\sqrt{2} + 14\sqrt{3} + \sqrt{6})/97 \simeq 1.372$ and monotonically increases to $(2 + \sqrt{2} + \sqrt{6})/4 \simeq 1.466$ as m_2/m_1 increases from 0 to $5\sqrt{2} - 4\sqrt{3} + 3\sqrt{6} - 7 \simeq 0.4912$, where $E_2 = \kappa$. Then, the upper limit

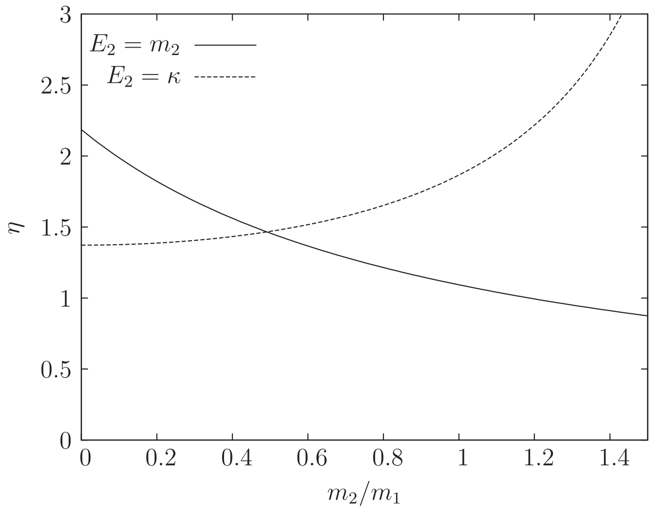

FIG. 3. The upper limit of the energy extraction efficiency for the upper limit of ejecta energy $E_3 = E_B$ as a function of the mass ratio m_2/m_1, where $E_1 = m_1$ and $m_3 = 0$ are chosen. The solid and dashed lines denote the efficiencies for $E_2 = m_2$ and $E_2 = \kappa$, respectively. If the mass ratio is smaller than 0.4913, we should adopt $E_2 = \kappa$, while if the ratio is greater than this value, we should adopt $E_2 = m_2$. The efficiency takes a maximum 1.466 at $m_2/m_1 = 0.4913$, where the two curves cross each other.

monotonically decreases to 0 as m_1/m_2 increases beyond this value, where $E_2 = m_2$. It becomes $(2+\sqrt{3}) \times (2-\sqrt{2})/2 \simeq 1.093$ at $m_1/m_2 = 1$.

In the end of the general analysis, it should be noted that the present mass, energy, and efficiency upper limits of the emission from the BSW collision are applicable even if product particles are more than two. This is because in such cases we can regard more than one product particles other than particle 3 as those produced as a result of the decay of particle 4. Thus, the present upper limits are unconditional in the sense that they are applicable irrespective of the details of the incident counterpart and the product particles.

V. UPPER LIMITS FOR SPECIFIC PHYSICAL REACTIONS

In this section, we specify physical reaction models and discuss the upper limits of the energy of the emitted particle and the energy extraction efficiency, based on the result obtained in Sec. IV.

A. Perfectly elastic collision

We first consider perfectly elastic collision of equal masses, i.e. $m_1 = m_2 = m_3 = m_4 = m_0$. We choose particle 1 as marginally bound for reference, i.e. $E_1 = m_0$. Then, from Eq. (4.9), the upper limit of the energy of particle 3 is given by

$$\lambda_+ = (7 - 4\sqrt{2})m_0 \simeq 1.343 m_0, \tag{5.1}$$

where $E_3 = \lambda_+$ is realized for $\delta_{(1)} = 0$. In fact, if $m_1 = m_3$, we can easily prove that the first term on the right-hand side of Eq. (4.14) is nonpositive because $\lambda_+ \geq E_1$. Then, the argument similar to that given in Sec. IV B applies. Since $m_2 = m_0 > (\lambda_+ - E_1)/(\sqrt{2}+1)$, we choose particle 2 as marginally bound, i.e. $E_2 = m_0$, and hence the upper limit of η for $E_3 = \lambda_+$ is given by

$$\eta \leq \frac{\lambda_+}{2m_0} = \frac{7 - 4\sqrt{2}}{2} \simeq 0.6716. \tag{5.2}$$

Therefore, we can obtain no net positive energy extraction. The above result will be discussed later in direct comparison with the claim in Ref. [6].

Next we assume that $m_1 = m_3$ and $m_2 = m_4$ but not $m_1 = m_2$. For $E_1 = m_1$, the upper limit of E_3 is given by

$$\lambda_+ = (7 - 4\sqrt{2})m_1. \tag{5.3}$$

We can optimize m_2 to $m_2 = (\lambda_+ - E_1)/(\sqrt{2}+1) = 2(5\sqrt{2} - 7)m_1 \simeq 0.1421 m_1$ so that we can obtain the upper limit of the energy extraction efficiency for $E_3 = \lambda_+$ as follows:

$$\frac{\lambda_+}{m_1 + m_2} = \frac{18\sqrt{2} + 11}{31} \simeq 1.176. \tag{5.4}$$

Therefore, net positive energy extraction is possible for perfectly elastic collision if the mass of the counterpart is in some range. The upper limit of the energy extraction efficiency becomes 117.6%, where the mass ratio is optimized.

B. Compton scattering

We here assume that particle 3 is massless. This is motivated by the fact that the unconditional energy upper limit E_B can be realized only if particle 3 is massless. If we consider the Compton scattering, we can identify either of particles 1 and 2 with a massless particle.

First we assume particle 1 is massless and hence $m_1 = m_3 = 0$ and $m_2 = m_4 = m_0$. Then, the upper limit of the energy of particle 3 is given by

$$\lambda_+ = E_1. \tag{5.5}$$

With $E_3 = E_1$ and $\delta_{(1)} = 0$, Eq. (4.14) yields $\delta_{(2)} = 0$. In other words, up to this order particles 1 and 2 just passed through each other and no energy nor angular momentum is exchanged. We cannot determine whether particle 3 can escape to infinity up to this order. Even if particle 3 can escape to infinity, we have no net positive energy extraction anyway.

Next, we assume particle 2 is massless and hence $m_1 = m_4 = m_0$, $m_2 = m_3 = 0$, and $E_1 = m_0$. In this case,

$$\lambda_+ = (2+\sqrt{3})(2-\sqrt{2})m_0 \simeq 2.186 m_0. \tag{5.6}$$

Since $m_2 = 0 < (\lambda_+ - E_1)/(\sqrt{2}+1)$, the upper limit of η for $E_3 = \lambda_+$ becomes

$$\eta \leq 1 + \frac{\lambda_+ - E_1}{\lambda_+ + E_1} = \frac{2(54 - 10\sqrt{2} + 14\sqrt{3} + \sqrt{6})}{97} \simeq 1.372. \quad (5.7)$$

This is comparable with the unconditional upper limit 1.466. The (inverse) Compton scattering between a subcritical photon and a critical massive particle is rather efficient as a collisional Penrose process.

C. Pair annihilation

We here consider pair annihilation of two equal masses into two massless particles. Then, $m_1 = m_2 = m_0$ and $m_3 = m_4 = 0$. We additionally assume $E_1 = m_0$ for reference. In this case, the upper limit of the energy of particle 3 is given by

$$\lambda_+ = (2 + \sqrt{3})(2 - \sqrt{2})m_0 \simeq 2.186 m_0. \quad (5.8)$$

In this case, since $m_2 = m_0 > (\lambda_+ - E_1)/(\sqrt{2} + 1)m_0$, we choose particle 2 as marginally bound, i.e. $E_2 = m_0$, and hence the upper limit of η is given by

$$\eta \leq \frac{(2 + \sqrt{3})(2 - \sqrt{2})}{2} \simeq 1.093. \quad (5.9)$$

Thus, net 9.3% of the total injected energy can be extracted. This result will be also discussed later in comparison with Ref. [6].

VI. DISCUSSION AND CONCLUSION

We have studied particle emission from the BSW collision and subsequent reaction, where a critical particle collides with a generic counterpart particle near the horizon of a maximally rotating Kerr black hole. Since the CM energy of the two particles can be arbitrarily high, the collision can produce very massive and/or energetic particles and one might speculate that such particles can potentially escape to infinity through a collisional Penrose process. We have however found that this is not the case. We cannot observe a particle much more massive nor much more energetic than the energy of the incident critical particle. This is qualitatively consistent with the earlier results [2–4,6].

We have derived the unconditional upper limits m_B and E_B of the mass and energy of the ejecta particle, respectively, which can be realized only if the emitted particle is massless. The ratio of E_B to E_1 the energy of incident critical particle takes a maximum $(2 + \sqrt{3})(2 - \sqrt{2}) \simeq 2.186$, for which the incident critical particle is massive and marginally bound. In general, the most energetic particle that escapes to infinity must be ejected inwardly on the production and subsequently bounces back outwardly at the turning point which is very close to the horizon due to the angular momentum which is slightly above the critical value. We have also determined the upper limit η_B of the energy extraction efficiency for the upper limit of ejecta energy from the near-horizon collision with an arbitrarily high CM energy. η_B is given by $(2 + \sqrt{2} + \sqrt{6})/4 \simeq 1.466$, which can be realized for the collision of two marginally bound massive particles with optimized mass ratio.

We have next analyzed perfectly elastic collision, Compton scattering, and pair annihilation. In all these cases, the energy of the emitted particle can be really greater than that of the injected critical particle. We have also found that net positive energy extraction is not possible for perfectly elastic collision of equal masses, while it is possible for perfectly elastic collision with optimized mass ratio, Compton scattering, and pair annihilation. In particular, the (inverse) Compton scattering of a subcritical photon by a critical massive particle is most efficient among these three reactions as a physically realistic process of energy extraction. Although the present analysis is restricted in the equatorial plane, it is unlikely that the result would be drastically changed even if we allow non-equatorial reactions.

Jacobson and Sotiriou (2010) [6] claim that, for the collision of two particles of equal mass m_0, the energy of the ejecta particle does not exceed $2m_0$ but drops to something just slightly above m_0 in the limit of infinite CM energy. The present result contradicts their claim. As we have shown, the energy of the ejecta particle can be $1.343m_0$ and $2.186m_0$ for perfectly elastic collision of two equal masses and for pair annihilation, respectively, in the limit of infinite CM energy. The latter gives the unconditional energy upper limit for the collision of two equal masses and enables net positive energy extraction. The disagreement of the claim in Ref. [6] with the present result is probably due to the strong assumption adopted in Ref. [6] that the four-momentum of the ejecta particle is parallel to that of the incident critical particle. We think that this assumption is not valid in estimating the energy of the emitted particle. See also Ref. [33].

The present result directly implies that when we consider gamma-ray emission from the pair annihilation of dark matter particles of mass m near the rapidly rotating Kerr black hole, the spectrum due to the BSW collision continues up to 218.6 MeV $(m/100\text{ MeV})$ and is cut off there. This is also the case for gamma-ray spectrum from the inverse Compton scattering by dark matter particles.

On the other hand, since the CM energy of particle collisions can be extremely high, high-energy reactions which are prohibited in low-energy collision may occur and leave their signatures in relatively low-energy gamma-ray spectrum in general. In this context, it should be noted that Cannoni *et al.* [34] discuss the possibility that colliding dark matter particles in the form of neutralinos may be gravitationally boosted near the supermassive black hole at the galactic center so that they can have enough collision energy to annihilate into a stau pair in some phenomenologically favored supersymmetric models. They also suggest the possibility that the signatures of the new channel of the reactions in gamma-ray spectrum might be discriminated by the Fermi-LAT satellite observation. They take into account the gravitational boost with the relative velocity is ~0.1–0.2 light speed, which exists also for a nonrotating black hole. The CM energy can be $2\sqrt{5}m_0$ at maximum in the former effect, while it can be $\sim 19m_0$ for $a/M = 0.998$ in the latter effect [9], where m_0 is the mass of the dark matter particle. This strongly suggests the channel of the dark matter pair annihilation may also be opened through the BSW process near a rapidly rotating black hole in some supersymmetric models, although the detailed analysis with fully general relativistic treatment is yet to be done.

The present analysis is restricted to a maximally rotating black hole, which is not expected to exist as an astrophysical object. It is interesting to study the upper limits of particle emission from the high-energy collision near a non-maximally rotating black hole. However, we can naturally expect that the upper limits of the emission do not change so drastically even if a/M is slightly below unity, although the maximum CM energy itself is sensitive to a/M. This is because the present upper limits for the maximal rotation are finite and determined by the spacetime geometry near the horizon and the metric there can change only smoothly as a/M increases to unity from below.

While the authors were finalizing the present paper, two papers [35,36] appeared on the arXiv, in which the upper limits of the mass, energy, and energy extraction efficiency are studied. Although the present result is consistent with the result of Ref. [35], not only does the present paper contain further new findings but also place the baseline for future research on this subject because of its systematic and analytical approach.

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APPENDIX A: COLLISION WITH A CIRCULARLY ORBITING PARTICLE

The expression for m_4^2 given by Eq. (4.13) contains a term which apparently diverges in the limit $E_1 \to m_1/\sqrt{3}$. To get a consistent approach, we here consider a massive particle circularly orbiting near the horizon because its energy approaches $m/\sqrt{3}$ and angular momentum asymptotically satisfies the critical condition in the near-horizon limit as we will see below. See Ref. [37] for circular orbits in the extreme Kerr spacetime in more general context. The energy and angular momentum of the circular orbit can be obtained by solving $V(r) = V'(r) = 0$. Putting $r = M/(1-\epsilon)$ and solving $V(r) = V'(r) = 0$ for E and $\tilde{L}$ order by order, we obtain

$$E = \frac{m}{\sqrt{3}}\left(1 + \frac{2}{3}\epsilon + \frac{1}{24}\epsilon^2\right) + O(\epsilon^3) \tag{A1}$$

and

$$\frac{\tilde{L}}{2E_1} = 1 + \frac{1}{4}\epsilon^2 + \frac{1}{16}\epsilon^3 + O(\epsilon^4). \tag{A2}$$

Assuming particle 1 belongs to this class, we find that $p_1^r = 0$ by definition, while

$$\begin{aligned} |p_4^r| = {} & (2E_2 - L_2) + [2(L_2 - E_2) + 2E_3(\delta_{(1)} - 1) + 2E_1]\epsilon \\ & + \left[\frac{(2E_2 - L_2)}{2} - 2(2\delta_{(1)} - \delta_2)E_3 \right. \\ & \left. - \frac{[(m_1/\sqrt{3}) + E_2 - E_3]^2 + m_4^2}{2(2E_2 - L_2)} + \frac{5}{6}\frac{m_1}{\sqrt{3}}\right]\epsilon^2 + O(\epsilon^3). \end{aligned} \tag{A3}$$

Then, Eq. (3.16) is not changed with $E_1 = m_1/\sqrt{3}$, while Eq. (4.13) is changed to

$$\begin{aligned} m_4^2 = {} & (2E_2 - \tilde{L}_2)\left[\frac{5}{3\sqrt{3}}m_1 - 4(2\delta_{(1)} - \delta_{(2)})E_3 \right. \\ & \left. + 2\sigma_3 \frac{E_3^2[1 - 4(2\delta_{(1)} - \delta_{(2)})(1 - \delta_{(1)})]}{\sqrt{E_3^2(3 - 8\delta_{(1)} + 4\delta_{(1)}^2) - m_3^2}}\right] \\ & + (E_2^2 + m_2^2) - \left(\frac{m_1}{\sqrt{3}} + E_2 - E_3\right)^2. \end{aligned} \tag{A4}$$

The apparently divergent term in Eq. (4.13) in the limit $E_1 \to m_1/\sqrt{3}$ is now replaced with a finite term.

APPENDIX B: PROOF FOR THE CONDITION ON THE ENERGY OF PARTICLE 2

First we calculate

$$\begin{aligned} & \left[\frac{\lambda_+^2}{\sqrt{3\lambda_+^2 - m_3^2}}\right]^2 - \left[\frac{E_1^2}{\sqrt{3E_1^2 - m_1^2}}\right]^2 \\ & = \frac{\lambda_+^4(3E_1^2 - m_1^2) - E_1^4(3\lambda_+^2 - m_3^2)}{(3\lambda_+^2 - m_3^2)(3E_1^2 - m_1^2)}. \end{aligned} \tag{B1}$$

For $m_3 = 0$, the numerator can be written as follows:

$$\begin{aligned} & \lambda_+^4(3E_1^2 - m_1^2) - E_1^4(3\lambda_+^2 - m_3^2) \\ & = [(3E_1^2 - m_1^2)\lambda_+^2 - 3E_1^4]\lambda_+^2 + E_1^4 m_3^2 = E_1^4 E_B^2 f(x), \end{aligned} \tag{B2}$$

where

$$f(x) = x^2\left(\frac{2-x}{2-\sqrt{3}}\right)^2 - 3,$$
$$x = \sqrt{3 - \frac{m_1^2}{E_1^2}}. \tag{B3}$$

For $E_1 \geq m_1$, we find $\sqrt{2} \leq x \leq \sqrt{3}$. $f(x)$ monotonically decreases in this domain and $f(\sqrt{3}) = 0$. Hence, $f(x) \geq 0$ for $\sqrt{2} \leq x \leq \sqrt{3}$. The condition (4.15) follows from Eq. (4.14) with $\delta_{(2)} \geq 0$, $\tilde{L}_2 < 2E_2$, $\lambda_+ \geq m_3$, and $m_4^2 \geq 0$. Q.E.D.

[1] M. Bañados, J. Silk, and S. M. West, Phys. Rev. Lett. **103**, 111102 (2009).
[2] T. Piran, J. Shaham, and J. Katz, Astrophys. J. **196**, L107 (1975).
[3] T. Piran and J. Shaham, Phys. Rev. D **16**, 1615 (1977).
[4] T. Piran and J. Shaham, Astrophys. J. **214**, 268 (1977).
[5] E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius, and U. Sperhake, Phys. Rev. Lett. **103**, 239001 (2009).
[6] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. **104**, 021101 (2010).
[7] A. A. Grib and Y. V. Pavlov, arXiv:1007.3222v1.
[8] A. A. Grib and Y. V. Pavlov, Gravitation Cosmol. **17**, 42 (2011).
[9] T. Harada and M. Kimura, Phys. Rev. D **83**, 024002 (2011).
[10] T. Harada and M. Kimura, Phys. Rev. D **83**, 084041 (2011).
[11] T. Harada and M. Kimura, Phys. Rev. D **84**, 124032 (2011).
[12] O. B. Zaslavskii, Phys. Rev. D **84**, 024007 (2011).
[13] M. Kimura, K.-i. Nakao, and H. Tagoshi, Phys. Rev. D **83**, 044013 (2011).
[14] T. Igata, T. Harada, and M. Kimura, Phys. Rev. D **85**, 104028 (2012).
[15] V. P. Frolov, Phys. Rev. D **85**, 024020 (2012).
[16] Y. Zhu, S.-F. Wu, Y.-X. Liu, and Y. Jiang, Phys. Rev. D **84**, 043006 (2011).
[17] S. W. Wei, Y. X. Liu, H. Guo, and C. E. Fu, Phys. Rev. D **82**, 103005 (2010).
[18] S. W. Wei, Y. X. Liu, H. T. Li, and F. W. Chen, J. High Energy Phys. 12 (2010) 066.
[19] C. Liu, S. Chen, and J. Jing, arXiv:1104.3225.
[20] W. Yao, S. Chen, C. Liu, and J. Jing, Eur. Phys. J. C **72**, 1898 (2012).
[21] O. B. Zaslavskii, Phys. Rev. D **82**, 083004 (2010).
[22] O. B. Zaslavskii, Zh. Eksp. Teor. Fiz. **92**, 635 (2010).
[23] M. Patil and P. S. Joshi, Classical Quantum Gravity **28**, 235012 (2011).
[24] M. Patil and P. S. Joshi, Phys. Rev. D **84**, 104001 (2011).
[25] M. Patil and P. S. Joshi, Phys. Rev. D **85**, 104014 (2012).
[26] M. Patil and P. S. Joshi, arXiv:1203.1803.
[27] M. Patil, P. S. Joshi, K. i. Nakao, and M. Kimura, arXiv:1108.0288.
[28] M. Bañados, B. Hassanain, J. Silk, and S. M. West, Phys. Rev. D **83**, 023004 (2011).
[29] A. J. Williams, Phys. Rev. D **83**, 123004 (2011).
[30] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).
[31] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[32] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004).
[33] A. A. Grib and Y. V. Pavlov, arXiv:1004.0913.
[34] M. Cannoni, M. E. Gomez, M. A. Perez-Garcia, and J. D. Vergados, Phys. Rev. D **85**, 115015 (2012).
[35] M. Bejger, T. Piran, M. Abramowicz, and F. Håkanson, arXiv:1205.4350.
[36] O. B. Zaslavskii, arXiv:1205.4410.
[37] D. Pugliese, H. Quevedo, and R. Ruffini, Phys. Rev. D **84**, 044030 (2011).