Flipped $SL(2, R)$ duality in five-dimensional supergravity

Shun'ya Mizoguchi[*] and Shinya Tomizawa[†]

Theory Center, Institute of Particle and Nuclear Studies, KEK, Tsukuba, Ibaraki, 305-0801, Japan

The dimensional reduction of the bosonic sector of five-dimensional minimal supergravity to a Lorentzian four-dimensional spacetime leads to a theory with a massless axion and a dilaton coupled to gravity and two $U(1)$ gauge fields and the dimensionally reduced equations of motion have $SL(2, \mathbb{R})/SO(2)$-duality invariance. In our previous work, utilizing the duality invariance, we formulated solution-generation techniques within five-dimensional minimal supergravity. In this work, by choosing a timelike Killing vector, we consider dimensional reduction to a four-dimensional Euclidean space, in which the field equations have $SL(2, \mathbb{R})/SO(1, 1)$ invariance. In the timelike case, we develop a new duality transformation technique, while in the spacelike case we have done that in the previous work. As an example, by applying it to the Rasheed solutions, we obtain rotating Kaluza-Klein black hole solutions in five-dimensional minimal supergravity. In general, in contrast to the spacelike case, the resulting dimensionally reduced solution includes the so-called NUT (Newman, Unti, and Tamburino) parameter, and therefore from a four-dimensional point of view, such a spacetime is not asymptotically flat. However, it is shown that in some special cases, it can describe ordinary Kaluza-Klein black holes.

I. INTRODUCTION

In modern string/supergravity theories and gauge theories, higher dimensional black holes and other extended black objects have played important roles. In particular, physics of black holes in the five-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theory has recently been one of the subjects of increasing interest since the discovery of the black ring [1–9] and other black object solutions with multiple horizons [10–14]. The five-dimensional EMCS theory describes the bosonic sector of five-dimensional minimal supergravity, which is one of the simplest supergravity sharing many common features with the 11-dimensional supergravity, and can be obtained as a certain low-energy limit of compactified string theory. In particular, hidden symmetries such dimensionally reduced theories possess are of technical importance since they enables us to construct nonlinear sigma models [15,16], which can be useful tools for the proofs of black hole uniqueness theorems [17–19] or for solution-generation of black holes. In fact, various types of black hole solutions in the EMCS theory have so far been found, with the help of the solution-generating techniques recently developed by using such nonlinear sigma models [16,20–25].

The dimensional reduction of the bosonic sector of five-dimensional minimal supergravity to four dimensions leads to a theory with a massless axion and a dilaton coupled to gravity and two $U(1)$ gauge fields with Chern-Simons coupling [15,26–28]. As was shown in Ref. [15], the field equations derived by the dimensional reduction

are invariant under the action of a global $SL(2, \mathbb{R})$ group, by which Maxwell's fields are related to Kaluza-Klein's electromagnetic fields. This so-called $SL(2, \mathbb{R})$-duality invariance enables us to generate a new solution in (the bosonic sector of) five-dimensional minimal supergravity by stating from a certain known solution in the same theory, as the $G_{2(+2)}$-duality invariance [15] does. In our previous work [29], we utilized a spacelike Killing vector for the dimensional reduction (hence the dimensionally reduced four-dimensional space is Lorentzian) and developed a formulation in which by using a certain known solution in five-dimensional pure gravity as a seed solution one can obtain new solutions in five-dimensional minimal supergravity.

One reason for our interest in developing new solution-generating techniques is the possibility that they might be used to generate the most general KK black hole solutions in five-dimensional minimal supergravity. In [30] it has been shown that such a solution is characterized by six parameters—the mass, angular momentum and electric/magnetic charges of the Kaluza-Klein gauge field and Maxwell field, respectively. So far, ones with some charges of them have been discovered by several authors [31–40]. In [29] we applied the spacelike Killing $SL(2, \mathbb{R})$ transformation to the Rasheed solutions, which are known to describe dyonic rotating black holes (from the four-dimensional point of view) of five-dimensional pure gravity and have four independent parameters; and we successfully obtained a new class of KK black hole solutions with *five* independent parameters, though we were not be able to find, within that framework, the ones with the maximal number ($= 6$) of parameters.

Therefore, in this paper, we focus on another $SL(2, \mathbb{R})$ symmetry that appears in a spacetime with a *timelike*

[*]mizoguch@post.kek.jp
[†]tomizawa@post.kek.jp

Killing vector. The dimensional reduction of the Lagrangian can be done mostly in parallel in both timelike and spacelike cases, except the sign flips of some terms that result in different coset spaces, $SL(2, \mathbb{R})/SO(2)$ if spacelike and $SL(2, \mathbb{R})/SO(1, 1)$ if timelike, just like the $SL(2, \mathbb{R})$ symmetries of dimensionally reduced pure Einstein theory [41–44]. This $SL(2, \mathbb{R})$ is, of course, a subgroup of the $G_{2(+2)}$ symmetry if the spacetime allows another spacelike Killing vector, and the two $SL(2, \mathbb{R})$ group actions do not commute but generate the whole $G_{2(+2)}$.

The purpose of this paper is to examine whether or not this timelike Killing $SL(2, \mathbb{R})$ can be used for generating six-parameter solutions. We will see that it cannot, unfortunately. We first present the $SL(2, \mathbb{R})$ transformation formulas in a unified way so that they can be used in both spacelike and timelike cases if the signs are appropriately chosen, and generalize our previous work [29] to the timelike case. Then, as an example, we apply it to the Rasheed solutions again to obtain rotating Kaluza-Klein black hole solutions in five-dimensional minimal supergravity. As will be shown later, in general the resulting (dimensionally reduced) spacetime geometry (which can be derived by the flip) has the so-called NUT (Newman, Unti, and Tamburino) parameter, and hence the four-dimensional reduced spacetime is not an asymptotic Minkowski spacetime. However, in some special cases, it can be shown that the NUT parameter vanishes and hence in that case, it can describe usual Kaluza-Klein black holes, i.e., asymptotically flat black holes from a four-dimensional point of view.

The remainder of this paper is organized as follows: In the next section, we will discuss our strategy for the solution-generation technique in both the spacelike and timelike cases. In particular, we will show that in the timelike case, the dimensionally reduced field equations have $SL(2, \mathbb{R})/SO(1, 1)$ invariance. In Sec. III, by acting the $SL(2, \mathbb{R})$ transformation on a certain seed solution, we write down some necessary formulas. In Sec. IV, we provide some brief review concerning the Rasheed solution, which we use as a seed in this paper. Also, applying actually this formalism to the Rasheed solutions, we present black hole solutions and study some basic properties, in particular, its asymptotics. Section V is devoted to summarizing our results and discussing our new method. In Appendix A, we apply the duality transformation to the asymptotically flat five-dimensional Myers-Perry black holes [47] and show that the solution obtained thereby coincides with the Cvetič-Youm solution [48]. In Appendix B, we clarify the precise relationship between the $SL(2, \mathbb{R})$-duality transformation and the nonlinear sigma model approach provided in [16].

II. $D = 4$ $SL(2, \mathbb{R})$ DUALITY WITH A TIMELIKE KILLING

In this section, we summarize the $SL(2, \mathbb{R})$ duality symmetry of five-dimensional minimal supergravity in the presence of a timelike Killing vector field. The reduction procedure is basically the same as the spacelike case, except the sign flips of some terms in the Lagrangian and the duality relations. They result in different coset spaces, $SL(2, \mathbb{R})/SO(2)$ if spacelike and $SL(2, \mathbb{R})/SO(1, 1)$ if timelike, just like Ehlers' or Matzner-Misner's $SL(2, \mathbb{R})$ symemtry of dimensionally reduced pure Einstein theory.

The conventions and notations are basically the same as those used in [29]. The Lagrangian is

$$\mathcal{L} = E^{(5)}\left(R^{(5)} - \frac{1}{4}F_{MN}F^{MN}\right)$$
$$- \frac{1}{12\sqrt{3}}\epsilon^{MNPQR}F_{MN}F_{PQ}A_R. \quad (1)$$

The vielbein $E_M^{(5)A}$ is related to the five-dimensional metric $G_{MN}^{(5)}$ as

$$G_{MN}^{(5)} = E_M^{(5)A}E_N^{(5)B}\eta_{AB}, \quad (2)$$

where

$$\eta_{AB} = \text{diag}(+1, +1, +1, +1, -1) \quad (3)$$

in the present case. ϵ^{MNPQR} is the densitized antisymmetric tensor that takes values ± 1.

As usual, we decompose the vielbein and gauge field as

$$E_M^{(5)A} = \begin{pmatrix} \rho^{-(1/2)}E_\mu^{(4)\alpha} & B_\mu\rho \\ 0 & \rho \end{pmatrix}, \quad (4)$$

$$A_M = (A_\mu, A_t), \quad (5)$$

where μ (α) is the four-dimensional curved (flat) index. Correspondingly, the five-dimensional coordinates x^M are grouped into (x^μ, t), and $\eta_{\alpha\beta} = \delta_{\alpha\beta}$. All the fields are assumed to be independent of t.

To make the $D = 4$ $SL(2, \mathbb{R})$ symmetry manifest, we need to dualize the Maxwell field A_μ [15,49] into

$$\tilde{A}_{\mu\nu} = -\rho(*F^{(4)})_{\mu\nu} - \frac{2}{\sqrt{3}}A_t F_{\mu\nu}^{(4)} + \frac{1}{\sqrt{3}}A_t^2 B_{\mu\nu}, \quad (6)$$

where

$$F_{\mu\nu}^{(4)} \equiv F'_{\mu\nu} + B_{\mu\nu}A_t, \quad (7)$$

$$F'_{\mu\nu} \equiv \partial_\mu A'_\nu - \partial_\nu A'_\mu, \quad (8)$$

$$A'_\mu \equiv A_\mu - B_\mu A_t, \quad (9)$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu. \quad (10)$$

We also define

$$\mathcal{G}_{\mu\nu} \equiv \begin{pmatrix} \tilde{A}_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \quad (11)$$

and

$$\mathcal{H}_{\mu\nu} \equiv \begin{pmatrix} \mathcal{H}_{\mu\nu}^{\tilde{A}} \\ \mathcal{H}_{\mu\nu}^{B} \end{pmatrix} \equiv m(*\mathcal{G})_{\mu\nu} + a\mathcal{G}_{\mu\nu}, \quad (12)$$

where

$$V^{-1}mV^{-1} = K - \tfrac{1}{2}(\Phi\Phi^*K + K\Phi^*\Phi) + \tfrac{1}{4}\Phi\Phi^{*2}K\Phi,$$
$$V^{-1}aV^{-1} = -\Phi^*K + \Phi + \tfrac{1}{2}(\Phi\Phi^{*2}K + K\Phi^{*2}\Phi)$$
$$- \tfrac{1}{3}\Phi\Phi^*\Phi - \tfrac{1}{4}\Phi\Phi^{*3}K\Phi, \quad (13)$$

$$V \equiv \begin{pmatrix} \rho^{-(1/2)} & 0 \\ 0 & \rho^{(3/2)} \end{pmatrix}, \quad \Phi \equiv \begin{pmatrix} 0 & \sqrt{3}\phi \\ \sqrt{3}\phi & 0 \end{pmatrix},$$
$$\Phi^* \equiv \begin{pmatrix} 2\phi & 0 \\ 0 & 0 \end{pmatrix}, \quad (14)$$

$$K \equiv (1 + \Phi^{*2})^{-1}, \quad \phi \equiv \frac{1}{\sqrt{3}}\rho^{-1}A_t. \quad (15)$$

Explicitly,

$$\mathcal{H}_{\mu\nu}^{\tilde{A}} = A_t B_{\mu\nu} - F_{\mu\nu}^{(4)} = -F'_{\mu\nu}, \quad (16)$$

$$\mathcal{H}_{\mu\nu}^{B} = \frac{A_t^3 B_{\mu\nu}}{3\sqrt{3}} - \frac{A_t^2 F_{\mu\nu}^{(4)}}{\sqrt{3}} - \rho A_t * F_{\mu\nu}^{(4)} + \rho^3 * B_{\mu\nu}. \quad (17)$$

Note the sign of the second term of (12). Also, two of the terms of a has changed their signs compared to the space-like case [29], while the matrix m has remained unchanged.

It is convenient to introduce

$$\mathcal{F}_{\mu\nu} \equiv \begin{pmatrix} \mathcal{G}_{\mu\nu} \\ \mathcal{H}_{\mu\nu} \end{pmatrix}, \quad (18)$$

so that the equations of motion and the Bianchi identities are expressed in a unified way:

$$d\mathcal{F} = 0, \quad \mathcal{F} \equiv \tfrac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu \wedge dx^\nu. \quad (19)$$

This means that $\mathcal{F}$ must be written as $d\mathcal{A}$ for some four-component column gauge potential vector $\mathcal{A} = \mathcal{A}_\mu dx^\mu$, where

$$\mathcal{A}_\mu \equiv \begin{pmatrix} \tilde{A}_\mu \\ B_\mu \\ -A'_\mu \\ \mathcal{H}_\mu^B \end{pmatrix}, \quad (20)$$

where $\tilde{A}_\mu$ and $\mathcal{H}_\mu^B$ are some gauge potentials that satisfy $\tilde{A}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$ and $\mathcal{H}_{\mu\nu}^B = \partial_\mu \mathcal{H}_\nu^B - \partial_\nu \mathcal{H}_\mu^B$, respectively.

It can be shown that $\mathcal{F}_{\mu\nu}$ satisfies

$$\mathcal{F}_{\mu\nu} = \mathcal{V}^{-1}\Omega\mathcal{V}(*\mathcal{F})_{\mu\nu}, \quad (21)$$

where

$$\mathcal{V} = \mathcal{V}_-\mathcal{V}_+, \quad \mathcal{V}_+ = \begin{pmatrix} V & \\ & V^{-1} \end{pmatrix},$$
$$\mathcal{V}_- = \exp\begin{pmatrix} & -\Phi^* \\ -\Phi & \end{pmatrix}, \quad \Omega = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}. \quad (22)$$

The scalar Lagrangian $\mathcal{L}_S$ can be written, using

$$\mathcal{R} = \Omega\mathcal{V}^{-1}\Omega\mathcal{V}, \quad (23)$$

or

$$\mathcal{R}' = \mathcal{V}^{-1}\Omega\mathcal{V} = \mathcal{R}'^{-1}, \quad (24)$$

as

$$\mathcal{L}_S = \frac{3}{40}E^{(4)}\text{Tr}\partial_\mu\mathcal{R}'\partial^\mu\mathcal{R}'. \quad (25)$$

The equations of motion and the Bianchi identity are invariant under

$$\mathcal{F}_{\mu\nu} \mapsto \Lambda^{-1}\mathcal{F}_{\mu\nu}, \quad (26)$$

$$\mathcal{V} \mapsto \mathcal{V}\Lambda, \quad (27)$$

$$\mathcal{R}' \mapsto \Lambda^{-1}\mathcal{R}'\Lambda, \quad (28)$$

where Λ is an $SL(2, \mathbb{R})$ matrix in the spin $-\tfrac{3}{2}$ (that is, **4**) representation generated by

$$E' \equiv \begin{pmatrix} & & 2 & \\ & & & 0 \\ & \sqrt{3} & & \\ \sqrt{3} & & & \end{pmatrix},$$
$$F' \equiv \begin{pmatrix} & & & \sqrt{3} \\ & & \sqrt{3} & \\ & 2 & & \\ 0 & & & \end{pmatrix}, \quad (29)$$
$$H' \equiv \begin{pmatrix} 1 & & & \\ & -3 & & \\ & & -1 & \\ & & & 3 \end{pmatrix}.$$

Since the $SL(2, \mathbb{R})$ transformation (26) is a rigid one, it can also be written at the gauge potential level:

$$\mathcal{A}_\mu \mapsto \Lambda^{-1}\mathcal{A}_\mu. \quad (30)$$

The point is that, once the gauge potential vector $\mathcal{A}_\mu$ can be computed for the seed solution, the new B_μ and $-A'_\mu$ fields can be found by simply a trivial matrix

multiplication. The nontrivial task is the computation of $\tilde{A}_\mu$ and $\mathcal{H}_\mu^B$, but the effort is reduced compared with the integrations of the potentials in the G_2 sigma model approach.

The scalar matrix $\mathcal{R}$ is defined as a 4×4 matrix in (23); it is more convenient to consider scalars in the defining representation of $SL(2, \mathbb{R})$ by using the Lie algebra isomorphism π:

$$\pi(E') = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi(F') = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$\pi(H') = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{31}$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a generic $SL(2, \mathbb{R})$ group element in the defining representation, where a, b, c, and d are all real numbers with $ad - bc = 1$. If $d \neq 0$, it can be written as

$$\begin{pmatrix} 1 & -\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\epsilon & 1 \end{pmatrix} \tag{32}$$

for some real δ, ϵ, and γ. If $d = 0$, on the other hand, this parametrization is singular. However, a generic such element

$$\begin{pmatrix} a & b \\ -b^{-1} & 0 \end{pmatrix}$$

can also be achieved by taking a limit α, β, $\gamma \to \infty$ in (32) with $\frac{\alpha}{\gamma} \equiv b$ and $\gamma + \frac{\alpha\beta}{\gamma} \equiv a$ fixed. Therefore, we consider $SL(2, \mathbb{R})$ transformations associated with group elements of the form

$$\Lambda = e^{-\delta E'} e^{(\log \gamma) H'} e^{-\epsilon F'}. \tag{33}$$

Moreover, since the Cartan algebra degree of freedom does not add a new parameter to the solutions, we set $\gamma = 1$ for simplicity and ignore the $e^{H'}$ factor [29]. (In fact, it also turns out that this $SL(2, \mathbb{R})$ transformation can add only one independent parameter, and δ may also be set to zero [29].) Hence

$$\Lambda = \begin{pmatrix} 3\delta^2\epsilon^2 + 4\delta\epsilon + 1 & \sqrt{3}\delta^2 & -\delta(3\delta\epsilon + 2) & -\sqrt{3}\epsilon(\delta\epsilon + 1)^2 \\ \sqrt{3}\epsilon^2 & 1 & -\sqrt{3}\epsilon & -\epsilon^3 \\ -\epsilon(3\delta\epsilon + 2) & -\sqrt{3}\delta & 3\delta\epsilon + 1 & \sqrt{3}\epsilon^2(\delta\epsilon + 1) \\ -\sqrt{3}\delta(\delta\epsilon + 1)^2 & -\delta^3 & \sqrt{3}\delta^2(\delta\epsilon + 1) & (\delta\epsilon + 1)^3 \end{pmatrix}, \tag{34}$$

$$\Lambda^{-1} = \begin{pmatrix} 3\delta\epsilon + 1 & \sqrt{3}\delta^2(\delta\epsilon + 1) & \delta(3\delta\epsilon + 2) & \sqrt{3}\epsilon \\ \sqrt{3}\epsilon^2(\delta\epsilon + 1) & (\delta\epsilon + 1)^3 & \sqrt{3}\epsilon(\delta\epsilon + 1)^2 & \epsilon^3 \\ \epsilon(3\delta\epsilon + 2) & \sqrt{3}\delta(\delta\epsilon + 1)^2 & 3\delta^2\epsilon^2 + 4\delta\epsilon + 1 & \sqrt{3}\epsilon^2 \\ \sqrt{3}\delta & \delta^3 & \sqrt{3}\delta^2 & 1 \end{pmatrix}, \tag{35}$$

$$\pi(\Lambda) = \begin{pmatrix} \delta\epsilon + 1 & -\delta \\ -\epsilon & 1 \end{pmatrix}. \tag{36}$$

Also we find

$$\pi(\mathcal{R}) = \begin{pmatrix} \rho^{-1} & -\frac{1}{\sqrt{3}}\rho^{-1}A_t \\ +\frac{1}{\sqrt{3}}\rho^{-1}A_t & -\frac{1}{3}\rho^{-1}A_t^2 + \rho \end{pmatrix}, \tag{37}$$

which transforms as

$$\pi(\mathcal{R}) \mapsto \pi(\Omega\Lambda^{-1}\Omega)\pi(\mathcal{R})\pi(\Lambda),$$
$$\pi(\Omega\Lambda^{-1}\Omega) = \begin{pmatrix} \delta\epsilon + 1 & \epsilon \\ \delta & 1 \end{pmatrix}. \tag{38}$$

III. TRANSFORMATION FORMULAS

The $SL(2, \mathbb{R})$-duality transformation requires that solutions should admit the existence of at least a single Killing isometry. In this paper, we assume that a spacetime admits two commuting Killing vector fields, timelike one $\partial/\partial t$ (at least at infinity), and spacelike one $\partial/\partial x^5$ and that each component of the spacetime metric and gauge potential oneform are independent of t and x^5. While in our previous work [29] we used the spacelike Killing vector $\partial/\partial x^5$ for $SL(2, \mathbb{R})$-duality transformation, where the five-dimensional metric is written as

$$ds^2 = \rho^2(dx^5 + B_\mu dx^\mu)^2 + \rho^{-1}ds_{(4)}^2, \tag{39}$$

we now rather use a timelike Killing vector $\partial/\partial t$ and hence one should complete the square by dt for a certain seed solution:

$$ds^2 = -\hat{\rho}^2(dt + \hat{B}_{\hat{\mu}}dx^{\hat{\mu}})^2 + \hat{\rho}^{-1}d\hat{s}^2_{(4)}. \qquad (40)$$

Here μ, ν, $\cdots$ runs indexes except x^5 and $\hat{\mu}$, $\hat{\nu}$, $\cdots$ runs ones except t. $ds_{(4)} = g^{(4)}_{\mu\nu}dx^\mu dx^\nu$ and $d\hat{s}_{(4)} = \hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}}$ are the four-dimensional metrics on the dimensionally reduced spacetime and space, respectively. In this paper, we call the operation *flip*. In general, this operation changes a set of the fields $(g^{(4)}_{\mu\nu}, B_\mu, A_\mu, \rho, A_5)$ to a set of different fields $(\hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}}, \hat{A}_{\hat{\mu}}, \hat{\rho}, \hat{A}_t)$, while the flip itself does not generate any new solutions, i.e., the solution described by the fields $(\hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}}, \hat{A}_{\hat{\mu}}, \hat{\rho}, \hat{A}_t)$ is the same as the one by $(g^{(4)}_{\mu\nu}, B_\mu, A_\mu, \rho, A_5)$. After the flip, one performs the $SL(2, \mathbb{R})$-duality transformation for the flipped seed solution and then can obtain, in general, a different solution in the bosonic sector of $D = 5$ minimal supergravity (we denote these field by *new*). Finally, one again flips the solution, i.e., one completes the square by dx^5 rather than dt for the obtained solution. Through this paper, we denote the flipped fields by attaching *hat*. To summarize, the procedure of obtaining new solutions by the series of transformations is as follows:

$$(g^{(4)}_{\mu\nu}, B_\mu, A_\mu, \rho, A_5) \rightarrow \text{Flip} \rightarrow (\hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}}, \hat{A}_{\hat{\mu}}, \hat{\rho}, \hat{A}_t)$$

$$\rightarrow SL(2, \mathbb{R}) \text{ duality transformation}$$

$$\rightarrow (\hat{g}^{(4)\text{new}}_{\hat{\mu}\hat{\nu}}, \hat{B}^{\text{new}}_{\hat{\mu}}, \hat{A}^{\text{new}}_{\hat{\mu}}, \hat{\rho}_{\text{new}}, \hat{A}^{\text{new}}_t)$$

$$\rightarrow \text{Flip}$$

$$\rightarrow (g^{(4)\text{new}}_{\mu\nu}, B^{\text{new}}_\mu, A^{\text{new}}_\mu, \rho_{\text{new}}, A^{\text{new}}_5).$$

Further, though the below formula, we assume that the spacetime also admits another spacelike Killing vector $\partial/\partial\phi$ that commutes with the other two Killing vectors. In general, this symmetry assumption is not necessarily required for our $SL(2, \mathbb{R})$-duality transformation. This is simply for later convenience and, actually, in the following section, we will apply our transformation to the Rasheed solutions that have this symmetry. In this case, it can be shown that a two-surface orthogonal to the three Killing vector fields are integrable [17,50]. For the integral two-surface, we use two coordinates (r, θ).

A. Flip

By completing the square by dt, we can easily obtain the flipped scalar fields $(\hat{\rho}, \hat{A}_t)$, $U(1)$ gauge fields $(\hat{B}_{\hat{\mu}}, \hat{A}_{\hat{\mu}})$ and four-dimensional metric $\hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}$. After the flip operation, the dilaton and axion fields can be written in the form:

$$\hat{\rho}^2 = -(\rho^2 B_t^2 + \rho^{-1}g^{(4)}_{tt}), \qquad (41)$$

$$\hat{A}_t = A_t. \qquad (42)$$

The gauge potential oneforms for the Kaluza-Klein $U(1)$ field and Maxwell $U(1)$ field are, respectively,

$$\hat{B}_{\hat{\mu}}dx^{\hat{\mu}} = -\frac{\rho^2 B_t B_\phi + \rho^{-1}g^{(4)}_{t\phi}}{\hat{\rho}^2}d\phi - \frac{\rho^2 B_t}{\hat{\rho}^2}dx^5, \quad (43)$$

$$\hat{A}_{\hat{\mu}} = A_{\hat{\mu}}. \qquad (44)$$

The four-dimensional metric is

$$d\hat{s}^2_{(4)} = -\frac{\rho}{\hat{\rho}}\Bigg[g^{(4)}_{tt}\bigg\{dx^5 + \bigg(B_\phi - \frac{g^{(4)}_{t\phi}}{g^{(4)}_{tt}}B_t\bigg)d\phi\bigg\}^2$$
$$-\frac{g^{(4)}_{\phi\phi}g^{(4)}_{tt} - g^{(4)2}_{t\phi}}{g^{(4)}_{tt}}\frac{\hat{\rho}^2}{\rho^2}d\phi^2\Bigg]$$
$$+\frac{\hat{\rho}}{\rho}(g^{(4)}_{rr}dr^2 + g^{(4)}_{\theta\theta}d\theta^2). \qquad (45)$$

B. $SL(2, \mathbb{R})$-duality transformation

In general, performing the $SL(2, \mathbb{R})$-duality transformation on the flipped fields (41)–(45) yields a different solution from the one obtained by starting from the unflipped (hence, original) fields. According to (38), the dilaton and axion fields for the new solution take the forms, respectively,

$$\hat{\rho}_{\text{new}} = \frac{\hat{\rho}}{\{1 + \epsilon(\delta + \frac{\hat{A}_t}{\sqrt{3}})\}^2 - \epsilon^2\hat{\rho}^2}, \qquad (46)$$

$$\hat{A}^{\text{new}}_t = \sqrt{3}\frac{(1 + \epsilon\delta + \epsilon\frac{\hat{A}_t}{\sqrt{3}})(\delta + \frac{\hat{A}_t}{\sqrt{3}}) - \epsilon\hat{\rho}^2}{\{1 + \epsilon(\delta + \frac{\hat{A}_t}{\sqrt{3}})\}^2 - \epsilon^2\hat{\rho}^2}. \qquad (47)$$

Also, by (26), the two $U(1)$ gauge fields are transformed to

$$\hat{B}^{\text{new}}_{\hat{\mu}} = \sqrt{3}\epsilon^2(1 + \delta\epsilon)\hat{\tilde{A}}_{\hat{\mu}} + (1 + \delta\epsilon)^3\hat{B}_{\hat{\mu}}$$
$$- \sqrt{3}\epsilon(1 + \delta\epsilon)^2\hat{A}'_{\hat{\mu}} + \epsilon^3\hat{\mathcal{H}}^B_{\hat{\mu}}, \qquad (48)$$

$$\hat{A}'^{\text{new}}_{\hat{\mu}} = -(3\delta\epsilon^2 + 2\epsilon)\hat{\tilde{A}}_{\hat{\mu}} - \sqrt{3}\delta(1 + \delta\epsilon)^2\hat{B}_{\hat{\mu}}$$
$$+ (3\delta^2\epsilon^2 + 4\delta\epsilon + 1)\hat{A}'_{\hat{\mu}} - \sqrt{3}\epsilon^2\hat{\mathcal{H}}^B_{\hat{\mu}}. \qquad (49)$$

Under the transformation, the four-dimensional metric is invariant:

$$\hat{g}^{(4)\text{new}}_{\hat{\mu}\hat{\nu}} = \hat{g}^{(4)}_{\hat{\mu}\hat{\nu}}. \qquad (50)$$

C. Flip again

Finally, in order to write the fields $(\hat{g}^{(4)\text{new}}_{\hat{\mu}\hat{\nu}}, \hat{B}^{\text{new}}_{\hat{\mu}}, \hat{A}^{\text{new}}_{\hat{\mu}}, \hat{\rho}_{\text{new}}, \hat{A}^{\text{new}}_t)$ in the standard Kaluza-Klein form (though this is not always required), we again must flip the new solution

by completing the square by dx^5. After performing the second flip, the dilaton and axion field ($\rho_{\text{new}}, A_5^{\text{new}}$) for the new solution are in the following forms, respectively:

$$\rho_{\text{new}}^2 = -(\hat{\rho}_{\text{new}})^2(\hat{B}_5^{\text{new}})^2 + (\hat{\rho}_{\text{new}})^{-1}\hat{g}_{55}^{\text{new}}, \quad (51)$$

$$A_5^{\text{new}} = \hat{A}_5^{\text{new}}. \quad (52)$$

The two $U(1)$ gauge fields are

$$B_\mu^{\text{new}}dx^\mu = -\frac{(\hat{\rho}_{\text{new}})^2(\hat{B}_5^{\text{new}})}{\rho_{\text{new}}^2}dt$$
$$-\frac{(\hat{\rho}_{\text{new}})^2(\hat{B}_5^{\text{new}})(\hat{B}_\phi^{\text{new}}) + (\hat{\rho}_{\text{new}})^{-1}\hat{g}_{5\phi}^{(4)\text{new}}}{\rho_{\text{new}}^2}d\phi, \quad (53)$$

$$A_\mu^{\text{new}} = \hat{A}_\mu^{\text{new}}. \quad (54)$$

The dimensionally reduced four-dimensional metric $ds_{(4)\text{new}}^2 = g_{\mu\nu}^{(4)\text{new}}dx^\mu dx^\nu$ is given by

$$ds_{(4)\text{new}}^2 = -\frac{\hat{\rho}_{\text{new}}}{\rho_{\text{new}}}\Bigg[\hat{g}_{55}^{(4)\text{new}}\Bigg\{dt + \Bigg(\hat{B}_\phi - \frac{\hat{g}_{5\phi}^{(4)\text{new}}}{\hat{g}_{55}^{(4)\text{new}}}\hat{B}_5^{\text{new}}\Bigg)d\phi\Bigg\}^2$$
$$-\frac{\hat{g}_{\phi\phi}^{(4)\text{new}}\hat{g}_{55}^{(4)\text{new}} - \hat{g}_{5\phi}^{(4)\text{new}2}}{\hat{g}_{55}^{(4)\text{new}}}\frac{\rho_{\text{new}}^2}{\hat{\rho}_{\text{new}}^2}d\phi^2\Bigg]$$
$$+\frac{\rho_{\text{new}}}{\hat{\rho}_{\text{new}}}(\hat{g}_{rr}^{(4)\text{new}}dr^2 + \hat{g}_{\theta\theta}^{(4)\text{new}}d\theta^2). \quad (55)$$

IV. APPLICATIONS

A. Rasheed solution

In the following subsection, using the $SL(2,\mathbb{R})$-transformation mentioned in the previous section, we will generate a rotating black hole solution, starting from the Rasheed solution [32]. Hence, in this section, we briefly review the Rasheed solutions in five-dimensional pure gravity. (We also apply our technique to asymptotically

flat black hole solutions such as the five-dimensional Myers-Perry solutions [47]. See Appendix A for this.) The metric of the Rasheed solution is given by

$$ds^2 = \frac{B}{A}(dx^5 + B_\mu dx^\mu)^2 + \sqrt{\frac{A}{B}}ds_{(4)}^2, \quad (56)$$

where the four-dimensional (dimensionally reduced) metric is given by

$$ds_{(4)}^2 = -\frac{f^2}{\sqrt{AB}}(dt + \omega_\phi^0 d\phi)^2 + \frac{\sqrt{AB}}{\Delta}dr^2$$
$$+ \sqrt{AB}d\theta^2 + \frac{\sqrt{AB}\Delta}{f^2}\sin^2\theta d\phi^2. \quad (57)$$

Here the functions $(A, B, C, \omega_\phi^0, \omega_\phi^5, f^2, \Delta)$ and the oneform B_μ [51] are

$$A = \Big(r - \frac{\Sigma}{\sqrt{3}}\Big)^2 - \frac{2P^2\Sigma}{\Sigma - \sqrt{3}M} + a^2\cos^2\theta$$
$$+ \frac{2JPQ\cos\theta}{(M + \Sigma/\sqrt{3})^2 - Q^2}, \quad (58)$$

$$B = \Big(r + \frac{\Sigma}{\sqrt{3}}\Big)^2 - \frac{2Q^2\Sigma}{\Sigma + \sqrt{3}M} + a^2\cos^2\theta$$
$$- \frac{2JPQ\cos\theta}{(M - \Sigma/\sqrt{3})^2 - P^2}, \quad (59)$$

$$C = 2Q(r - \Sigma/\sqrt{3}) - \frac{2PJ\cos\theta(M + \Sigma/\sqrt{3})}{(M - \Sigma/\sqrt{3})^2 - P^2}, \quad (60)$$

$$\omega_\phi^0 = \frac{2J\sin^2\theta}{f^2}\Big[r - M$$
$$+ \frac{(M^2 + \Sigma^2 - P^2 - Q^2)(M + \Sigma/\sqrt{3})}{(M + \Sigma/\sqrt{3})^2 - Q^2}\Big], \quad (61)$$

$$\omega_\phi^5 = \frac{2P\Delta\cos\theta}{f^2} - \frac{2QJ\sin^2\theta[r(M - \Sigma/\sqrt{3}) + M\Sigma/\sqrt{3} + \Sigma^2 - P^2 - Q^2]}{f^2[(M + \Sigma/\sqrt{3})^2 - Q^2]}, \quad (62)$$

$$\Delta = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2, \quad (63)$$

$$f^2 = r^2 - 2Mr + P^2 + Q^2 - \Sigma^2 + a^2\cos^2\theta, \quad (64)$$

$$B_\mu dx^\mu = \frac{C}{B}dt + \Big(\omega_\phi^5 + \frac{C}{B}\omega_\phi^0\Big)d\phi, \quad (65)$$

where B_μ describes the electromagnetic vector potential derived by dimensional reduction to four dimensions. Here

the constants, (M, P, Q, J, Σ), mean the mass, Kaluza-Klein magnetic charge, Kaluza-Klein electric charge, angular momentum along four dimensions and dilaton charge, respectively, which are parameterized by the two parameters $(\hat{\alpha}, \hat{\beta})$

$$M = \frac{(1 + \cosh^2\hat{\alpha}\cosh^2\hat{\beta})\cosh\hat{\alpha}}{2\sqrt{1 + \sinh^2\hat{\alpha}\cosh^2\hat{\beta}}}M_k, \quad (66)$$

$$\Sigma = \frac{\sqrt{3}\cosh\hat{\alpha}(1 - \cosh^2\hat{\beta} + \sinh^2\hat{\alpha}\cosh^2\hat{\beta})}{2\sqrt{1 + \sinh^2\hat{\alpha}\cosh^2\hat{\beta}}}M_k, \quad (67)$$

$$Q = \sinh\hat{\alpha}\sqrt{1 + \sinh^2\hat{\alpha}\cosh^2\hat{\beta}}\,M_k, \quad (68)$$

$$P = \frac{\sinh\hat{\beta}\cosh\hat{\beta}}{\sqrt{1 + \sinh^2\hat{\alpha}\cosh^2\hat{\beta}}}M_k, \quad (69)$$

$$J = \cosh\hat{\beta}\sqrt{1 + \sinh^2\hat{\alpha}\cosh^2\hat{\beta}}\,aM_k. \quad (70)$$

Note that all the above parameters are not independent since they are related through the equation

$$\frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M} = \frac{2\Sigma}{3}, \quad (71)$$

and the constant M_k is written in terms of these parameters

$$M_k^2 = M^2 + \Sigma^2 - P^2 - Q^2. \quad (72)$$

The constant J is also related to a by

$$J^2 = a^2\frac{[(M + \Sigma/\sqrt{3})^2 - Q^2][(M - \Sigma/\sqrt{3})^2 - P^2]}{M^2 + \Sigma^2 - P^2 - Q^2}. \quad (73)$$

The dilaton and axion fields for the Rasheed solution are, respectively,

$$\rho = \sqrt{\frac{B}{A}}, \qquad A_5 = 0. \quad (74)$$

The Kaluza-Klein gauge field and Maxwell field are, respectively,

$$B_\mu dx^\mu = \frac{C}{B}dt + \left(\omega^5_\phi + \frac{C}{B}\omega^0_\phi\right)d\phi, \qquad A_\mu dx^\mu = 0. \quad (75)$$

B. Flipped Rasheed solution

By completing the square for the time coordinate t, the metric of the flipped Rasheed solution is obtained:

$$ds^2 = -\frac{Af^2 - C^2}{AB}(dt + \hat{B}_{\hat{\mu}}dx^{\hat{\mu}})^2 + \sqrt{\frac{AB}{Af^2 - C^2}}d\hat{s}^2_{(4)}, \quad (76)$$

where the gauge potential oneform for Kaluza-Klein $U(1)$ gauge field is

$$\hat{B}_{\hat{\mu}}dx^{\hat{\mu}} = \frac{-BC}{Af^2 - C^2}dx^5 + \left(\frac{-BC}{Af^2 - C^2}\omega^5_\phi + \omega^0_\phi\right)d\phi, \quad (77)$$

and the four-dimensional metric is

$$d\hat{s}^2_{(4)} = \hat{\rho}\left[\frac{Bf^2}{Af^2 - C^2}(dx^5 + \omega^5{}_\phi d\phi)^2 + \frac{A\Delta}{f^2}\sin^2\theta d\phi^2 + A\left(\frac{dr^2}{\Delta} + d\theta^2\right)\right]. \quad (78)$$

From this, the dilaton and axion read

$$\hat{\rho} = \sqrt{\frac{Af^2 - C^2}{AB}}, \quad (79)$$

$$\hat{A}_t = 0. \quad (80)$$

The gauge potential oneform for the Maxwell field is

$$\hat{A}_{\hat{\mu}} = 0. \quad (81)$$

C. Transformed Rasheed solutions

Applying the $SL(2, \mathbb{R})$-duality transformation to the flipped Rasheed solution, we can obtain new Kaluza-Klein black hole solutions in $D = 5$ minimal supergravity. By putting $\hat{A}_t = 0$ in Eqs. (46) and (47), the two scalar fields for new solutions are written as

$$\hat{\rho}_{\text{new}} = \frac{\hat{\rho}}{(1 + \epsilon\delta)^2 - \epsilon^2\hat{\rho}^2}, \quad (82)$$

$$\hat{A}_t^{\text{new}} = \sqrt{3}\frac{(1 + \epsilon\delta)\delta - \epsilon\hat{\rho}^2}{(1 + \epsilon\delta)^2 - \epsilon^2\hat{\rho}^2}. \quad (83)$$

Putting $\hat{\tilde{A}}_{\hat{\mu}} = \hat{A}'_{\hat{\mu}} = 0$ in Eqs. (48) and (49), we immediately obtain the two vector fields as

$$B_{\hat{\mu}}^{\text{new}} = (1 + \delta\epsilon)^3 B_{\hat{\mu}} + \epsilon^3\hat{\mathcal{H}}^B_{\hat{\mu}}, \quad (84)$$

$$\hat{A}_{\hat{\mu}}^{\text{new}} = [(1 + \delta\epsilon)^3\hat{A}_t^{\text{new}} - \sqrt{3}\delta(1 + \delta\epsilon)^2]B_{\hat{\mu}} + (\epsilon^3\hat{A}_t^{\text{new}} - \sqrt{3}\epsilon^2)\hat{\mathcal{H}}^B_{\hat{\mu}}, \quad (85)$$

where $\hat{\mathcal{H}}^B_{\hat{\mu}}$ can be obtained by actually integrating Eq. (17) as

$$\hat{\mathcal{H}}^B_5 = \frac{-2PQ - 2J\cos\theta}{A}, \quad (86)$$

$$\hat{\mathcal{H}}^B_\phi = \left(b_0 r + \frac{(c_2 r^2 + c_1 r + c_0)\cos\theta + d_3 r^3 + d_2 r^2 + d_1 r + d_0}{A}\right), \quad (87)$$

where the constants $b_0, c_0, c_1, c_2, d_0, d_1, d_2, d_3$ are defined by

$$b_0 = -d_3 = \frac{2JP(M + \Sigma/\sqrt{3})}{a^2[(M - \Sigma/\sqrt{3})^2 - P^2]}, \quad (88)$$

$$c_0 = \frac{1}{2}Q\bigg[-4a^2 - \frac{4\Sigma^2}{3} + \frac{8\sqrt{3}MP^2}{-\sqrt{3}M+\Sigma} + (24J^2P^2(54M^6 + 63\sqrt{3}M^5\Sigma - 9M^4(6P^2 + 12Q^2 - 13\Sigma^2)$$
$$+ 6\sqrt{3}M^3\Sigma(-12P^2 - 15Q^2 + 11\Sigma^2) - (3Q^2 - \Sigma^2)(9Q^2(P^2 + Q^2) - 3(2P^2 + 5Q^2)\Sigma^2 + 5\Sigma^4)$$
$$+ 3M^2(27Q^2(P^2 + Q^2) - 18(2P^2 + 3Q^2)\Sigma^2 + 28\Sigma^4) + \sqrt{3}M\Sigma(45Q^4 - 66Q^2\Sigma^2 + 19\Sigma^4$$
$$+ 6P^2(9Q^2 - 4\Sigma^2))))[81a^2[(M - \Sigma/\sqrt{3})^2 - P^2][(M + \Sigma/\sqrt{3})^2 - Q^2]^3]^{-1}\bigg], \tag{89}$$

$$c_1 = \frac{4Q\Sigma}{\sqrt{3}} - \frac{4J^2P^2Q(M + \Sigma/\sqrt{3})}{a^2[(M - \Sigma/\sqrt{3})^2 - P^2][(M + \Sigma/\sqrt{3})^2 - Q^2]}, \tag{90}$$

$$c_2 = -2Q, \tag{91}$$

$$d_0 = [2JP(81\sqrt{3}a^2M^2Q^2(-2M^2 + P^2 + Q^2) + 81M(a^2(3M^4 + M^2Q^2 - 2Q^2(P^2 + Q^2)) + 2P^2(2M^4$$
$$- 2M^2(P^2 + Q^2) + Q^2(P^2 + Q^2)))\Sigma + 27\sqrt{3}(3a^2M^4 + 2M^6 - 4M^2P^4 - (2M^4 + M^2P^2 + 2P^4$$
$$+ a^2(M - P)(M + P))Q^2 + (a^2 + M^2 - 2P^2)Q^4)\Sigma^2 - 27M(6a^2M^2 + M^4 - 4M^2P^2 - 4P^4$$
$$- (7a^2 + M^2 - 6P^2)Q^2 + 2Q^4)\Sigma^3 + 9\sqrt{3}(-6a^2M^2 + M^4 + 12M^2P^2 + 4P^4$$
$$- (5a^2 + M^2 - 11P^2)Q^2 + Q^4)\Sigma^4 + 9M(3a^2 + 2M^2 - 8P^2 + 7Q^2)\Sigma^5 - 3\sqrt{3}(-3a^2 + 8M^2 + 12P^2 + 5Q^2)\Sigma^6$$
$$- 3M\Sigma^7 + 5\sqrt{3}\Sigma^8)][81\sqrt{3}a^2(M - \Sigma/\sqrt{3})^2[(M - \Sigma/\sqrt{3})^2 - P^2][(M + \Sigma/\sqrt{3})^2 - Q^2]]^{-1} \tag{92}$$

$$d_1 = [2JP(27\sqrt{3}a^2M^2(M - Q)(M + Q) + 54M(a^2M^2 + 2M^4 + Q^4 - M^2(P^2 + 2Q^2))\Sigma$$
$$+ 9\sqrt{3}(3M^4 + M^2(2P^2 - 3Q^2) + Q^2(a^2 - 2(2P^2 + Q^2)))\Sigma^2 - 18M(a^2 - 3M^2 - 5P^2 + 2Q^2)\Sigma^3$$
$$+ 3\sqrt{3}(-a^2 + 8M^2 + 6P^2 + 11Q^2)\Sigma^4 - 30M\Sigma^5 - 11\sqrt{3}\Sigma^6)][-27\sqrt{3}a^2(M - \Sigma/\sqrt{3})$$
$$\times [(M - \Sigma/\sqrt{3})^2 - P^2][(M + \Sigma/\sqrt{3})^2 - Q^2]]^{-1}, \tag{93}$$

$$d_2 = -[2JP(-18M^4 - 9Q^2(P^2 + Q^2) - 15\sqrt{3}M^3\Sigma + 3(2P^2 + 7Q^2)\Sigma^2 - 7\Sigma^4 + 3M^2(6(P^2 + Q^2) - 11\Sigma^2)$$
$$+ 3\sqrt{3}M\Sigma(4P^2 + 5(Q - \Sigma)(Q + \Sigma)))][9a^2[(M - \Sigma/\sqrt{3})^2 - P^2][(M + \Sigma/\sqrt{3})^2 - Q^2]]^{-1}. \tag{94}$$

D. Flipped transformed Rasheed solution

By flipping the metric and Maxwell's $U(1)$ field again according to Sec. III C, one can read off the (Lorentzian) metric and Kaluza-Klein's $U(1)$ field and Maxwell's $U(1)$ field for the four-dimensional dimensionally reduced spacetime.

E. Asymptotics

Now we investigate asymptotics of the obtained solution. It turns out that the solutions do not have Kaluza-Klein asymptotics in a usual sense since after the flip the $t\phi$ component of the dimensionally reduced four-dimensional metric (55) at infinity $r \to \infty$ behaves as

$$g_{t\phi}^{(4)\text{new}} \simeq \frac{1}{N^{3/2}}(c + 2Q\epsilon^2 \cos\theta) + \mathcal{O}(r^{-1}), \tag{95}$$

where the constants N and c_ϕ are

$$N = (1 + \delta\epsilon)^2 - \epsilon^2, \tag{96}$$

$$c = -[2JP\epsilon^3(18M^4 + 9Q^2(P^2 + Q^2) + 9\sqrt{3}M^3\Sigma$$
$$- 3(2P^2 + 5Q^2)\Sigma^2 + 5\Sigma^4 - 3M^2(6(P^2 + Q^2) - 5\Sigma^2)$$
$$+ 3\sqrt{3}M\Sigma(-4P^2 - 3Q^2 + 3\Sigma^2))]$$
$$\times [a^2(3M^2 - 3P^2 - 2\sqrt{3}M\Sigma + \Sigma^2)$$
$$\times (3M^2 - 3Q^2 + 2\sqrt{3}M\Sigma + \Sigma^2)]^{-1}. \tag{97}$$

The constant term vanishes under the coordinate transformation $t - c \to t$ but the term proportional to $\cos\theta$ does not vanish (hence this is not a simply gauge). The existence of this term means that the dimensionally reduced spacetime has the so-called NUT parameter. Though as a result, for our solutions the dimensionally reduced four-dimensional spacetimes are generally not asymptotically Minkowskian, if and only if $Q\epsilon = 0$; this NUT parameter vanishes and our solutions describe usual Kaluza-Klein black holes.

V. SUMMARY AND DISCUSSION

In this paper, in the choice of a timelike Killing vector, we have performed dimensional reduction to a four-dimensional Euclidean space and have also shown in that case the field equations are invariant under $SL(2, \mathbb{R})/SO(1, 1)$ transformation. In the timelike case, we have also developed a new solution-generation technique using the duality transformation, as we have done in [29] for the space-like case. As an example, by applying this transformation to the Rasheed solutions, we have obtained rotating Kaluza-Klein black hole solutions in five-dimensional minimal supergravity. In general, in contrast to the spacelike cases, the resulting dimensionally reduced solution includes the so-called NUT parameter and for this reason, in general, the dimensionally reduced spacetime is not asymptotically flat. However, in some special cases (for instance, when the electric charge Q for the Kaluza-Klein $U(1)$ field vanishes), it can describe ordinary Kaluza-Klein black holes. Finally, it would be interesting to examine if the signature change of spacetime [46,52–55], observed in various timelike T dualities, also occurs in this single timelike Killing case.

ACKNOWLEDGMENTS

APPENDIX A: MYERS-PERRY BLACK HOLES

The flipped metric of the five-dimensional Myers-Perry solutions is given by

$$
\begin{aligned}
ds^2 = &-\left(1 - \frac{r_0^2}{\varrho^2}\right)\left(dt - \frac{r_0^2 a}{\varrho^2 - r_0^2}\sin^2\theta d\phi - \frac{r_0^2 b}{\varrho^2 - r_0^2}\cos^2\theta d\psi\right)^2 \\
&+\left(x + a^2 + \frac{r_0^2 a^2}{\varrho^2 - r_0^2}\sin^2\theta\right)\sin^2\theta d\phi^2 \\
&+\left(x + b^2 + \frac{r_0^2 b}{\varrho - r_0^2}\cos^2\theta\right)\cos^2\theta \\
&+2\frac{r_0^2 ab}{\varrho^2 - r_0^2}\sin^2\theta\cos^2\theta d\phi d\psi \\
&+\frac{\varrho^2}{4\Delta}dx^2 + \varrho^2 d\theta^2,
\end{aligned} \tag{A1}
$$

where

$$
\begin{aligned}
\varrho^2 &= x + a^2\cos^2\theta + b^2\sin^2\theta, \\
\Delta &= (x + a^2)(x + b^2) - r_0^2 x.
\end{aligned} \tag{A2}
$$

The dilaton and axion are, respectively,

$$
\hat{\rho} = \sqrt{1 - \frac{r_0^2}{\varrho^2}}, \qquad \hat{A}_t = 0, \tag{A3}
$$

and the Kaluza-Klein's and Maxwell's $U(1)$ fields are

$$
\begin{aligned}
\hat{B}_{\hat{\mu}}dx^{\hat{\mu}} &= -\frac{r_0^2 a}{\varrho^2 - r_0^2}\sin^2\theta d\phi - \frac{r_0^2 b}{\varrho^2 - r_0^2}\cos^2\theta d\psi, \\
\hat{A}_{\hat{\mu}}dx^{\hat{\mu}} &= 0.
\end{aligned} \tag{A4}
$$

Integrating Eq. (17), we obtain

$$
\hat{H}^B_{\hat{\mu}}dx^{\hat{\mu}} = \frac{br_0^2(x + a^2)}{(a^2 - b^2)\varrho^2}d\phi - \frac{ar_0^2(x + b^2)}{(a^2 - b^2)\varrho^2}d\psi. \tag{A5}
$$

Consider the rescale of the coordinates:

$$
\frac{dt}{N} \rightarrow dt, \qquad Ndx \rightarrow dx, \tag{A6}
$$

the redefinition of the parameters:

$$
N^{1/2}r_0 \rightarrow r_0, \quad N^{1/2}a \rightarrow a, \quad N^{1/2}b \rightarrow b, \tag{A7}
$$

where we have defined $N \equiv \gamma^2 - \epsilon^2$ ($\gamma = 1 + \delta\epsilon$). Further, transform the coordinates

$$
dt + \tilde{\epsilon}^3 \frac{r_0^2 b}{a^2 - b^2}d\phi - \tilde{\epsilon}^3 \frac{r_0^2 a}{a^2 - b^2}d\psi \rightarrow dt, \tag{A8}
$$

where $\tilde{\epsilon} \equiv \epsilon/N^{1/2}$ and $\tilde{\gamma} = \gamma/N^{1/2}$. After the redefinition and coordinate transformations, the metric [derived by the $SL(2, R)$-duality transformation] takes the following form:

$$
\begin{aligned}
ds^2 = &-\frac{1}{[\tilde{\gamma}^2 - \tilde{\epsilon}^2(1 - r_0^2/\varrho^2)]^2}\left(1 - \frac{r_0^2}{\varrho^2}\right)\left[dt - r_0^2\left(\frac{\tilde{\gamma}^3 a}{\varrho^2 - r_0^2} - \frac{\tilde{\epsilon}^3 b}{\varrho^2}\right)\sin^2\theta d\phi - r_0^2\left(\frac{\tilde{\gamma}^3 b}{\varrho^2 - r_0^2} - \frac{\tilde{\epsilon}^3 a}{\varrho^2}\right)\cos^2\theta d\psi\right]^2 \\
&+ [\tilde{\gamma}^2 - \tilde{\epsilon}^2(1 - r_0^2/\varrho^2)]\left[\left(x + a^2 + \frac{r_0^2 a^2}{\varrho^2 - r_0^2}\sin^2\theta\right)\sin^2\theta d\phi^2 + \left(x + b^2 + \frac{r_0^2 b}{\varrho - r_0^2}\cos^2\theta\right)\cos^2\theta\right. \\
&\left. + 2\frac{r_0^2 ab}{\varrho^2 - r_0^2}\sin^2\theta\cos^2\theta d\phi d\psi + \frac{\varrho^2}{4}dx^2 + \varrho^2 d\theta^2\right].
\end{aligned} \tag{A9}
$$

This coincides with the metric form of the Cvetič-Youm solution [48] that was (re)derived by the $G_{2(+2)}$ transformation in Ref. [16]. (Note that $\tilde{\gamma} = c$ and $\tilde{\epsilon} = -s$.) See Appendix B for the relationship between the two, $SL(2, R)$ and $G_{2(+2)}$, transformations.

APPENDIX B: RELATION TO THE HARRISON TRANSFORMATION IN THE $G_{2(+2)}$ DUALITY

In this appendix we clarify how the timelike $SL(2, R)$ transformation, investigated in this paper, is embedded into $G_{2(+2)}$ if the given seed solution allows another spacelike Killing vector $\frac{\partial}{\partial z}$. A similar analysis for the case when the two Killing vectors are both spacelike was already done in [27].

We decompose the four-dimensional metric (vielbein) and gauge field as

$$E_\mu^{(4)\alpha} = \begin{pmatrix} e^\phi E_m^{(3)a} & C_m e^{-\phi} \\ 0 & e^{-\phi} \end{pmatrix}, \tag{B1}$$

$$A_\mu = (A_m, A_z), \tag{B2}$$

and write

$$E_M^{(5)A} = \begin{pmatrix} e^{-1} E_m^{(3)a} & B_m^i e_i^{\bar{a}} \\ 0 & e_i^{\bar{a}} \end{pmatrix}, \tag{B3}$$

$$A_M = (A_m, A_i), \tag{B4}$$

$$\eta_{AB} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}, \qquad \eta_{\bar{a}\bar{b}} = \text{diag}(+1, -1), \tag{B5}$$

$$e_i^{\bar{a}} = \begin{pmatrix} \rho^{-(1/2)} e^{-\phi} & \rho B_z \\ 0 & \rho \end{pmatrix}, \qquad e = \det e_i^{\bar{a}}, \tag{B6}$$

$$B_m^i = (C_m, B_m).$$

Here m (a) is the three-dimensional curved (flat) index, $i = t, z$ and $\bar{a} = 1, 2$. All the fields are assumed to be independent of t and z. Then the reduced Lagrangian reads

$$\mathcal{L} = E^{(3)}\bigg(R^{(3)} + \frac{1}{4} \partial_m g^{ij} \partial^m g_{ij} - e^{-2} \partial_m e \partial^m e - \frac{1}{2} g^{ij} \partial_m A_i \partial^m A_j - \frac{1}{4} e^2 g_{ij} B_{mn}^i B^{jmn}$$

$$- \frac{1}{4} e^2 F_{mn}^{(3)} F^{(3)mn} - \frac{1}{2\sqrt{3}} E^{(3)-1} \epsilon^{mnp} \epsilon^{ij} F_{mn} \partial_p A_i A_j \bigg), \tag{B7}$$

where $g_{ij} = e_i^{\bar{a}} \eta_{\bar{a}\bar{b}} e_j^{\bar{b}}$ and $F_{mn}^{(3)} = F_{mn} - 2B_{[n}^i \partial_{m]} A_i$. The vector fields B_m^i and A_m are dualized by adding the Lagrange multiplier terms

$$\mathcal{L}_{\text{Lag.mult.}}^{5\to3} = -\tfrac{1}{2} \epsilon^{mnp} (F_{mn}' \partial_p \varphi + B_{mn}^i \partial_p \psi_i) \tag{B8}$$

and completing the squares. Up to duality relations we have

$$\mathcal{L} + \mathcal{L}_{\text{Lag.mult.}}^{5\to3} = E^{(3)}\bigg(R^{(3)} + \frac{1}{4} \partial_m g^{ij} \partial^m g_{ij} - e^{-2} \partial_m e \partial^m e - \frac{1}{2} g^{ij} \partial_m A_i \partial^m A_j + \frac{1}{2} e^{-2}\bigg(\partial_m \varphi - \frac{1}{\sqrt{3}} \epsilon^{ij} A_i \partial_m A_j \bigg)$$

$$\times \bigg(\partial^m \varphi - \frac{1}{\sqrt{3}} \epsilon^{kl} A_k \partial^m A_l \bigg) + \frac{1}{2} e^{-2} g^{ij} \bigg(\partial_m \psi_i - A_i \partial_m \varphi + \frac{1}{3\sqrt{3}} \epsilon^{kl} A_i A_k \partial_m A_l \bigg)$$

$$\times \bigg(\partial^m \psi_j - A_j \partial^m \varphi + \frac{1}{3\sqrt{3}} \epsilon^{k'l'} A_j A_{k'} \partial^m A_{l'} \bigg)\bigg). \tag{B9}$$

The duality relations are

$$F_{mn}^{(3)} = -e^{-2} E^{(3)-1} \epsilon_{mn}^p \bigg(\partial_p \varphi - \frac{1}{\sqrt{3}} \epsilon^{ij} A_i \partial_p A_j \bigg), \tag{B10}$$

$$B_{mn}^i = -e^{-2} E^{(3)-1} \epsilon_{mn}^p \bigg(\partial_p \psi_i - A_i \partial_p \varphi + \frac{1}{3\sqrt{3}} \epsilon^{kl} A_i A_k \partial_p A_l \bigg).$$

From (B9) the target space metric can be read off, in terms of a matrix $(\mathbf{g})_{ij} = g_{ij}$ and vectors $(\vec{A})_i = A_i$ and $(\vec{\psi})_i = \psi_i$, as

$$ds_{\text{target}}^2 = \frac{1}{4} \text{Tr}(\mathbf{g}^{-1} d\mathbf{g})^2 + e^{-2} de^2 + \frac{1}{2} d\vec{A}^T \mathbf{g}^{-1} d\vec{A} - \frac{1}{2} e^{-2} \bigg(d\varphi - \frac{1}{\sqrt{3}} \epsilon^{ij} A_i dA_j \bigg)^2$$

$$- \frac{1}{2} e^{-2} \bigg(d\vec{\psi} - \vec{A}\bigg(d\varphi - \frac{1}{3\sqrt{3}} \epsilon^{ij} A_i dA_j \bigg)\bigg)^T \mathbf{g}^{-1} \bigg(d\vec{\psi} - \vec{A}\bigg(d\varphi - \frac{1}{3\sqrt{3}} \epsilon^{kl} A_k dA_l \bigg)\bigg). \tag{B11}$$

Comparing (B11) with the target space metric of Ref. [16], Eq. (77), the translation rules are $\mathbf{g} \to \lambda$, $e \to \tau^{(1/2)}$, $\varphi \to \mu$ and $\vec{A} \to \sqrt{3}\psi$ and $\vec{\psi} \to V$.

To reveal its group theoretical structure, we introduce a set of $G_{2(+2)}$ generators h_1, h_2, E^i_j ($1 \le i \ne j \le 3$), E^i ($1 \le i \le 3$), and E^*_i ($1 \le i \le 3$), satisfying [27]

$$
\begin{aligned}
&[h_i, h_j] = 0, \quad [h_i, E^j_k] = \delta^j_i E^i_k - \delta^j_{i+1} E^{i+1}_k - \delta^i_k E^j_i + \delta^{i+1}_k E^j_{i+1}, \quad [h_i, E^j] = \delta^j_i E^i - \delta^j_{i+1} E^{i+1}, \\
&[h_i, E^*_j] = -(\delta^i_j E^*_i - \delta^{i+1}_j E^*_{i+1}), \quad [E^i_j, E^k_l] = \delta^k_j E^i_l - \delta^i_l E^k_j, \quad [E^i_j, E^k] = \delta^k_j E^i, \\
&[E^i_j, E^*_k] = -\delta^i_k E^*_j, \quad [E^i, E^j] = -2\sum_{k}^{3} \epsilon^{ijk} E^*_k, \quad [E^*_i, E^*_j] = +2\sum_{k}^{3} \epsilon_{ijk} E^k, \quad [E^i, E^*_j] = 3E^i_j \text{ if } i \ne j, \\
&[E^1, E^*_1] = 2h_1 + h_2, \quad [E^2, E^*_2] = -h_1 + h_2, \quad [E^3, E^*_3] = -h_1 - 2h_2,
\end{aligned}
\tag{B12}
$$

where ϵ^{ijk} and ϵ_{ijk} are totally antisymmetric tensors with $\epsilon^{123} = \epsilon_{123} = +1$ (The sign convention for ϵ_{123} was -1 in [27]). This is the realization due to Fruedenthal, which shows the close relationship between the two exceptional Lie groups E_8 and G_2 [27,56]. Then the $G_{2(+2)}$ group element

$$
\mathcal{V}^{(3)} = \exp(-(\log e^{\bar{1}}_1)h_1 - (\log e^{\bar{1}}_1 e^{\bar{2}}_2)h_2)\exp(-e^2_1 e^2_2 E^1_2)\exp(\psi_i E^i_3)\exp\left(-\frac{1}{\sqrt{3}}A_i E^i\right)\exp\left(\frac{1}{\sqrt{3}}\varphi E^*_3\right)
\tag{B13}
$$

gives rise to a right invariant vector field

$$
\begin{aligned}
\partial_m \mathcal{V}^{(3)}\mathcal{V}^{(3)-1} = {} & \frac{1}{2}\partial_m(\phi + \log\rho)h_1 + \frac{1}{2}\partial_m(\phi - \log\rho)h_2 - e^{(\phi/2)}\rho^{(3/2)}\partial_m B_z E^1_2 - \frac{1}{\sqrt{3}}e^i_{\bar{a}}\partial_m A_i E^{\bar{a}} \\
& + e^{-1}\left(\frac{1}{\sqrt{3}}\partial_m\varphi - \frac{1}{3}\epsilon^{ij}A_i\partial_m A_j\right)E^*_3 + e^{-1}e_{\bar{a}}{}^i\left(\partial_m\psi_i - A_i\partial_m\varphi + \frac{1}{3\sqrt{3}}\epsilon^{kl}A_i A_k \partial_m A_l\right)E^{\bar{a}}_3.
\end{aligned}
\tag{B14}
$$

To obtain the reduced Lagrangian (B9), we define the symmetric space involution τ, which is an automorphism and decomposes the $G_{2(+2)}$ Lie algebra into its eigenspaces:

$$
\begin{aligned}
G_{2(+2)} = \mathbb{H} \oplus \mathbb{K}, \qquad &\mathbb{H} = \{X \in G_{2(+2)}|\tau(X) = +X\}, \\
&\mathbb{K} = \{X \in G_{2(+2)}|\tau(X) = -X\}.
\end{aligned}
\tag{B15}
$$

In the present case, $\mathbb{H}$ is defined to be a subspace spanned by

$$
E^1_2 + E^2_1, \qquad E^1_3 + E^3_1, \qquad E^2_3 - E^3_2, \qquad E^1 - E^*_1, \qquad E^2 + E^*_2 \quad \text{and } E^3 + E^*_3,
\tag{B16}
$$

and $\mathbb{K}$ is by

$$
E^1_2 - E^2_1, \qquad E^1_3 - E^3_1, \qquad E^2_3 + E^3_2, \qquad E^1 + E^*_1, \qquad E^2 - E^*_2, \qquad E^3 - E^*_3, \qquad h_1 \quad \text{and} \quad h_2.
\tag{B17}
$$

The reduced Lagrangian (B9) can be obtained (with a suitable overall constant normalization factor) by projecting (B14) onto $\mathbb{K}$ and taking trace of the square. $\mathbb{H}$ is the Lie algebra of the $SO(2, 2) \sim SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ subgroup, and hence $\mathbb{K}$ the Lie algebra of the coset space $G_{2(+2)}/SO(2, 2)$. According to the general prescription, the coset representative is

$$
\mathcal{R}^{(3)} = \tau(\mathcal{V}^{(3)-1})\mathcal{V}^{(3)},
\tag{B18}
$$

then $E^{(3)} \text{Tr}\partial_m \mathcal{R}^{(3)-1}\partial^m \mathcal{R}^{(3)}$ is automatically proportional to (B9). Note that the coset representative can be chosen, as in [16], to be a symmetric matrix, which can be obtained by multiplying a suitable constant matrix to $\mathcal{R}^{(3)}$ from, say, the right, but it is not necessary.

The timelike Killing $SL(2, \mathbb{R})$ we considered in this paper acts as a group multiplication to the $SL(2, \mathbb{R})$ generated by E^2, E^*_2, and $-h_1 + h_2$. On the other hand, using the dictionary given below (B11), one can identify that these generators are represented in [16] as

$$E^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \qquad E_2^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_2 - h_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(B19)

and the matrix $\mathcal{C}$ in [16] for the Harrison rotation

$$\mathcal{C} = \begin{pmatrix} c^2 & 0 & 0 & s^2 & 0 & 0 & \sqrt{2}sc \\ 0 & c & 0 & 0 & 0 & s & 0 \\ 0 & 0 & c & 0 & -s & 0 & 0 \\ s^2 & 0 & 0 & c^2 & 0 & 0 & \sqrt{2}sc \\ 0 & 0 & -s & 0 & c & 0 & 0 \\ 0 & s & 0 & 0 & 0 & c & 0 \\ \sqrt{2}sc & 0 & 0 & \sqrt{2}sc & 0 & 0 & c^2 + s^2 \end{pmatrix}, \qquad c = \cosh\alpha, \qquad s = \sinh\alpha \qquad (B20)$$

can be written as $\exp(-\alpha(E^2 + E_2^*))$, and hence belong to the timelike Killing $SL(2,\mathbb{R})$. This explains why we have obtained the Cvetič-Youm solution in Appendix A. Note that $\exp(-\alpha(E^2 + E_2^*))$ is not the same as the $SL(2,\mathbb{R})$ group element used in Appendix A; this redundancy of the $SL(2,\mathbb{R})$ group element was already reported in the spacelike Killing case [29].

[1] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002).

[2] A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005.

[3] T. Mishima and H. Iguchi, Phys. Rev. D **73**, 044030 (2006).

[4] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. Lett. **93**, 211302 (2004).

[5] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, Phys. Rev. D **71**, 024033 (2005).

[6] I. Bena and N. P. Warner, Adv. Theor. Math. Phys. **9**, 667 (2005).

[7] I. Bena, P. Kraus, and N. P. Warner, Phys. Rev. D **72**, 084019 (2005).

[8] J. P. Gauntlett and J. B. Gutowski, Phys. Rev. D **71**, 045002 (2005).

[9] S. S. Yazadjiev, Phys. Rev. D **73**, 104007 (2006).

[10] H. Iguchi and T. Mishima, Phys. Rev. D **75**, 064018 (2007).

[11] H. Elvang and P. Figueras, J. High Energy Phys. 05 (2007) 050.

[12] K. Izumi, Prog. Theor. Phys. **119**, 757 (2008).

[13] H. Elvang and M. J. Rodriguez, J. High Energy Phys. 04 (2008) 045.

[14] H. Elvang, R. Emparan, and P. Figueras, J. High Energy Phys. 02 (2005) 031.

[15] S. Mizoguchi and N. Ohta, Phys. Lett. B **441**, 123 (1998).

[16] A. Bouchareb, G. Clement, C-M. Chen, D. V. Gal'tsov, N. G. Scherbluk, and T. Wolf, Phys. Rev. D **76**, 104032 (2007); **78**, 029901(E) (2008).

[17] S. Tomizawa, Y. Yasui, and A. Ishibashi, Phys. Rev. D **79**, 124023 (2009).

[18] S. Tomizawa, Y. Yasui, and A. Ishibashi, Phys. Rev. D **81**, 084037 (2010).

[19] S. Tomizawa, Phys. Rev. D **82**, 104047 (2010).
[20] S. Giusto, S. F. Ross, and A. Saxena, J. High Energy Phys. 12 (2007) 065.
[21] G. Compere, S. Buyl, E. Jamsin, and A. Virmani, Classical Quantum Gravity **26**, 125016 (2009).
[22] G. Compere, S. Buyl, S. Stotyn, and A. Virmani, J. High Energy Phys. 11 (2010) 133.
[23] S. Tomizawa, Y. Yasui, and Y. Morisawa, Classical Quantum Gravity **26**, 145006 (2009).
[24] D. V. Gal'tsov and N. G. Scherbluk, Phys. Rev. D **78**, 064033 (2008).
[25] D. V. Gal'tsov and N. G. Scherbluk, Phys. Rev. D **79**, 064020 (2009).
[26] A. H. Chamseddine and H. Nicolai, Phys. Lett. B **96**, 89 (1980).
[27] S. Mizoguchi and G. Schröder, Classical Quantum Gravity **17**, 835 (2000).
[28] S. Mizoguchi, Phys. Lett. B **523**, 351 (2001).
[29] S. Mizoguchi and S. Tomizawa, Phys. Rev. D **84**, 104009 (2011).
[30] S. Tomizawa, Phys. Rev. D **82**, 104047 (2010).
[31] P. Dobiasch and D. Maison, Gen. Relativ. Gravit. **14**, 231 (1982).
[32] D. Rasheed, Nucl. Phys. **B454**, 379 (1995).
[33] H. Ishihara and K. Matsuno, Prog. Theor. Phys. **116**, 417 (2006).
[34] T. Nakagawa, H. Ishihara, K. Matsuno, and S. Tomizawa, Phys. Rev. D **77**, 044040 (2008).
[35] S. Tomizawa, H. Ishihara, K. Matsuno, and T. Nakagawa, Prog. Theor. Phys. **121**, 823 (2009).
[36] S. Tomizawa and A. Ishibashi, Classical Quantum Gravity **25**, 245007 (2008).
[37] S. Tomizawa, arXiv:1009.3568.
[38] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, Classical Quantum Gravity **20**, 4587 (2003).
[39] D. Gaiotto, A. Strominger, and X. Yin, J. High Energy Phys. 02 (2006) 023.
[40] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, J. High Energy Phys. 08 (2005) 042.
[41] J. Ehlers, Dissertation, University Hamburg, 1957.
[42] R. A. Matzner and C. W. Misner, Phys. Rev. **154**, 1229 (1967).
[43] P. Breitenlohner, D. Maison, and G. W. Gibbons, Commun. Math. Phys. **120**, 295 (1988).
[44] Timelike reductions of string and *M* theories were studied in [45,46], where in the latter the signature change in timelike *T* duality was first discovered.
[45] C. M. Hull and B. Julia, Nucl. Phys. **B534**, 250 (1998).
[46] C. M. Hull, J. High Energy Phys. 07 (1998) 021.
[47] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
[48] M. Cvetič and D. Youm, Nucl. Phys. **B476**, 118 (1996).
[49] E. Cremmer and B. Julia, Nucl. Phys. **B159**, 141 (1979).
[50] R. Emparan and H. S. Reall, Phys. Rev. D **65**, 084025 (2002).
[51] To avoid confusion with the Maxwell field, we denote in this paper the Kaluza-Klein vector field as B_μ, instead of $2A_\mu^{(\text{Rasheed})}$ used in the original article [32].
[52] A. Keurentjes, Nucl. Phys. **B697**, 302 (2004).
[53] A. Keurentjes, J. High Energy Phys. 11 (2004) 034.
[54] S. de Buyl, L. Houart, and N. Tabti, J. High Energy Phys. 06 (2005) 084.
[55] L. Houart, A. Kleinschmidt, J. Lindman Hornlund, D. Persson, and N. Tabti, J. High Energy Phys. 08 (2009) 098.
[56] S. Mizoguchi, Nucl. Phys. **B528**, 238 (1998).