PHYSICAL REVIEW D **86,** 024017 (2012)

Linking covariant and canonical loop quantum gravity: New solutions to the Euclidean scalar constraint

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(Received 6 November 2011; published 9 July 2012)

It is often emphasized that spin-foam models could realize a projection on the physical Hilbert space of canonical loop quantum gravity. As a first test, we analyze the one-vertex expansion of a simple Euclidean spin foam. We find that for fixed Barbero-Immirzi parameter $\gamma = 1$, the one-vertex amplitude in the Kaminski, Kisielowski, and Lewandowski prescription annihilates the Euclidean Hamiltonian constraint of loop quantum gravity [T. Thiemann, Classical Quantum Gravity **15**, 839 (1998).]. Since, for $\gamma = 1$, the Lorentzian part of the Hamiltonian constraint does not contribute, this gives rise to new solutions of the Euclidean theory. Furthermore, we find that the new states only depend on the diagonal matrix elements of the volume. This seems to be a generic property when applying the spin-foam projector.

DOI: 10.1103/PhysRevD.86.024017 PACS numbers: 04.60.Ds

I. INTRODUCTION

A. Motivation

One major problem when quantizing gravity is the constrained algebra, which completely determines the theory, and background independence. Canonical loop quantum gravity (LQG) [1–3] follows the ideas of Dirac [4] for quantizing constrained systems and preserves background independence. The kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ of LQG is spanned by spin-network functions living on semianalytic closed graphs embedded in a 3-dimensional spatial hypersurface Σ of a 4-dimensional manifold $\mathcal{M}$. Diffeomorphism and gauge constraints can be embedded via a group averaging procedure. The remaining constraint (Hamiltonian) is more complicated. Even though a quantization of the latter has been found [5,6], the structure of the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is not fully understood until now.

To circumvent the problems of the canonical theory, Reisenberger and Rovelli [7] introduced a covariant formulation of quantum gravity, the so-called spin-foam model [8,9]. This model is mainly based on the observation that the Holst action for general relativity (GR) [10] defines a constrained BF theory. The strategy is first to quantize discrete BF theory and then to implement the so-called simplicity constraints. The main building block of the model is a linear two-complex κ embedded into 4-dimensional space-time $\mathcal{M}$ whose boundary is given by an initial and final (gauge-invariant) spin network, ψ_i and ψ_f, living on the initial and final spatial hypersurface of a foliation of $\mathcal{M}$. The physical information is encoded in the spin-foam amplitude

$$Z[\kappa] = \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v \times \mathcal{B}, \tag{1.1}$$

where $\mathcal{A}_f$, $\mathcal{A}_e$, and $\mathcal{A}_v$ are the amplitudes associated to the internal faces, edges, and vertices[1] of κ, and $\mathcal{B}$ contains the boundary amplitudes. Each spin foam can be thought of as a generalized Feynman diagram contributing to the transition amplitude from an ingoing spin network to an outgoing spin network. By summing over all possible two-complexes, one obtains the complete "transition amplitude" between ψ_i and ψ_f.

Unfortunately, the simplicity constraint is second-class, and the procedure how to implement it is still under debate [11]. Nevertheless, substantial progress has been achieved during the last few years [12]. Especially, the introduction of a new vertex amplitude by Engle, Pereira, Rovelli, and Livine; independently, by Freidel and Krasnov [13]; and the introduction of an abstract model [14] led to a major breakthrough.

Instead of considering spin foams as a "sum over histories," one could equally well think of spin foams as some group averaging procedure to implement the Hamiltonian constraint in the canonical formalism (see Refs. [7,15]). Suppose we have a family of first-class constraints $(\hat{C}_I)_{I\in\mathcal{I}}$ which form a Lie algebra. Generically, the point zero does not lie in the point spectrum of the constraint operators, and therefore the eigenvectors cannot form the entire solution space. To obviate this problem, one has to consider generalized eigenvectors $l \in \mathcal{D}^*_{\text{kin}}$ in the algebraic dual of a dense domain of $\mathcal{H}_{\text{kin}}$ such that

$$[(\hat{C}_I)'l](\psi) := l(\hat{C}_I^\dagger \psi) = 0 \qquad \forall\ I \in \mathcal{I} \text{ and } \psi \in \mathcal{D}_{\text{kin}}, \tag{1.2}$$

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[1] In the following, we will call edges and vertices in the boundary links and, respectively, nodes to distinguish between the two-complex and the graph.

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where $(\hat{C}_I)'$ is the dual operator on $\mathcal{D}^*_{\text{kin}}$. The space of generalized solutions $\mathcal{D}^*_{\text{phys}}$ is a proper subspace of $\mathcal{D}^*_{\text{kin}}$. In order to construct a physical Hilbert space, one considers $\mathcal{D}^*_{\text{phys}}$ as the algebraic dual of a dense subspace $\mathcal{D}_{\text{phys}} \subset \mathcal{H}_{\text{phys}}$ so that all observables are densely defined in $\mathcal{H}_{\text{phys}}$. The inner product on $\mathcal{H}_{\text{phys}}$ is chosen such that adjoints in the physical scalar product represent adjoints in the kinematical one. It can be systematically constructed by an antilinear map, called a rigging map,

$$\eta\colon \mathcal{D}_{\text{kin}} \to \mathcal{D}^*_{\text{kin}}, \tag{1.3}$$

such that

$$\langle \eta[\phi] | \eta[\psi] \rangle_{\text{phys}} := \eta[\phi](\psi) \qquad \phi, \psi \in \mathcal{D}_{\text{kin}} \tag{1.4}$$

and

$$\hat{O}'\eta[\phi] = \eta[\hat{O}\phi] \qquad \forall\ \phi \in \mathcal{D}_{\text{kin}}. \tag{1.5}$$

The physical Hilbert space is subsequently defined by the completion of $\mathcal{D}_{\text{phys}} := \eta(\mathcal{D}_{\text{kin}}) \setminus \ker(\eta)$.[2] Strictly speaking, such a construction only works for closed, first-class constraints. But the constraint algebra in GR is open with structure functions instead of structure constants. Nevertheless, it is often emphasized that spin foams could provide such a rigging map even though one starts with a different action and constraint algebra than in the canonical approach. If this is indeed the case, then the physical inner product would be given by

$$\langle \phi | \psi \rangle_{\text{phys}} = \sum_{\kappa\colon\ \psi \to \phi} Z[\kappa], \tag{1.6}$$

and the rigging map would correspond (schematically) to

$$\eta[\psi] = \sum_{\phi \in \mathcal{H}_{\text{kin}}} \sum_{\kappa\colon\ \psi \to \phi} Z[\kappa] \langle \phi |. \tag{1.7}$$

Since all constraints are satisfied in $\mathcal{H}_{\text{phys}}$, the so-defined physical scalar product must obey

$$\langle \psi_{\text{out}} | \hat{C}^\dagger | \psi_{\text{in}} \rangle_{\text{phys}} = \sum_{\phi \in \mathcal{H}_{\text{kin}}} \sum_{\kappa\colon\ \psi_{\text{out}} \to \phi} Z[\kappa] \langle \phi | \hat{C}^\dagger | \psi_{\text{in}} \rangle_{\text{kin}} = 0 \tag{1.8}$$

for all $\psi_{\text{out}}, \psi_{\text{in}} \in \mathcal{H}_{\text{kin}}$. This is clearly the case for the Gauss constraint because the boundary of a spin foam constitutes a gauge-invariant spin network. The diffeomorphism constraint is harder to deal with since spin foams are defined on a discretization of space-time and break diffeomorphism invariance. But in the abstract formulation, the amplitudes do not depend on the embedding, and one can implement the constraint by restricting on equivalence classes of spin networks. Whether the Hamiltonian constraint also obeys Eq. (1.8) depends crucially on the definition of the vertex amplitudes. On the other hand, many results, for example the definition of a propagator and in spin foam cosmology [16–18], obtained so far in covariant LQG are based on the assumption that the Engle, Pereira, Rovelli and Levine (EPRL) and Freidel and Krasnov (FK) amplitude defines a physical scalar product. Thus, testing Eq. (1.8) provides a consistency check for the canonical as well as covariant approach towards quantum gravity.

[2] For more details on the construction of a rigging map see, e.g., [2].

B. Outline

As a first test for Eq. (1.8), we consider an easy spin-foam amplitude and show that

$$\sum_{\phi} Z[\kappa] \langle \phi | \hat{H}_{\mathfrak{n}} | \psi_{\text{in}} \rangle = 0, \tag{1.9}$$

where κ is a two-complex with only one internal vertex such that ϕ is a spin network induced on the boundary of κ, and $\hat{H}_{\mathfrak{n}}$ is the Hamiltonian constraint acting on the node $\mathfrak{n}$.

In Sec. II, we briefly review the quantization of the Hamiltonian constraint [5] and compute the action on three- and four-valent nodes by employing graphical calculus. Recall that the full constraint $\mathcal{C} = -[H + (s - \gamma^2)H_L]$ can be decomposed into its Lorentzian and Euclidean part, H_L and H, respectively, where γ is the Barbero-Immirzi parameter and s is the signature of the metric. We restrict the analysis to the Euclidean sector with $s = 1$, $\gamma = 1$ so that $\mathcal{C}$ reduces to the Euclidean part only and choose a tetrahedral regularization of the latter as proposed in Ref. [5]. In this regularization, the constraint $\hat{H}$ acts locally on nodes and creates a new link connecting two pairwise distinct links adjacent to the same node.

We will summarize the construction of the spin-foam amplitude in Sec. III.

In the subsequent section, we evaluate the spin-foam amplitude for a two-complex κ with only one internal vertex and boundary $\partial\kappa = \psi_{\text{out}} \cup \phi$, such that κ is a tube $\psi_{\text{out}} \times [0, 1]$ with an additional face between the internal vertex and the new link created by $\hat{H}$ (see Fig. 1).

In Sec. IV, we show that the one-vertex amplitude annihilates the Hamiltonian constraint by employing basic summation identities of $6j$ symbols, when acting on three and four-valent nodes. For an n-valent node, the sum (1.9)

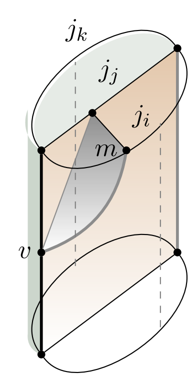


FIG. 1 (color online). Two-complex κ with on internal vertex.

is a sum over spin networks based on $\binom{n}{2}$ different graphs. Remarkably, each partial sum over spin networks based on the same graph vanishes. This shows that the solutions constructed via the spin-foam method build a proper subset of $\mathcal{H}_{\rm phys}$. As an important side result, we find that $\sum_\phi Z[\kappa]\langle\phi|$ selects only those matrix elements of $\hat{H}$ which depend on the diagonal matrix elements of the volume.

Section V contains a summary of our results and gives an outlook to open questions.

II. HAMILTONIAN CONSTRAINT

A primary quantum version of the Hamiltonian constraint operator was introduced by Rovelli and Smolin [19]. The operator proved to act only on the nodes of a spin-network function. But it was divergent on general states. Later [20], it was shown that the following two properties are crucial in order to obtain a well-defined finite Hamiltonian operator in the background-independent context:

(i) the operator needs to be a density (more precisely, a three-form)

(ii) diffeomorphism invariance trivializes the limit when the regulator is removed from the operator.

The first requirement forces us to use a nonpolynomial version of the constraint whose quantization is much more involved. After many efforts [21–23], it was suggested [5,6] to express the inverse triad e as the Poisson bracket between the volume V and the holonomy h of the Ashtekar connection A: $e \sim h^{-1}[h, V]$. This trick made it possible to construct a Hamiltonian with the above properties which can be regularized on a given triangulation T of the space manifold. (For criticism, see Ref. [24]). In the following two sections, we will review the basic construction of $\mathcal{C}$ as proposed in Ref. [5] in order to clarify the model and our notation. The reader familiar to the framework can easily skip the next two sections.

A. Hamiltonian constraint

The classical Hamiltonian constraint is

$$\mathcal{C} = -\frac{2}{\kappa}\,\mathrm{Tr}[({}^{(\gamma)}F - (\gamma^2 - s)K\wedge K)\wedge e], \tag{2.1}$$

where $e = e^i_a\tau_i dx^a$ is the inverse triad, K the extrinsic curvature, ${}^{(\gamma)}F$ the curvature of the Ashtekar connection with real Immirzi parameter γ, and s the signature. In the following, we choose units such that $\kappa/2 := \frac{8\pi G}{c^3} = 1$.

The constraint can be split into its "Euclidean" part $H = \mathrm{Tr}[F\wedge e]$ and Lorentzian part $H_L = C - H$. Following Ref. [5], we can rewrite (2.1) by using

$$e^i_a = 2\{A^i_a(x), V\} \quad K^i_a = 2\{A^i_a(x), K\} \quad K = 2\{H, V\}, \tag{2.2}$$

where V is the volume of an arbitrary region Σ containing the point x. Smearing the constraints with lapse function $N(x)$ gives

$$H[N] = \int_\Sigma d^3x N(x) H(x) = -2\int_\Sigma N\,\mathrm{Tr}(F\wedge\{A, V\}) \tag{2.3}$$

$$H_L[N] = \int_\Sigma d^3x N(x) H_L(x) = -(\gamma^2 - s)\int_\Sigma N\,\mathrm{Tr}(\{A,\{H,V\}\}\wedge\{A,\{H,V\}\}\wedge\{A,V\}). \tag{2.4}$$

This expression requires a regularization in order to obtain a well-defined operator on $\mathcal{H}_{\rm kin}$. Up to now, there exists many different proposals (see, e.g., Refs. [1,25]). We will follow the original proposal [5] and use a triangulation T of the manifold Σ into elementary tetrahedra with analytic links adapted to the graph Γ of an arbitrary spin network. For each pair of links e_i and e_j incident at a node $\mathfrak{n}$ of Γ, we choose semianalytic arcs a_{ij} such that the end points s_{e_i}, s_{e_j} are interior points of e_i and e_j, respectively, and $a_{ij}\cap\Gamma = \{s_{e_i}, s_{e_j}\}$. The arc s_i is the segment of e_i from $\mathfrak{n}$ to s_i, and s_i, s_j and a_{ij} generate a triangle $\alpha_{ij} := s_i \circ a_{ij} \circ s_j^{-1}$. Three (nonplanar) links define a tetrahedron (see Fig. 2). Now, we can decompose Eq. (2.3) into a sum of one term per each tetrahedron of the triangulation

$$H[N] = \sum_{\Delta\in T} -2\int_\Delta d^3x N\epsilon^{abc}\,\mathrm{Tr}(F_{ab}\{A_c, V\}). \tag{2.5}$$

Define the classical *regularized* Hamiltonian constraint

$$H_T[N] := \sum_{\Delta\in T} H_\Delta[N]. \tag{2.6}$$

The connection A and the curvature are regularized as usual by the holonomy $h_s := h[s] \in \mathrm{SU}(2)$ (in the fundamental representation $m = 1/2$) along the segments s_i and along the loop α_{ij}, respectively. This yields

$$H_\Delta[N] := -\tfrac{2}{3}N(\mathfrak{n})\epsilon^{ijk}\,\mathrm{Tr}[h_{\alpha_{ij}}h_{s_k}\{h^{-1}_{s_k}, V\}] \tag{2.7}$$

and converges to the Hamiltonian constraint (2.5) if the triangulation is sufficiently fine. The expression (2.6) can finally be promoted to a quantum operator, since volume and holonomy have corresponding well-defined operators in LQG. The lattice spacing of the triangulation T which

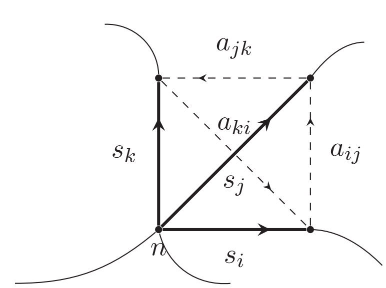


FIG. 2. An elementary tetrahedron $\Delta\in T$ constructed by adapting it to a graph Γ which underlies a cylindrical function.

acts as a regularization parameter can be removed in a suitable operator topology, see Ref. [5] for details.

Remarks:

(i) In Ref. [26], it was pointed out that the operator can be immediately generalized by replacing the trace in Eq. (2.5) with a trace in an arbitrary irreducible representation m: $\mathrm{Tr}_m[U] = \mathrm{Tr}[R^{(m)}(U)]$ where $R^{(m)}$ is a matrix representation of $U \in \mathrm{SU}(2)$. Equation (2.7) can thus be replaced by

$$H^m_\Delta[N] := \frac{N(\mathfrak{n})}{N_m^2} \epsilon^{ijk} \mathrm{Tr}[h^{(m)}_{\alpha_{ij}} h^{(m)}_{s_k} \{h^{(m)-1}_{s_k}, V\}], \quad (2.8)$$

$N_m^2 = \mathrm{Tr}_m[\tau^i \tau^i] = -(2m+1)m(m+1)$ and $h^{(m)} = R^{(m)}(h)$. As shown in Ref. [26], this converges to $H[N]$ as well.

(ii) The Lorentzian part of the constraint can be regularized by a similar method.

B. Properties

In this section, we will summarize the important properties of the Euclidean Hamiltonian constraint.

It is immediate to see that when acting on a spin-network state, the operator reduces to a sum over terms each acting on individual nodes. Acting on nodes of valence n, the operator gives

$$\hat{H}^m_\Gamma[N]\psi_\Gamma = \frac{i}{\hbar} \sum_{\mathfrak{n}\in\mathcal{N}(\Gamma)} \sum_{\mathfrak{n}(\Delta)=\mathfrak{n}} \frac{p_\Delta}{E(\mathfrak{n})} \hat{H}^m_\Delta[N]\psi_\Gamma, \quad (2.9)$$

where H^m_Δ is the quantum version of (2.8), $\mathcal{N}(\Gamma)$ is the set of nodes of Γ, and

$$E(\mathfrak{n}) = \binom{n}{3}$$

is the number of unordered triples of links adjacent to $\mathfrak{n}$. The second sum is a sum over tetrahedra with a node at $\mathfrak{n}$ and not intersecting with other nodes of Γ. Moreover, $p_\Delta = 1$, whenever Δ is a tetrahedron having three edges coinciding with three links of the spin-network state, which meets at the node $\mathfrak{n}$, otherwise, $p_\Delta = 0$.

On diffeomorphism-invariant states $\phi \in \mathcal{H}_{\mathrm{phys}} \subset \mathcal{H}^*_{\mathrm{kin}}$, the regulator dependence drops out trivially because two operators $\hat{H}$ and $\hat{H}'$, which are related by a refinement of the triangulation, differ only in the size of the loops α_{ij}. Therefore, the resulting states are in the same equivalence class, and

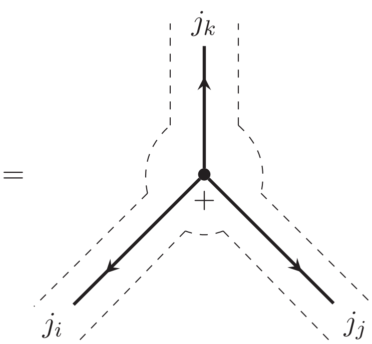
$$[\hat{H}^\dagger \phi](\psi) := \langle \phi, \hat{H}\psi\rangle = \langle \phi, \hat{H}'\psi\rangle, \quad (2.10)$$

in $\mathcal{H}_{\mathrm{diff}}$. This proves that the Hamiltonian constraint on diffeomorphism-invariant states is independent from the refinement of the triangulation.

The action $\hat{H}(N)$ on a spin-network state $T_{\Gamma,\vec{j},\vec{c}}$ defined on a graph Γ results in a finite linear combination of spin-network states defined on graphs Γ_I, where $\Gamma \subset \Gamma_I$ and $a_I := \Gamma_I - \Gamma$ is produced by one of the arcs $a_{ij}(\Delta)$, which carries spin $j_I = m$. The new nodes are called *extraordinary*. In some cases, it can happen that links connecting the original node with the new extraordinary nodes carry trivial representation if this is allowed by the recoupling conditions. Extraordinary nodes are, at most, trivalent and intersections of precisely two analytic curves $c, c' \subset \Gamma$; that is, $\mathfrak{n} = c \cap c'$, such that $\mathfrak{n}$ is an endpoint of c but not of c'. A link e of a graph Γ is called extraordinary provided that its endpoints $\mathfrak{n}_1$, $\mathfrak{n}_2$ are both extraordinary nodes. Furthermore, those links are adverse to a node $\mathfrak{n}$ of γ which is incident to at least three links s_1, s_2, s_3 with linearly independent tangents at $\mathfrak{n}$, such that s_1/s_2 connect $\mathfrak{n}$ and $\mathfrak{n}_1/\mathfrak{n}_2$. We will call $\mathfrak{n}$ the typical node associated with e. All links produced by $\hat{H}$ are extraordinary. Since the volume operator annihilates coplanar nodes and gauge invariant nodes of valence three (only true for the Ashtekar-Lewandowski version, see Ref. [1]), H does not act on extraordinary nodes.

C. Action on a trivalent node

Let us now compute the action of the operator $\mathcal{H}^m_\Delta$ on a trivalent node where all links are outgoing, following Refs. [25,26]. Denote a trivalent node by $|\mathfrak{n}(j_i, j_j, j_k)\rangle \equiv |\mathfrak{n}_3\rangle$, whereas j_i, j_j, j_k are the spins of the adjacent links e_i, e_j, e_k:

$|\mathfrak{n}_3\rangle =$  . (2.11)

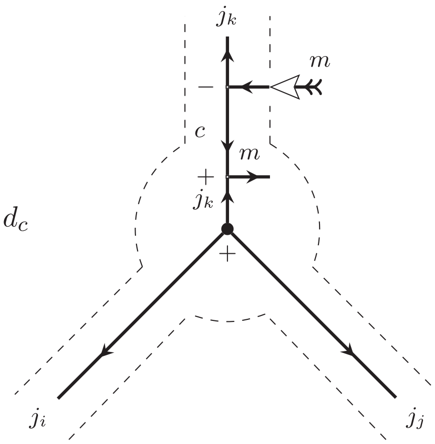
Note, the links are also labeled by group elements with orientations indicated by the arrows. In order to simplify the graphics, we only displayed the node and its adjacencies. Furthermore, everything contained in the dashed circle belongs to the node, and everything between the dashed lines belong to the same link.

When quantizing expression (2.8), the holonomies and the volume are replaced by their corresponding operators, and the Poisson bracket is replaced by a commutator. Since the volume operator vanishes on a gauge-invariant trivalent node, we only need to compute

$$\hat{H}^m_\Delta|\mathfrak{n}_3\rangle = N_\mathfrak{n} \epsilon^{ijk} \mathrm{Tr} \times \left(\frac{\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}]}{2} \hat{h}^{(m)}[s_k] \hat{V} \hat{h}^{(m)}[s_k^{-1}] \right) |\mathfrak{n}_3\rangle, \quad (2.12)$$

where all (global) constants have been absorbed in the lapse function $N_{\mathfrak{n}}$. The operator $\hat{h}^{(m)}[s_k^{-1}]$, corresponding to the holonomy along a segment s_k with reversed orientation, acts by multiplication with $R^{(m)}(h_{s_k^{-1}})$ along s_k. The matrix $R^{(m)}$ can be recoupled using Eqs. (A12) and (A10). Thus, $h^{(m)}[s_k^{-1}]$ creates a free index in the m representation located at the node (inside the dashed circle), making it non-gauge-invariant, and a new node on the link e_k:

$$\hat{h}^{(m)}[s_k^{-1}]\ |\mathfrak{n}_3\rangle \ = (-1)^{2m} \sum_c d_c \qquad (2.13)$$



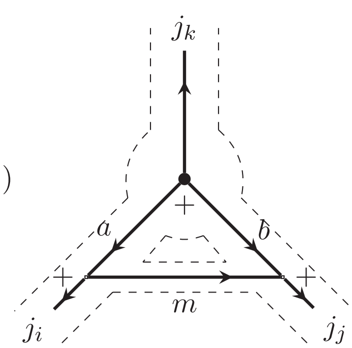
where $d_c = 2c + 1$ is the dimension of c. The range of the sum over the spin c is determined by the Clebsch-Gordan conditions, and the little flag represents the group element $h_{s_k}^{-1}$.

The volume operator now acts on a trivalent non-gauge-invariant node with a virtual link in j_k representation. The matrix elements of the volume operator [27–29] have been computed in Refs. [30,31], and the results have been applied to the Hamiltonian constraint operator in Refs. [26,32].

The operators $h^{(m)}[\alpha_{ij}]h^{(m)}[s_k]$ and $h^{(m)}[\alpha_{ji}]h^{(m)}[s_k]$ add open loops with opposite orientations α_{ij} and α_{ji}, where we fix α_{ij} to be oriented counterclockwise. Like above, the representations living on the same link can be recoupled. The m trace connects the free ends of the open loops to the two open links in $h^{(m)}[s_k^{-1}]|\mathfrak{n}\rangle$, taking into account the orientations. Finally, one has to use (A17):

$$\mathrm{Tr}\left(\frac{h^{(m)}[\alpha_{ij}] - h^{(m)}[\alpha_{ji}]}{2}\, h^{(m)}[s_k]\, V\, h^{(m)}[s_k^{-1}]\right) |\mathfrak{n}(j_i, j_j, j_k)\rangle$$

$$= \sum_{a,b} A^{(m)}(j_i, a|j_j, b|j_k) \qquad (2.14)$$



where the range of the sums over a, b is determined by the Clebsch-Gordan conditions[3] and

[3]The link m cannot be removed with Eq. (A17) since this is a pure recoupling identity, but m also carries a group element.

$$A^{(m)}(j_i,a|j_j,b|j_k) := \sum_c \lambda_a^{mj_i}\lambda_b^{mj_j}\lambda_c^{mj_k}(-)^{j_i+j_j-j_k} d_a d_b d_c \sum_{\beta(j_i,j_j,m,c)} V_{j_k}{}^{\beta}(j_i,j_j,m,c)\Big[\lambda_c^{m\beta}(-)^{a+j_j+c}\begin{Bmatrix} a & j_j & c \\ \beta & m & j_i\end{Bmatrix}\begin{Bmatrix} a & b & j_k \\ m & c & j_j\end{Bmatrix} - \lambda_c^{mj_k}(-)^{b+j_i+c}\begin{Bmatrix} j_i & b & c \\ m & \beta & j_j\end{Bmatrix}\begin{Bmatrix} a & b & j_k \\ c & m & j_i\end{Bmatrix}\Big]. \tag{2.15}$$

The sign factors are due to the chosen orientation which has to be respected when applying the recoupling identities (see Appendix A) and can be manipulated by realizing that $(-)^{2a+2b+2c} = 1$ if a, b, c fulfill the Clebsch-Gordan conditions. The summation index $\beta = \beta(j_i, j_j, m, c)$, which appears due to the nondiagonal action of the volume operator, ranges on the values which are determined by the simultaneous admissibility of the trivalent nodes $\{j_i, j_j, \beta\}$ and $\{m, c, \beta\}$. If $m = \frac{1}{2}$, then the volume operator acts diagonally, and $\beta = j_k$.[4]

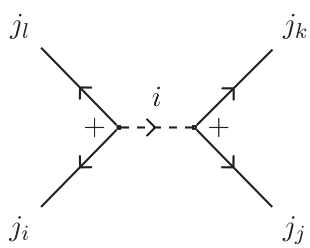
The complete action of the operator on a trivalent state $|\mathfrak{n}(j_i, j_j, j_k)\rangle$ can be obtained by contracting the trace part (2.14) with ϵ^{ijk}. Thus, $\hat{H}$ projects on a linear combination of three spin networks which differ by exactly one new link labeled by m between each couple of the "old" links at the node.

D. Action on a 4-valent node

The computation for a 4-valent node $|\mathfrak{n}_4\rangle$ is similar to the previous:

$$\hat{H}^m_\Delta|\mathfrak{n}_4\rangle = N_\mathfrak{n}\epsilon^{ijk}\,\mathrm{Tr}\Big(\frac{\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}]}{2}\hat{h}^{(m)}[s_k][\hat{V}, \hat{h}^{(m)}[s_k^{-1}]]\Big)|\mathfrak{n}_4\rangle. \tag{2.16}$$

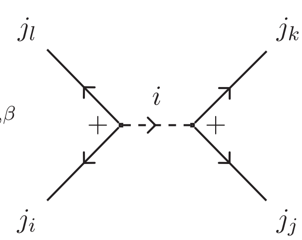
In the subsequent calculation, we fix all links to be outgoing from the node

$$|\mathfrak{n}_4\rangle = \sqrt{d_i}$$ 

, (2.17)

where i labels the intertwiner (inner link). Furthermore, we fix the orientation of the loop α_{ij} in Eq. (2.16) to be counterclockwise. The part $\mathrm{Tr}(\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}]\hat{V})|\mathfrak{n}_4\rangle$ vanishes since the volume does not modify the representations but the trace is taken in the representation space and $\mathrm{Tr}(\hat{h}^{(m)}[\alpha_{ij}] - \hat{h}^{(m)}[\alpha_{ji}]) = 0$.

For the other part, the holonomy $\hat{h}^{(m)}[s_k^{-1}]$ changes the valency of the node, and the volume subsequently acts on the 5-valent non-gauge-invariant node. Graphically, this corresponds to

$$\hat{V}\ \hat{h}^{(m)}[s_k^{-1}]\ |\mathfrak{n}_4\rangle = (-1)^{2m}\sum_c d_c \sum_{\beta,\gamma} V_{i,j_k}^{\ \gamma,\beta}$$ 

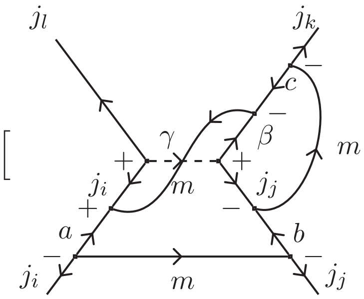
. (2.18)

Finally one arrives at

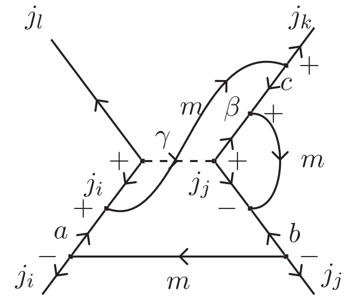
[4]We get a correction of sign factors compared to Ref. [25]. This correction is necessary in order that the action of $\mathrm{Tr}(F_{ij})$ vanishes.

$$\mathrm{Tr}\left(\frac{\hat{h}^{(m)}[\alpha_{ij}]-\hat{h}^{(m)}[\alpha_{ji}]}{2}\,\hat{h}^{(m)}[s_k]\,\hat{V}\,\hat{h}^{(m)}[s_k^{-1}]\right)|\mathfrak{n}_4\rangle =$$

$$\sum_{a,b,c} d_a d_b d_c \sum_{\beta,\gamma} V_{i,j_k}^{\gamma,\beta}\ \Big[$$



$-$

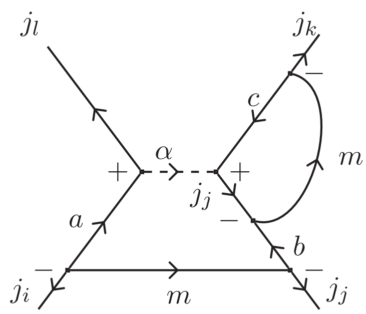


$$\Big]. \qquad (2.19)$$

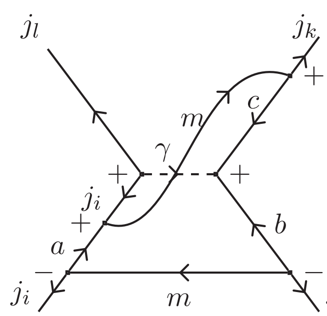
This can be simplified using Eqs. (B1) and (A17):

$$\hat{H}^m_\Delta\,|\mathfrak{n}_4\rangle = \sum_{a,b,c} d_a d_b d_c \sum_{\beta,\gamma} V_{i,j_k}^{\gamma,\beta}\Big[(-1)^{2j_i+a+j_l-j_j-\beta+m}$$

$$\sum_\alpha d_\alpha \begin{Bmatrix}\gamma & \alpha & m\\ a & j_i & j_l\end{Bmatrix}\begin{Bmatrix}\gamma & \alpha & m\\ c & \beta & j_j\end{Bmatrix}$$



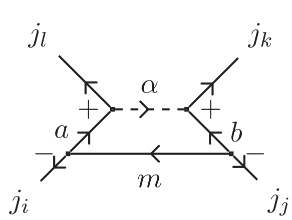
$$-(-1)^{\gamma+j_j+c+m}\begin{Bmatrix}\gamma & j_j & \beta\\ m & c & b\end{Bmatrix}$$



$$\Big] \qquad (2.20)$$

$$= \sum_{a,b,c} d_a d_b d_c \sum_{\beta,\gamma} V_{i,j_k}^{\gamma,\beta}\ (-1)^{2j_i+j_l+j_j+a+m}\sum_\alpha d_\alpha\Big[(-1)^{\alpha+\beta-c-j_j}\left\{\begin{smallmatrix}\gamma & \alpha & m\\ a & j_i & j_l\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\gamma & \alpha & m\\ c & \beta & j_j\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\alpha & j_j & c\\ m & j_k & b\end{smallmatrix}\right\}$$

$$-(-1)^{c+\gamma-b-j_k}\left\{\begin{smallmatrix}\gamma & j_j & \beta\\ m & c & b\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\gamma & \alpha & m\\ a & j_i & j_l\end{smallmatrix}\right\}\left\{\begin{smallmatrix}\gamma & \alpha & m\\ j_k & c & b\end{smallmatrix}\right\}\Big]$$



$$.$$

The sign factors are due to the chosen orientation and can be manipulated as above.

III. SPIN FOAM

A. The model

In this section, we briefly recall the definition of Euclidean spin-foam models as suggested by Kaminski, Kisielowski, and Lewandowski [14] and clarify our notation. Since we are only interested in the evaluation of a spin-foam amplitude, we choose a combinatorial definition of the model.

Consider an oriented two-complex κ defined as the union of the set of faces (2-cells) $\mathcal{F}$, edges (1-cells) $\mathcal{E}$, and vertices (0-cells) $\mathcal{V}$ such that every edge e is a 1-face[5] of at least one face f (notation: $e \in \partial f$) and every vertex v is a 0-face of at least one edge e (notation: $v \in \partial e$). We call edges which are contained in more than one face f internal and denote the set of all internal edges by $\mathcal{E}_{\rm int}$. Vice versa, all vertices adjacent to more than one internal edge are also called internal, and we denote the set of these vertices by $\mathcal{V}_{\rm int}$. The boundary $\partial\kappa$ is the union of all external vertices (called "nodes") $\mathfrak{n} \notin \mathcal{V}_{\rm int}$ and external edges (called "links") $l \notin \mathcal{E}_{\rm int}$. If $\partial\kappa$ forms a closed but possibly disconnected graph and the orientation of $e \in \partial\kappa$ agrees with the orientation induced by the unique face f, $e \in \partial f$ (we say f is ingoing to e), then κ is called a proper foam. In the following, we will only consider proper foams.

A spin foam is a triple (κ, ρ_f, I_e) consisting of a proper foam whose faces are labeled by irreducible representations of a Lie-group G [here, SO(4)] and whose internal edges are labeled by intertwiners I. This induces a spin-network structure $\partial(\kappa, \rho_{l_f}, I_{\mathfrak{n}_e})$ on the boundary of κ. In the following, we will denote the pair (v, f) such that $v \in \partial f$ by v_f and analogously for all other pairings e_v, e_f, etc. Furthermore, ∂v is the set of all faces f_v and edges e_v.

Suppose κ is a foam without boundary. Following Ref. [34], we label each edge $e \in \kappa$ by a group element $e \rightarrow U_e \in {\rm SO}(4)$, such that

$$U_e = g_{e_{s(e)}} g^{-1}_{e_{t(e)}} \tag{3.1}$$

where $s(e)/t(e)$ is the source/target of e. For each pair (v, f) with $v \cap f = v$ and edges $e \cap e' = v$, e, $e' \in \partial f$, we define

$$g_{f_v} := (g^{-1}_{e_v} g_{e'_v})^{\epsilon_{e_f}} \tag{3.2}$$

where $\epsilon_{e_f} = \pm$ according to the orientations. With this definition, the BF partition function can be rewritten as

$$Z^{BF}[\kappa] = \int_{\rm SO(4)} dg_{f_v} \prod_{f\in\kappa} \delta\Big(\prod_{v\in\partial f} g_{f_v}\Big) \times \prod_{f_v} \underbrace{\int dg_{e_v}\, \delta(g^{-1}_{f_v} g_{e_v} g^{-1}_{e'_v})}_{\mathcal{A}_v(g_{f_v})}. \tag{3.3}$$

[5] For a definition of complex, see, e.g., Ref. [33].

Note, $\mathcal{A}_v(g_{fv})$ defines an SO(4) invariant function on the graph Γ_v induced on the boundary of the vertex v [34]. As it is well-known, the boundary Hilbert space $\mathcal{H}_v$ is spanned by (normalized) spin-network functions $T^{BF}_{\Gamma_v,\rho,I}(g_f)$[6]

$$\mathcal{A}_v(g_f) = \sum_{\rho_f, I_e} \prod_{f\in\partial v} \sqrt{\dim\rho_f}\, {\rm Tr}_v\Big(\bigotimes_{e\in\partial v} I^\dagger_e\Big) T^{BF}_{\Gamma_v,\rho,I}(g_f). \tag{3.4}$$

Locally, ${\rm SO}(4) \sim {\rm SU}(2)\times{\rm SU}(2)$ which implies $\rho_{\rm SO(4)} = \rho^+_{\rm SU(2)} \otimes \rho^-_{\rm SU(2)}$ and $T^{\rm BF}_{\Gamma_v,\rho,I}(g_f) = T_{\Gamma_v,j^+,\iota^+}(g^+_f) \otimes T_{\Gamma_v,j^-,\iota^-}(g^-_f)$.

In the EPRL model [13], the simplicity constraint is imposed weakly. Consequentially, we have to restrict $\mathcal{A}_v(g_f)$ to the EPRL subspace $\mathcal{H}^{\rm EPRL}_v$ spanned by the functions

$$T^E_{\Gamma_v,j_f,\iota_e} = \prod_{f_v} \sqrt{d_{j^+_{f_v}} d_{j^-_{f_v}}} \prod_{e_v} \Big[\iota_e^{A_{e1}\cdots A_{eF}} \prod_{f\in\partial v} C^{m^+_{e_f} m^-_{e_f}}_{A_{e_f}} \Big] \times \prod_{(e,f)\in\partial v} [\epsilon^{n^+_{e_f} n^+_{e'_f}} \epsilon^{n^-_{e_f} n^-_{e'_f}} R^{j^+_f}_{m^+_{e_f} n^+_{e_f}}(g^+_{e_f}) R^{j^+_f}_{m^-_{e_f} n^-_{e_f}}(g^-_{e_f})] \tag{3.5}$$

with $j^\pm \equiv \frac{|\gamma\pm 1|}{2} j$. Furthermore, $R^j(g)$ denotes a Wigner matrix, $C^{m^+_e m^-_E}_{A_e}$ a Clebsch-Gordan coefficient, and $\epsilon^{n^+_e n^-_e}$ represent the unique two-valent intertwiners of SU(2) (see Ref. [34]).

It follows immediately that

$$\mathcal{A}^E_v(g_f) = \sum_{j_f,\iota_e} \langle T^E_{\Gamma_v,j_f,\iota_e} | \mathcal{A}_v \rangle T^E_{\Gamma_v,j_f,\iota_e}(g_f) := \sum_{j_f,\iota_e} \Big(\prod_{f\in\partial v} \sqrt{d_{j^+_f} d_{j^-_f}}\Big) \mathcal{A}^E_v(j_f, \iota_e) T^E_{\Gamma_v,j_f,\iota_e}(g_f) \tag{3.6}$$

defines the EPRL vertex amplitude with

$$A^E_v(j_f, \iota_e) = \sum_{\iota^+_e,\iota^-_e} {\rm Tr}_v\Big(\bigotimes_{e_v} (\iota^+_{e_v} \otimes \iota^-_{e_v})^\dagger\Big) \prod_{e_v} f^{\iota_{e_v}}_{\iota^+_{e_v},\iota^-_{e_v}} \tag{3.7}$$

where $f^{\iota_e}_{\iota^+_e,\iota^-_e}$ are the well known fusion coefficients [13]. At last, one has to replace Eq. (3.6) in Eq. (3.3) to obtain the full transition amplitude of the EPRL model. Expanding the delta function in Eq. (3.3) in terms of spin-network function and integrating over the group elements gives

$$Z[\kappa] = \sum_{j_f,\iota_e} \prod_f d_{j^+_f} d_{j^-_f} \prod_v \mathcal{A}^E_v(j_f, \iota_e). \tag{3.8}$$

[6] The links l_f bounding the face f are labeled by irreducible representations ρ_f, and nodes n_e bounding the edge e are labeled by intertwiners I_e as usual.

Remark—In order to evaluate the fusion coefficients $f^{\iota_e}_{\iota_e^+,\iota_e^-}$ by graphical calculus, it is convenient to work with $3j$ symbols instead of Clebsch-Gordan coefficients. When replacing the Clebsch-Gordan coefficients, we have to multiply by an overall factor $\prod_e \prod_{f_e} \sqrt{2j_{f_e}+1}$.

B. Spin-foam projector

Instead of using spin foams as a tool to compute "transition" amplitudes between spin networks, one is tempted to interpret spin foams as a projector onto the physical Hilbert space. Given any couple of ingoing and outgoing kinematical states $\psi_{\rm out}$, $\psi_{\rm in}$, the Physical scalar product can be formally defined by

$$\langle \psi_{\rm out} | \psi_{\rm in} \rangle_{\rm phys} := [\eta(\psi_{\rm out})](\psi_{\rm in}) \tag{3.9}$$

where η is a projector (Rigging map) onto the Kernel of the Hamiltonian constraint. Suppose that the transition amplitude Z

$$\langle \psi_{\rm out} | Z | \psi_{\rm in} \rangle := [\eta(\psi_{\rm out})](\psi_{\rm in}) \tag{3.10}$$

can be expressed in terms of a sum of spin foams (κ, ρ, ι) with boundary $\partial(\kappa, \rho, \iota) = \psi_{\rm out} \bigcup \psi_{\rm in}$. To realize that, we first have to reconsider Eq. (3.3) for a foam κ with nonempty boundary $\partial\kappa \neq \varnothing$. Then,[7]

$$Z[\kappa] = \int_{{\rm SO(4)}^{\mathcal{V}_{\rm int}}} dg_{f_v} \prod_{f \in \kappa} \delta\Big(\prod_{v \in \partial f} g_{f_v} g_l\Big) \prod_{v \in \mathcal{V}_{\rm int}} \mathcal{A}_v(g_{fv}) \tag{3.11}$$

where

$$g_l = \begin{cases} h_l & \text{if } f \cap \partial\kappa = l \\ 1 & \text{otherwise} \end{cases}. \tag{3.12}$$

Equation (3.11) can be interpreted as a function on the boundary graph $\partial\kappa$. That is to say

$$Z[\kappa] = \sum_{j_f, \iota_e} \prod_f d_{j_f^+} d_{j_f^-} \prod_{v \in \mathcal{V}_{\rm int}} \mathcal{A}_v^E(j_f, \iota_e) \times \sum_{j_l, \iota_n} \Big(\prod_{l \in \partial\kappa^{(1)}} \frac{1}{\sqrt{d_{j_l^+} d_{j_l^-}}} \Big) T^E_{\partial\kappa, j_l, \iota_n}(h_l) \tag{3.13}$$

in the EPRL sector. Here, $\partial\kappa^{(1)}$ is the set of boundary links. Unfortunately, Eq. (3.13) defines an SO(4) spin-network function while the kinematical Hilbert space of the canonical theory is spanned by SU(2) functions. It is, however, easy to resolve that problem: when restricting the boundary elements $h_l \in {\rm SU(2)} \subset {\rm SO(4)}$, then $T^E_{\partial\kappa, j_l, \iota_n}(h_l)$ is a true SU(2) spin-network function. Indeed,

$$T^E_{\partial\kappa, j_l, \iota_n}(h_l) = \Big(\prod_{l \in \partial\kappa} \sqrt{\frac{d_{j_l^+} d_{j_l^-}}{d_{j_l}}} \Big) |S(\partial\kappa, j, \iota)\rangle_N \tag{3.14}$$

where $|S\rangle_N$ is a normalized spin-network function on SU(2) (see Appendix C). This finally implies

$$\langle \psi_{\rm out} | Z | \psi_{\rm in} \rangle = \sum_{j_f, \iota_e} \prod_f d_{j_f^+} d_{j_f^-} \prod_{l \in \partial\kappa^{(1)}} \frac{1}{\sqrt{d_{j_l}}} \prod_{v \in \mathcal{V}_{\rm int}} \mathcal{A}_v^E(j_f, \iota_e). \tag{3.15}$$

In the next section, we will compute an easy example of such an amplitude.

IV. NEW SOLUTIONS TO THE EUCLIDEAN HAMILTONIAN CONSTRAINT

In the following, we compute new solutions to the Euclidean Hamiltonian constraint by employing spin-foam methods. We show that

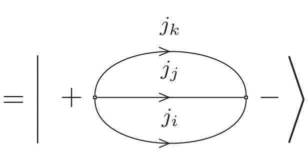
$$\sum_\phi \langle \psi_{\rm out} | Z[\kappa] | \phi \rangle \langle \phi | \hat{H}^{(m)} | \psi_{\rm in} \rangle = 0 \tag{4.1}$$

in the Euclidean sector with $\gamma = 1$ and $s = 1$, where κ is an easy 2-complex with only one internal vertex.

A. Trivalent nodes

Consider the simplest possible case given by an initial and final state $|\Theta\rangle$, characterized by two trivalent nodes joined by three links:

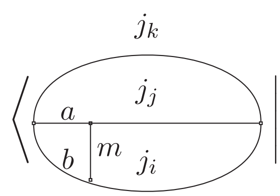
$|\Theta(j_i, j_j, j_k)\rangle =$



(4.2)

As shown in Sec. II, the only states produced by the Hamiltonian $\hat{H}^{(m)}$ acting on a node are given by a linear combination of spin networks which differ from the original one by the presence of an extraordinary link. In particular, the term $\langle s | \hat{H}^{(m)} | \Theta(j_i, j_j, j_k) \rangle$, will be nonvanishing only if $\langle s |$ is of the kind

$\langle s | =$



(4.3)

The simplest two-complex $\kappa(\Theta, s)$ with only one internal vertex defining a cobordism between $|\Theta\rangle$ and $|s\rangle$ is a tube $\Theta \times [0, 1]$ with an additional face between the internal vertex and the new link m (see Fig. 1).

The computation of Eq. (3.13) for $\kappa(\Theta, s)$ is straightforward when using graphical calculus. Since the space of trivalent intertwiners is one-dimensional and all labelings

[7]If we would also integrate over group elements in the boundary, then $Z^{BF} = \int \delta(F)$ would become singular.

j_f are fixed by the states $|s\rangle$, $|\Theta\rangle$, the first sum in Eq. (3.13) is trivial. Thus,

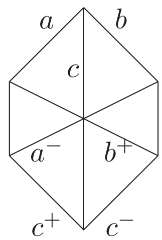
$$\langle\Theta|Z[\kappa]|s\rangle := W_E(\kappa,\Theta,s) = \mathcal{A}_f\mathcal{B}\mathcal{A}_v^E(j_f,\iota_e) \qquad (4.4)$$

where $\mathcal{A}_f = \prod_f d_{j_f^+}d_{j_f^-}$ are the face amplitudes and $\mathcal{B} = \prod_{l\in\partial\kappa^{(1)}}\frac{1}{\sqrt{d_{j_l}}}$ are the boundary amplitudes. The evaluation of the trace in A_v^E is equivalent to evaluating the boundary spin network Γ_v of the vertex v [34] at 1. The reader can easily be convinced that $\Gamma_v = s$, and therefore with Eq. (A17), it gives

$$\mathrm{Tr}_v\Big(\bigotimes_e \iota_e^+\iota_e^-\Big) = (-)^{j_i^++j_j^+-j_k^+}(-)^{j_i^-+j_j^--j_k^-} \times \begin{Bmatrix} j_i^+ & j_j^+ & j_k^+ \\ b^+ & a^+ & m^+ \end{Bmatrix}\begin{Bmatrix} j_i^- & j_j^- & j_k^- \\ b^- & a^- & m^- \end{Bmatrix} \qquad (4.5)$$

where the sign factor is due to the orientation of s [see Eq. (2.14)]. The fusion coefficients contribute four 9_j symbols since

$$f^{\iota_e}_{\iota_e^+,\iota_e^-} = \sqrt{d_ad_bd_c}\left[a^+ \quad \text{(diagram)} \quad b^-\right] = \sqrt{d_ad_bd_c}\begin{Bmatrix} a & b & c \\ a^+ & b^+ & c^+ \\ a^- & b^- & c^- \end{Bmatrix}, \qquad (4.6)$$



where the dimension factors come from the replacement of Clebsch-Gordan coefficients by $3j$ symbols (see Sec. III A). The full amplitude is

$$\begin{aligned} W_E(\kappa,\Theta,s) = {}& \mathcal{A}_f\mathcal{A}_e\mathcal{B}(-)^{j_i^++j_j^+-j_k^+}(-)^{j_i^-+j_j^--j_k^-} \\ &\times \begin{Bmatrix} j_i^+ & j_j^+ & j_k^+ \\ b^+ & a^+ & m^+ \end{Bmatrix}\begin{Bmatrix} j_i^- & j_j^- & j_k^- \\ b^- & a^- & m^- \end{Bmatrix} \\ &\times \begin{Bmatrix} j_i & j_j & j_k \\ j_i^+ & j_j^+ & j_k^+ \\ j_i^- & j_j^- & j_k^- \end{Bmatrix}\begin{Bmatrix} j_i & a & m \\ j_i^+ & a^+ & m^+ \\ j_i^- & a^- & m^- \end{Bmatrix} \\ &\times \begin{Bmatrix} j_j & b & m \\ j_j^+ & b^+ & m^+ \\ j_j^- & b^- & m^- \end{Bmatrix}\begin{Bmatrix} a & b & j_k \\ a^+ & b^+ & j_k^+ \\ a^- & b^- & j_k^- \end{Bmatrix} \end{aligned} \qquad (4.7)$$

with $\mathcal{A}_e = d_{j_i}d_{j_j}d_{j_k}d_ad_bd_m$. Let us fix $\gamma = 1$; then, $j^+ = j$ and $j^- = 0$, and Eq. (4.7) reduces to

$$W_E(\kappa,\Theta,s)|_{\gamma=1} = (d_ad_bd_m)^{1/2}(-)^{j_i+j_j-j_k}\begin{Bmatrix} j_i & j_j & j_k \\ b & a & m \end{Bmatrix} \qquad (4.8)$$

where we have used

$$\begin{Bmatrix} a & b & c \\ a & b & c \\ 0 & 0 & 0 \end{Bmatrix} = \frac{1}{\sqrt{d_ad_bd_c}}. \qquad (4.9)$$

With the previous results and Eq. (2.15), we are now able to compute Eq. (4.1). Note that in Eq. (2.14), the new created links labeled by a, b, m are not normalized, but the spin-foam amplitude has been constructed such that $|s\rangle$ is normalized. Taking the scalar product $\langle s|H|\Theta\rangle$ gives, therefore, an additional factor $\frac{1}{\sqrt{d_ad_bd_c}}$. This yields[8]

$$\begin{aligned} \sum_s W_E(\kappa,s,\Theta)|_{\gamma=1}\langle s|\hat{H}^{(m)}|\Theta\rangle = {}& \sum_{a,b}\begin{Bmatrix} j_i & j_j & j_k \\ b & a & m \end{Bmatrix}\sum_c \lambda_a^{mj_i}\lambda_b^{mj_j}\lambda_c^{mj_k}d_ad_bd_c\sum_{\beta(j_i,j_j,m,c)}V_{j_k}{}^{\beta}(j_i,j_j,m,c) \\ &\times\left[\lambda_c^{m\beta}(-)^{a+j_j+c}\begin{Bmatrix} a & j_j & c \\ \beta & m & j_i \end{Bmatrix}\begin{Bmatrix} a & b & j_k \\ m & c & j_j \end{Bmatrix} - \lambda_c^{mj_k}(-)^{b+j_i+c}\begin{Bmatrix} j_i & b & c \\ m & \beta & j_j \end{Bmatrix}\begin{Bmatrix} a & b & j_k \\ c & m & j_i \end{Bmatrix}\right] \\ &+[j_k\leftrightarrow j_i]+[j_k\leftrightarrow j_j]. \end{aligned} \qquad (4.10)$$

The last two terms are equivalent to the first term when exchanging $j_k \leftrightarrow j_j$ and j_i, respectively, and correspond to the other extraordinary links. In fact, the EPRL spin foam reduces just to the SU(2) BF amplitude which is just the single $6j$ left in the first line. Now, using the definition of a $9j$ in terms of three $6j$'s (A18), Eq. (4.10) becomes

[8]with Lapse function $N_{\mathrm{n}} = 1$

$$\sum_s W_E(\kappa, s, \Theta)|_{\gamma=1}\langle s|\hat{H}^{(m)}|\Theta\rangle = \sum_c d_c \sum_{\beta(j_i,j_j,m,c)} V_{j_k}{}^{\beta}(j_i, j_j, m, c) \times \Bigg[\sum_b d_b \lambda_b^{mj_j}(-)^{m+j_i+j_j+c}\lambda_c^{mj_k}\lambda_c^{m\beta}\begin{Bmatrix} j_i & j_j & \beta \\ j_k & m & c \\ j_j & b & m \end{Bmatrix} - \sum_a d_a \lambda_a^{mj_i}(-1)^{m+j_j+j_i+c}\begin{Bmatrix} m & c & \beta \\ a & m & j_i \\ j_i & j_k & j_j \end{Bmatrix}\Bigg] + [j_k \leftrightarrow j_i] + [j_k \leftrightarrow j_j]. \tag{4.11}$$

The $9j$'s involved in this expression can be reordered using the permutation symmetries (A20) and (A19) giving

$$\sum_c d_c \sum_{\beta(j_i,j_j,m,c)} V_{j_k}{}^{\beta}(j_i, j_j, m, c) \times \Bigg[\sum_b d_b (-)^{2\beta}\begin{Bmatrix} j_i & \beta & j_j \\ j_k & c & m \\ j_j & m & b \end{Bmatrix} - \sum_a d_a (-)^{j_k+\beta}\begin{Bmatrix} j_j & j_k & j_i \\ \beta & c & m \\ j_i & m & a \end{Bmatrix}\Bigg] = \sum_c d_c (-1)^{2j_k} \sum_{\beta(j_i,j_j,m,c)} V_{j_k}{}^{\beta}(j_i, j_j, m, c)\left[\frac{\delta_{\beta j_k}}{d_{j_k}} - \frac{\delta_{\beta j_k}}{d_{j_k}}\right] = 0. \tag{4.12}$$

In the last expression, we have used the summation identity (A21). Equation (4.12) shows that the states $|s\rangle_{\text{phys}} = \sum_s W_E(\kappa, \Theta, s)|_{\gamma=1}\langle s|$ are solutions of the (Euclidean) Hamiltonian constraint if s is of the form (4.3). However, each term depending on one of the three graphs which differ by its extraordinary link vanishes separately. This suggest that the solution we have constructed is very likely not the most arbitrary solution for trivalent nodes.

Remarks

(i) *The role of the volume*—It is noteworthy that the spin-foam amplitude selects only those terms which depend on the diagonal elements on the volume. The consequences of this behavior are manifold. First, it simplifies the calculation since we do not have to evaluate the volume explicitly. If $m = 1/2$, this would not be a problem since then the volume is already diagonal and can be computed easily [31]. But if $m \neq 1/2$, or in higher-valent cases, the structure of the volume operator is very complicated and is the major obstacle for computing solutions of the Hamiltonian. Indeed, we show in the next section that the above property carries over to higher-valent nodes and therefore enables us to compute more solutions.

On the other side, this behavior supports the conjecture that the states constructed are not the most arbitrary solutions but only a special class. Looking more closely at the spin-foam amplitude, this is hardly surprising. When setting the Barbero-Immirzi parameter $\gamma = 1$, we restrict to BF theory (in the spin-foam framework). The Hamiltonian of BF theory is essentially given by the curvature F, and the only part of $\hat{H}^{(m)}$ influencing the spin-network structure of $|s\rangle_{\text{phys}}$ is again the curvature; the volume just yields an overall factor. This shows to some extent the consistency between the models.

(ii) *Arbitrary cobordism*—The result (4.12) is obviously not sensitive to the orientations of Θ and s, respectively, since a change in the orientation would give the same sign factor in $A^{(m)}(j_i, a|j_j, b|j_k)$ as in $W_E(\kappa, \Theta, s)$. The crucial ingredient of $W_E(\kappa, \Theta, s)$ is the appearance of the $6j$ symbol [see Eqs. (4.10) and (4.11)]. Thus, Eq. (4.1) also vanishes if we consider a more general complex κ' as long as $W_E(\kappa', \psi, s)$ still depends on the same $6j$ symbol and the rest does not depend on the spins a, b. For example, we could work with a cobordism between an arbitrary state ψ and s such that all faces of κ wind up in the same internal vertex (see Fig. 3).

B. Four-valent nodes

Let us now turn to the case with $\psi_{\text{in}} = \psi_{\text{out}} = |\mathfrak{n}_4\rangle$ where

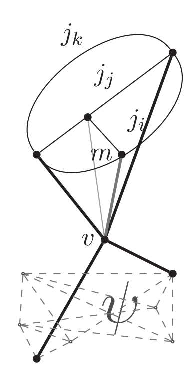
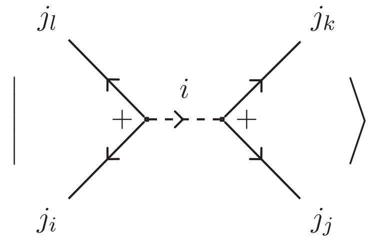
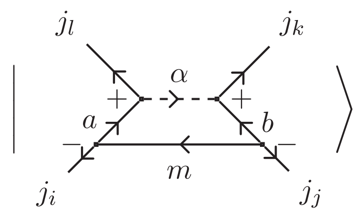


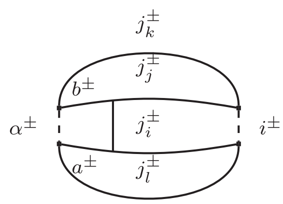
FIG. 3. Two-complex κ' with a single internal vertex and arbitrary ψ.

$|\mathfrak{n}_4\rangle =$  . (4.13)

The matrix element $\langle s|\hat{H}^{(m)}|\mathfrak{n}_4\rangle$ is nonvanishing iff $|s\rangle$ is of the form

$|s\rangle =$  . (4.14)

We choose again an easy complex κ of the form of Fig. 1 with one additional face j_l. The vertex trace in Eq. (3.7) can be evaluated by graphical calculus:

$\mathrm{Tr}_v(\bigotimes_{e_v} \iota_e^\pm) =$  (4.15)

$$= (-)^{b^\pm - a^\pm + \alpha^\pm + m^\pm + j_l^\pm - j_j^\pm}\, d_{i^\pm} \left\{ \begin{matrix} \alpha^\pm & i^\pm & m^\pm \\ j_i^\pm & a^\pm & j_l^\pm \end{matrix} \right\} \left\{ \begin{matrix} \alpha^\pm & i^\pm & m^\pm \\ j_j^\pm & a^\pm & j_k^\pm \end{matrix} \right\}.$$

The fusion coefficients $f^{\iota_e}_{\iota_e^+ \iota_e^-}$ give two $9j$ symbols for the two trivalent edges and two $15j$ symbols for the two four-valent edges. As in the above section, the fusion coefficients reduce to 1 when setting $\gamma = 1$. When taking the scalar product (3.15), the internal links labeling the intertwiner can be in principle treated like the real links, and we obtain

$$W_E(\kappa, \mathfrak{n}_4, s) = \sqrt{d_a d_b d_m} (-)^{b-a+\alpha+m+j_l-j_j} \begin{Bmatrix} \alpha & i & m \\ j_i & a & j_l \end{Bmatrix} \begin{Bmatrix} \alpha & i & m \\ j_j & b & j_k \end{Bmatrix}. \tag{4.16}$$

Taking the scalar product with the Hamiltonian as in Eq. (4.10) yields

$$\sum_s W_E(\kappa, \mathfrak{n}_4, s)\langle s|\hat{H}^{(m)}|\mathfrak{n}_4\rangle = \sum_{a,b,c} d_a d_b d_c \sum_\alpha d_\alpha (-)^{b-\alpha} \begin{Bmatrix} \alpha & i & m \\ j_i & a & j_l \end{Bmatrix} \begin{Bmatrix} \alpha & i & m \\ j_j & b & j_k \end{Bmatrix}$$
$$\times \sum_{\beta,\gamma} V_{i,j_k}{}^{\gamma,\beta} \Bigg[(-1)^{\alpha+\beta-c-j_j} \begin{Bmatrix} \gamma & \alpha & m \\ a & j_i & j_l \end{Bmatrix} \begin{Bmatrix} \gamma & \alpha & m \\ c & \beta & j_j \end{Bmatrix} \begin{Bmatrix} \alpha & j_j & c \\ m & j_k & b \end{Bmatrix}$$
$$- (-1)^{c+\gamma-j_k-b} \begin{Bmatrix} \gamma & j_j & \beta \\ m & c & b \end{Bmatrix} \begin{Bmatrix} \gamma & \alpha & m \\ a & j_i & j_l \end{Bmatrix} \begin{Bmatrix} \gamma & \alpha & m \\ j_k & c & b \end{Bmatrix} \Bigg] \tag{4.17}$$

where we have used $(-1)^{2a+2j_l+2b} = 1$. Summing over a and using the orthogonality relation (A15) and $(-1)^{2j_i+2a+2m} = 1$ gives

$$\sum_s W_E(\kappa, \mathfrak{n}_4, s)\langle s|\hat{H}^{(m)}|\mathfrak{n}_4\rangle = \sum_{b,c} d_b d_c \sum_\alpha d_\alpha \sum_{\beta,\gamma} V_{i,j_k}{}^{\gamma,\beta} \delta_{i,\gamma} \times \Bigg[(-1)^{\beta-b-c-j_j-2m} \begin{Bmatrix} \alpha & i & m \\ j_j & b & j_k \end{Bmatrix} \begin{Bmatrix} \gamma & \alpha & m \\ c & \beta & j_j \end{Bmatrix} \begin{Bmatrix} \alpha & j_j & c \\ m & j_k & b \end{Bmatrix}$$
$$- (-1)^{c+i+\alpha-j_k+2m} \begin{Bmatrix} \alpha & i & m \\ j_j & b & j_k \end{Bmatrix} \begin{Bmatrix} \gamma & j_j & \beta \\ m & c & b \end{Bmatrix} \begin{Bmatrix} \gamma & \alpha & m \\ j_k & c & b \end{Bmatrix} \Bigg]. \tag{4.18}$$

Note, the three $6j$'s in the two terms of Eq. (4.17) define a $9j$ summing over the indices α and b, respectively:

$$\sum_c d_c \sum_\beta V_{i,j_k}{}^{i,\beta}\left[\sum_b d_b(-1)^{b+\beta-c-j_j+2b+2m}\begin{Bmatrix} m & c & \beta \\ b & m & j_j \\ j_j & j_k & i \end{Bmatrix} - \sum_\alpha d_\alpha(-1)^{c+i-j_k+\alpha+2m}\begin{Bmatrix} i & \alpha & m \\ j_j & i & j_k \\ \beta & m & c \end{Bmatrix}\right]$$
$$= \sum_c d_c \sum_\beta V_{i,j_k}{}^{i,\beta}\left[\sum_b d_b(-1)^{2\beta+j_k+\gamma+j_j}\begin{Bmatrix} \gamma & j_k & j_j \\ \beta & c & m \\ j_j & m & b \end{Bmatrix} - \sum_\alpha d_\alpha(-1)^{2j_k+\beta+\gamma+j_j}\begin{Bmatrix} j_j & j_k & \gamma \\ \beta & c & m \\ \gamma & m & \alpha \end{Bmatrix}\right]. \quad (4.19)$$

In the second line, we used the permutation symmetry (see the Appendix). With Eq. (A21), we obtain the final result:

$$\sum_c d_c \sum_\beta V_{i,j_k}{}^{i,\beta}(-1)^{3j_k+\gamma+j_j}\left[\frac{\delta_{\beta j_k}}{d_{j_k}} - \frac{\delta_{\beta j_k}}{d_{j_k}}\right] = 0. \quad (4.20)$$

As for the trivalent vertex, the spin-foam amplitude just takes those elements into account which depend on the diagonal volume elements. Therefore, the above calculation can be easily extended to n-valent nodes since the curvature always acts locally on three links while the influence of the volume on the rest of the internal links is unimportant.

V. CONCLUSIONS

LQG is grounded on two parallel constructions: the canonical and the covariant ones. One of the bigger missing theoretical ingredients of this road to quantum gravity is the relation between these two. In this paper, we were mainly concerned with the following questions: The EPRL-FK with the Kaminski, Kisielowski, and Lewandowski extension shares the same kinematics of LQG; do they share also the same *dynamics*? Can we really use the EPRL-FK as defining the Physical Hilbert Space? A first step to find an answer to that important question is to construct a simple spin-foam amplitude which annihilates the Hamiltonian constraint as argued in Eq. (4.1). Indeed, we found that in the euclidean sector with signature $s = 1$ and Barbero-Immirzi parameter $\gamma = 1$, the Euclidean Hamiltonian constraint is annihilated by a spin-foam amplitude $Z[\kappa]$ where κ is a simple two-complex with only one internal vertex. Even though we considered only a very special case, this has some important consequences: First, even neglecting their spin-foam origin, the one-vertex amplitudes of BF theory are *new explicit analytic solutions of the Hamiltonian theory* and represent a proper subspace of the physical Hilbert space.

Second, the equation (4.10) vanishes for each triple of edges; this means that the $6j$ symbol associated to every face is annihilated by the Euclidean scalar constraint. This is a generalization of the work by Bonzom-Freidel in the context of 3D gravity. In Ref. [35], they found that the $6j$ (a physical state in 3D) is annihilated by a suitable quantization of the 3D scalar constraint $F = 0$ rewritten, following Ref. [36], as EEF. Their result holds exactly only for the choice $m = 1$ (even if a generalization to higher spin, involving a proper redefinition of the quantum constraint, is discussed; see Ref. [35]). Here, we showed that the $6j$ symbols one obtains in the one-vertex expansion annihilates for arbitrary spin m the complete nonpolynomial density constraint $\frac{EEF}{\sqrt{\det E}}$.

It was already pointed out at the end of Sec. IV A that the spin-foam amplitude picks only the diagonal elements of the volume. On the one hand, this behavior proves to be very useful when computing (4.1) for higher-valent nodes; on the other hand, this indicates that the solutions are not the most arbitrary solutions but are closely related to BF solutions, which is not surprising since setting $\gamma = 1$ in the spin-foam model yields BF theory. The only solutions of $3 + 1$ BF theory are the ones that satisfy ${}^{3d}F = 0$ where ${}^{3d}F$ is the curvature on the 3D hypersurface Σ of the $3 + 1$ splitting. A natural question is then if the gravitational solutions presented in this article correspond to the BF's ones. This question can be reformulated asking if our solutions are proportional to a formal distribution "$\delta({}^{3d}F)$" forcing the states to have everywhere a flat connection, namely,

$${}_{\rm phys}\langle s| \sim \text{“}\delta({}^{3d}F)\text{” such that } {}_{\rm phys}\langle s|{}^{3d}F s'\rangle = 0 \ \ \forall\, s' \in \mathcal{H}_{\rm kin}.$$

The answer is no, because one can choose loops to regularize the curvature operator for which our solution is not annihilated by ${}^{3d}F$. Instead, flatness is only imposed on the two-dimensional faces bounded by a loop α at the vertex v which yields the extraordinary link. Even if the new solutions would be flat everywhere, they are. strictly speaking. no topological solutions because ${}^{3d}F = 0$ is not a sufficient criterion for topological solutions in GR due to the equations of motions. Therefore, the role of the spatial topology is rather obscure also because the foams we considered are not dual to proper triangulations of the four-dimensional bulk.

Of course many open questions remain, e.g.:

(i) The general case $\gamma \neq 1$ and the Lorentzian signature model—in this case, there are indications (work in progress) that seem to suggest the use of projected spin networks [37].

(ii) The relation between the geometric structure of the n-j's involved in the amplitude, the Hamiltonian, and the simplicial geometry [38] deserves also to be investigated, for example, along the lines of Ref. [39].

(iii) The relation with other regularization [25] and quantization programs, e.g. [40] for the canonical theory and the EPRL-FK can be analyzed along the same lines described here.

ACKNOWLEDGMENTS

A. Z. wants to thank "Universität Bayern e.V." for financial support. E. A. wishes to thank V. Bonzom and L. Freidel for useful discussions and a clarification of their construction during a visit to Perimeter Institute.

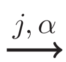
APPENDIX A: GRAPHICAL CALCULUS

In order to compute the matrix elements of the Hamiltonian constraint operator as well as the vertex amplitudes in the spin-foam model, one has to make extensive use of recoupling identities for SU(2). In this context, graphical calculus can be very useful to keep track of indices and sign. In this appendix, we summarize the graphical methods used in the main text.

1. Basic elements

Our convention is applicable in pure recoupling theory as well as in computations involving group elements. The convention is mainly based on Ref. [41], including some minor improvements.

(i) *Irreducible Representation:* Multiplication with an orthonormal vector in the j representation of SU(2) is represented by

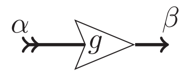


$= |j, \alpha\rangle$

$= \langle j, \alpha|\,,$ (A.1)

where the italic letters $j \in \frac{1}{2}\mathbb{N}$ label the irreducible representation of SU(2), and Greek letters $-j \geq \alpha \leq j$ represent magnetic quantum numbers. To avoid an unnecessary cumulation of labelings, we will suppress the label j, α if there is no danger of confusion.

(ii) *Wigner-R matrix*:

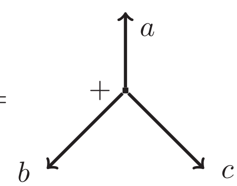


$= R^{\alpha}{}_{\beta}(g)\,\cdot$ (A.2)

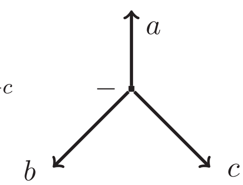
Note: α transforms in the dual representation while β transforms in the standard representation. Thus, α must be contracted with an intertwiner $\iota^{\cdots}{}_{\ldots\alpha\ldots}$ while β gets contracted by the dual $\iota^{\ldots\beta\ldots}{}_{\ldots}$.

(iii) *Wigner 3j Symbol*:

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} =$$



$= (-1)^{a+b+c}$



. (A.3)

The $+$ sign marks counterclockwise orientation while the $-$ sign marks clockwise orientation.

(iv) *Dualization:*

$=$


$$= \begin{pmatrix} j \\ \beta \;\; \alpha \end{pmatrix} = (-1)^{j+\beta}\delta_{\beta,-\alpha} \qquad \text{(A.4)}$$

$=$


$$= \begin{pmatrix} j \\ \alpha \;\; \beta \end{pmatrix} = (-1)^{j-\beta}\delta_{\alpha,-\beta}\;\cdot \qquad \text{(A.5)}$$

This implies:

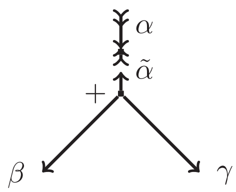
$= (-1)^{2j}$

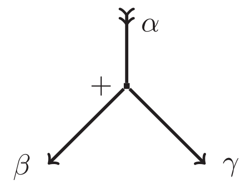

(A.6)

$$= (-1)^{2j}\delta_{\alpha,\beta}\ \cdot \qquad \text{(A.8)}$$


α
α̃
β
$$= (-1)^{2j}\delta_{\alpha,\beta}\ \cdot \qquad \text{(A.8)}$$

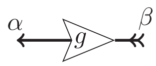
Thus,



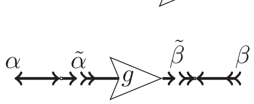
$$= (-1)^{j_a-\alpha}\begin{pmatrix} j_a & j_b & j_c \\ -\alpha & \beta & \gamma \end{pmatrix} =$$


$$\text{(A.9)}$$

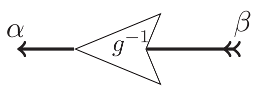
and



$$=: \overline{R_\alpha{}^\beta(g)} =$$

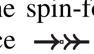
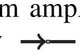


$$= (-1)^{\alpha-\beta}R_{-\alpha}{}^{-\beta}(g) = \qquad \text{(A.10)}$$

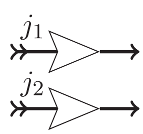


$$= R^\beta{}_\alpha(g^{-1})\cdot \qquad \text{(A.11)}$$

2. Recoupling and simplifications of graphs

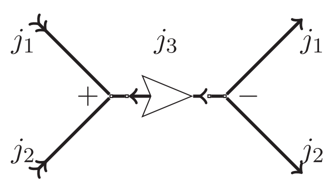
This subsection will give an overview of the basic recoupling identities needed for the evaluation of the Hamiltonian constraint and the spin-foam amplitudes. In the following, we will replace  by  on closed lines to simplify the graphs. Furthermore, the labels will be suppressed whenever possible. In the following, denote the dimension of j by $d_j = 2j + 1$.

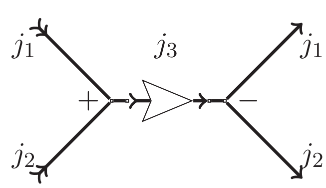
(i) *Basic recoupling*



$$= (R^{j_1}(g))^\alpha{}_{\tilde\alpha}(R^{j_2}(g))^\beta{}_{\tilde\beta}$$

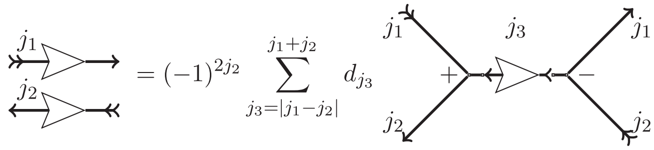
$$= \sum_{j_3=|j_1-j_2|}^{j_1+j_2} d_{j_3} \sum_{\gamma,\tilde\gamma=-j_3}^{j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \tilde\alpha & \tilde\beta & \tilde\gamma \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha & \beta & \gamma \end{pmatrix}\overline{(R^{j_3}(g))^\gamma{}_{\tilde\gamma}}$$

$$= \sum_{j_3=|j_1-j_2|}^{j_1+j_2} d_{j_3} \qquad \text{(A.12)}$$


$$= (-1)^{2j_3}\sum_{j_3=|j_1-j_2|}^{j_1+j_2} d_{j_3}$$


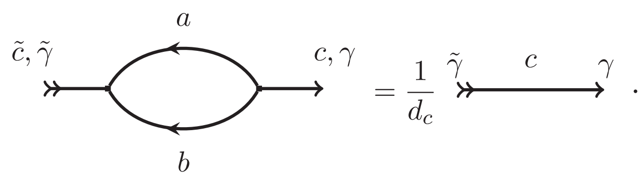
.

Similarly, one finds



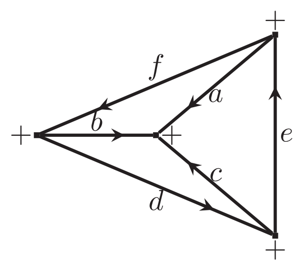
(ii) *First orthogonality relation*

$$\sum_{\alpha,\beta} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b & \tilde{c} \\ \alpha & \beta & \tilde{\gamma} \end{pmatrix} = \frac{1}{d_c} \delta_{c,\tilde{c}} \delta^{\tilde{\gamma}}{}_{\gamma}$$



(A.13)

(iii) $6j$ *symbol*



$$= \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} . \tag{A.14}$$

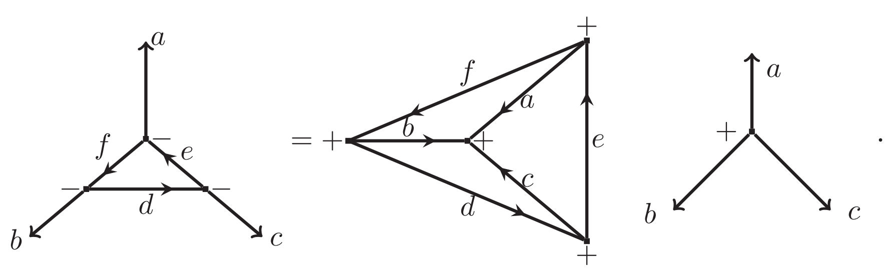
(iv) *Second orthogonality relation*

$$\sum_f d_m d_f \begin{Bmatrix} a & b & f \\ d & c & e \end{Bmatrix} \begin{Bmatrix} a & c & m \\ d & b & f \end{Bmatrix} = \delta_{em}. \tag{A15}$$

(v) *Summation*—The following identity is a crucial ingredient for the computation of the matrix elements of the Hamiltonian operator:

$$\sum_{\delta,\epsilon,\phi} (-1)^{d+e+f-\delta-\epsilon-\phi} \begin{pmatrix} d & e & c \\ -\delta & \epsilon & \gamma \end{pmatrix} \begin{pmatrix} e & f & a \\ -\epsilon & \phi & \alpha \end{pmatrix} \begin{pmatrix} f & d & b \\ -\phi & \delta & \beta \end{pmatrix} = \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}. \tag{A16}$$

Graphically, this identity can be encoded in



(A.17)

(vi) $9j$ *symbol*
Definition of a $9j$ symbol in terms of $6j$'s:

$$\sum_x d_x(-1)^{2x}\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix}\begin{Bmatrix} e & f & x \\ a & b & r \end{Bmatrix} = \begin{Bmatrix} a & f & r \\ d & q & e \\ p & c & b \end{Bmatrix}. \tag{A18}$$

Permutation symmetry:

(1)

$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = \epsilon \begin{Bmatrix} j_{1i} & j_{1j} & j_{1k} \\ j_{2i} & j_{2j} & j_{2k} \\ j_{3i} & j_{3j} & j_{3k} \end{Bmatrix}. \tag{A19}$$

(2)

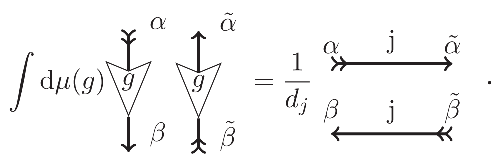
$$\begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} = \epsilon \begin{Bmatrix} j_{i1} & j_{i2} & j_{i3} \\ j_{j1} & j_{j2} & j_{j3} \\ j_{k1} & j_{k2} & j_{k3} \end{Bmatrix} \tag{A20}$$

with $\epsilon = 1$ for even permutations and $\epsilon = (-1)^R$ with $R = \sum_{ij} j_{ij}$ for odd permutations.

Summation Identity:

$$\sum_x d_x \begin{Bmatrix} a & b & e \\ c & d & f \\ e & f & x \end{Bmatrix} = \frac{\delta_{bc}}{d_b}\theta(a, b, e)\theta(b, d, f). \tag{A21}$$

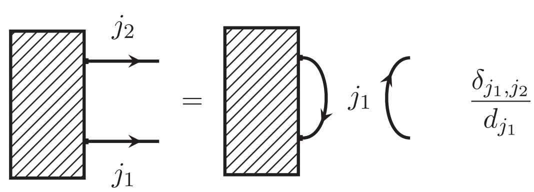
(vii) *Integration*—The matrix elements of the irreducible representations of SU(2) provide an orthonormal basis in the space of square integrable functions of SU(2) with respect to the Haar measure $\mu(g)$. This can be visualized by the following diagram:



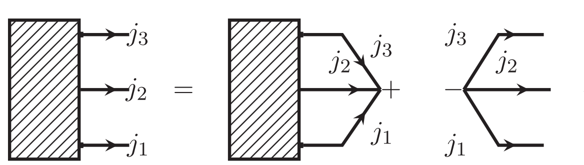
(A.22)

The visualization of the integral over higher tensor products works analogously.

(viii) *Simplify Graphs*—From Eq. (A22), follow some useful rules for simplifying graphs. In the following, completely contracted graphs are symbolized by dashed boxes.



(A.23)



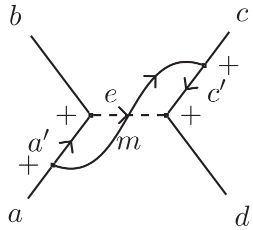
(A.24)

The simplification of graphs with more than three edges can be obtained by applying Eqs. (A12) and (A23) and Eq. (A24), respectively.

(ix) *Sign-Manipulation*—Suppose the three irreducible representations a, b, c obey the Clebsch-Gordan conditions; then, $(a + b + c) \in \mathbb{N}$, and thus $(-1)^{2a+2b+2c} = 1$. This identity is crucial in many calculations in order to simplify the signs or add missing signs of the form $(-1)^{2a}$. Since $a \in \frac{1}{2}\mathbb{N}$, we also have $(-1)^{3a} = (-1)^{-a}$.

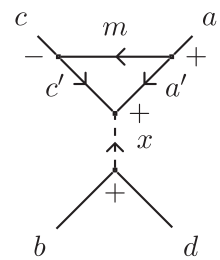
APPENDIX B: GRASPING

A relevant formula used for the computation of the matrix elements of the Hamiltonian in the 4-valent case can be deduced by the double grasping operators on 4-valent nodes, computed in Ref. [16]:

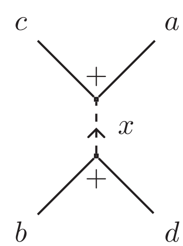


$$=$$

$$= \sum_x (-1)^{a'+d+e+x} d_x \begin{Bmatrix} b & d & x \\ c' & a' & e \end{Bmatrix}$$

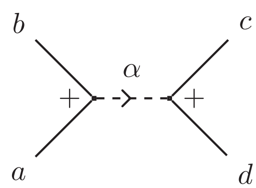


$$= \sum_x (-1)^{a+c'+x+m} (-1)^{a'+d+e+x} d_x \begin{Bmatrix} b & d & x \\ c' & a' & e \end{Bmatrix} \begin{Bmatrix} c' & a' & x \\ a & c & m \end{Bmatrix} \tag{B.1}$$

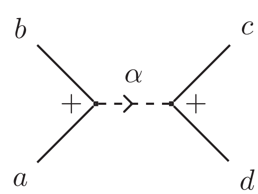


$$= \sum_x (-1)^{a+c'+x+m} (-1)^{a'+d+e+x} d_x \sum_\alpha d_\alpha (-1)^{a+d+\alpha+x}.$$

$$\cdot \begin{Bmatrix} b & d & x \\ c' & a' & e \end{Bmatrix} \begin{Bmatrix} c' & a' & x \\ a & c & m \end{Bmatrix} \begin{Bmatrix} b & a & \alpha \\ c & d & x \end{Bmatrix}$$



$$= (-1)^{b+a-c-d} \sum_\alpha d_\alpha \begin{Bmatrix} e & \alpha & m \\ a & a' & b \end{Bmatrix} \begin{Bmatrix} e & \alpha & m \\ c & c' & d \end{Bmatrix}$$



APPENDIX C: NORMALIZATION OF THE SPIN-NETWORK STATES

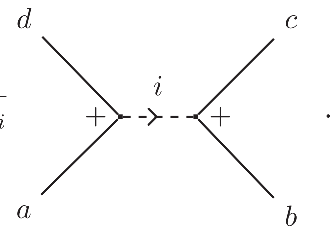
Following Ref. [2], we define a spin network $S = (\Gamma, j_l, i_n)$ as given by a graph Γ with a given orientation (or ordering of the links) with L links and N nodes, and by a representation j_l associated to each link and an intertwiner i_n to each node. As a functional of the connection, a spin-network state is given by

$$\Psi_S[A] = \langle A|S\rangle \equiv (\otimes_l R^{j_l}(h[A, \gamma_l]))_\llcorner (\otimes_n \mathrm{i}_n) \tag{C1}$$

where $_\llcorner$ indicates the contraction with the intertwiners and $R^{j_l}(h[A, \gamma_l])$ is the j_l representation of the holonomy group element $h[A, \gamma_l]$ along the curve γ_l of the gravitation field connection A. The scalar product on $\mathcal{H}_{\text{kin}}$ is defined via the Ashtekar-Lewandowski measure:

$$\langle S|S'\rangle = \int d\mu_{AL} \overline{\Psi_S[A]}\Psi_S[A] = \delta_{S,S'} \prod_{e\in\mathcal{E}} \frac{1}{d_{j_e}} \prod_{v\in\mathcal{V}} \mathrm{Tr}(\iota_v^* \iota_v) \tag{C2}$$

where $\mathcal{E}$ is the set of links and $\mathcal{V}$ is the set of vertices of Γ. Throughout this paper, we used normalized intertwiners such that $\mathrm{Tr}(\iota_v^* \iota_v) = 1$. For a trivalent node, this requirement is trivially fulfilled if we use Wigner $3j$ symbols. A higher-valent node can be decomposed into trivalent nodes by introducing virtual links. For example, for a 4-valent node, we obtain

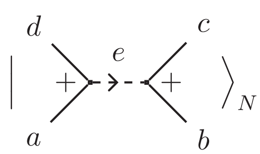
$\sqrt{d_i}$ 

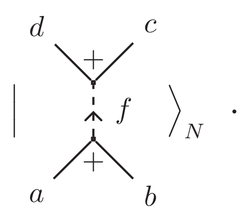
. (C.3)

If, additionally, we multiply Eq. (C2) by

$$\prod_{e\in\mathcal{E}} \sqrt{d_{j_e}},$$

we obtain a normalized state $|S\rangle_N$ with respect to Eq. (C2). Consequentially, the recoupling theorem applied to normalized spin-network state yields



$$= \sum_f \sqrt{d_e}\sqrt{d_f}(-1)^{b+c+e+f} \begin{Bmatrix} a & b & f \\ d & c & e \end{Bmatrix}$$ 

. (C.4)

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