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New exact solutions of quadratic curvature gravity

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It is a known fact that the Kerr-Schild type solutions in general relativity satisfy both exact and linearized Einstein field equations. We show that this property remains valid also for a special class of the Kerr-Schild metrics in arbitrary dimensions in generic quadratic curvature theory. In addition to the anti-de Sitter (AdS) wave (or Siklos) metric which represents plane waves in an AdS background, we present here a new exact solution, in this class, to the quadratic gravity in D dimensions which represents a spherical wave in an AdS background. The solution is a special case of the Kundt metrics belonging to spacetimes with constant curvature invariants.

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I. INTRODUCTION

Whatever the full UV-finite quantum gravity is, its successful low energy limit, general relativity (GR), is based on the Riemannian geometry. In this context finding exact Riemannian spacetimes as solutions to Einstein's equations (with or without a cosmological constant and/or sources) has evolved to be a fine art on its own. There are at least two books [1,2] that compile and classify these spacetimes, discuss their physical interpretations and present techniques of finding solutions. Like any other low energy theory, GR is expected to receive corrections at high energies built on more powers of curvature starting with the quadratic gravity which is the subject of this work. Even though much has been studied in quadratic gravity theories, compared to Einstein's theory very little is known about the exact solutions in generic D dimensions ($D = 3$ and $D = 4$ are somewhat special as we shall discuss below). There has been a revival of interest in quadratic gravity theories because of three recent enticing developments: a specific quadratic gravity model in $(2 + 1)$ dimensions dubbed as the new massive gravity (NMG) [3] provided the first example of a parity invariant nonlinear unitary theory with massive gravitons in its perturbative spectrum. The second development was the introduction of "critical gravity" [4,5] built from the Ricci scalar, the square of the Weyl tensor and a tuned cosmological constant that has the same perturbative spectrum as the Einstein's theory with an improved UV behavior. The third one is the observation that with Neumann boundary conditions on the metric non-Einstein solutions of the conformal gravity are eliminated and the theory reduces to the cosmological Einstein's gravity in $D = 4$ dimensions [6]. All these developments in quadratic curvature gravity theories prompted us to study systematically some exact solutions of these theories.

In this work, we will present special Kundt type radiating solutions [7,8] to quadratic gravity theories in generic D dimensions. This will be a D-dimensional generalization of the works in three dimensions [9,10].[1] Subclasses of Kundt metrics in various forms have also been studied as solutions of topologically massive gravity [12,13] in [9,10,14–20]. In D dimensions, the anti-de Sitter (AdS)-wave metric (also called the Siklos metric [21,22]) which is a Kundt metric of Type N with a cosmological constant was shown to be a solution of the quadratic curvature theories [23] generalizing the result in $D = 3$ [24]. All Einstein spacetimes of Type N solve this theory exactly in D dimensions [25,26]. It is a known fact that in $D = 4$ all Einstein spaces solve quadratic theory exactly. Critical quadratic gravity has genuinely new solutions with asymptotically non-AdS geometry that has logarithmic behavior in Poincaré and global coordinates [23,27]. It is important here to note that the works of Coley *et al.* [7,8,28–32] on the classification of pseudo-Riemannian spacetimes, on spacetimes with constant invariants and on Kundt spacetimes in general relativity have attracted many researchers [9,19,20,33] to use them in higher order curvature theories in arbitrary dimensions. Another important point is that all those metrics solving higher order curvature theories belong to both Kundt and Kerr-Schild classes, [1,34,35].

The layout of the paper is as follows: In the next section, we discuss the Kerr-Schild class of metrics in AdS backgrounds possessing some special properties. These properties are so effective that some tensorial quantities, like Ricci and Riemann tensors become linear in the metric "perturbation" around the AdS background. In the third section, we show that the full quadratic gravity field equations reduce to a fourth order linear partial differential

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[1]In [11], for $D = 3$, Kundt type solutions of NMG [9,10] are used to generate solutions of $f(R_{\mu\nu})$ theories which naturally includes the generic quadratic curvature theory.

equation. We give a new exact solution which we call a spherical-AdS wave that has asymptotically AdS and asymptotically non-AdS; i.e. Log mode behavior just like the previously found AdS wave. In Sec. IV, we show that the same class solve the linearized quadratic gravity field equations. We delegate the details of the computations to the Appendices.

II. A SPECIAL CLASS OF KERR-SCHILD METRICS

Let us take a D-dimensional metric in the Kerr-Schild form [34,35]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu, \tag{1}$$

where $\bar{g}_{\mu\nu}$ is the metric of the AdS space and V is a function of spacetime (see [36] for some properties of the Kerr-Schild metrics with generic backgrounds and see also [26,37] with an AdS background). The vector $\lambda^\mu = g^{\mu\nu}\lambda_\nu$ is assumed to be null; i.e. $\lambda_\mu\lambda^\mu = g_{\mu\nu}\lambda^\mu\lambda^\nu = 0$ and geodesic $\lambda^\mu\nabla_\mu\lambda_\rho = 0$. These two assumptions imply

$$\bar{g}_{\mu\nu}\lambda^\mu\lambda^\nu = 0, \qquad \lambda_\mu = \bar{g}_{\mu\nu}\lambda^\nu, \qquad \lambda^\mu\bar{\nabla}_\mu\lambda_\rho = 0,$$

where the barred covariant derivative is with respect to $\bar{g}_{\mu\nu}$. The inverse metric can be found as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - 2V\lambda^\mu\lambda^\nu. \tag{2}$$

Writing the metric in the form (1) will help us in explicitly observing the fact that the solutions of the field equations of the quadratic gravity are also solutions of the linearized field equations of the theory with $h_{\mu\nu} \equiv 2V\lambda_\mu\lambda_\nu$. AdS wave or Siklos spacetimes are in this class with the line element

$$\begin{aligned} ds^2 &= \frac{1}{k^2z^2}\left(-dt^2 + dx^2 + \sum_{m=1}^{D-3}(dx^m)^2 + dz^2\right) \\ &\quad + 2V(t, x, x_m, z)\lambda_\mu\lambda_\nu dx^\mu \otimes dx^\nu \\ &= \frac{1}{k^2z^2}\left(2dudv + \sum_{m=1}^{D-3}(dx^m)^2 + dz^2\right) \\ &\quad + 2V(u, x_m, z)du^2, \end{aligned} \tag{3}$$

where in the second line we have used the null coordinates defined as $u = (x + t)/\sqrt{2}$, $v = (x - t)/\sqrt{2}$ and chosen $\lambda_\mu dx^\mu = du$ and $\lambda^\mu\partial_\mu V = 0$ that is V does not depend on v. The constant k^2 is related to the cosmological constant as $-k^2 = \frac{2\Lambda}{(D-1)(D-2)}$. With these assumptions, λ^μ becomes divergence free (nonexpanding) with respect to the full and background metrics namely $\nabla_\mu\lambda^\mu = \bar{\nabla}_\mu\lambda^\mu = 0$, and the Ricci scalar turns out to be a constant given as $R = -D(D-1)k^2$. Besides being nonexpanding, it is possible to show that λ^μ is a shear-free, $\nabla^\mu\lambda^\nu\nabla_{(\mu}\lambda_{\nu)} = 0$, and nontwisting, $\nabla^\mu\lambda^\nu\nabla_{[\mu}\lambda_{\nu]} = 0$, vector. As λ_μ is a null vector which is nonexpanding, shear-free and nontwisting, AdS wave is a Kundt spacetime by definition. Furthermore, the Weyl tensor satisfies the following property

$$C_{\alpha\beta\gamma\sigma}\lambda^\sigma = 0, \tag{4}$$

therefore, λ_μ is a null direction of the Weyl tensor. In $D = 4$, (4) is equivalent to the metric being of Type N. Note that λ_μ is *not* a Killing vector, but $\zeta_\mu \equiv \frac{1}{z^2}\lambda_\mu$ is a null Killing vector. Recently, it was shown that the AdS-wave metric (3) solves the quadratic gravity field equations in D dimensions provided that the function V satisfies a fourth order linear partial differential equation which was solved in the most general setting [23].

In this work, we present a new Kundt solution of the quadratic gravity field equations in D dimensions which is also in the Kerr-Schild form (1) as the AdS wave. The new solution is similar to the AdS-wave metric in form, but with a different λ_μ which dramatically changes the spacetime. To reach the new metric, let us rewrite the background AdS in the spherical coordinates which turns the full metric to

$$\begin{aligned} ds^2 &= \frac{1}{k^2z^2}\left[-dt^2 + \sum_{m=1}^{D-2}(dx^m)^2 + dz^2\right] \\ &\quad + 2V\lambda_\mu\lambda_\nu dx^\mu \otimes dx^\nu \\ &= \frac{1}{k^2r^2\cos^2\theta}\left[-dt^2 + dr^2 + r^2d\Omega^2_{D-2}\right] \\ &\quad + 2V\lambda_\mu\lambda_\nu dx^\mu \otimes dx^\nu, \end{aligned} \tag{5}$$

where $d\Omega^2_{D-2}$ is the metric on the unit sphere in $(D-2)$ dimensions. Here, note that since $z > 0$, one needs to constrain θ in the interval $0 \le \theta < \pi/2$. In the spherical coordinates, boundary of AdS ($z \to 0$) can be reached with the limits $r \to 0$ or/and $\theta \to \pi/2$. One can define the null coordinates as $u \equiv \frac{1}{\sqrt{2}}(r + t)$ and $v \equiv \frac{1}{\sqrt{2}}(r - t)$, then (5) becomes

$$\begin{aligned} ds^2 &= \frac{2}{k^2(u+v)^2\cos^2\theta}\left[2dudv + \frac{(u+v)^2}{2}d\Omega^2_{D-2}\right] \\ &\quad + 2V(u, \Omega_{D-2})du^2, \\ &= \frac{1}{k^2\cos^2\theta}\left(\frac{4dudv}{(u+v)^2} + d\Omega^2_{D-2}\right) \\ &\quad + 2V(u, \Omega_{D-2})du^2, \end{aligned} \tag{6}$$

where we have again chosen $\lambda_\mu dx^\mu = du$ and $\lambda^\mu\partial_\mu V = 0$. With these assumptions, once again $\nabla_\mu\lambda^\mu = \bar{\nabla}_\mu\lambda^\mu = 0$. This metric can be recast in other coordinates as

(1) Cartesian:

$$\begin{aligned} ds^2 &= \frac{1}{k^2z^2}\left[-dt^2 + \sum_{m=1}^{D-2}(dx^m)^2 + dz^2\right] \\ &\quad + 2V(\lambda_\mu dx^\mu)^2, \end{aligned} \tag{7}$$

where

$$\lambda_\mu = \left(1, \frac{x^m}{r}, \frac{z}{r}\right),$$
$$m = 1, 2, \ldots, D-2; \quad r^2 = z^2 + \sum_{m=1}^{D-2} (x^m)^2. \tag{8}$$

Here, we note that an infinite boost in the $(t - x^1)$ plane reduces this metric to the AdS-wave metric (3).

(2) Another form of the above metric can be given as

$$ds^2 = dr^2 + \frac{4\cosh^2 kr}{k^2(u+v)^2} du dv + \frac{\sinh^2 kr}{k^2} d\Omega^2_{D-3} + 2V(u, r, \Omega_{D-3}) du^2. \tag{9}$$

This form was given in [28,33] as an example of Kundt spacetimes with constant curvature invariants. There exists no null Killing vector field of this spacetime. $D = 3$ case of this form of the metric was given [10,17] as the most general Type-N solution of the three-dimensional NMG.

The AdS-wave metric (3) and the spherical-wave metric (6) have the following (not necessarily independent) properties which define the Kerr-Schild-Kundt (KSK) class:

(1) $\bar{g}_{\mu\nu}$ is the metric of the AdS space, $g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu\lambda_\nu$ is the full metric.

(2) The vector $\lambda^\mu = g^{\mu\nu}\lambda_\nu$ assumed to have the properties of being null $\lambda_\mu\lambda^\mu = g_{\mu\nu}\lambda^\mu\lambda^\nu = 0$ and geodesic $\lambda^\mu\nabla_\mu\lambda_\rho = 0$.

(3) V is a function of spacetime assumed to satisfy $\lambda^\mu\partial_\mu V = 0$. This assumption has wonderful implications together with the assumption $\nabla_\mu\lambda^\mu = \bar{\nabla}_\mu\lambda^\mu = 0$. With these assumptions, Riemann and Ricci tensors become linear in V and the scalar curvature becomes constant.

(4) $\nabla_\mu\lambda_\nu = \lambda_{(\mu}\xi_{\nu)}$, where $\xi^\mu\lambda_\mu = 0$.[2]

(5) λ_μ is nonexpanding, $\nabla_\mu\lambda^\mu = 0$, shear-free, $\nabla^\mu\lambda^\nu\nabla_{(\mu}\lambda_{\nu)} = 0$, and nontwisting, $\nabla^\mu\lambda^\nu\nabla_{[\mu}\lambda_{\nu]} = 0$ which are implied by the fourth property. Note that one can replace the full covariant derivative and the metric with the background covariant derivative and the background metric in these relations, namely $\bar{\nabla}^\mu\lambda^\nu\bar{\nabla}_{[\mu}\lambda_{\nu]} = 0$, etc.

These properties are useful in calculating various tensorial quantities. Here, we note the results of the relevant computations and delegate some to Appendix B. The Riemann tensor of (1) after using some of the properties listed above reduces to

$$R^\mu{}_{\alpha\nu\beta} = \bar{R}^\mu{}_{\alpha\nu\beta} + \bar{\nabla}_\nu\Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta\Omega^\mu{}_{\alpha\nu}, \tag{10}$$

where

$$\begin{aligned}\bar{\nabla}_\nu\Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta\Omega^\mu{}_{\alpha\nu} = {} & 2\lambda_\alpha\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial^\mu V - 2\lambda^\mu\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\alpha V \\ & + \lambda_{[\nu}\xi_{\beta]}(\lambda_\alpha\partial^\mu V - \lambda^\mu\partial_\alpha V \\ & + \lambda_\alpha\xi^\mu V) + (\lambda_\alpha\xi^\mu - \lambda^\mu\xi_\alpha) \\ & \times\lambda_{[\nu}\partial_{\beta]}V + 2V\lambda^\mu \\ & \times(\lambda_\alpha\bar{\nabla}_{[\nu}\xi_{\beta]} - \lambda_{[\nu}\bar{\nabla}_{\beta]}\xi_\alpha),\end{aligned} \tag{11}$$

where the background part reads $\bar{R}_{\mu\alpha\nu\beta} = -k^2(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\alpha\nu})$ and the remaining part is linear in V. The property (4) leads to

$$R^\rho{}_{\mu\nu\alpha}\lambda_\rho = \frac{R}{D(D-1)}(\lambda_\alpha g_{\mu\nu} - \lambda_\nu g_{\mu\alpha}). \tag{12}$$

For the class of Kerr-Schild-Kundt metrics, the Ricci tensor follows from (10) as

$$R_{\mu\nu} = -(D-1)k^2 g_{\mu\nu} - \rho\lambda_\mu\lambda_\nu, \tag{13}$$

where

$$\rho \equiv \bar{\Box}V + 2\xi_\mu\partial^\mu V + \tfrac{1}{2}V\xi_\mu\xi^\mu - 2Vk^2(D-2), \tag{14}$$

where $\bar{\Box} \equiv \bar{\nabla}^\rho\bar{\nabla}_\rho$ and $\lambda^\mu\partial_\mu\rho = 0$ and the Ricci scalar is $R = -D(D-1)k^2$. It is amusing to see that the metric solves the cosmological Einstein equations in the presence of a null fluid in all dimensions as long as $T_{\mu\nu} = \rho\lambda_\mu\lambda_\nu$, but our task is to show that the same metric solves the vacuum field equations of the quadratic gravity.

Using the properties listed above of the new metric we find the following tensors that we shall need in the field equations of the most general quadratic gravity;

$$\Box R_{\mu\nu} = -\bar{\Box}(\rho\lambda_\mu\lambda_\nu), \tag{15}$$

or in another form

$$\begin{aligned}\Box R_{\mu\nu} = {} & -\lambda_\mu\lambda_\nu(\bar{\Box}\rho + 2\xi_\sigma\partial^\sigma\rho + \tfrac{1}{2}\rho\xi_\sigma\xi^\sigma \\ & - 2\rho k^2(D-1)),\end{aligned} \tag{16}$$

and

$$R^\rho_\mu R_{\rho\nu} = (D-1)^2k^4 g_{\mu\nu} + 2(D-1)k^2\rho\lambda_\mu\lambda_\nu, \tag{17}$$

$$R_{\mu\alpha\nu\beta}R^{\alpha\beta} = (D-1)^2k^4 g_{\mu\nu} + (D-2)k^2\rho\lambda_\mu\lambda_\nu, \tag{18}$$

$$R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} = 2(D-1)k^4 g_{\mu\nu} + 4k^2\rho\lambda_\mu\lambda_\nu. \tag{19}$$

III. A NEW SOLUTION OF THE QUADRATIC GRAVITY

The action of the quadratic gravity is

$$I = \int d^Dx\sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma(R^2_{\mu\nu\sigma\rho} - 4R^2_{\mu\nu} + R^2)\right]. \tag{20}$$

[2]Symmetrization is done as usual; i.e. $2A_{(\mu}B_{\nu)} \equiv A_\mu B_\nu + A_\nu B_\mu$.

The (source-free) field equations were given in [38,39] as

$$\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}\right)+2\alpha R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(2\alpha+\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R+2\gamma\Big[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}-2R_{\mu\sigma}R_\nu{}^{\sigma}-\frac{1}{4}g_{\mu\nu}(R^2_{\tau\lambda\sigma\rho}-4R^2_{\sigma\rho}+R^2)\Big]+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\beta\left(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}=0. \tag{21}$$

Using (13)–(19) in (21), one obtains

$$\frac{\Lambda-\Lambda_0}{2\kappa}+f\Lambda^2=0,\quad \Lambda\equiv-\frac{(D-1)(D-2)}{2}k^2,\qquad f\equiv(D\alpha+\beta)\frac{(D-4)}{(D-2)^2}+\gamma\frac{(D-3)(D-4)}{(D-1)(D-2)}, \tag{22}$$

as a trace equation, and the remaining traceless equation is a fourth order equation,

$$(\beta\bar{\Box}+c)(\rho\lambda_\mu\lambda_\nu)=0, \tag{23}$$

where

$$c\equiv\frac{1}{\kappa}+\frac{4\Lambda D}{D-2}\alpha+\frac{4\Lambda}{D-1}\beta+\frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma. \tag{24}$$

As noted before, AdS wave [23] solves (23). Now, let us find the second solution that is the spherical-AdS-wave metric (6). This can be achieved by obtaining a fourth order scalar equation on V

$$(\mathcal{O}-M^2)\mathcal{O}V(u,\Omega_{D-2})=0, \tag{25}$$

where

$$M^2\equiv-\frac{c}{\beta}+2k^2,\qquad \mathcal{O}\equiv\bar{\Box}-2k^2\sin2\theta\partial_\theta-2k^2(D-2-\sin^2\theta). \tag{26}$$

To reach (25), we have calculated ρ for the spherical-AdS wave which is $\rho=\mathcal{O}V$. It is important to notice that there are two different types of solutions to (25). The first type solution is $V=V_1+V_2$ where V_1 is a solution to the quadratic partial differential equation (PDE)

$$\mathcal{O}V_1(u,\Omega_{D-2})=0, \tag{27}$$

which is also a solution of the cosmological Einstein's theory, ($\rho=0$), and V_2 is a solution to again a quadratic PDE

$$(\mathcal{O}-M^2)V_2(u,\Omega_{D-2})=0. \tag{28}$$

As long as $M^2\neq0$, $V=V_1+V_2$ is the most general solution to the fourth order PDE (25). But, when $M^2=0$, then the equation becomes

$$\mathcal{O}^2V(u,\Omega_{D-2})=0, \tag{29}$$

and new solutions arise which represent the non-Einstein solutions of the critical gravity. To get the solutions, let us employ the separation of variables technique as $V(u,\Omega_{D-2})=F(u,\theta)G(u,\Omega_{D-3})$ where $G(u,\Omega_{D-3})$ is the function defined on the $(D-3)$-dimensional unit sphere. For a scalar function $\Phi(u,\theta,\Omega_{D-3})$, let us calculate $\bar{\nabla}^\rho\bar{\nabla}_\rho\Phi(u,\theta,\Omega_{D-3})$ for the background AdS metric

$$d\bar{s}^2=\frac{4dudv}{k^2\cos^2\theta(u+v)^2}+\frac{1}{k^2\cos^2\theta}d\Omega^2_{D-2}, \tag{30}$$

which corresponds to $V=0$ in (6):

$$\bar{\nabla}^\rho\bar{\nabla}_\rho\Phi(u,\theta,\Omega_{D-3})=2\bar{g}^{vu}\bar{\nabla}_v\partial_u\Phi(u,\theta,\Omega_{D-3})+\bar{g}^{\Omega_i\Omega_i}\bar{\nabla}_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3}), \tag{31}$$

where Ω_i represents the angular coordinates on S^{D-2} which includes the θ direction. Using the results in Appendix C, the first term yields

$$2\bar{g}^{vu}\bar{\nabla}_v\partial_u\Phi(u,\theta,\Omega_{D-3})=2k^2\sin\theta\cos\theta\partial_\theta\Phi(u,\theta,\Omega_{D-3}). \tag{32}$$

On the other hand, the second term can be written as

$$\begin{aligned}&\bar{g}^{\Omega_i\Omega_i}\bar{\nabla}_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3})\\&=\bar{g}^{\Omega_i\Omega_i}\partial_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3})\\&\quad-\bar{g}^{\Omega_i\Omega_i}\bar{\Gamma}^{\Omega_j}_{\Omega_i\Omega_i}\partial_{\Omega_j}\Phi(u,\theta,\Omega_{D-3})\\&\quad-\bar{g}^{\Omega_i\Omega_i}\bar{\Gamma}^{u}_{\Omega_i\Omega_i}\partial_u\Phi(u,\theta,\Omega_{D-3}),\end{aligned} \tag{33}$$

In Appendix C, it is shown that $\bar{\Gamma}^u_{\Omega_i\Omega_i}=0$; therefore, the last term vanishes. Then, let us calculate the remaining terms in (33) which corresponds to the box operator acting on a scalar function with the following metric conformal to the metric $\eta_{\Omega_i\Omega_j}$ (not to be confused with the flat metric) on the round S^{D-2} sphere:

$$ds^2=\frac{1}{k^2\cos^2\theta}d\Omega^2_{D-2}\Rightarrow\bar{g}_{\Omega_i\Omega_j}=\omega^{-2}\eta_{\Omega_i\Omega_j},\qquad \omega\equiv k\cos\theta. \tag{34}$$

The Christoffel connection of $\bar{g}_{\Omega_i\Omega_j}$ is related to the Christoffel connection of $\eta_{\Omega_i\Omega_j}$ via the usual conformal transformations

$$\bar{\Gamma}^{\mu}_{\alpha\beta}=(\Gamma^\mu_{\alpha\beta})_{S^{D-2}}-\delta^\mu_\alpha\partial_\beta\ln\omega-\delta^\mu_\beta\partial_\alpha\ln\omega+\eta_{\alpha\beta}\eta^{\mu\sigma}\partial_\sigma\ln\omega, \tag{35}$$

Using this result in $\bar{g}^{\Omega_i\Omega_i}\bar{\nabla}_{\Omega_i}\partial_{\Omega_i}\Phi$, one obtains

$$\bar{g}^{\Omega_i\Omega_i}\bar{\nabla}_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3}) = \omega^2[\eta^{\Omega_i\Omega_i}\partial_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3}) - \eta^{\Omega_i\Omega_i}(\Gamma^{\Omega_j}_{\Omega_i\Omega_i})_{S^{D-2}}\partial_{\Omega_j}\Phi(u,\theta,\Omega_{D-3})] + \omega^2[2\eta^{\theta\theta}\delta^{\Omega_j}_{\theta}\partial_\theta\ln\omega - \eta^{\Omega_i\Omega_i}\eta_{\Omega_i\Omega_i}\eta^{\Omega_j\theta}\partial_\theta\ln\omega] \times\partial_{\Omega_j}\Phi(u,\theta,\Omega_{D-3}), \quad (36)$$

where the operator in the first square bracket is the Laplace-Beltrami operator on S^{D-2} which can be recursively written as

$$\Delta_{S^{D-2}}\Phi(u,\theta,\Omega_{D-3}) = \frac{1}{\sin^{D-3}\theta}\frac{\partial}{\partial\theta}\left(\sin^{D-3}\theta\frac{\partial\Phi(u,\theta,\Omega_{D-3})}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\Delta_{S^{D-3}}\Phi(u,\theta,\Omega_{D-3}) = \left(\frac{\partial^2}{\partial\theta^2} + (D-3)\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{S^{D-3}}\right) \times\Phi(u,\theta,\Omega_{D-3}). \quad (37)$$

Collecting (36) and (37), one arrives at

$$\bar{g}^{\Omega_i\Omega_i}\bar{\nabla}_{\Omega_i}\partial_{\Omega_i}\Phi(u,\theta,\Omega_{D-3}) = k^2\cos^2\theta\left(\frac{\partial^2}{\partial\theta^2} + (D-3)\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{S^{D-3}}\right) \times\Phi(u,\theta,\Omega_{D-3}) + k^2(D-4) \times\sin\theta\cos\theta\partial_\theta\Phi(u,\theta,\Omega_{D-3}). \quad (38)$$

Finally, one has

$$\bar{\Box}\Phi(u,\theta,\Omega_{D-3}) = k^2\cos^2\theta\frac{\partial^2\Phi(u,\theta,\Omega_{D-3})}{\partial\theta^2} + k^2[(D-3) \times\cot\theta + \sin\theta\cos\theta]\frac{\partial\Phi(u,\theta,\Omega_{D-3})}{\partial\theta} + k^2\cot^2\theta\Delta_{S^{D-3}}\Phi(u,\theta,\Omega_{D-3}). \quad (39)$$

This result is sufficient for us to carry out the separation of variables. Let us first focus on the Einstein modes satisfying (27). Using (39) for $V(u,\Omega_{D-2}) = F(u,\theta)G(u,\Omega_{D-3})$, one has two decoupled equations

$$\cos^2\theta\frac{\partial^2F(u,\theta)}{\partial\theta^2} + [(D-3)\cot\theta - 3\sin\theta\cos\theta]\frac{\partial F(u,\theta)}{\partial\theta} - [2(D-2-\sin^2\theta) + a^2(u)\cot^2\theta]F(u,\theta) = 0, \quad (40)$$

$$(\Delta_{S^{D-3}} + a^2(u))G(u,\Omega_{D-3}) = 0, \quad (41)$$

where a^2 is an arbitrary function of u. Both of these equations can be solved exactly for $a^2 \neq 0$: (40) has a solution in terms of hypergeometric functions and (41) in terms of spherical harmonics on S^{D-3} [40]. Since the most general solution is not particularly illuminating to depict here for the sake of simplicity let us concentrate on $D = 4$, for which one has

$$F(u,\theta) = \frac{c_1(u)}{a}\left(\tan\frac{\theta}{2}\right)^a\sec\theta(a+\sec\theta) + \frac{c_2(u)}{(a^2-1)}\left(\tan\frac{\theta}{2}\right)^{-a}\sec\theta(a-\sec\theta), \quad (42)$$

$$G(u,\phi) = c_3(u)\cos(a\phi) + c_4(u)\sin(a\phi). \quad (43)$$

Here, one of the functions $c_i(u)$ can be set to 1 without loss of generality, if it is not zero. Note that $a = 0$ and $a^2 = 1$ are the special values for which the solutions can be obtained as

(i) $D = 4$ and $a = 0$:

$$F(u,\theta) = c_1(u)\sec^2\theta + c_2(u)\left(\cos\theta + \log\left[\tan\left(\frac{\theta}{2}\right)\right]\right)\sec^2\theta, \quad (44)$$

$$G(u,\phi) = c_3(u) + c_4(u)\phi. \quad (45)$$

More explicitly, the solution reads

$$V(u,\theta,\phi) = \frac{1}{\cos^2\theta}\left[1 + c_2(u)\left(\cos\theta + \log\left[\tan\left(\frac{\theta}{2}\right)\right]\right)\right] \times(c_3(u) + c_4(u)\phi). \quad (46)$$

Let us investigate the near-boundary behavior of this metric by defining $x \equiv \pi/2 - \theta$ and finding the asymptotic form for small x. In order to have complete comparison with the AdS-wave boundary behavior, one needs to expand up to $O(x^4)$ which yields

$$F(u,x) \sim \frac{1}{x^2}\left[1 + \frac{1}{3}x^2 + c_2(u)x^3 + O(x^4)\right]. \quad (47)$$

Here, the leading order represents the asymptotically AdS spacetime just like the AdS wave; while the next-to-leading order shows that the spherical-AdS wave approaches to AdS spacetime more slowly than the AdS-wave which exactly behaves as

$$V_{\text{AdS-wave}}(u,x) = \frac{1}{x^2}[1 + c_2(u)x^3]. \quad (48)$$

(ii) $D = 4$ and $a^2 = 1$ is also a simple solution which we depict here:

$$F(u,\theta) = c_1(u)\sec\theta\tan\theta + c_2(u)\csc\theta\left(\log\left[\tan\left(\frac{\theta}{2}\right)\right] - \sec\theta + \text{arctanh}[\cos\theta]\sec^2\theta\right), \quad (49)$$

$$G(u,\phi) = c_3(u)\cos(\phi) + c_4(u)\sin(\phi). \quad (50)$$

Clearly, the solutions of (28), which we call massive

modes, have the same functional form as the Einstein modes in (42) and (43). In order to obtain the massive modes explicitly, the only thing one should do is to replace a in (42) with $\sqrt{a^2+M^2}$.

Now, let us focus on the non-Einstein solutions of the $M^2=0$ case with the field equation (29) corresponding to the critical gravity. We are interested in the spherical-wave solutions which spoil the asymptotically AdS nature of the spacetime. Thus, in order to study the near-boundary behavior, it is enough to study the θ dependence of the metric function V by studying the square of the operator appearing in the θ equation (40) as acting on $V(u,\theta)$ as

$$\Big[\cos^2\theta\frac{\partial^2}{\partial\theta^2}+[(D-3)\cot\theta-3\sin\theta\cos\theta]\frac{\partial}{\partial\theta} - 2(D-2-\sin^2\theta)\Big]^2 V(u,\theta)=0. \tag{51}$$

Besides the homogeneous solutions (44), the particular solution of the equation

$$\Big[\cos^2\theta\frac{\partial^2}{\partial\theta^2}+[(D-3)\cot\theta-3\sin\theta\cos\theta]\frac{\partial}{\partial\theta} - 2(D-2-\sin^2\theta)\Big]V(u,\theta) = \frac{1}{\cos^2\theta}\Big[1+c_2(u)\Big(\cos\theta+\log\Big[\tan\Big(\frac{\theta}{2}\Big)\Big]\Big)\Big], \tag{52}$$

also provide a solution to (51). As the $1/x^2$ part of (48) gives rise to the Log mode which changes the boundary behavior in the AdS-wave case, one may expect that $1/\cos^2\theta$ part of the homogeneous solution (44), having the same near-boundary behavior, should give rise to the Log mode of the spherical-AdS wave. This expectation is confirmed by investigating the asymptotic behavior of the particular solution for the source with $c_2(u)=0$ which can be found as

$$V_p(u,\theta)=\frac{\log[\tan\theta]}{3\cos^2\theta}. \tag{53}$$

Again with the definition $x\equiv\pi/2-\theta$, the asymptotic form of (53) for small x becomes

$$V_p(u,\theta)\sim-\frac{1}{3x^2}\log x+O(1), \tag{54}$$

which is same as the exact form of the Log mode of the AdS wave. With the asymptotic behavior (54), the Log mode associated with the spherical-AdS wave changes the asymptotically AdS nature of the spacetime in the same way as the AdS wave.

Since the solutions we have found in this section are also solutions of the linearized field equations as we show below, these metrics constitute new explicit solutions for the Einstein and non-Einstein (Log mode) excitations of the critical gravity besides the previously studied AdS-wave solution [23,27].

IV. LINEARIZED FIELD EQUATIONS AS EXACT FIELD EQUATIONS

Once one recognizes the fact that the curvature tensors, (10) and (13), and the two tensors appearing in the field equations, (15)–(19), are linear in the metric function V for the Kerr-Schild-Kundt class of metrics defined as

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+2V\lambda_\mu\lambda_\nu,\qquad \lambda^\mu\partial_\mu V=0, \qquad \bar{\nabla}_\mu\lambda_\nu=\lambda_{(\mu}\xi_{\nu)},\qquad \lambda_\mu\xi^\mu=0, \tag{55}$$

one realizes that the exact field equations of the quadratic curvature gravity reduce to the linearized field equations in the metric perturbation $h_{\mu\nu}\equiv g_{\mu\nu}-\bar{g}_{\mu\nu}=2V\lambda_\mu\lambda_\nu$ for the KSK class (55). Even though this is straight forward to see, let us analyze this observation in a little more detail for the sake of completeness. First of all, for a generic metric perturbation $h_{\mu\nu}$, the linearized field equations corresponding to the field equations of the quadratic curvature gravity (21) has the form [38,39,41]

$$c\mathcal{G}^L_{\mu\nu}+(2\alpha+\beta)\Big(\bar{g}_{\mu\nu}\bar{\Box}-\bar{\nabla}_\mu\bar{\nabla}_\nu+\frac{2\Lambda}{D-2}\bar{g}_{\mu\nu}\Big)R^L +\beta\Big(\bar{\Box}\mathcal{G}^L_{\mu\nu}-\frac{2\Lambda}{D-1}\bar{g}_{\mu\nu}R^L\Big)=0, \tag{56}$$

where the parameter c is defined in (24), and $\mathcal{G}^L_{\mu\nu}$, R_L represent the linearized cosmological Einstein tensor and the linearized scalar curvature, respectively, which have the forms

$$\mathcal{G}^L_{\mu\nu}=R^L_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}R^L-\frac{2\Lambda}{D-2}h_{\mu\nu}, \tag{57}$$

$$R^L_{\mu\nu}=\frac{1}{2}(\bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\nu\sigma}+\bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\mu\sigma}-\bar{\Box}h_{\mu\nu}-\bar{\nabla}_\mu\bar{\nabla}_\nu h), \qquad R^L=-\bar{\Box}h+\bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\sigma\mu}-\frac{2\Lambda}{D-2}h. \tag{58}$$

Here, $R^L_{\mu\nu}$ is the linearized Ricci tensor, and Λ is the effective cosmological constant corresponding to the AdS background and satisfies the field equation (22).

After describing the linearized field equations and the linearized quantities for generic $h_{\mu\nu}$, let us focus on the KSK class where $h_{\mu\nu}=2V\lambda_\mu\lambda_\nu$ and after this point $h_{\mu\nu}$ represents the metric perturbation defined for the KSK class. First thing to notice is that $h_{\mu\nu}$ satisfies $h=0$ and $\nabla_\mu h^{\mu\nu}=0$; therefore, the nontrivial part of $h_{\mu\nu}$ is its transverse-traceless part which represents the (massive and/or massless) spin-2 excitations. For tranverse-traceless $h_{\mu\nu}$, the linearized field equations take the form

$$(\beta\bar{\Box}+c)\mathcal{G}^L_{\mu\nu}=0, \tag{59}$$

where

$$\mathcal{G}^L_{\mu\nu}=R^L_{\mu\nu}-\frac{2\Lambda}{D-2}h_{\mu\nu}=R^L_{\mu\nu}+k^2(D-1)h_{\mu\nu}. \tag{60}$$

Now, let us compare (59) with the quadratic curvature gravity field equation for the KSK class (23). From (13), one can find the linearized Ricci tensor for the KSK class as

$$R^L_{\mu\nu} = -\rho\lambda_\mu\lambda_\nu - k^2(D-1)h_{\mu\nu}, \tag{61}$$

therefore, $\mathcal{G}^L_{\mu\nu}$ is just $\mathcal{G}^L_{\mu\nu} = -\rho\lambda_\mu\lambda_\nu$. As a result, the field equations of the exact theory and the linearized field equations are equivalent for the KSK class of metrics which includes the AdS wave [23] and the spherical-AdS-wave metrics presented above. Note that not all solutions of (59) taken as a linear equation of *generic* perturbation $h_{\mu\nu}$ solve the full nonlinear theory. Such linear solutions were studied in [42,43].

V. FURTHER RESULTS AND CONCLUSIONS

We have defined a new subclass of metrics in the Kerr-Schild-Kundt class for which the null vector λ^μ has a symmetric covariant derivative, namely $\nabla_\mu\lambda_\nu = \lambda_{(\mu}\xi_{\nu)}$ (note that λ^μ is not a recurrent vector; therefore, our subclass does not have the special holonomy group $\mathrm{Sim}(n-2)$ discussed in [29]). Up to now two explicit metrics in this class as solutions to quadratic gravity theories has been shown to exist. One of them is the previously found AdS-wave metric [23], and the other one which we called spherical-AdS wave was presented above. The latter solution is a generalization of the $D = 3$ solution of new massive gravity given in [10,17]. Just like the AdS wave, the spherical-AdS wave has Log modes which do not asymptote to the AdS space [23,27]. As of now, it is not clear if these two metrics exhaust the class of Kerr-Schild-Kundt metrics having a null vector with a symmetric-covariant derivative or there are some other.

In this work, even though we have concentrated in the quadratic gravity theories both for the sake simplicity and for recent activity in quadratic gravity theories, the class of metrics that we have studied has rather remarkable properties which make them potential solutions to a large class of theories that are built on arbitrary contractions of the Riemann tensor whose Lagrangian is given as $f(g^{\mu\nu}, R_{\mu\nu\rho\sigma})$ along the lines of [11]. Leaving the details for another work [44], let us summarize the curvature properties of the Kerr-Schild-Kundt class having a null vector with a symmetric-covariant derivative:

(1) These metrics describe spacetimes with constant scalar invariants built form the contractions of the Riemann tensor, but not its covariant derivative, denoted as CSI_0 [28], for example $R = -D(D-1)k^2$, $R^\mu_\nu R^\nu_\mu = D(D-1)^2k^4$, $R_{\mu\alpha\beta\gamma}R^{\mu\alpha\beta\gamma} = 2D(D-1)k^4$.

(2) All symmetric second rank tensors built from the contractions of the Riemann tensor are linear in $\lambda_\mu\lambda_\nu$ for example see (17)–(19). This property implies property 1 above. This property is also sufficient to show that this class of metrics also solve the Lovelock theory [44].

(3) Related to property 2, these metrics linearize the field equations. For example,

$$\begin{aligned}\Box R_{\mu\nu} &= \bar{\Box} R_{\mu\nu} \\ &= -\lambda_\mu\lambda_\nu[\bar{\Box}\rho + 2\xi_\mu\partial^\mu\rho + \tfrac{1}{2}\rho\xi_\mu\xi^\mu \\ &\quad - 2\rho k^2(D-2)].\end{aligned} \tag{62}$$

We expect that similar properties hold for symmetric two tensors built from the covariant derivatives of the Riemann tensor, namely $[(\nabla^{(m)}_\gamma R_{\mu\nu\rho\sigma})^n]_{\alpha\beta} = a(k^2)g_{\alpha\beta} + b(\rho)\lambda_\alpha\lambda_\beta$, which is consistent with the boost weight decomposition of the Riemann tensor and its derivatives [45] This would lead to the result that these metrics could solve all geometric theories.

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APPENDIX A: DEFINITION OF ξ_ν

Let us discuss the symmetric-covariant derivative of the vector λ^μ, $\nabla_\mu\lambda_\nu = \lambda_{(\mu}\xi_{\nu)}$. Here, $\lambda_\mu\xi^\mu = 0$ should hold in order to have λ^μ as a null geodesic. Besides, note that $\nabla_\mu\lambda_\nu = \bar{\nabla}_\mu\lambda_\nu$ (see Appendix B). One can take the AdS background metric in the canonical form as

$$d\bar{s}^2 = \frac{1}{k^2z^2}\left[-dt^2 + \sum_{m=1}^{D-2}(dx^m)^2 + dz^2\right], \tag{A1}$$

where $z > 0$ and $z \to 0$ represents the AdS boundary. The Christoffel connection of (A1), which is in the form $\bar{g}_{\mu\nu} = \omega^{-2}\eta_{\mu\nu}$ where $\omega(z) = kz$, can be calculated with the usual conformal transformations as

$$\bar{\Gamma}^{\ \mu}_{\alpha\beta} = \frac{1}{z}\eta_{\alpha\beta}\delta^\mu_z - \frac{1}{z}(\delta^\mu_\alpha\delta^z_\beta + \delta^\mu_\beta\delta^z_\alpha). \tag{A2}$$

With this result, $\bar{\nabla}_\mu\lambda_\nu$ becomes

$$\bar{\nabla}_\mu\lambda_\nu = \partial_\mu\lambda_\nu - \frac{1}{z}\eta_{\mu\nu}\lambda_z + \frac{1}{z}(\lambda_\mu\delta^z_\nu + \lambda_\nu\delta^z_\mu). \tag{A3}$$

Note that the last term in the parenthesis is already in the form where $\lambda_{(\mu}\xi_{\nu)}$. Therefore, the first two terms should take a form

$$\partial_\mu\lambda_\nu - \frac{1}{z}\eta_{\mu\nu}\lambda_z = a\lambda_\mu\lambda_\nu + \lambda_\mu\zeta_\nu + \lambda_\nu\zeta_\mu. \tag{A4}$$

Now, let us define ξ_μ for the AdS-wave and the spherical-AdS wave metrics. For AdS-wave metric, λ_μ has the form

$$\lambda_\mu dx^\mu = \frac{1}{\sqrt{2}}(dt + dx), \tag{A5}$$

in the canonical coordinates of AdS, and one has

$$\bar{\nabla}_\mu \lambda_\nu = \frac{1}{z}(\lambda_\mu \delta^z_\nu + \lambda_\nu \delta^z_\mu) \Rightarrow \xi_\mu = \frac{2}{z}\delta^z_\mu. \tag{A6}$$

For the spherical-AdS wave, one has

$$\lambda_\mu dx^\mu = dt + \sum_{m=1}^{D-2} \frac{x^m}{r} dx^m + \frac{z}{r} dz,$$
$$r^2 = \sum_{m=1}^{D-2} (x^m)^2 + z^2, \tag{A7}$$

and $\bar{\nabla}_\mu \lambda_\nu$ becomes

$$\bar{\nabla}_\mu \lambda_\nu = -\frac{1}{r}\lambda_\mu \lambda_\nu + \frac{1}{r}\delta^t_\mu \lambda_\nu + \frac{1}{r}\delta^t_\nu \lambda_\mu + \frac{1}{z}(\lambda_\mu \delta^z_\nu + \lambda_\nu \delta^z_\mu) \tag{A8}$$

therefore,

$$\xi_\mu = -\frac{1}{r}\lambda_\mu + \frac{2}{r}\delta^t_\mu + \frac{2}{z}\delta^z_\mu. \tag{A9}$$

APPENDIX B: CURVATURE TENSORS OF THE KERR-SCHILD METRIC

In this section, we obtain the forms of the Riemann and Ricci tensors, and the scalar curvature for the Kerr-Schild metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V\lambda_\mu \lambda_\nu, \tag{B1}$$

where $\bar{g}_{\mu\nu}$ is the metric of the AdS spacetime, the vector λ^μ is null and geodesic for both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$;

$$\lambda_\mu \lambda^\mu = g_{\mu\nu}\lambda^\mu \lambda^\nu = \bar{g}_{\mu\nu}\lambda^\mu \lambda^\nu = 0, \tag{B2}$$

$$\lambda^\mu \nabla_\mu \lambda_\rho = \lambda^\mu \bar{\nabla}_\mu \lambda_\rho = 0, \tag{B3}$$

and, finally, V is a function of spacetime which is assumed to satisfy $\lambda^\mu \partial_\mu V = 0$.[3] The Christoffel connection of $g_{\mu\nu}$ has the form

$$\Gamma^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} + \Omega^\mu{}_{\alpha\beta}, \tag{B4}$$

where $\bar{\Gamma}^\mu_{\alpha\beta}$ is the Christoffel connection of the background metric $\bar{g}_{\mu\nu}$, and the terms linear in V collected in $\Omega^\mu_{\alpha\beta}$ which can be written as

$$\Omega^\mu{}_{\alpha\beta} = \bar{\nabla}_\alpha(V\lambda^\mu \lambda_\beta) + \bar{\nabla}_\beta(V\lambda^\mu \lambda_\alpha) - \bar{\nabla}^\mu(V\lambda_\alpha \lambda_\beta). \tag{B5}$$

One can easily show that $\Omega^\mu{}_{\alpha\beta}$ satisfies the properties

$$\Omega^\mu{}_{\mu\beta} = 0, \quad \lambda_\mu \Omega^\mu{}_{\alpha\beta} = 0, \quad \lambda^\alpha \Omega^\mu{}_{\alpha\beta} = 0, \tag{B6}$$

which have the important implication that the covariant derivative of λ^μ reduces to the covariant derivative with respect to the background AdS metric, namely

$$\nabla_\mu \lambda_\rho = \bar{\nabla}_\mu \lambda_\rho. \tag{B7}$$

With (B4), the Riemann tensor has the form

$$R^\mu{}_{\alpha\nu\beta} = \bar{R}^\mu{}_{\alpha\nu\beta} + \bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta \Omega^\mu{}_{\alpha\nu} + \Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha} - \Omega^\mu{}_{\beta\sigma}\Omega^\sigma{}_{\nu\alpha}, \tag{B8}$$

where $\bar{R}^\mu{}_{\alpha\nu\beta}$ is the Riemann tensor of the AdS spacetime having the form

$$\bar{R}^\mu{}_{\alpha\nu\beta} = -k^2(\delta^\mu_\nu \bar{g}_{\alpha\beta} - \delta^\mu_\beta \bar{g}_{\alpha\nu}). \tag{B9}$$

Contraction of the Riemann tensor with two λ^μ vectors has the simple form

$$\lambda_\mu \lambda^\nu R^\mu{}_{\alpha\nu\beta} = \lambda_\mu \lambda^\nu \bar{R}^\mu{}_{\alpha\nu\beta} = k^2 \lambda_\alpha \lambda_\beta,$$
$$\lambda^\alpha \lambda^\nu R^\mu{}_{\alpha\nu\beta} = \lambda^\alpha \lambda^\nu \bar{R}^\mu{}_{\alpha\nu\beta} = -k^2 \lambda^\mu \lambda_\beta. \tag{B10}$$

Using (B6), one can obtain the Ricci tensor from (B8) as

$$R_{\alpha\beta} = \bar{R}_{\alpha\beta} + \bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta} - \Omega^\mu{}_{\beta\sigma}\Omega^\sigma{}_{\mu\alpha}, \tag{B11}$$

where the Ricci tensor of the AdS spacetime is $\bar{R}_{\alpha\beta} = -k^2(D-1)\bar{g}_{\alpha\beta}$. The last term can be written in the form

$$\Omega^\mu{}_{\beta\sigma}\Omega^\sigma{}_{\mu\alpha} = -4V^2 \lambda_\alpha \lambda_\beta (\bar{\nabla}_{[\mu}\lambda_{\sigma]})\bar{\nabla}^\mu \lambda^\sigma, \tag{B12}$$

therefore the Ricci tensor with down indices is quadratic in V. However, it is well-known that the Ricci tensor with up-down indices, $R^\rho_\beta = g^{\rho\alpha}R_{\alpha\beta}$, is linear in V for a metric in the Kerr-Schild form [48];

$$R^\rho_\beta = \bar{R}^\rho_\beta - 2V\lambda^\rho \lambda^\alpha \bar{R}_{\alpha\beta} + \bar{g}^{\rho\alpha}\bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta}. \tag{B13}$$

Finally, the scalar curvature is a constant having a value which is equal to the background one;

$$R = \bar{R} = -D(D-1)k^2. \tag{B14}$$

1. Curvature tensors of the Kerr-Schild-Kundt class

Up to now, we consider the Kerr-Schild metrics for which λ^μ is a null geodesic as usual. On the other hand, the AdS-wave and spherical-AdS-wave metrics belong to the class of Kerr-Schild-Kundt metrics for which the vector λ^μ satisfies the property

[3]The exposition until Appendix B 2 is rather standard. Here, we provide self-contained presentation on curvature tensors of the Kerr-Schild metric (B1) satisfying $\lambda^\mu \partial_\mu V = 0$ in addition to the generally assumed properties (B2) and (B3). See [46,47] for Kerr-Schild metrics having the property (B2) with a flat background and [36] for Kerr-Schild satisfying (B2) and (B3) for generic backgrounds and generic V.

$$\nabla_\mu \lambda_\nu = \lambda_{(\mu}\xi_{\nu)}, \qquad \xi^\mu \lambda_\mu = 0. \tag{B15}$$

Note that due to $\xi^\mu \lambda_\mu = 0$, one has $\xi^\mu = g^{\mu\nu}\xi_\nu = \bar{g}^{\mu\nu}\xi_\nu$. The nonexpanding, $\nabla_\mu \lambda^\mu = 0$, shear-free, $\nabla^\mu \lambda^\nu \nabla_{(\mu}\lambda_{\nu)} = 0$, and nontwisting, $\nabla^\mu \lambda^\nu \nabla_{[\mu}\lambda_{\nu]} = 0$, nature of the vector λ^μ simply follows from (B15) which means the Kerr-Schild metric is a member of the Kundt class by definition. Immediate implications of (B15) are

$$\xi^\mu \nabla_\mu \lambda_\nu = \xi^\mu \nabla_\nu \lambda_\mu = \tfrac{1}{2}\lambda_\nu \xi^\mu \xi_\mu, \tag{B16}$$

and

$$\bar{\nabla}_\nu (\xi^\mu \lambda_\mu) = 0 \Rightarrow \lambda^\mu \bar{\nabla}_\nu \xi_\mu = -\xi^\mu \bar{\nabla}_\nu \lambda_\mu. \tag{B17}$$

Using the Ricci identity in the form $[\bar{\nabla}_\mu, \bar{\nabla}_\nu]\lambda^\mu = \bar{R}_{\nu\sigma}\lambda^\sigma$ together with $\bar{\nabla}_\mu \lambda^\mu = 0$, one can obtain

$$\bar{\Box}\lambda_\nu = -k^2(D-1)\lambda_\nu, \tag{B18}$$

and explicitly calculating the left-hand side yields the relation

$$\lambda^\mu \bar{\nabla}_\mu \xi_\nu = -\lambda_\nu[\bar{\nabla}_\mu \xi^\mu + \tfrac{1}{2}\xi^\mu \xi_\mu + 2k^2(D-1)] \tag{B19}$$

that is used in the calculations below.

In order to study the curvature tensors, first one should find the $\Omega^\mu{}_{\alpha\beta}$ part of the Christoffel connection which is linear in V, and it becomes

$$\Omega^\mu{}_{\alpha\beta} = -\lambda_\alpha \lambda_\beta \partial^\mu V + 2\lambda^\mu \lambda_{(\alpha}\partial_{\beta)}V + 2V\lambda^\mu \lambda_{(\alpha}\xi_{\beta)}. \tag{B20}$$

Note that contraction of the vector ξ^μ with $\Omega^\mu{}_{\alpha\beta}$ yields

$$\begin{aligned} \xi_\mu \Omega^\mu{}_{\alpha\beta} &= -\lambda_\alpha \lambda_\beta \xi_\mu \partial^\mu V, \\ \xi^\alpha \Omega^\mu{}_{\alpha\beta} &= \lambda^\mu \lambda_\beta (\xi^\alpha \partial_\alpha V + V \xi^\alpha \xi_\alpha), \end{aligned} \tag{B21}$$

so $\nabla_\mu \xi_\rho \neq \bar{\nabla}_\mu \xi_\rho$. Now, using (B20), we can calculate $\bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta}$ and $\Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha}$ for the KSK class. First, $\bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta}$ can be obtained as

$$\begin{aligned} \bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta} = &-\lambda_\alpha \lambda_\beta (\bar{\nabla}_\nu \partial^\mu V + \xi_\nu \partial^\mu V) - \lambda_\nu \lambda_{(\alpha}\xi_{\beta)}\partial^\mu V \\ &+ 2\lambda^\mu \lambda_{(\alpha}\bar{\nabla}_{\beta)}\partial_\nu V + 2\lambda^\mu \lambda_{(\alpha}\xi_{\beta)}\partial_\nu V \\ &+ 2V\lambda^\mu \lambda_{(\alpha|}\bar{\nabla}_\nu \xi_{|\beta)} + (2\lambda^\mu \xi_\nu + \xi^\mu \lambda_\nu) \\ &\times (V\lambda_{(\alpha}\xi_{\beta)} + \lambda_{(\alpha}\partial_{\beta)}V) \\ &+ \lambda^\mu \lambda_\nu (V\xi_\alpha \xi_\beta + \xi_{(\alpha}\partial_{\beta)}V). \end{aligned} \tag{B22}$$

Then, the linear in V terms in the Riemann tensor becomes

$$\begin{aligned} \bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta \Omega^\mu{}_{\alpha\nu} &= 2\lambda_\alpha \lambda_{[\nu}\bar{\nabla}_{\beta]}\partial^\mu V - 2\lambda^\mu \lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\alpha V \\ &+ \lambda_{[\nu}\xi_{\beta]}(\lambda_\alpha \partial^\mu V - \lambda^\mu \partial_\alpha V + \lambda_\alpha \xi^\mu V) \\ &+ (\lambda_\alpha \xi^\mu - \lambda^\mu \xi_\alpha)\lambda_{[\nu}\partial_{\beta]}V \\ &+ 2V\lambda^\mu (\lambda_\alpha \bar{\nabla}_{[\nu}\xi_{\beta]} - \lambda_{[\nu}\bar{\nabla}_{\beta]}\xi_\alpha). \end{aligned} \tag{B23}$$

Second, the term $\Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha}$ has the form

$$\Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha} = -\lambda^\mu \lambda_\alpha \lambda_\beta \lambda_\nu (\partial^\sigma V)(V\xi_\sigma + \partial_\sigma V). \tag{B24}$$

Note that $\Omega^\mu{}_{\nu\sigma}\Omega^\sigma{}_{\beta\alpha}$ is symmetric in ν and β indices; therefore, the quadratic in V terms in the Riemann tensor cancel each other due to antisymmetry in ν and β. Thus, the Riemann tensor for the KSK class is linear in V and has the form

$$R^\mu{}_{\alpha\nu\beta} = \bar{R}^\mu{}_{\alpha\nu\beta} + \bar{\nabla}_\nu \Omega^\mu{}_{\alpha\beta} - \bar{\nabla}_\beta \Omega^\mu{}_{\alpha\nu}, \tag{B25}$$

where the last two terms are given in (B23). Now, let us discuss the contractions of the Riemann tensor with one λ^μ vector. By using (B6), (B15), and (B21), one can show that

$$\begin{aligned} \lambda_\mu R^\mu{}_{\alpha\nu\beta} = \lambda_\mu \bar{R}^\mu{}_{\alpha\nu\beta}, \qquad \lambda^\alpha R^\mu{}_{\alpha\nu\beta} = \lambda^\alpha \bar{R}^\mu{}_{\alpha\nu\beta}, \\ \lambda^\nu R^\mu{}_{\alpha\nu\beta} = \lambda^\nu \bar{R}^\mu{}_{\alpha\nu\beta} - 2k^2 V \lambda^\mu \lambda_\alpha \lambda_\beta, \end{aligned} \tag{B26}$$

where the last one is implied by either one of the previous two results. After using (B9), one can also have

$$\lambda_\mu R^\mu{}_{\alpha\nu\beta} = \frac{R}{D(D-1)}(\lambda_\nu g_{\alpha\beta} - \lambda_\beta g_{\alpha\nu}), \tag{B27}$$

where the right-hand side can also be written in terms of background quantities, and the other two contractions follow similarly. On the other hand, one can calculate $\lambda^\nu R^\mu{}_{\alpha\nu\beta}$ explicitly by using (B6), (B15), (B22), (B21), and (B19), as

$$\begin{aligned} \lambda^\nu R^\mu{}_{\alpha\nu\beta} = \lambda^\nu \bar{R}^\mu{}_{\alpha\nu\beta} - 2V\lambda^\mu \lambda_\alpha \lambda_\beta (\bar{\nabla}_\nu \xi^\nu \\ + \tfrac{1}{4}\xi_\nu \xi^\nu + 2k^2(D-1)), \end{aligned} \tag{B28}$$

which together with (B26) implies

$$\bar{\nabla}_\nu \xi^\nu + \tfrac{1}{4}\xi_\nu \xi^\nu + k^2(2D-3) = 0. \tag{B29}$$

This relation can be verified explicitly for the AdS-wave and the spherical-AdS wave cases.

In order to calculate the Ricci tensor, one needs to calculate $\bar{\nabla}_\mu \Omega^\mu{}_{\alpha\beta}$. One may follow two routes: directly computing it from (B22) by using (B19) and (B29) or using the following result obtained by use of the Ricci identity;

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha (V\lambda^\mu \lambda_\beta) = -k^2 D V \lambda_\alpha \lambda_\beta, \tag{B30}$$

with the original form of the $\Omega^{\mu}{}_{\alpha\beta}$ in (B5). Then, one can obtain the Ricci tensor as

$$R_{\alpha\beta} = -k^2(D-1)g_{\alpha\beta} - \rho\lambda_\alpha\lambda_\beta, \tag{B31}$$

where

$$\rho \equiv \bar{\Box}V + 2\xi_\mu\partial^\mu V + \tfrac{1}{2}V\xi_\mu\xi^\mu - 2Vk^2(D-2), \tag{B32}$$

or

$$R_{\alpha\beta} = -k^2(D-1)g_{\alpha\beta} - (\bar{\Box} + 2k^2)(V\lambda_\alpha\lambda_\beta). \tag{B33}$$

Two forms of the Ricci tensor imply

$$\bar{\Box}(V\lambda_\alpha\lambda_\beta) = (\rho - 2Vk^2)\lambda_\alpha\lambda_\beta. \tag{B34}$$

It is possible to verify this relation by explicitly calculating the left-hand side by using (B18). Besides, one can easily show that the scalar curvature is constant, since the linear part of the Ricci tensor is in the form $R^L_{\alpha\beta} \sim \lambda_\alpha\lambda_\beta$.

Finally, let us show that the KSK metrics satisfy $C_{\mu\alpha\nu\beta}\lambda^\beta = 0$ where the Weyl tensor is defined as

$$C_{\mu\alpha\nu\beta} \equiv R_{\mu\alpha\nu\beta} - \frac{2}{D-2}(g_{\mu[\nu}R_{\beta]\alpha} - g_{\alpha[\nu}R_{\beta]\mu}) + \frac{2}{(D-1)(D-2)}Rg_{\mu[\nu}g_{\beta]\alpha}. \tag{B35}$$

Using (B26), $g_{\mu\nu} - \bar{g}_{\mu\nu} \sim \lambda_\mu\lambda_\nu$ and $R_{\mu\nu} - \bar{R}_{\mu\nu} \sim \lambda_\mu\lambda_\nu$, it can be shown that $C_{\mu\alpha\nu\beta}\lambda^\beta$ reduces to $\bar{C}_{\mu\alpha\nu\beta}\lambda^\beta$ where $\bar{C}_{\mu\alpha\nu\beta} = 0$; therefore, one has

$$C_{\mu\alpha\nu\beta}\lambda^\beta = \bar{C}_{\mu\alpha\nu\beta}\lambda^\beta = 0. \tag{B36}$$

2. Two tensors in the field equations

In order to find the field equations of the quadratic curvature gravity for the KSK metrics (B15), one needs to obtain the form of the two tensors $R^\rho_\mu R_{\rho\nu}$, $R_{\mu\alpha\nu\beta}R^{\alpha\beta}$, $R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}$ and $\Box R_{\mu\nu}$ for this class of metrics. By using (B31), the term $R^\rho_\mu R_{\rho\nu}$ can easily be calculated as

$$R^\rho_\mu R_{\rho\nu} = (D-1)^2k^4g_{\mu\nu} + 2(D-1)k^2\rho\lambda_\mu\lambda_\nu. \tag{B37}$$

The term $R_{\mu\alpha\nu\beta}R^{\alpha\beta}$ is also rather simple: after using (B31) and (B26), one has

$$R_{\mu\alpha\nu\beta}R^{\alpha\beta} = (D-1)^2k^4g_{\mu\nu} + (D-2)k^2\rho\lambda_\mu\lambda_\nu. \tag{B38}$$

Then, moving to $R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}$ whose calculation is straightforward, but time consuming. It is better to calculate $R^\mu{}_{\sigma\nu\beta}R_\gamma{}^{\sigma\nu\beta} = R^{\mu\sigma}{}_{\nu\beta}R^{\nu\beta}{}_{\gamma\sigma}$ which can be written as

$$R^{\mu\sigma}{}_{\nu\beta}R^{\nu\beta}{}_{\gamma\sigma} = R^{\nu\beta}{}_{\gamma\sigma}\bar{g}^{\sigma\alpha}R^\mu{}_{\alpha\nu\beta} - 2V\bar{R}^{\nu\beta}{}_{\gamma\sigma}\lambda^\sigma\lambda^\alpha R^\mu{}_{\alpha\nu\beta}, \tag{B29}$$

where (B26) is used and the first term explicitly has the form

$$\begin{aligned} R^{\nu\beta}{}_{\gamma\sigma}\bar{g}^{\sigma\alpha}R^\mu{}_{\alpha\nu\beta} = {} & R^{\nu\beta}{}_{\gamma\sigma}\bar{g}^{\sigma\alpha}\{\bar{R}^\mu{}_{\alpha\nu\beta} + 2\lambda_\alpha\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial^\mu V \\ & - 2\lambda^\mu\lambda_{[\nu}\bar{\nabla}_{\beta]}\partial_\alpha V + \lambda_{[\nu}\xi_{\beta]} \\ & \times(\lambda_\alpha\partial^\mu V - \lambda^\mu\partial_\alpha V + \xi^\mu\lambda_\alpha V) \\ & + (\lambda_\alpha\xi^\mu - \lambda^\mu\xi_\alpha)\lambda_{[\nu}\partial_{\beta]}V + 2V\lambda^\mu \\ & \times(\lambda_\alpha\bar{\nabla}_{[\nu}\xi_{\beta]} - \lambda_{[\nu}\bar{\nabla}_{\beta]}\xi_\alpha)\}. \end{aligned} \tag{B40}$$

Since the terms in $R^\mu{}_{\alpha\nu\beta}$ which are linear in V involve either λ_α or λ_ν or λ_β, using again (B26) yields

$$R^{\nu\beta}{}_{\gamma\sigma}\bar{g}^{\sigma\alpha}R^\mu{}_{\alpha\nu\beta} = \bar{R}^{\nu\beta}{}_{\gamma\sigma}\bar{g}^{\sigma\alpha}R^\mu{}_{\alpha\nu\beta} + (R^{\nu\beta}{}_{\gamma\sigma})_L\bar{R}^{\mu\sigma}{}_{\nu\beta}, \tag{B41}$$

where $(R^{\nu\beta}{}_{\gamma\sigma})_L \equiv R^{\nu\beta}{}_{\gamma\sigma} - \bar{R}^{\nu\beta}{}_{\gamma\sigma}$. With this result and (B9), $R^{\mu\sigma}{}_{\nu\beta}R^{\nu\beta}{}_{\gamma\sigma}$ becomes

$$\begin{aligned} R^{\mu\sigma}{}_{\nu\beta}R^{\nu\beta}{}_{\gamma\sigma} &= \bar{R}^{\nu\beta}{}_{\gamma\sigma}R^{\mu\sigma}{}_{\nu\beta} + (R^{\nu\beta}{}_{\gamma\sigma})_L\bar{R}^{\mu\sigma}{}_{\nu\beta} \\ &= -2k^2[R^\mu_\gamma + (R^\mu_\gamma)_L], \end{aligned} \tag{B42}$$

where $(R^\mu_\gamma)_L = -\rho\lambda^\mu\lambda_\gamma$ from (B31). As a result, one obtains

$$R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} = 2(D-1)k^4g_{\mu\nu} + 4k^2\rho\lambda_\mu\lambda_\nu. \tag{B43}$$

Finally, let us study the term $\Box R_{\mu\nu}$, and from (B31) it immediately becomes $\Box R_{\mu\nu} = -\Box(\rho\lambda_\mu\lambda_\nu)$. Then, since $\nabla_\mu\lambda_\rho = \bar{\nabla}_\mu\lambda_\rho$, $\Box\lambda^\mu = \bar{\Box}\lambda^\mu$ and $\Box\rho = \bar{\Box}\rho$, one has

$$\Box R_{\mu\nu} = -\bar{\Box}(\rho\lambda_\mu\lambda_\nu). \tag{B44}$$

In Appendix B 1, we have discussed the explicit calculation of $\bar{\Box}(V\lambda_\mu\lambda_\nu)$ which becomes

$$\begin{aligned} \bar{\Box}(V\lambda_\mu\lambda_\nu) = {} & \lambda_\mu\lambda_\nu(\bar{\Box}V + 2\xi_\sigma\partial^\sigma V + \tfrac{1}{2}V\xi_\sigma\xi^\sigma \\ & - 2Vk^2(D-1)), \end{aligned} \tag{B45}$$

and in deriving this relation $\lambda^\mu\partial_\mu V = 0$ is used. One can show that $\lambda^\mu\partial_\mu\rho = 0$ (note that $\lambda^\sigma\bar{\nabla}_\sigma\xi_\mu \sim \lambda_\mu$), then the same relation also holds for ρ. Hence, one has

$$\begin{aligned} \Box R_{\mu\nu} = {} & -\lambda_\mu\lambda_\nu(\bar{\Box}\rho + 2\xi_\sigma\partial^\sigma\rho + \tfrac{1}{2}\rho\xi_\sigma\xi^\sigma \\ & - 2\rho k^2(D-1)). \end{aligned} \tag{B46}$$

APPENDIX C: SPHERICAL-ADS WAVE COMPUTATIONS

Let us have the AdS metric in the coordinates

$$d\bar{s}^2 = \frac{4dud\upsilon}{k^2\cos^2\theta(u+\upsilon)^2} + \frac{1}{k^2\cos^2\theta}d\Omega^2_{D-2}. \tag{C1}$$

Then, some components of the Christoffel connection for this metric are

$$\bar{\Gamma}^u_{uu} = -\frac{2}{u+v}, \quad \bar{\Gamma}^u_{uv} = 0, \quad \bar{\Gamma}^u_{u\theta} = \tan\theta,$$
$$\bar{\Gamma}^u_{\theta\theta} = 0 \quad \bar{\Gamma}^v_{uv} = 0, \quad \bar{\Gamma}^v_{\theta\theta} = 0,$$
$$\bar{\Gamma}^\theta_{uv} = -\frac{2\tan\theta}{(u+v)^2}, \quad \bar{\Gamma}^\theta_{\theta\theta} = \tan\theta,$$
$$\bar{\Gamma}^\theta_{\Omega_i\Omega_i} = -k^2\bar{g}_{\Omega_i\Omega_i}\cot\theta, \quad \bar{\Gamma}^u_{\Omega_i\Omega_i} = 0, \quad \bar{\Gamma}^v_{\Omega_i\Omega_i} = 0. \tag{C2}$$

where Ω_i denotes the angular coordinates of $d\Omega^2_{D-2}$ other than θ. Now, let us first discuss the form of $\bar{\nabla}_\mu\lambda_\nu$;

$$\bar{\nabla}_\mu\lambda_\nu = -\bar{\Gamma}^u_{\mu\nu} = -\lambda_\mu\lambda_\nu\bar{\Gamma}^u_{uu} - \lambda_\mu\bar{\Gamma}^u_{u\theta}\delta^\theta_\nu - \delta^\theta_\mu\bar{\Gamma}^u_{\theta u}\lambda_\nu \tag{C3}$$

and one has

$$\xi_\nu \equiv -\bar{\Gamma}^u_{uu}\lambda_\nu - 2\bar{\Gamma}^u_{u\theta}\delta^\theta_\nu. \tag{C4}$$

Finally, one can calculate ρ as

$$\rho = \bar{\Box}V - 4k^2\sin\theta\cos\theta\partial_\theta V - 2k^2(D-2-\sin^2\theta)V. \tag{C5}$$

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