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Entropy increase during physical processes for black holes in Lanczos-Lovelock gravity

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We study quasistationary physical process for black holes within the context of Lanczos-Lovelock gravity. We show that the Wald entropy of the stationary black holes in Lanczos-Lovelock gravity monotonically increases for quasistationary physical processes in which the horizon is perturbed by the accretion of positive energy matter and the black hole ultimately settles down to a stationary state. This result reinforces the physical interpretation of Wald entropy for Lanczos-Lovelock models and takes a step towards proving the analogue of the black hole area increase theorem in a wider class of gravitational theories.

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Pioneering work by Bekenstein [1], Hawking [2], Davies [3], and Unruh [4] in the 1970s showed that there is a consistent manner in which one can associate thermodynamical variables with horizons in general relativity (GR). This association gave substance to the formal connection between the laws of black hole dynamics and thermodynamics.

A natural question is whether this analogy is a peculiar property of GR or a robust feature of any generally covariant theory of gravity. Pursuing this line of thought, Wald and collaborators [5,6] established the equilibrium state version of first law for black holes for any arbitrary diffeomorphism invariant theory of gravity. Comparing the form of a differential identity with the first law of thermodynamics, the entropy of the black hole was expressed as an integral over a space-like cross section of the horizon of a local geometric quantity and is identified with the Noether charge of Killing isometry that generates the horizon.

The standard results in the case of general relativity concerning the entropy of horizons rely, in one way or another, on the fact that the entropy is proportional to the horizon area. This proportionality does not hold for the Wald entropy in more general theories, and therefore it is quite intriguing that many of the results connecting gravitational dynamics to horizon thermodynamics still allow a natural generalization to a more general class of models.

Recent work suggests that this connection may indicate a far deeper truth regarding the nature of gravity viz. that it could be an emergent phenomena like, for example, fluid mechanics [7,8]. Studies show that this correspondence, between gravitational dynamics and horizon thermodynamics, transcends general relativity and holds true for a much wider class of theories called the Lanczos-Lovelock models of gravity [9]. These are the only natural generalization of Einstein's theory to higher dimension if we insist that the equations of motion should not be of degree higher than 2. The Lanczos-Lovelock gravity is also free from perturbative ghosts [10] and admits consistent initial value formulation. As a result, Lanczos-Lovelock theories can be thought of as a natural extension of general relativity in higher dimensions. On the other hand, while Lanczos-Lovelock models show remarkable structural similarity with Einstein's theory, the form of the horizon entropy in the Lanczos-Lovelock models is quite complicated and in general entropy is not proportional to any simple geometric quantity.

Implicit in the investigations which use the Wald entropy in these theories is the assumption that the entropy associated with a horizon behaves like ordinary thermodynamic entropy. But, the equilibrium state version of first law for black holes, established by Wald and collaborators [5,6], requires the existence of a stationary black hole with regular bifurcation surface. As a result, from the equilibrium state version of first law, it is not immediately clear whether the Wald entropy always increases under physical processes, except for black holes in GR, in which the "area theorem" asserts that area of a black hole cannot decrease in any process provided null energy condition holds for the matter fields [11]. The area theorem, in turn, follows from the Raychaudhuri equation and crucially depends on the contracted Einstein's equation $R_{ab}k^ak^b = 8\pi T_{ab}k^ak^b$ where k^a is the tangent to the horizon. Since the entropy of black holes is no longer proportional to area in Lanczos-Lovelock models of gravity, there is no obvious assurance that the entropy still obeys an increase theorem. As a result, the question of validity of the second law of black hole thermodynamics for arbitrary theory of gravity remains an unresolved issue. Except for the case of $f(R)$ gravity [12], there is no proof of the analog of Hawking's area theorem beyond GR. In the quasistationary case, an argument for

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second law valid for all diffeomorphism invariant gravity theories was given in [12], but it is based on the assumption that the stationary comparison version of the first law implies the physical process version for quasistationary processes.

For the thermodynamic interpretation to be valid, we would expect horizon entropy to increase when a black hole in the Lanczos-Lovelock model participates in some physical process, like, e.g., accretion of matter. Recently, a direct proof of the physical process version of first law is proposed for Einstein-Gauss-Bonnet (EGB) gravity [13], which establishes that the net change of black hole entropy during a physical process is positive as long as matter satisfies null energy condition.

In this paper, we investigate this question for general Lanczos-Lovelock models and show that during a physical process, the Wald entropy of stationary black holes in general Lanczos-Lovelock gravity monotonically increases, provided the matter stress energy tensor obeys null energy condition. As a result, not only is the net change of the entropy is positive, but the entropy is increasing at every cross section of the horizon. Therefore, our result provides a crucial step towards (possibly) proving an analogue of the area theorem in Lanczos-Lovelock models.

Let us start with a brief review of the properties of stationary, nonextremal, Killing horizons. (We adopt the metric signature $(-,+,+,+,\ldots)$ and our sign conventions are the same as those of [14].) In a D-dimensional spacetime, the event horizon is a null hypersurface $\mathcal{H}$ parametrized by an affine parameter λ. The vector field $k^a=(\partial_\lambda)^a$ is tangent to the horizon and obeys geodesic equation. All $\lambda=$ constant slices are space like and foliate the horizon. Any point p on such slices have coordinates $\{\lambda, x^A\}$ where x^A, $(A=2,\cdots,D)$ are the coordinates of a point on $\lambda=0$ slice connected with p by a horizon generator. We can construct a basis with the vector fields, $\{k^a, l^a, e^a_A\}$ where l^a is a second null vector such that $l^a k_a=-1$. The induced metric on any slice is $\gamma_{ab}=g_{ab}+2k_{(a}l_{b)}$ and $k^a\gamma_{ab}=0=l^a\gamma_{ab}$. The change of the induced metric from one slice to another can be obtained from the metric evolution equation [14],

$$\mathcal{L}_k\gamma_{ab}=2\left(\sigma_{ab}+\frac{\theta}{(D-2)}\gamma_{ab}\right), \tag{1}$$

where σ_{ab} is the shear and θ is the expansion of the horizon. If the event horizon is also a Killing horizon [15], i.e., the horizon generators are the orbits of a Killing field $\xi^a=(\partial/\partial v)^a$, which is null on the horizon, then the surface gravity κ of the horizon is defined as $\xi^a\nabla_a\xi^b=\kappa\xi^b$. For stationary spacetimes with a Killing horizon, both the expansion and shear vanish, and using the Raychaudhuri equation and the evolution equation for shear, we obtain [14,17],

$$R_{ab}k^ak^b=\xi^a\gamma^b_i\gamma^c_j\gamma^d_kR_{abcd}=k^ak^c\gamma^b_m\gamma^d_nR_{abcd}=0. \tag{2}$$

Note that, in order to derive these relationships, we have only used the fact that the horizon is stationary Killing horizon with zero expansion and shear without any further symmetry.

We would like to consider the situation when a stationary black hole is perturbed by a weak matter stress energy tensor and ultimately settle down to a stationary state in the asymptotic future. Since the black hole is stationary in the asymptotic future, the vector field ξ^a is an exact Killing vector at late times. The accretion process is assumed to be slow such that all changes of the dynamical fields are first order in some suitable bookkeeping parameter ϵ and that we can neglect all viscous effects. More specifically, we assume that, $\theta\sim\sigma_{ab}\sim\mathcal{O}(\epsilon)$.

In GR, a concrete example of such a physical process is a black hole of mass M slowly accreting matter for a finite time and ultimately settle down to a stationary state. Then a linearized version of the Raychaudhuri equation gives,

$$\frac{d\theta}{d\lambda}\approx -R_{ab}k^ak^b=-8\pi T_{ab}k^ak^b, \tag{3}$$

where we have used Einstein's equation to get the second equality. If the matter stress tensor satisfies null energy condition, i.e., $T_{ab}k^ak^b\geq 0$, the rate of change of the expansion is negative on any slice prior to the asymptotic future. Since the expansion vanishes in the future, the generators must have positive expansion during the accretion process. As a result, the area is monotonically increasing in the physical process. Note that, the result is crucially dependent on the field equation. As a result, the monotonicity of the horizon area is only valid in case of GR. Our aim is to prove a same statement for the Wald entropy during a dynamical change of the black holes in Lanczos-Lovelock gravity.

We shall now turn our attention to the features of Lanczos-Lovelock gravity. As discussed before, a natural generalization of the Einstein-Hilbert Lagrangian is provided by the Lanczos-Lovelock Lagrangian, which is the sum of dimensionally extended Euler densities,

$$\mathcal{L}^D=\sum_{m=0}^{[(D-1)/2]}\alpha_m\mathcal{L}^D_m, \tag{4}$$

where the α_m are arbitrary constants and $\mathcal{L}^D_m$ is the mth order Lanczos-Lovelock term given by,

$$\mathcal{L}^D_m=\frac{1}{16\pi}\sum_{m=0}^{[(D-1)/2]}\frac{1}{2^m}\delta^{a_1b_1\ldots a_mb_m}_{c_1d_1\ldots c_md_m}R^{c_1d_1}_{a_1b_1}\cdots R^{c_md_m}_{a_mb_m}, \tag{5}$$

where $R^{cd}{}_{ab}$ is the D-dimensional curvature tensor and the generalized alternating tensor $\delta^{\cdots}_{\cdots}$ is totally antisymmetric in both sets of indices. The Einstein-Hilbert Lagrangian is a special case of Eq. (5) when $m=1$. The field equation of the Lanczos-Lovelock theory is, $G_{ab}/(16\pi)+\alpha_mE_{(m)ab}=(1/2)T_{ab}$ where,

$$E^{i}_{(m)j} = -\frac{1}{16\pi}\frac{1}{2^{m+1}}\delta^{ia_1b_1\ldots a_mb_m}_{jc_1d_1\ldots c_md_m}R^{c_1d_1}{}_{a_1b_1}\cdots R^{c_md_m}{}_{a_mb_m}, \quad (6)$$

and $m \geq 2$. For convenience, we have written the GR part (i.e., for $m = 1$) separately so that the GR limit can be easily verified by setting all α_m's to zero.

Spherically symmetric black hole solutions in Lanczos-Lovelock gravity were derived in [18,19] and the Wald entropy associated with a stationary Killing horizon is [20–22],

$$S = \frac{1}{4}\int \rho\sqrt{\gamma}dA, \quad (7)$$

where the entropy density

$$\rho = \left(1 + \sum_{m=2}^{[(D-1)/2]} 16\pi m\alpha_m {}^{(D-2)}L_{(m-1)}\right). \quad (8)$$

The integration is over $(D-2)$-dimensional space-like cross section of the horizon and ${}^{(D-2)}L_{(m-1)}$ is the intrinsic $(m-1)$th Lanczos-Lovelock scalar of the horizon cross section. We would like to prove that this entropy always increases when a black hole is perturbed by a weak matter stress energy tensor of $\mathcal{O}(\epsilon)$ provided the matter obeys null energy condition.

The change in entropy is [12],

$$\Delta S = \frac{1}{4}\int_{\mathcal{H}}\left(\frac{d\rho}{d\lambda} + \theta\rho\right)d\lambda\sqrt{\gamma}dA. \quad (9)$$

We define a quantity Θ as,

$$\Theta = \left(\frac{d\rho}{d\lambda} + \theta\rho\right). \quad (10)$$

In case of GR, Θ is equal to the expansion parameter of the null generators. But, in case of a general gravity theory, Θ is the rate of change of the entropy associated with a infinitesimal portion of horizon (see Ref. [12] for similar construction in $f(R)$ gravity). We would like to prove that given null energy condition holds, Θ is positive on any slice in a physical process. To proceed further, we note that the change of the $(D-2)$-dimensional scalar ${}^{(D-2)}L_{(m-1)}$ can be thought of due to the change in the intrinsic metric. Then, we can calculate this change by using the standard result of variation of Lanczos-Lovelock scalar. The variation of ${}^{(D-2)}L_{(m-1)}$ simply gives the equations of motion of $(m-1)$th order Lanczos-Lovelock term in $(D-2)$ dimensions. Therefore, for a general Lanczos-Lovelock gravity, we can write

$$\begin{aligned}\frac{d\rho}{d\lambda} &= \sum_{m=2}^{[(D-1)/2]} 16\pi m\alpha_m k^a\nabla_a({}^{(D-2)}L_{(m-1)}) \\ &= -\sum_{m=2}^{[(D-1)/2]} 16\pi m\alpha_m {}^{(D-2)}\mathcal{R}^{ab}_{(m-1)}\mathcal{L}_k\gamma_{ab},\end{aligned} \quad (11)$$

where we have ignored a surface term which does not contribute because the sections of the horizon are compact surfaces without boundaries. ${}^{(D-2)}\mathcal{R}_{ab}$ is the generalization of Ricci tensor for $(m-1)$-Lanczos-Lovelock gravity and is given by [23],

$$\begin{aligned}{}^{(D-2)}\mathcal{R}^{a}_{b(m-1)} = \frac{1}{16\pi}\frac{(m-1)}{2^m} \\ \times \delta^{a_1b_1\ldots a_mb_m}_{bc_1d_1\ldots c_md_m}{}^{(D-2)}R^{ad_1}_{a_1b_1}\cdots{}^{(D-2)}R^{c_md_m}_{a_mb_m}.\end{aligned} \quad (12)$$

Then using Eq. (1), we obtain,

$$\begin{aligned}\Theta = \theta + 16\pi\sum_{m=2}^{[D-1)/2]}\alpha_m\Bigg[-2\Bigg(\frac{{}^{(D-2)}\mathcal{R}_{(m-1)}\theta}{(D-2)} \\ + {}^{(D-2)}\mathcal{R}^{ab}_{(m-1)}\sigma_{ab}\Bigg) + \theta^{(D-2)}L_{(m-1)}\Bigg].\end{aligned} \quad (13)$$

We would like to study the rate of change of Θ along the congruence using the Raychaudhuri equation and the evolution equation of shear [14]. We are only interested in quantities first order in perturbation over a background stationary spacetime. Therefore, when we encounter a product of two quantities X and Y, to extract the part linear in perturbation, we will always express such a product as,

$$XY \approx X^{(B)}Y^{(P)} + X^{(P)}Y^{(B)}, \quad (14)$$

where $X^{(B)}$ is the value of the quantity X evaluated on the stationary background, and $X^{(P)}$ is the perturbed value of X linear in perturbation. Note that, on the stationary background, the Raychaudhuri equation demands $R^{(B)}_{ab}k^ak^b = 0$ and since $T^{(B)}_{ab}k^ak^b = 0$, we have $E^{(B)}_{(m)ab}k^ak^b = 0$. Also, to simplify the calculation, we use diffeomorphism freedom to make the null geodesic generators of the event horizon of the perturbed black hole coincide with the null geodesic generators of the background stationary black hole [24].

Using the perturbation scheme mentioned above and the evolution equation of θ and σ_{ab} to linear order as $d\theta/d\lambda \approx -R^{(P)}_{ab}k^ak^b$ and $d\sigma_{ab}/d\lambda \approx C^{(P)}_{acdb}k^ck^d$ and further using the relationships in Eq. (2) for the background, the evolution equation of Θ to linear order in perturbation can be written as

$$\frac{d\Theta}{d\lambda} = -8\pi T_{ab}k^ak^b + \mathcal{D}_{ab}k^ak^b, \quad (15)$$

in which we have defined

$$\begin{aligned}\mathcal{D}_{ab}k^ak^b = \sum_{m=2}^{[(D-1)/2]} 16\pi\alpha_m[E^{(P)}_{(m)ab}k^ak^b \\ + 2m^{(D-2)}E^{(B)ab}_{(m-1)}R^{(P)}_{acbd}k^ck^d].\end{aligned} \quad (16)$$

Here, we have used expression of the perturbed Weyl tensor in terms of curvature and Ricci tensors and the relation ${}^{(D-2)}E_{(m)ab} = {}^{(D-2)}\mathcal{R}_{(m)ab} - (1/2)\gamma_{ab}{}^{(D-2)}L_{(m)}$. We will next prove that the first order part of $\mathcal{D}_{ab}k^a k^b$ vanishes identically. To show this, let us start with the first term in Eq. (16) and write its first order perturbed part as

$$E^{(P)}_{(m)ab}k^a k^b = -\frac{m}{16\pi}\frac{1}{2^{m+1}} \times \delta^{ia_1b_1\ldots a_{m-1}b_{m-1}ab}_{jc_1d_1\ldots c_{m-1}d_{m-1}cd} R^{(B)c_1d_1}_{a_1b_1}\cdots R^{(P)cd}_{ab}k^j k_i. \quad (17)$$

Then, we first expand the background curvature tensors in the basis $\{k^a, N^a, \gamma^a_b\}$ on the horizon and use Eq. (2). We also use the fact that due to the antisymmetry of the generalized alternating tensor $\delta^{\cdots}_{\cdots}$, any component of a curvature tensor along the direction of the generator of the horizon in the expression of $E^{(P)}_{(m)ab}k^a k^b$ will not contribute. These constrains ensure that the only surviving contribution will be from the transverse components and we will finally obtain,

$$E^{(P)}_{(m)ab}k^a k^b = -\frac{m}{16\pi}\frac{1}{2^{m+1}} \times \delta^{iA_1B_1\ldots ab}_{jC_1D_1\ldots cd}\,{}^{(D-2)}R^{(B)C_1D_1}_{A_1B_1}\cdots R^{(P)cd}_{ab}k^j k_i, \quad (18)$$

where we have the fact that for stationary spacetimes [25]

$$\gamma^m_a\gamma^n_b\gamma^c_p\gamma^d_q R^{(B)pq}_{mn} \equiv^{\mathcal{H}}\ {}^{(D-2)}R^{(B)cd}_{ab}, \quad (19)$$

which holds for any space-like cross section of the stationary horizon. Next, we use the technique in [23] to write the alternating tensor in a factorized form as,

$$\delta^{iA_1B_1\ldots A_{m-1}B_{m-1}ab}_{jC_1D_1\ldots C_{m-1}D_{m-1}cd}R^{(P)cd}_{ab}k^j k_i = -4\delta^i_c\delta^a_j\delta^{A_1B_1\ldots A_{m-1}B_{m-1}b}_{C_1D_1\ldots C_{m-1}D_{m-1}d}R^{(P)cd}_{ab}k^j k_i. \quad (20)$$

Using this, we finally get,

$$E^{(P)}_{(m)ab}k^a k^b = -2m\,{}^{(D-2)}E^{(B)ab}_{(m-1)}R^{(P)}_{acbd}k^c k^d. \quad (21)$$

Equation (21) immediately shows that the first order part of $\mathcal{D}_{ab}k^a k^b$ vanishes identically and we finally arrive at,

$$\frac{d\Theta}{d\lambda} = -8\pi T_{ab}k^a k^b + \mathcal{O}(\epsilon^2). \quad (22)$$

Equation (22) shows that if the null energy condition holds, the rate of change of Θ is always negative during a slow classical dynamical process (i.e., ignoring the terms which are higher order in the perturbation), which perturbs the black hole and leads to a new stationary state. Since the final state is assumed to be stationary, both θ and σ and as a consequence, Θ vanishes in the asymptotic future. Hence, we can use the same argument as with the expansion parameter in case of GR to conclude that Θ must be positive at every slice during the physical process. As a result, we conclude that the horizon entropy of black holes in Lanczos-Lovelock gravity is a monotonically increasing function during any quasistationary physical process, i.e.,

$$\frac{dS}{d\lambda} \geq 0, \quad (23)$$

which is what we set out to prove.

In case of a dynamical scenario, it is possible to write down several candidates for the black hole entropy beyond GR [26], such that all the expressions have the same stationary limit. We have actually chosen a particular expression and the validity of Eq. (23) favors such a choice. In fact, in Ref. [6], a local and geometrical prescription for the entropy of dynamical black holes is proposed. This proposal is based on a boost invariant construction and agrees with the Wald's Noether charge formula for stationary black holes and their perturbations. Interestingly, for Lanczos-Lovelock gravity, the entropy expression used in this work matches with the expression obtained from the boost invariant construction. Consequently, our result provides a strong justification in favor of the prescription for dynamical entropy as proposed in Ref. [6]. This may also be important to decide the right candidate for the entropy of nonstationary black holes for non Lanczos-Lovelock gravity models.

From a quantum gravitational point of view, a natural interpretation of the black hole entropy is that it counts the microstates of the black hole. For any reasonable theory of gravity, which has stable black holes, the density of states must be of the form $\exp(S_W)$, where S_W is the corresponding Wald entropy. For example, in the context of string theory, it has been shown at least for extremal and near-extremal black holes, that the microscopic computations exactly match with the Wald formula [27,28]. Hence, it is quite desirable that the Wald entropy satisfies an increase theorem.

Some obvious further investigations suggested by this work are the following: first, one would like to relax the quasistationarity physical process assumption and calculate the full change of the Wald entropy along the horizon to understand the validity of classical second law for Lanczos-Lovelock models. The possible conclusions crucially depend on the signs of the higher order terms in Eq. (22). As in case of GR, if all the higher order terms are negative, this would imply that Θ has to decrease monotonically. Further, it cannot be negative on any cross section of the horizon, which otherwise will lead to $\Theta \to -\infty$ and hence the existence of a caustic on the event horizon which is prohibited by the fact that the event horizon is future complete [29]. Then, to avoid the contradiction, Θ must be positive on any arbitrary slice of the horizon which would lead to classical second law for Lanczos-Lovelock gravity.

The second issue worth exploring is the following: a crucial assumption in our derivation is that there exists a quasistationary physical process in which the black hole ultimately settles down to a final stationary state. Although such an assumption is quite reasonable, one should not overlook an extreme possibility that the black holes in a general Lanczos-Lovelock gravity may not be stable under a small perturbation. While the linear stability around flat spacetime of the Einstein-Gauss-Bonnet gravity is demonstrated in [30], the positive energy theorem has not been extended to a general Lovelock theory and even if such an extension is possible, there may be other instabilities. This requires further investigation.

Finally, we would like to note that the techniques used in this work are specific to Lanczos-Lovelock gravity. As a result, it would be worthwhile to find a general approach which can answer whether classical second law holds in a physical process for *any* diffeomorphism invariant gravity theory or applies to a special class of action functionals.

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