Alternate $1/N_c$ expansions and SU(3) breaking from baryon lattice results

Aleksey Cherman[*]

Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge CB3 0WB, United Kingdom

Thomas D. Cohen[†]

Maryland Center for Fundamental Physics, Department of Physics, University of Maryland,
College Park, Maryland 20742-4111, USA

Richard F. Lebed[‡]

Department of Physics, Arizona State University, Tempe, Arizona 85287-1504, USA
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A combined expansion in the number of QCD colors $1/N_c$ and SU(3) flavor-breaking parameter ϵ has long been known to provide an excellent accounting for the mass spectrum of the lightest spin-$\frac{1}{2}$, $\frac{3}{2}$ baryons when the quarks are taken to transform under the fundamental $SU(N_c)$ representation, and in the final step $N_c \rightarrow 3$ and ϵ is set to its physical value ~ 0.3. Subsequent work shows that placing quarks in the two-index antisymmetric $SU(N_c)$ representation leads to quantitatively equally successful mass relations. Recent lattice simulations allow for varying the value of ϵ and confirm the robustness of the original $1/N_c$ relations. In this paper we show that the same conclusion holds for the antisymmetric quarks and demonstrate that the mass relations also hold under alternate prescriptions for identifying physical baryons with particular members of the large N_c multiplets.

I. INTRODUCTION

QCD with three colors and realistic quark masses has no expansion parameters to allow perturbative calculations of low-energy observables from first principles. Soon after the discovery of QCD, however, 't Hooft observed [1] that non-Abelian gauge theories such as QCD simplify in the limit that the number of colors N_c tends to infinity, suggesting that one could try to compute observables in an expansion in the small parameter $1/N_c$. The notion of large N limits has been extraordinarily fruitful for studies of the formal aspects of gauge theories, and many of the qualitative predictions of the $1/N_c$ expansion are reasonable when compared to experimental results.

Unfortunately, because of the lack of a solution of large N_c QCD in four dimensions, only a limited number of quantitative predictions has been obtained from the $1/N_c$ expansion. Despite some initial skepticism [2] about the phenomenological relevance of the properties of baryons at large N_c [3], these successes have turned out to involve primarily large N_c baryons rather than mesons. The most powerful tool for extracting such predictions has been the contracted $SU(2N_f)$ spin-flavor symmetry that emerges at large N_c for baryons [4–9]. When combined with the SU(3) flavor-breaking expansion, the $SU(2N_f)$ spin-flavor symmetry was shown to imply relations between baryon masses, even including isospin splittings [10,11], and these relations show a good fit to the experimental baryon mass

[*]a.cherman@damtp.cam.ac.uk
[†]cohen@physics.umd.edu
[‡]Richard.Lebed@asu.edu

spectra. Indeed, these relations are among the strongest pieces of evidence that the large N_c expansion is necessary to understand real-world baryon spectroscopy, since the combined $1/N_c$ and flavor-breaking expansion predictions fit the data much better than the predictions from flavor breaking alone. The robustness of baryon mass relations has recently been confirmed by showing they continue to hold to predicted uncertainties when tested on baryon mass spectra obtained from lattice simulations as the size of the SU(3) flavor breaking is varied [12].

However, recent theoretical developments have revealed that the predictions resulting from the $SU(2N_f)$ symmetry are not quite unique. The reason is the existence of more than one phenomenologically viable way to take a large N_c limit starting from $N_c = 3$. The limit advocated in the seminal papers of 't Hooft [1] and Witten [3] assumed that quarks are in the fundamental (F) representation for all $N_c \geq 3$, while keeping the number of flavors N_f fixed. We refer to this as the *large N_c^{F} limit*. However, one can obtain a different large N_c limit by taking the N_f quarks to be in the two-index antisymmetric (AS) representation of $SU(N_c)$ for $N_c \geq 3$ [13–16]; we call this the *large N_c^{AS} limit*. The reason for two possible extrapolations to general N_c from $N_c = 3$ is that the AS and F representations are isomorphic for $N_c = 3$.

The large N_c^{AS} limit also implies the emergence of an $SU(2N_f)$ spin-flavor symmetry, and hence it also results in baryon mass relation predictions, but with different $1/N_c$ suppression factors than the large N_c^{F} limit. Curiously, it turns out that the experimental data appear to be consistent with both large N_c limits to within the experimental and

theoretical errors [17]. Apparently, *some* $1/N_c$ expansion is necessary to fit the data, but it is not possible to decide which large N_c limit provides a better guide for baryon mass spectra. The $SU(2N_f)$ symmetry also makes predictions for magnetic moments of baryons, and here the large N_c^{F} expansion is a better guide to the data than the large N_c^{AS} expansion [18].

An additional complication leads to possible ambiguities in large N_c predictions for baryon properties. At large N_c, color antisymmetrization allows baryons to have many more than 3 quarks, and consequently many more baryon species appear than at $N_c = 3$; one needs a prescription to match the baryons that can exist at $N_c = 3$ to the baryons seen at large N_c. With more than two flavors, no unique prescription exists, and one finds at least two natural extrapolations of baryons with valence strange quarks at $N_c = 3$ to large N_c. With one choice one obtains large N_c baryons with $\sim N_c^1$ strange quarks, while with the other one obtains only $\sim N_c^0$ strange quarks. The baryon properties are different depending upon the prescription one adopts, and this is a serious phenomenological challenge in comparing to data.

In this paper, we address both of these subtleties in the context of the large N_c baryon mass relations. First, we use the lattice data employed by Jenkins *et al.* [12] to compare the predictions of the large N_c^{F} and large N_c^{AS} limits as a function of the strength of $SU(3)$ breaking. We find that they both remain consistent with the data. Second, we discuss the ambiguities in matching the $N_c = 3$ and large N_c flavor representations of baryons. We show that remarkably, while individual baryon masses are sensitive to these ambiguities, the baryon mass relations are not. It is tempting to speculate that this may in some sense be the reason for the robustness and phenomenological success of the large N_c baryon mass relations.

The organization of this paper is as follows. Section II presents a brief review of the operator methods by which large N_c baryon phenomenology is performed. Section III presents an analysis of the baryon mass lattice results in light of the large N_c^{AS} expansion and shows that this expansion remains phenomenologically just as relevant for this observable as does the large N_c^{F} limit. The robustness of baryon mass relations under different prescriptions for treating strangeness is discussed in Sec. IV, and Sec. V offers concluding remarks.

II. BARYONS AT LARGE N_c

The conventional phenomenological operator analysis of large N_c baryons is based upon the use of three specific properties: (i) the large number of valence quarks in the baryon [which is N_c in the large N_c^{F} expansion and $N_c(N_c - 1)/2$ in the large N_c^{AS} expansion] and their detailed combinatorics in the baryon wave function [3,19], (ii) the 't Hooft scaling [1] $g \sim N_c^{-1/2}$ of the QCD coupling constant required to obtain a nontrivial large N_c limit, and

(iii) a ground-state multiplet whose states are completely symmetric under the combined spin-flavor symmetry, and whose $N_c = 3$, $N_f = 3$ case is the $SU(6)$ **56**-plet [5,8,9]. Ambiguities related to the identification of the physical baryon states with particular states within the large N_c multiplets are addressed in Sec. IV. The nonvalence (gluon and sea quark) degrees of freedom enter into the analysis only indirectly: Since the physical baryons fill specific spin and flavor representations based upon the quantum numbers of valence quarks, the entire baryon wave function may be written in terms of interpolating fields that carry the same quark spin/flavor/color quantum numbers and that encompass the full baryon wave function, thus providing a rigorous footing to the concept of constituent quarks [20].

Operators that describe the interactions among the component quarks of the baryon are classified in terms of their transformation properties under the spin-flavor symmetry [7] and the number n of quark fields appearing in the interaction, hence defining an *n-body operator*. Upon including both the appropriate quark combinatorics and the 't Hooft scaling, one finds that n-body operators are generically suppressed by a factor $1/N_c^n$ ($1/N_c^{2n}$) in the N_c^{F} (N_c^{AS}) counting. However, obtaining the full $1/N_c$ suppression factor requires one to account for two other sources: First, one must also consider the possibility that contributions from the quarks *add coherently* in the baryon matrix elements, which introduces compensatory combinatoric factors of N_c that may change the power counting. Second, certain combinations of operators may give matrix elements that form a linearly dependent set when evaluated on the baryon multiplet in question; such a linear dependence may occur due to an exact operator identity (for example, in the case of Casimir operators) or due to the symmetry properties of a particular multiplet on which the analysis is performed (for example, the vanishing of an antisymmetric tensor acting upon the completely symmetric ground-state multiplet) [7]. In addition, a particular combination of operators acting upon a particular multiplet might produce a result that is subleading in the $1/N_c$ expansion compared to each of the component operators, which is termed an *operator demotion*. Once this reorganization is complete, one is left with a linearly independent set of operators, each one of which produces matrix elements with a well-defined power counting in $1/N_c$ acting upon a particular baryon multiplet, and this set carries precisely the same dimension as the space of independent baryon observables for the multiplet. One sees that the operators and independent observables form equivalent bases for the baryons, and the operators may then be assembled into an effective Hamiltonian for any baryon observable such that the operators form a hierarchy in powers of $1/N_c$.

To see how these properties work in detail, we begin by defining operators with specific transformation properties under the spin-flavor symmetry. The operators in the

adjoint representation are labeled J^i, T^a, and G^{ia}, respectively, in Refs. [5,10,11]:

$$J^i \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes \mathbb{1} \right) q^\alpha,$$

$$T^a \equiv q_\alpha^\dagger \left(\mathbb{1} \otimes \frac{\lambda^a}{2} \right) q^\alpha, \tag{1}$$

$$G^{ia} \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes \frac{\lambda^a}{2} \right) q^\alpha,$$

where $\mathbb{1}$ is the identity matrix, σ^i are Pauli matrices in spin space, λ^a is the usual Gell-Mann matrix in flavor space, and α sums over all the quarks in the baryon. Color indices do not appear explicitly in this expression since the baryon ground-state multiplet is completely symmetric under the combined spin-flavor symmetry and therefore is completely antisymmetric under color, so that the operators are well-defined for both large $N_c^{\rm F}$ and large $N_c^{\rm AS}$. All operators that have nonvanishing matrix elements on baryon states are expressible as polynomials in J^i, T^a, and G^{ia} (with suitable contractions of spin-flavor indices), and such a polynomial of nth degree is an n-body operator. Since the physical baryons have $N_c = 3$ quarks, any such polynomial beyond cubic order applied to these baryons gives matrix elements linearly dependent upon those of lower-order operators, which means that such operators are ignorable; the $1/N_c$ series for any finite given value of N_c terminates after providing a complete set of independent operators, which does not extend beyond the N_c-body level.

An n-body operator requires an n-quark interaction, which in turn implies $2n$ factors of the QCD coupling g, to give the $1/N_c^n$ suppression for the large $N_c^{\rm F}$ limit, as discussed above. One might naively expect the same suppression factor for the large $N_c^{\rm AS}$ case, but one finds, as argued in [21] and systematically verified in [22], that the

necessity of maintaining the color-singlet nature of the large $N_c^{\rm AS}$ baryon effectively makes the appropriate effective expansion parameter $1/N_c^2$. Therefore, in the effective large $N_c^{\rm F}$ baryon Hamiltonian, the operator T^a appears multiplied by an explicit factor of $1/N_c$ compared to the spin-flavor symmetric operator $\mathbb{1}$ that has $O(N_c^1)$ matrix elements [while, in large $N_c^{\rm AS}$, the corresponding factors are $1/N_c^2$ and $O(N_c^2)$]. Consider now the mass operator T^8; its matrix elements, which naively merely count strange quarks, are actually given in large $N_c^{\rm F}$ by

$$\langle T^8 \rangle = \frac{1}{2\sqrt{3}} (N_c - 3N_s). \tag{2}$$

Here we see a coherent $O(N_c^1)$ contribution that seems to upset the large N_c counting; however, note that it is the same for all baryons and therefore simply provides an additional contribution to the leading-order spin-flavor symmetric mass operator $\mathbb{1}$. In the above terminology, $T^8 - \mathbb{1}/2\sqrt{3}$ is a demoted operator. The operator T^8 also breaks SU(3)$_{\rm flavor}$ and therefore requires an explicit prefactor of ϵ. When one repeats this analysis for a complete set of linearly independent operators, one finds that each operator contributes to a unique baryon mass combination. To be specific, the mass Hamiltonian when isospin breaking is suppressed reads [10]

$$M = c_{(0)}^{1,0} N_c \mathbb{1} + c_{(2)}^{1,0} \frac{1}{N_c} J^2 + c_{(1)}^{8,0} \epsilon T^8 + c_{(2)}^{8,0} \frac{\epsilon}{N_c} \{J^i, G^{i8}\}$$

$$+ c_{(3)}^{8,0} \frac{\epsilon}{N_c^2} \{J^2, T^8\} + c_{(2)}^{27,0} \frac{\epsilon^2}{N_c} \{T^8, T^8\}$$

$$+ c_{(3)}^{27,0} \frac{\epsilon^2}{N_c^2} \{T^8, \{J^i, G^{i8}\}\} + c_{(3)}^{64,0} \frac{\epsilon^3}{N_c^2} \{T^8, \{T^8, T^8\}\}, \tag{3}$$

where the coefficients c are $O(N_c^0)$, and the nontrivial matrix elements of the baryon operators are given by

$$O_{(2)}^{8,0} \equiv \frac{\epsilon}{N_c} \{J^i, G^{i8}\} \rightarrow \frac{1}{2\sqrt{3}} \cdot \frac{\epsilon}{N_c} \left[3I(I+1) - J(J+1) - \frac{3N_s}{2} \left(\frac{N_s}{2} + 1 \right) \right],$$

$$O_{(3)}^{27,0} \equiv \frac{\epsilon^2}{N_c^2} \{T^8, \{J^i, G^{i8}\}\} \rightarrow \frac{\epsilon^2}{N_c^2} \cdot \frac{1}{6} (N_c - 3N_s) \left[3I(I+1) - J(J+1) - 3\frac{N_s}{2} \left(\frac{N_s}{2} + 1 \right) \right],$$

$$O_{(1)}^{8,0} \equiv \epsilon T^8 \rightarrow \epsilon \cdot \frac{1}{2\sqrt{3}} (N_c - 3N_s), \tag{4}$$

$$O_{(2)}^{27,0} \equiv \frac{\epsilon^2}{N_c} \{T^8, T^8\} \rightarrow \frac{1}{6} \cdot \frac{\epsilon^2}{N_c} (N_c - 3N_s)^2,$$

$$O_{(3)}^{64,0} \equiv \frac{\epsilon^3}{N_c^2} \{T^8, \{T^8, T^8\}\} \rightarrow \frac{\epsilon^3}{N_c^2} \cdot \frac{1}{6\sqrt{3}} (N_c - 3N_s)^3,$$

where the explicit suppressions of ϵ or $1/N_c$ have here been absorbed into the operator definitions. The operators of Eq. (3) define the combinations M_i in Table I, including the $1/N_c$ and ϵ suppression factors. For example, the

combination M_2 is associated with the operator ϵT^8 considered above.

Such an analysis is not unique to the $1/N_c$ expansion; all that is required is a finite multiplet of states under some

symmetry and a perturbative parameter that suppresses some of the independent operators acting upon the multiplet. For example, since an operator with matrix elements linear in strangeness breaks $SU(3)_{\text{flavor}}$ symmetry by transforming as an **8**, an operator transforming as a **27** ($\subset$ **8** $\otimes$ **8**) does not occur until second order in flavor breaking, and this operator must be associated with a doubly suppressed flavor-breaking mass combination. Indeed, when evaluated for the $N_c = 3$ baryon octet, this mass combination turns out to be just the one that appears in the Gell-Mann–Okubo relation, $2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0$, where X_0 indicates the isospin average of the X baryon isomultiplet masses.

III. THE $1/N_c^{\text{F}}$ AND $1/N_c^{\text{AS}}$ EXPANSIONS

Our previous results [17] show that the predictions of both the large N_c^{F} and large N_c^{AS} analyses fit the experimental spectrum of $I = 0$ baryon masses comparably well. In the real world, one can take the $SU(3)_{\text{flavor}}$-breaking parameter to be, e.g., $\epsilon = (m_K^2 - m_\pi^2)/\Lambda_\chi^2 \approx 0.226$ ($\Lambda_\chi \approx$ 1 GeV indicating the scale of chiral symmetry breaking); however, lattice calculations of baryon spectra allow one to move away from the physical value of ϵ, and Jenkins *et al.* [12] demonstrated that the predictions of the large N_c^{F} expansion continue to accommodate the data well even as ϵ is varied over the range $(0, 0.26)$. In this section we compare the predictions of the $1/N_c^{\text{AS}}$ expansion to the lattice data.

First, let us briefly review how these comparisons are defined in Refs. [12,17]. Each mass combination M_i from Table I (our $M_{0,...,7}$ corresponding, respectively, to $M_{1,...,8}$ in Ref. [12]) is associated with a dimensionless ratio $R_i \equiv M_i/(M_i'/2)$, where M_i' is defined to be the same combination of masses as in M_i, but with each coefficient replaced with its absolute value. These ratios therefore compare the size of M_i as extracted from lattice simulations or experiment relative to the appropriately weighted average M_i' of the masses that enter the mass combinations M_i, which makes R_i scale-independent quantities (but note that our R_i's differ from those defined in [12]). We then compute (as in [17]) the ratio of each R_i to its corresponding

theoretical suppression S_i, including the appropriate $SU(3)_{\text{flavor}}$ and N_c factors as listed in Table I, and plot finally the *accuracy factors* $A_i \equiv \log_3(R_i/S_i)$. The logarithm appears in order to distinguish different integer-power suppressions, while the base 3 is used to separate each factor of $N_c = 3$ by one unit. In particular, deviations from $R_i/S_i \sim 1$ to either, e.g., $R_i/S_i = 1/2$ or $R_i/S_i = 2$ are considered equally significant. If the suppression factors in the measured quantities agree with those predicted theoretically, one expects that the A_i should lie roughly in the range $-1 \lesssim A_i \lesssim 1$.

Figure 1 shows the accuracy factors A_i for each mass combination as a function of the strength of $SU(3)_{\text{flavor}}$ breaking ϵ, along with uncertainties reported in [12]. An examination of the plots shows that the results we found at the physical value of ϵ in [17] regarding the viability of both large N_c limits also apply at generic values of ϵ. The predictions of both large N_c expansions fit the data equally well (i.e., their points cluster in $A_i \in [-1, 1]$), and both these large N_c predictions give a much better fit than predictions based solely on $SU(3)_{\text{flavor}}$ breaking. Furthermore, each A_i appears to possess a smooth $\epsilon \to 0$ limit. However, evidence of nonanalytic corrections in ϵ (which arise in chiral perturbation theory) has recently been identified in baryon lattice simulations [23].

IV. SU(3) SYMMETRY AND BARYONS AT LARGE N_c

The existence of lattice calculations for baryon masses with varying degrees of SU(3) breaking helps to put into stark focus some of the underlying challenges for the phenomenology of the physical $N_c = 3$ world, since one can now analytically investigate the behavior of large N_c baryons at more than just a single point. The crux of the remaining difficulty is that many more baryon states necessarily appear at large N_c than at $N_c = 3$. Accordingly, one needs some method for relating the baryons at $N_c = 3$ to particular baryons in the large N_c world. To keep the discussion focused in this section, we analyze the situation assuming that the quarks are in the fundamental representation. An analogous argument follows for the case of

TABLE I. Baryon mass combinations and their theoretical suppression factors. The 0 subscript indicates an average over isospin states, which are termed $I = 0$ baryon masses.

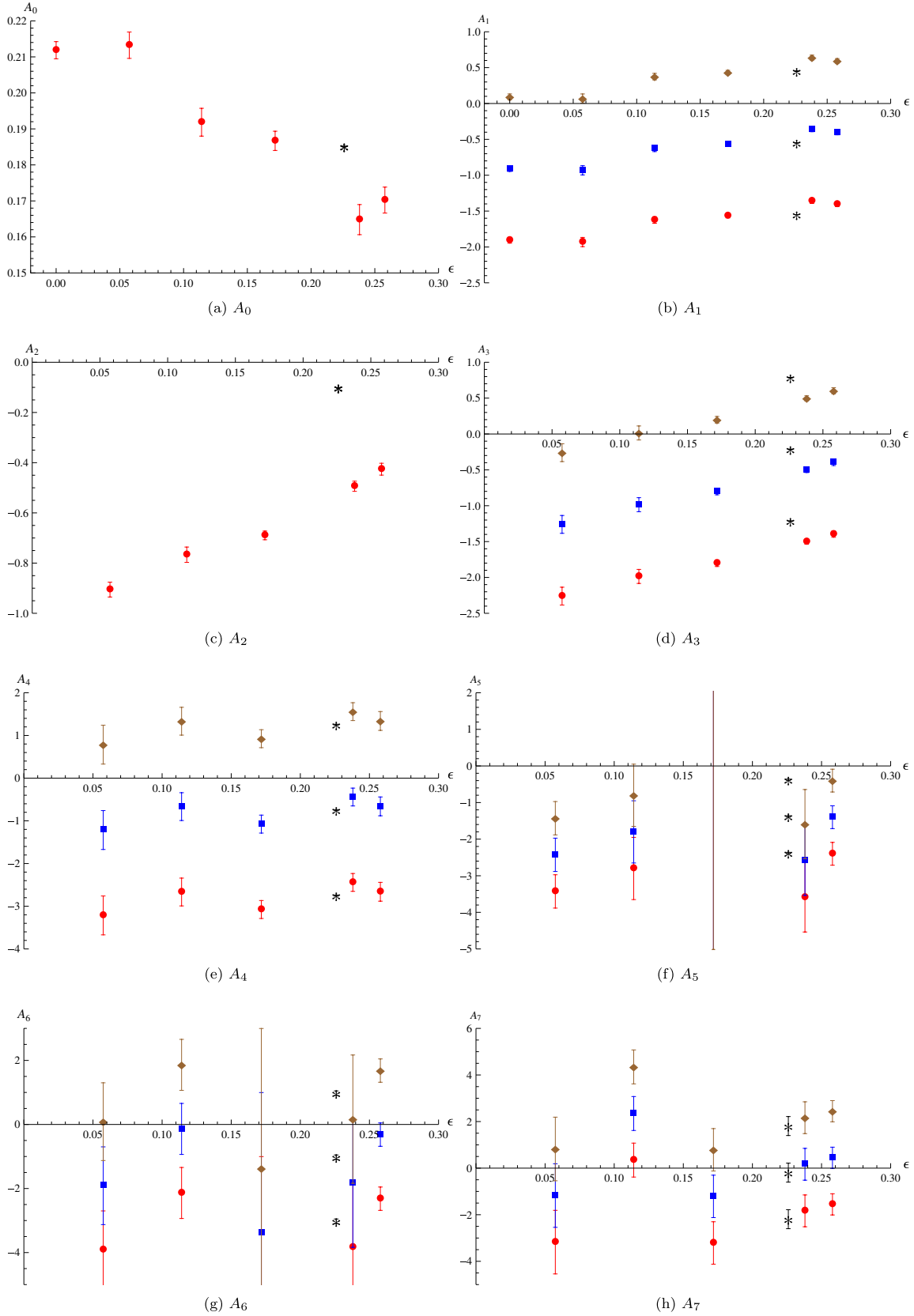|	Mass combination	Large N_c^{F} suppression	Large N_c^{AS} suppression
M_0	$25(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) - 4(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$	N_c	N_c^2
M_1	$5(2N_0 + 3\Sigma_0 + \Lambda + 2\Xi_0) - 4(4\Delta_0 + 3\Sigma_0^* + 2\Xi_0^* + \Omega)$	$1/N_c$	$1/N_c^2$
M_2	$5(6N_0 - 3\Sigma_0 + \Lambda - 4\Xi_0) - 2(2\Delta_0 - \Xi_0^* - \Omega)$	ϵ	ϵ
M_3	$N_0 - 3\Sigma_0 + \Lambda + \Xi_0$	ϵ/N_c	ϵ/N_c^2
M_4	$(-2N_0 - 9\Sigma_0 + 3\Lambda + 8\Xi_0) + 2(2\Delta_0 - \Xi_0^* - \Omega)$	ϵ/N_c^2	ϵ/N_c^4
M_5	$35(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 4(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$	ϵ^2/N_c	ϵ^2/N_c^2
M_6	$7(2N_0 - \Sigma_0 - 3\Lambda + 2\Xi_0) - 2(4\Delta_0 - 5\Sigma_0^* - 2\Xi_0^* + 3\Omega)$	ϵ^2/N_c^2	ϵ^2/N_c^4
M_7	$\Delta_0 - 3\Sigma_0^* + 3\Xi_0^* - \Omega$	ϵ^3/N_c^2	ϵ^3/N_c^4

FIG. 1 (color online). Plots of the accuracy factors A_i defined in the text. Black stars denote A_i extracted from experimental values, red circles are evaluated with an N_c suppression factor of 1, corresponding to breaking of SU(3)$_{\text{flavor}}$ only (no N_c expansion), blue squares are evaluated with N_c^{F} suppression factors, and gold diamonds are evaluated with N_c^{AS} suppression factors.

quarks in the antisymmetric representation, with appropriate replacements of N_c^2 for N_c.

For the case of two degenerate flavors, this identification is obvious. The standard large N_c analysis [4–9] produces ground-state multiplet baryons with $I = J$ differing in mass at $O(1/N_c)$ [for $J = O(1)$], and with $I = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. One naturally identifies baryons with $I = \frac{1}{2}, \frac{3}{2}$ with the analogous N, Δ states, respectively, at $N_c = 3$. Large N_c baryons with $I \geq \frac{5}{2}$ are taken to be artifacts of the large N_c world and not relevant at $N_c = 3$. However, for three degenerate flavors the analogue of this construction cannot be carried out: *All* representations of flavor SU(3) for baryons have dimensions of $O(N_c^2)$ as $N_c \to \infty$. This raises a problem: Which baryons at large N_c are analogous to ones at $N_c = 3$? Clearly, one must declare that almost all of the members in any SU(3) flavor multiplet are large N_c artifacts, but what principle should one use to choose?

The issue is complicated by the fact that in nature the three flavors are *not* degenerate; explicit SU(3) violations occur due to the mass difference between strange and nonstrange quarks. One parametrizes the scale of this SU(3) breaking in the Hamiltonian or in the meson sector by some dimensionless parameter ϵ. At finite but large N_c, a typical baryon in an SU(3) multiplet has $O(N_c)$ strange quarks and a contribution from the SU(3)-violating term of size ϵN_c. As N_c becomes sufficiently large, $\epsilon N_c \gg 1$, and the notion of "small violations" of SU(3) becomes complicated. Indeed, one might easily imagine that the behavior of states might be qualitatively different for varying values of ϵN_c. Thus, one might worry that the behavior of states at $\epsilon N_c \ll 1$ could be qualitatively different than that for $\epsilon N_c \sim 1$ or $\epsilon N_c \gg 1$. The ability to use the lattice to see how states behave when ϵ is varied can therefore play an important role in clarifying the issues. The fact that the mass relations of Table I are quite robust and work from zero SU(3) breaking to fairly strong breaking provides a real clue as to what is happening. As seen below, the explanation is that the mass relations themselves are quite special in being quite insensitive to the ambiguities.

Let us return to the question of how to identify states at $N_c = 3$ with particular states in the large N_c world, and do so without prejudice from the lattice data. For simplicity, we consider the case where N_c is an odd multiple of three. In principle, one can find an infinite number of prescriptions to do this. Two of them are rather natural: The first one—and the one typically used—is to identify a baryon at $N_c = 3$ with a large N_c baryon of the same strangeness, total isospin, and third component of isospin. This identification is particularly natural if ϵN_c is comparatively large, in that one focuses on the lowest-lying states in the large N_c multiplet. In effect, one adopts a counting prescription in N_s; the strangeness is treated as being $O(1)$, and we refer to this identification as the *S prescription*. Such an approach has a natural analogue in the treatment of

topological solitons such as the Skyrme model, the so-called bound-state approach of Callan and Klebanov [24]. In that scheme, one begins by treating a nucleon as an SU(2) Skyrmion and then considers a strange baryon as a bound state of a kaon and an SU(2) Skyrmion, a doubly strange one as two kaons bound to a nucleon, etc.

However, the approach suggested by the S prescription presents a difficulty. Consider the exact SU(3) flavor limit, in which the u, d, and s quarks should appear on the same footing, which in turn means that I spin, U spin, and V spin should be treated equivalently. This symmetry is badly violated at large N_c by the identification of states discussed above. At $N_c = 3$, the 18 states N, Λ, Σ, Ξ, Δ, Σ^*, Ξ^*, and Ω associated with the octet and decuplet clearly treat I spin, U spin, and V spin on the same footing. For example, the third component of isospin varies between $-\frac{3}{2}$ and $+\frac{3}{2}$ for these states, as do the third components of the U spin and V spin. Now consider baryons at large but finite N_c, using the S prescription; the third component of isospin still varies between $-\frac{3}{2}$ and $+\frac{3}{2}$, but the third components of U spin and V spin are radically different as N_c becomes large: They vary from $(N_c - 9)/4$ to $(N_c + 3)/4$. This result conflicts with the fundamental idea underlying SU(3) flavor symmetry, that all flavors are created equal.

One particular identification of states avoids this problem: Instead of identifying each state at large N_c with the state at $N_c = 3$ of the same strangeness, as in the S prescription, one can identify large N_c states with $N_c = 3$ states of the same *hypercharge*. We refer to this as the *Y prescription*. Note that, for general N_c, the relationship between hypercharge and number of strange quarks for a state of baryon number B reads

$$Y = \frac{N_c B}{3} + N_s. \quad (5)$$

Thus, at large N_c a state with $S = O(1)$ has $Y = O(N_c)$, and conversely, a state that has $Y = O(1)$ has $S = O(N_c)$. Using this new identification, the 18 states of the octet and the decuplet all have I_3, U_3, and V_3 eigenvalues between $-\frac{3}{2}$ and $+\frac{3}{2}$. It is perhaps not surprising that using the hypercharge to identify states rather than the strangeness does a better job in preserving the symmetry between u, d, and s quarks for this class of states since hypercharge is a traceless generator of SU(3)$_{\text{flavor}}$ but strangeness is not.

The S and Y prescriptions are easy to distinguish in pictorial form. In Fig. 2 we exhibit the weight diagram for the SU(3) representation corresponding to the spin-$\frac{3}{2}$ baryon multiplet [the spin-$\frac{1}{2}$ representation is similar, but has two sites on the top row and $(N_c + 1)/2$ sites on the long sides]. In the S prescription, the analogues to the $N_c = 3$ baryons appear in the top rows (minimum N_s) of the diagram, while in the Y prescription they coincide with the sites nearest the centroid ($Y = 0$) of the diagram.

The two prescriptions have important physical differences. If one focuses on the mass of a typical baryon M in
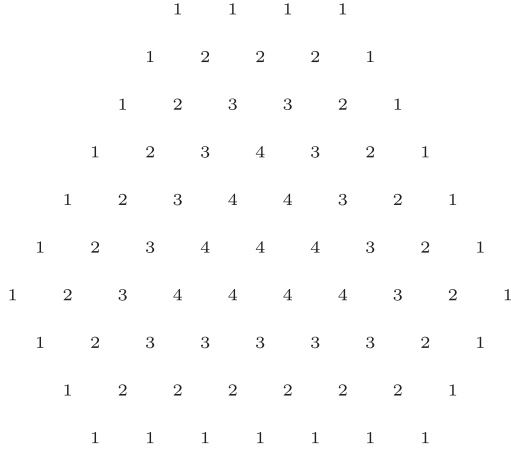
```
            1    1    1    1
         1    2    2    2    1
      1    2    3    3    2    1
   1    2    3    4    3    2    1
1    2    3    4    4    3    2    1
   1    2    3    4    4    4    3    2    1
1    2    3    4    4    4    4    3    2    1
   1    2    3    3    3    3    3    2    1
      1    2    2    2    2    2    2    1
         1    1    1    1    1    1    1
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FIG. 2. Weight diagram for the ground-state spin-$\frac{3}{2}$ representation of SU(3), adapted from Ref. [7]. The numbers indicate the multiplicity of states at a given site. While this literal diagram corresponds to $N_c = 15$, it is easily generalized to arbitrary odd N_c by extending the longer sides to $(N_c - 1)/2$ sites.

the octet or decuplet and considers its dependence on the three quark masses, one finds

$$\text{Prescription } S: \quad \frac{dM}{dm_s} = N_s = O(N_c^0),$$

$$\frac{dM}{dm_{u,d}} = N_{u,d} = O(N_c^1), \qquad (6)$$

$$\text{Prescription } Y: \quad \frac{dM}{dm_{u,d,s}} = N_{u,d,s} = O(N_c^1).$$

Note that the rate of change of the baryon mass with m_s is qualitatively different in the two approaches. Since this rate also equals $\langle B| \int d^3x \bar{s}s |B\rangle$, the scalar strangeness content of the baryon behaves differently in the two approaches, while $\langle B| \int d^3x \bar{u}u |B\rangle \sim \langle B| \int d^3x \bar{d}d |B\rangle = O(N_c)$ in either approach. Given that the two prescriptions are physically quite different at large N_c, it should be clear that they actually correspond to distinct $1/N_c$ expansions for the baryon masses. One then faces the obvious

question: Which one of these expansions is more phenomenologically useful in describing the world of $N_c = 3$?

It is noteworthy that Ref. [10], where the mass relations of Table II were first derived, makes the explicit assumption that the strangeness—rather than the hypercharge—is of order unity, which is consistent with prescription S. The fact that the lattice suggests the $N_c = 3$ mass relations are robust—holding qualitatively over a fairly wide ride range of the scale of SU(3) breaking ϵ—may then seem to suggest that prescription S is the phenomenologically relevant choice. Such a result may seem a bit puzzling, since it is plausible that prescription Y, which unlike prescription S treats u, d, and s on an equal footing, is more sensible than prescription S very near the SU(3) limit.

However, this result produces no real puzzle. The robustness of the mass relations tells us more about the nature of the relations themselves than about which prescription is better. Except for M_0, the relations all involve the *differences* of masses rather than masses themselves, and while the individual masses are sensitive to the choice of prescription, we argue that this sensitivity cancels completely in the mass relations to the order that the relations hold. Thus, the relations are themselves quite robust, and this fact appears to be responsible for the relations holding at the predicted level of accuracy, even for widely varying values of SU(3) breaking. Additional evidence for the robustness of the mass analysis under an alternate treatment of the flavor quantum numbers appears in Ref. [25].

To see how the cancellations arise, recall from Sec. II how the mass relations are derived from operator relations. Assuming exact isospin symmetry and using quarks in the fundamental representation gives the mass Hamiltonian in Eq. (3), whose operators have matrix elements given in Eq. (4). Using these expressions, the mass Hamiltonian can be recast in the equivalent two forms:

$$M = \sum_i a_i^S O_i^S = \sum_i a_i^Y O_i^Y, \qquad (7)$$

where the a coefficients are linear combinations of the c coefficients and can be shown to be of order unity; the operators are defined in Appendix to have the simple

TABLE II. The eight operators given in Eq. (7), including the N_c and ϵ dependence. Prescriptions S and Y are defined in the text.

Operator	Prescription S	Prescription Y
O_1	N_c	N_c
O_2	$\frac{1}{N_c}J(J+1)$	$\frac{1}{N_c}J(J+1)$
O_3	ϵN_s	$-\epsilon Y$
O_4	$\frac{\epsilon}{N_c}[I(I+1) - \frac{N_s}{2}(\frac{N_s}{2}+1)]$	$\frac{\epsilon}{N_c}[I(I+1) - \frac{Y}{2}(\frac{Y}{2}-1)]$
O_5	$\frac{\epsilon}{N_c^2}N_s J(J+1)$	$-\frac{\epsilon}{N_c^2}Y J(J+1)$
O_6	$\frac{\epsilon^2}{N_c}N_s^2$	$\frac{\epsilon^2}{N_c}Y^2$
O_7	$\frac{\epsilon^2}{N_c^2}N_s[I(I+1) - \frac{N_s}{2}(\frac{N_s}{2}+1)]$	$-\frac{\epsilon^2}{N_c^2}Y[I(I+1) - \frac{Y}{2}(\frac{Y}{2}-1)]$
O_8	$\frac{\epsilon^3}{N_c^2}N_s^3$	$-\frac{\epsilon^3}{N_c^2}Y^3$

matrix elements given in Table II. For simplicity, the leading N_c dependence and ϵ dependence are included as part of the operators O_i^S and O_i^Y. The superscripts S and Y indicate which of the two prescriptions is used in the N_c counting.

The two sets of operators have one very important property: All operators of type S at a given order in ϵ and $\frac{1}{N_c}$ can be written as linear combinations of operators of type Y at equal or subleading order in both ϵ and $\frac{1}{N_c}$. Thus, for example, O_4^S, whose matrix elements are $O(\epsilon/N_c)$, is expressible in terms of O_1^Y, O_3^Y, and O_4^Y, which are all of equal or subleading order in both ϵ and $\frac{1}{N_c}$. It is straightforward to explicitly show that this property holds for all of the operators, and this result essentially amounts to a statement about the decomposition of any high-order tensor into an irreducible piece plus subleading traces. Similarly, all operators of type Y at a given order in ϵ and $\frac{1}{N_c}$ can be written as linear combinations of operators of type S at equal or subleading order in both ϵ and $\frac{1}{N_c}$.

Let us now recall how the mass relations are obtained. One finds linear combinations of the masses among the eight independent baryons N, Λ, Σ, Ξ, Δ, Σ^*, Ξ^*, and Ω that vanish when acted on by all operators of an equal or subleading order in both ϵ and $\frac{1}{N_c}$. Each such combination gives a mass relation that holds up to the given order in ϵ and $\frac{1}{N_c}$. However, since any operator of type S can be written as a superposition of operators of type Y at equal and subleading order, any mass relation that holds for prescription S also holds for prescription Y, and vice versa.

In terms of Eq. (7), our result indicates that $a_i^S = a_i^Y +$ equal or subleading order in ϵ and $1/N_c$. One additional point deserves mention: If the correction is of equal order, one might fear that only one of the two prescriptions gives fully hierarchical coefficients, the other one merely "maintaining the status quo." In the example of O_4 given above, $O_4^S = O_4^Y - \frac{1}{6}O_3^Y +$ subleading order, so one might expect that either a_4^S is no smaller than a_3^S or a_4^Y is no smaller than a_3^Y, at least parametrically. But in fact, one can check that the numerical hierarchy using the physical baryon masses supports the result that both a_4's are $O(N_c) \sim 3$ smaller than their corresponding a_3's, owing to the smallness of the explicit $-\frac{1}{6}$ coefficient.

To summarize, individual masses depend sensitively on one's choice of prescription, but the mass relations do not. They hold at the stated order of accuracy regardless of how one chooses to identify baryons at large N_c with $N_c = 3$ baryons. In particular, the validity of the mass relations does not depend upon the scale of ϵN_c; rather they only depend on both ϵ and $1/N_c$ to be separately small. The fact that lattice results uphold the mass relations at the expected levels of accuracy, even as ϵN_c varies widely, presumably reflects this fact.

Ultimately, this behavior might teach us an important lesson about the applicability of large N_c operator analysis

to baryons with three flavors. To find quantities that discriminate between possible prescription choices, one presumably needs to consider quantities sensitive to the absolute number of quarks of various types in a given state, rather than their relative number between different baryon states. The one place in our analysis that this distinction might be possible is the common mass accuracy parameter A_0. However, a linear fit to the points in Fig. 1(a) gives $A_0 = 0.215$–0.177ϵ. Since $0.177/0.215$ is neither large nor small compared to unity, one cannot distinguish decisively between $M_0 \sim N_c + \epsilon$ (S prescription) or $M_0 \sim N_c + \epsilon N_c$ (Y prescription).

V. CONCLUSIONS

Lattice simulations provide a unique window into understanding nonperturbative physics and why our universe chooses one unique solution out of many possibilities, particularly since the simulations allow one to explore universes in which the underlying parameters (quark masses, for example) can be chosen at will. Another very recent example studies baryon masses by varying the literal numerical value of N_c [26]. Such results provide otherwise unattainable insights into strongly interacting systems.

In this work we have seen that lattice simulations of baryons over a range of $SU(3)_{\text{flavor}}$-breaking parameter ϵ provide a spectrum of masses not only explainable in terms of a large N_c QCD expansion (as seen in previous work), but are in fact agnostic as to whether the quarks fill the fundamental or two-index antisymmetric representation of $SU(N_c)$. Moreover, either expansion does a much better job accounting for the mass spectrum than including only the ϵ dependence and ignoring all factors of N_c. This result greatly extends the scope of our previous work, which reached the same conclusion but only at the physically realized value of ϵ.

We also addressed the interesting question of why this conclusion should hold as $\epsilon \rightarrow 0$, the $SU(3)$-symmetric point, when the baryon states in these analyses are assumed to have $O(1)$ strange quarks and $O(N_c)$ u and d quarks—a highly $SU(3)$-asymmetric configuration. The resolution appears to be that mass differences, which constitute the bulk of the large N_c baryon results, are insensitive to whether one works in a prescription that favors minimizing either the number of s quarks (S prescription) or the difference between the number of u, d, and s quarks (Y prescription). An examination of observables sensitive not to differences of quarks but rather to their collective effect is necessary to resolve this remarkable ambiguity.

APPENDIX: RELATIONSHIPS BETWEEN OPERATOR BASES

The precise relationships between the operators given in Eq. (4) and the operators $O_i^{S,Y}$ of the S and Y prescriptions that give the matrix elements listed in Table II are

$$O_1^{S,Y} = O_{(0)}^{1,0}, \qquad O_2^{S,Y} = O_{(2)}^{1,0}, \qquad O_3^{S} = -\frac{2}{\sqrt{3}} O_{(1)}^{8,0} + \frac{\epsilon}{3} O_{(0)}^{1,0}, \qquad O_3^{Y} = -\frac{2}{\sqrt{3}} O_{(1)}^{8,0}, \qquad O_4^{S} = \frac{2}{\sqrt{3}} O_{(2)}^{8,0} + \frac{\epsilon}{3} O_{(2)}^{1,0},$$

$$O_4^{Y} = \frac{2}{\sqrt{3}} O_{(2)}^{8,0} + \frac{\epsilon}{3} O_{(2)}^{1,0} + \frac{\epsilon}{6} \left(\frac{1}{6} + \frac{1}{N_c}\right) O_{(0)}^{1,0} - \frac{1}{3\sqrt{3}} O_{(1)}^{8,0}, \qquad O_5^{S} = -\frac{1}{\sqrt{3}} O_{(3)}^{8,0} + \frac{\epsilon}{3} O_{(2)}^{1,0}, \qquad O_5^{Y} = -\frac{1}{\sqrt{3}} O_{(3)}^{8,0},$$

$$O_6^{S} = \frac{2}{3} O_{(2)}^{27,0} - \frac{4\epsilon}{3\sqrt{3}} O_{(1)}^{8,0} + \frac{\epsilon^2}{9} O_{(0)}^{1,0}, \qquad O_6^{Y} = \frac{2}{3} O_{(2)}^{27,0}, \qquad O_7^{S} = -\frac{2}{3} O_{(3)}^{27,0} - \frac{\epsilon}{3\sqrt{3}} O_{(3)}^{8,0} + \frac{2\epsilon}{3\sqrt{3}} O_{(2)}^{8,0} + \frac{\epsilon^2}{9} O_{(2)}^{1,0},$$

$$O_7^{Y} = -\frac{2}{3} O_{(3)}^{27,0} + \frac{1}{9} O_{(2)}^{27,0} - \frac{\epsilon}{3\sqrt{3}} O_{(3)}^{8,0} - \left(\frac{1}{6} + \frac{1}{N_c}\right) \frac{\epsilon}{3\sqrt{3}} O_{(1)}^{8,0}, \qquad O_8^{S} = -\frac{2}{3\sqrt{3}} O_{(3)}^{64,0} + \frac{2\epsilon}{3} O_{(2)}^{27,0} - \frac{2\epsilon^2}{3\sqrt{3}} O_{(1)}^{8,0} + \frac{\epsilon^3}{27} O_{(0)}^{1,0},$$

$$O_8^{Y} = -\frac{2}{3\sqrt{3}} O_{(3)}^{64,0}. \tag{A1}$$

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