

$\gamma^* \gamma^* \rightarrow \eta_c$ transition form factor

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We study the $\gamma^* \gamma^* \rightarrow \eta_c$ transition form factor, $F_{\eta_c \gamma \gamma}(Q_1^2, Q_2^2)$, with the local-duality (LD) version of QCD sum rules. We analyze the extraction of this quantity from two different correlators, $\langle PVV \rangle$ and $\langle AVV \rangle$, with P , A , and V being the pseudoscalar, axial-vector, and vector currents, respectively. The QCD factorization theorem for $F_{\eta_c \gamma \gamma}(Q_1^2, Q_2^2)$ allows us to fix the effective continuum thresholds for the $\langle PVV \rangle$ and $\langle AVV \rangle$ correlators at large values of $Q^2 = Q_2^2$ and some fixed value of $\beta \equiv Q_1^2/Q_2^2$. We give arguments that, in the region $Q^2 \geq 10\text{--}15 \text{ GeV}^2$, the effective threshold should be close to its asymptotic value such that the LD sum rule provides reliable predictions for $F_{\eta_c \gamma \gamma}(Q_1^2, Q_2^2)$. We show that, for the experimentally relevant kinematics of one real and one virtual photon, the result of the LD sum rule for $F_{\eta_c \gamma}(Q^2) \equiv F_{\eta_c \gamma \gamma}(0, Q^2)$ may be well approximated by the simple monopole formula $F_{\eta_c \gamma}(Q^2) = 2e_c^2 N_c f_P (M_V^2 + Q^2)^{-1}$, where f_P is the η_c decay constant, e_c^2 is the c -quark charge, and the parameter M_V lies in the mass range of the lowest $\bar{c}c$ vector states.

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I. INTRODUCTION

The processes $\gamma^* \gamma^* \rightarrow P$, with $P = \pi^0, \eta, \eta', \eta_c$, are of great interest for our understanding of QCD and of the meson structure. The corresponding amplitude

$$\langle \gamma^*(q_1) \gamma^*(q_2) | P(p) \rangle = i \epsilon_{\varepsilon_1 \varepsilon_2 q_1 q_2} F_{P\gamma\gamma}(q_1^2, q_2^2) \quad (1.1)$$

contains only one invariant form factor, $F_{P\gamma\gamma}(q_1^2, q_2^2)$, which is one of the simplest hadronic form factors in QCD. We shall address the general situation when both photons are virtual: $q_i^2 = -Q_i^2$, $Q_i^2 \geq 0$, $i = 1, 2$. From the experimental perspective, the most interesting kinematical configuration is when one of the photons is almost real and the other has virtuality Q^2 . For this special case, we use the notation $F_{P\gamma}(Q^2) \equiv F_{P\gamma\gamma}(Q_1^2 = 0, Q_2^2 = Q^2)$. The form factor $F_{P\gamma}(Q^2)$ has been the subject of detailed experimental [1–7] and theoretical investigations (for recent references, see [8–16]). A QCD factorization theorem predicts the behavior of the form factor at asymptotically large momentum transfers [17]:

$$F_{P\gamma\gamma}(Q_1^2, Q_2^2) = 2e_c^2 \int_0^1 \frac{d\xi \phi_P^{\text{ass}}(\xi)}{Q_1^2 \xi + Q_2^2 (1 - \xi)}, \quad (1.2)$$

$$\phi_P^{\text{ass}}(\xi) = 6f_P \xi(1 - \xi),$$

which gives for $Q^2 = Q_2^2$, $\beta \equiv Q_1^2/Q_2^2$, and $0 \leq \beta \leq 1$ (without loss of generality, we denote the larger virtuality by Q_2^2):

$$F_{P\gamma\gamma}(Q_1^2, Q_2^2) = \frac{6e_c^2 f_P}{Q^2} I(\beta),$$

$$I(\beta) = \frac{1 + 2\beta \log \beta - \beta^2}{(1 - \beta)^3}, \quad (1.3)$$

$$I(0) = 1, \quad I(1) = 1/3.$$

In the pion case, setting $Q_1^2 = 0$ and $Q_2^2 = Q^2$, this result reduces to the asymptotic behavior $Q^2 F_{\pi\gamma}(Q^2) \rightarrow \sqrt{2} f_\pi$ [17], with $f_\pi = 0.130 \text{ GeV}$. Similar relations follow for the mesons η and η' after taking particle mixing into account [18,19].

Within errors, this saturation property is indeed found for the η and η' form factors. However, large- Q^2 data up to $Q^2 = 35 \text{ GeV}^2$ from *BABAR* [4] indicate that $Q^2 F_{\pi\gamma}(Q^2)$ does not saturate at large Q^2 but increases further. No compelling theoretical explanation of the qualitatively different behavior of the $\pi\gamma$ form factor compared to the $\eta\gamma$ and $\eta'\gamma$ form factors has been proposed. As concluded in [8,12,14,15], the behavior of the $\pi\gamma$ form factor is hard to explain in QCD. Moreover, the *BABAR* findings for $F_{\pi\gamma}$ require $O(1/s)$ duality-violating corrections between the hadron and the QCD spectral densities [20]. Very recently, Belle [7] presented their results on the $F_{\pi\gamma}$ form factor that are in fact compatible with QCD factorization.

Another, particularly interesting process, is the transition $\gamma^* \gamma^* \rightarrow \eta_c$. Here, one expects that, for the case of massive quarks, the onset of the factorization regime is, compared to the case of massless quarks, delayed to higher Q^2 . The details of the form factor behavior provide valuable information on the interplay of perturbative and non-perturbative QCD at intermediate and large momentum transfers.

In recent publications [15], we analyzed the $P\gamma$ form factors for light mesons, making use of QCD sum rules in their local-duality (LD) limit [21]. We have given arguments that the LD sum rules provide already for Q^2 larger than a few GeV^2 , reliable predictions for the $F_{P\gamma}$ form factors of light pseudoscalars with an accuracy increasing very fast with Q^2 .

The goal of this analysis is two-fold: First, we discuss the subtleties of the formulation of an LD model for transition form factors for the case of massive quarks. Second, we apply our LD model to the case of the $\gamma^* \gamma^* \rightarrow \eta_c$ form factor. The paper is organized as follows: Section II briefly recalls results for the various two-point functions that may be used for the extraction of decay constants of heavy $\bar{c}c$ pseudoscalars. In Sec. III, we present the dispersion representations for $\langle PVV \rangle$ and $\langle AVV \rangle$ three-point functions and discuss the procedure of obtaining the $F_{P\gamma\gamma}(Q_1^2, Q_2^2)$ form factor from these correlators. We also give our predictions for $F_{P\gamma}(Q^2)$ in a broad range of Q^2 . Section IV summarizes our conclusions.

II. TWO-POINT FUNCTIONS OF AXIAL AND PSEUDOSCALAR CURRENTS

For the case of massive quarks, one may consider, on an equal footing, the $\langle AA \rangle$, $\langle AP \rangle$, and $\langle PP \rangle$ correlators, where $A_\mu \equiv \bar{\psi} \gamma_\mu \gamma_5 \psi$ and $P \equiv i \bar{\psi} \gamma_5 \psi$ denote the axial-vector and pseudoscalar currents, respectively.

The $\langle AA \rangle$ correlator involves two independent Lorentz structures; to leading order in the strong coupling α_s it reads

$$(p^2 g_{\mu\nu} - p_\mu p_\nu) \frac{N_c}{16\pi^2} \int_{4m^2}^{\infty} \frac{ds}{s - p^2} \frac{4}{3} \left(1 - \frac{4m^2}{s}\right)^{3/2} + p_\mu p_\nu \frac{N_c}{16\pi^2} \int_{4m^2}^{\infty} \frac{ds}{s - p^2} \frac{8m^2}{s} \sqrt{1 - \frac{4m^2}{s}}. \quad (2.1)$$

We consider the sum rule for the longitudinal part $\langle AA \rangle_L$ of the correlator $\langle AA \rangle$, which contains the contribution of the pseudoscalar mesons on its hadronic side. The dispersion representations for the other correlators are also well known [22,23]. After application of the Borel transformation, one finds

$$\begin{aligned} \langle AA \rangle_L: f_P^2 e^{-M^2\tau} + \text{excited states} \\ = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{\infty} \frac{ds}{s} e^{-s\tau} \sqrt{1 - \frac{4m^2}{s}} [1 + O(\alpha_s)] \\ + \text{power corrections}, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \langle AP \rangle: f_P^2 M^2 e^{-M^2\tau} + \text{excited states} \\ = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{\infty} ds e^{-s\tau} \sqrt{1 - \frac{4m^2}{s}} [1 + O(\alpha_s)] \\ + \text{power corrections}, \end{aligned} \quad (2.3)$$

$$\begin{aligned} \langle PP \rangle: f_P^2 M^4 e^{-M^2\tau} + \text{excited states} \\ = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{\infty} ds s e^{-s\tau} \sqrt{1 - \frac{4m^2}{s}} [1 + O(\alpha_s)] \\ + \text{power corrections}. \end{aligned} \quad (2.4)$$

The sum rules for $\langle AP \rangle$ and $\langle PP \rangle$ may be obtained from the $\langle AA \rangle$ sum rule by taking the first and second τ derivatives, respectively. Thus, considering any of these correlators leads to equivalent results for the case of massive quarks, once proper subtractions are performed.

Implementing quark-hadron duality in the usual way, i.e., as a low-energy cut on the perturbative contribution to the correlator, and setting $\tau = 0$ (LD limit)—in which case all nonperturbative power corrections vanish—the resulting expressions for the decay constants take the form

$$\langle AA \rangle_L: f_P^2 = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{s_{\text{eff}}^{\text{AA}}} \frac{ds}{s} \sqrt{1 - \frac{4m^2}{s}}, \quad (2.5)$$

$$\langle AP \rangle: f_P^2 = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{s_{\text{eff}}^{\text{AP}}} \frac{ds}{M^2} \sqrt{1 - \frac{4m^2}{s}}, \quad (2.6)$$

$$\langle PP \rangle: f_P^2 = \frac{N_c m^2}{2\pi^2} \int_{4m^2}^{s_{\text{eff}}^{\text{PP}}} ds \frac{s}{M^4} \sqrt{1 - \frac{4m^2}{s}}. \quad (2.7)$$

Obviously, the effective thresholds $s_{\text{eff}}^{\text{AA}}$, $s_{\text{eff}}^{\text{AP}}$, and $s_{\text{eff}}^{\text{PP}}$ must be (slightly) different from each other.

It will be useful to recall that in the chiral limit, $m = 0$, the situation is qualitatively different from the massive-quark case: In the chiral limit, the $\langle AA \rangle$ correlator is transverse and contains only one Lorentz structure, $g_{\mu\nu} - p_\mu p_\nu / p^2$. The corresponding invariant amplitude contains the contribution of the Goldstone whereas excited pseudoscalars decouple from the axial current in the chiral limit [22]. Unlike the case of massless quarks, massive ground-state pseudoscalars do not contribute to the transverse Lorentz structure of the $\langle AA \rangle$ correlator of the axial currents of massive quarks [24].

III. LD MODEL FOR THE $\gamma^*(Q_1) \gamma^*(Q_2) \rightarrow P$ TRANSITION FORM FACTOR

The $\gamma^* \gamma^* \rightarrow P$ form factor may be extracted from two different correlators, namely, from $\langle PVV \rangle$ and from $\langle AVV \rangle$.

A. Transition form factor from the three-point function $\langle PVV \rangle$

Let us start with the amplitude for two-photon production from the vacuum $|0\rangle$, induced by the pseudoscalar current $j^5(x) = i \bar{\psi}(x) \gamma_5 \psi(x)$, with $\varepsilon_{1,2}$ denoting the photon polarization vectors:

$$\begin{aligned} \langle \gamma(q_1) \gamma(q_2) | j^5(x=0) | 0 \rangle &= T_{\alpha\beta}(p|q_1, q_2) \varepsilon_1^\alpha \varepsilon_2^\beta, \\ p &\equiv q_1 + q_2. \end{aligned} \quad (3.1)$$

The amplitude $T_{\alpha\beta}$ is obtained from the vacuum expectation value of the T product of one pseudoscalar and two vector currents and will be called the $\langle PVV \rangle$ amplitude.

The decomposition of the amplitude contains only one invariant form factor F_5 :

$$T_{\alpha\beta}(p|q_1, q_2) = \epsilon_{\alpha\beta q_1 q_2} F_5(p^2, q_1^2, q_2^2). \quad (3.2)$$

To one-loop accuracy, this form factor satisfies the spectral representation (see, e.g., [23])

$$F_5(p^2, q_1^2, q_2^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - p^2 - i0} \Delta_5(s, q_1^2, q_2^2),$$

$$\Delta_5(s, q_1^2, q_2^2) = \frac{N_c e_c^2 m}{2\pi} \frac{1}{\lambda^{1/2}(s, q_1^2, q_2^2)} \times \log \left(\frac{s - q_1^2 - q_2^2 + \lambda^{1/2}(s, q_1^2, q_2^2) \sqrt{1 - 4m^2/s}}{s - q_1^2 - q_2^2 - \lambda^{1/2}(s, q_1^2, q_2^2) \sqrt{1 - 4m^2/s}} \right), \quad (3.3)$$

where $\lambda \equiv (s - q_1^2 - q_2^2)^2 - 4q_1^2 q_2^2$. The two-loop radiative corrections to $\Delta_5(s, q_1^2, q_2^2)$ have been calculated for massive quarks and one virtual and one real photon and have been found to vanish [25].

We now perform the usual steps of the method of QCD sum rules [22]: calculate $T_{\alpha\beta}(p|q_1, q_2)$ by inserting hadronic intermediate states, perform the Borel transform ($p^2 \rightarrow \tau$), implement duality as a low-energy cut on the corresponding Borelized spectral representation [22], and go to the LD limit by setting $\tau = 0$ [21]. This brings us to the representation for the $P\gamma\gamma$ form factor

$$F_{P\gamma\gamma}(q_1^2, q_2^2) = \frac{2m}{M^2 f_P} \int_{4m^2}^{s_{\text{eff}}} \frac{ds}{\pi} \Delta_5(s, q_1^2, q_2^2). \quad (3.4)$$

In order to obtain the form factor, we have to fix s_{eff} . Finding reliable criteria for fixing effective thresholds is

a rather subtle and difficult problem that has been investigated in great detail in [26].

In general, the effective threshold depends on all external kinematical variables, in our case q_1^2 and q_2^2 . We consider both momenta as spacelike and different from each other, $q_2^2 = -Q^2$ and $q_1^2 = -\beta Q^2$; therefore, we have $s_{\text{eff}}(\beta, Q^2)$. At large Q^2 and fixed β the effective threshold can be determined by matching the LD expression (3.4) to the factorization theorem for the form factor (1.3). How to proceed at smaller Q^2 will be discussed in Sec. III C.

For any finite effective threshold s_{eff} , the form factor behaves like $1/Q^2$ as demanded by pQCD. However, the spectral density of the three-point function $\Delta_5(s, q_1^2, q_2^2)$ does not reduce to the product $I(\beta)\rho(s)/Q^2$ with a β -independent function $\rho(s)$. This means that, in order to reproduce correctly the pQCD asymptotics, the effective threshold should depend on β . The result of a numerical computation of the exact effective threshold that provides the correct matching of the LD form factor at $Q^2 \rightarrow \infty$ to the asymptotic pQCD form factor (1.3) is shown in Fig. 1. In practice, the β dependence of the threshold is not very strong.

Let us present the explicit behavior of the $\gamma^* \gamma^* \rightarrow P$ form factor for the two boundary values $\beta = 1$ and $\beta = 0$ of β :

- (1) For $\beta = 1$, $Q_1^2 = Q_2^2 = Q^2$, and $Q^2 \rightarrow \infty$, we find

$$Q^2 F_{P\gamma\gamma}(Q^2, Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{2e_c^2}{f_P} \frac{N_c}{8\pi^2} \left(\frac{2m}{M} \right)^2 \times \int_{4m^2}^{s_{\text{eff}}(1, Q^2 \rightarrow \infty)} ds \sqrt{1 - \frac{4m^2}{s}}; \quad (3.5)$$

thus, the effective threshold $s_{\text{eff}}(1, Q^2 \rightarrow \infty)$ should be chosen equal to $s_{\text{eff}}^{\text{AP}}$ of the two-point sum rule (2.7) for $\langle AP \rangle$.

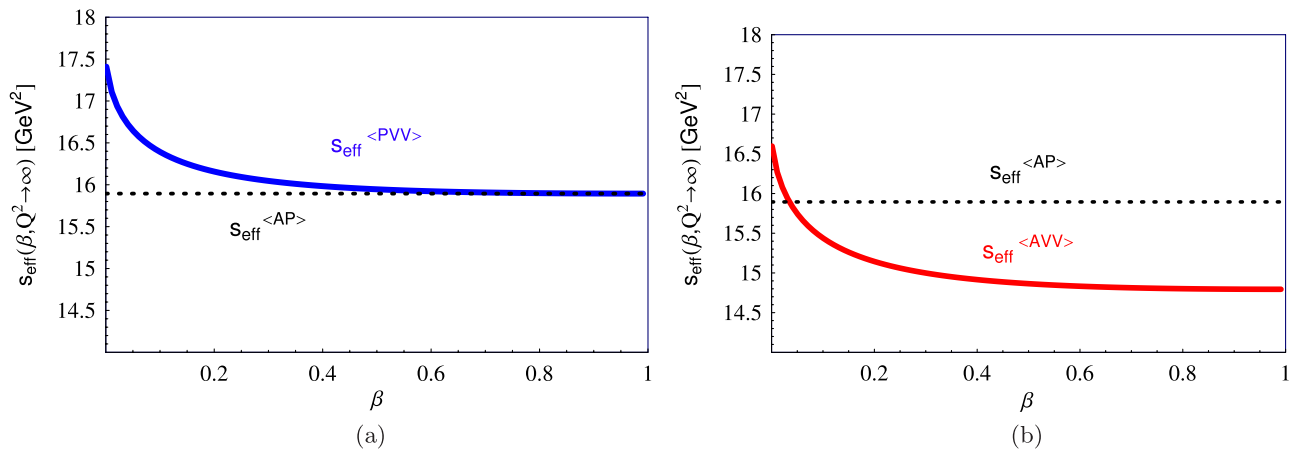


FIG. 1 (color online). Exact effective threshold $s_{\text{eff}}(\beta) \equiv s_{\text{eff}}(\beta, Q^2 \rightarrow \infty)$, $Q^2 = Q_2^2$, $\beta \equiv Q_1^2/Q_2^2$, calculated by matching the LD form factor at large Q^2 to the asymptotic pQCD form factor, for the LD $\gamma^* \gamma^* \rightarrow P$ form factor arising from the three-point correlation functions for (a) $\langle PVV \rangle$ and (b) $\langle AVV \rangle$ (solid lines). The effective threshold for the $\langle AP \rangle$ correlation function is indicated by the dashed line.

(2) For $\beta = 0$, $Q_1^2 = 0$, and $Q_2^2 = Q^2 \rightarrow \infty$, we get

$$Q^2 F_{P\gamma\gamma}(0, Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{e_c^2}{f_P} \frac{N_c}{4\pi^2} \left(\frac{2m}{M}\right)^2 \times \int_{4m^2}^{s_{\text{eff}}(0, Q^2 \rightarrow \infty)} ds \log\left(\frac{1+v}{1-v}\right),$$

$$v \equiv \sqrt{1 - \frac{4m^2}{s}}. \quad (3.6)$$

Matching to the pQCD result requires

$$f_P^2 = \frac{N_c}{24\pi^2} \left(\frac{2m}{M}\right)^2 \int_{4m^2}^{s_{\text{eff}}(0, Q^2 \rightarrow \infty)} ds \log\left(\frac{1+v}{1-v}\right). \quad (3.7)$$

Obviously, the effective threshold $s_{\text{eff}}(0, Q^2 \rightarrow \infty)$ does not coincide with any of the effective thresholds for the various two-point functions discussed in Sec. II.

B. Transition form factor from the three-point function $\langle AVV \rangle$

Next, we consider the amplitude for two-photon production from the vacuum $|0\rangle$, induced by the axial-vector current $j_\mu^5(x) = \bar{q}(x)\gamma_\mu\gamma_5 q(x)$ of quarks q of a single flavor:

$$\langle \gamma(q_1)\gamma(q_2) | j_\mu^5(x=0) | 0 \rangle = T_{\mu\alpha\beta}(p|q_1, q_2) \varepsilon_1^\alpha \varepsilon_2^\beta, \quad (3.8)$$

$$p \equiv q_1 + q_2.$$

The amplitude $T_{\mu\alpha\beta}$ is obtained from the vacuum expectation value of the T product of one axial-vector and two vector currents and will be called the $\langle AVV \rangle$ amplitude. The structure of this amplitude compatible with gauge invariance is

$$T_{\mu\alpha\beta}(p|q_1, q_2) = -p_\mu \epsilon_{\alpha\beta q_1 q_2} iF_0 + (q_1^2 \epsilon_{\mu\alpha\beta q_2} - q_{1\alpha} \epsilon_{\mu q_1 \beta q_2}) iF_1 + (q_2^2 \epsilon_{\mu\beta\alpha q_1} - q_{2\beta} \epsilon_{\mu q_2 \alpha q_1}) iF_2. \quad (3.9)$$

The form factor F_0 involves the contribution of the pseudoscalar meson of interest; it can be cast into the form (cf. [27])

$$F_0(p^2, q_1^2, q_2^2) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - p^2 - i0} \Delta_0(s, q_1^2, q_2^2), \quad (3.10)$$

with the one-loop spectral density

$$\Delta_0(s, q_1^2, q_2^2) = \frac{m^2(q_1^2 + q_2^2 - s)}{\lambda} \Delta + \frac{q_1^2 q_2^2}{\lambda^2} [(q_1^2 - q_2^2)^2 + (q_1^2 + q_2^2)s - 2s^2] \Delta + \frac{1}{2\lambda^2} \{ (q_1^2 - q_2^2)^2 (q_1^2 + q_2^2) - 2[(q_1^2)^2 - 4q_1^2 q_2^2 + (q_2^2)^2]s + (q_1^2 + q_2^2)s^2 \} \sigma. \quad (3.11)$$

Here,

$$\lambda \equiv (s - q_1^2 - q_2^2)^2 - 4q_1^2 q_2^2, \quad \Delta \equiv \frac{1}{\pi\sqrt{\lambda}} \log\left(\frac{s - q_1^2 - q_2^2 + \sqrt{\lambda}\sqrt{1 - 4m^2/s}}{s - q_1^2 - q_2^2 - \sqrt{\lambda}\sqrt{1 - 4m^2/s}}\right), \quad (3.12)$$

$$\sigma \equiv \frac{1}{\pi} \sqrt{1 - \frac{4m^2}{s}}.$$

Note that Δ and σ are the spectral densities of the triangle and two-point loop diagrams with scalar particles of mass m in the loop, respectively. One can check that

$$\int_{4m^2}^{\infty} \frac{ds}{\pi} \Delta_0(s, q_1^2, q_2^2) = -\frac{1}{2\pi^2}, \quad (3.13)$$

independently of q_1^2 , q_2^2 , and m^2 ; thus this integral represents the axial anomaly [27].

Performing the same steps as in the previous section, we obtain the following LD expression for the $P\gamma\gamma$ form factor:

$$F_{P\gamma\gamma}(q_1^2, q_2^2) = \frac{1}{f_P} \int_{4m^2}^{\bar{s}_{\text{eff}}} \frac{ds}{\pi} \Delta_0(s, q_1^2, q_2^2). \quad (3.14)$$

Thus, even for massive fermions the form factor is related to the low-energy part—the contribution below the relevant effective threshold \bar{s}_{eff} —of the axial-anomaly integral [27].

The two-loop radiative corrections to the $\langle AVV \rangle$ correlator vanish. This has been checked for arbitrary virtualities of both photons in the chiral limit [28] and for one real and one virtual photon for massive quarks [25]. Multiloop radiative corrections to the spectral density Δ_0 are unknown but expected to be nonzero [15]. Nevertheless, the one-loop spectral density Δ_0 of (3.11) yields a reliable result for the invariant amplitude F_0 in (3.9) for not too small photon virtualities. The principal uncertainty of the extracted $P\gamma\gamma$ transition form factor arises from the implementation of quark-hadron duality as a low-energy cut on the spectral representation (3.14).

The effective threshold for the $\langle AVV \rangle$ correlator is denoted by $\bar{s}_{\text{eff}}(\beta, Q^2)$ and depends on $q_2^2 = -Q^2$ and $q_1^2 = -\beta Q^2$. As in the $\langle PVV \rangle$ case, at large Q^2 the threshold can be fixed by matching the LD result (3.14) to the pQCD asymptotics (1.3). For massless and massive quarks

slightly different pictures arise, so we consider below these two cases separately.

1. Chiral limit

For massless quarks of a single flavor, the spectral density takes in the limit $Q_2^2 \equiv Q^2 \rightarrow \infty$, $\beta = Q_1^2/Q_2^2$ kept fixed, the following form:

$$\Delta_0(s, Q_1^2, Q_2^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{e_c^2 N_c I(\beta)}{2\pi Q^2}. \quad (3.15)$$

Consequently,

$$F_{P\gamma\gamma}(Q_1^2, Q_2^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{2e_c^2 N_c f_P I(\beta)}{Q^2} \frac{\bar{s}_{\text{eff}}(\beta, Q^2 \rightarrow \infty)}{4\pi^2 f_P^2}. \quad (3.16)$$

Therefore, choosing a β -independent threshold $\bar{s}_{\text{eff}}(\beta, Q^2 \rightarrow \infty) = 4\pi^2 f_P^2$ reproduces the correct pQCD asymptotics of the form factor for any value of β . Recall that to order α_s this threshold coincides with the effective threshold of the LD sum rule for the two-point $\langle AA \rangle$ function for massless quarks

$$f_P^2 = \frac{N_c}{12\pi^2} \int_0^{\bar{s}_{\text{eff}}} ds \left[1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right]. \quad (3.17)$$

The LD model for the transition form factor at finite Q^2 arises if we assume that, for all not too small Q_1^2 and Q_2^2 , the form factor may be well described by the LD expression (3.14) with $\bar{s}_{\text{eff}} = 4\pi^2 f_P^2$. The form factor at $Q_1^2 = Q_2^2 = 0$ is related to the axial anomaly; interestingly, this relation is satisfied for any $\bar{s}_{\text{eff}}(Q_1^2, Q_2^2)$ [15]. Thus, the LD sum rule with constant $\bar{s}_{\text{eff}} = 4\pi^2 f_P^2$ provides for *all* Q_1^2 and Q_2^2 the form factor $F_{P\gamma\gamma}(Q_1^2, Q_2^2)$ consistent with all rigorous constraints. However, explicit calculations show that for $Q^2 \leq 2\text{--}4 \text{ GeV}^2$ the exact effective threshold differs from its LD value [15].¹

2. Massive quarks

In this case, quark-mass corrections destroy the nice picture one has in the chiral limit: Requiring that, for large Q^2 , the LD expression reproduces the correct pQCD asymptotics yields a β -dependent effective threshold \bar{s}_{eff} that differs from the effective thresholds of the two-point correlators in Sec. II. Figure 1 presents the exact threshold $\bar{s}_{\text{eff}}(\beta, Q^2 \rightarrow \infty)$.

The resulting explicit expressions for the two boundary values $\beta = 1$ and $\beta = 0$ of β are given below:

¹Setting the effective threshold equal to $4\pi^2 f_\pi^2$ in the LD sum rule for the elastic pion form factor leads to the correct pQCD asymptotics of $F_\pi(Q^2)$ for $Q^2 \rightarrow \infty$. The $F_\pi(Q^2)$ data for low Q^2 , however, indicate that the exact threshold at small Q^2 deviates from its LD value [15].

(1) For $\beta = 1$, one finds

$$F_{P\gamma\gamma}(Q^2, Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{4e_c^2 N_c f_P}{6Q^2} \frac{1}{4\pi^2 f_P^2} \times \int_{4m^2}^{\bar{s}_{\text{eff}}(1, Q^2 \rightarrow \infty)} ds \left(1 + \frac{2m^2}{s} \right) \sqrt{1 - \frac{4m^2}{s}}. \quad (3.18)$$

Therefore, in order to reproduce the correct pQCD asymptotics, we have to require the following relation for the effective threshold:

$$f_P^2 = \frac{1}{4\pi^2} \int_{4m^2}^{\bar{s}_{\text{eff}}(1, Q^2 \rightarrow \infty)} ds \left(1 + \frac{2m^2}{s} \right) \sqrt{1 - \frac{4m^2}{s}}. \quad (3.19)$$

(2) For $\beta = 0$, that is, for $Q_1^2 = 0$ and $Q_2^2 = Q^2 \rightarrow \infty$, one finds, at leading order in $1/Q^2$,

$$F_{P\gamma\gamma}(0, Q^2) \xrightarrow{Q^2 \rightarrow \infty} \frac{2e_c^2 N_c f_P}{Q^2} \frac{1}{4\pi^2 f_P^2} \times \int_{4m^2}^{\bar{s}_{\text{eff}}(0, Q^2 \rightarrow \infty)} ds \sqrt{1 - \frac{4m^2}{s}}. \quad (3.20)$$

Matching to pQCD requires

$$f_P^2 = \frac{1}{4\pi^2} \int_{4m^2}^{\bar{s}_{\text{eff}}(0, Q^2 \rightarrow \infty)} ds \sqrt{1 - \frac{4m^2}{s}}. \quad (3.21)$$

C. Effective threshold at finite Q^2 and predictions for $F_{\eta_c\gamma}(Q^2)$

Matching the LD outcomes for the form factor to the result of the QCD factorization theorem allows us to determine the effective thresholds at large Q^2 . In order to obtain predictions for the form factor at finite Q^2 , we have to understand the behavior of the effective threshold as a function of Q^2 . The LD *model* for the form factor for all Q^2 is obtained by assuming that, for all not too small Q^2 , $s_{\text{eff}}(\beta, Q^2) = s_{\text{eff}}(\beta, Q^2 \rightarrow \infty)$.

For the case of massless quarks, the above assumption appears rather natural since the effective threshold found by matching LD to pQCD at large Q^2 does not depend on β . For massive quarks, the effective threshold at large Q^2 turns out to be β dependent. Therefore, it may not seem obvious that the assumption of a β -dependent but Q^2 -independent effective threshold provides a good approximation to the exact effective threshold.

We have tested this assumption in the case of a non-relativistic quantum-mechanical potential model since there the exact form factor may be computed by solving the Schrödinger equation and thus the exact effective threshold may be calculated. For the $\bar{c}c$ pseudoscalar,

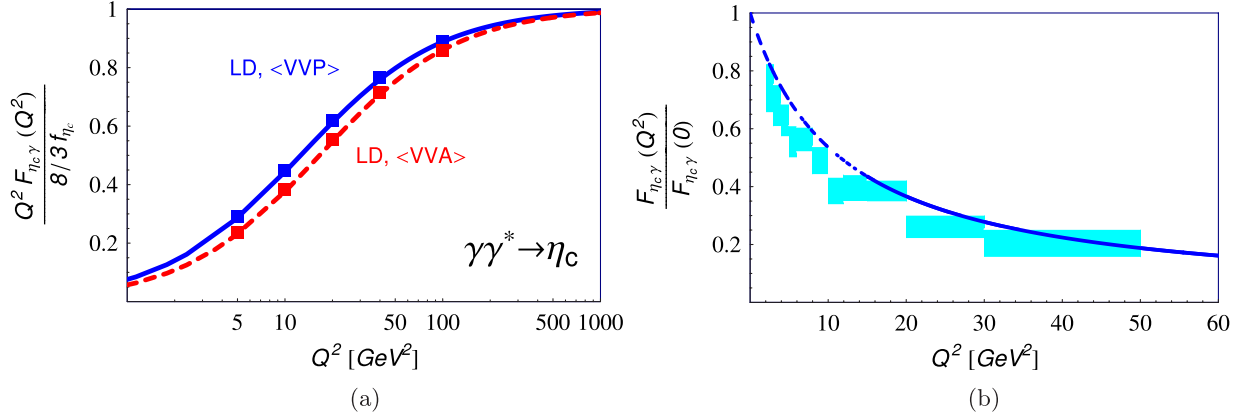


FIG. 2 (color online). Form factor $F_{\eta_c \gamma}(Q^2) \equiv F_{\eta_c \gamma \gamma}(0, Q^2)$ for the transition $\gamma\gamma^* \rightarrow \eta_c$: (a) Predictions from LD sum rules for the correlators $\langle PVV \rangle$ (blue solid line) and $\langle AVV \rangle$ (red dashed line). Blue and red boxes show a monopole fit $F_{\eta_c \gamma}(Q^2) = F_{\eta_c \gamma}(0) / (1 + Q^2/M_V^2)$ to the predictions of the LD model, with $F_{\eta_c \gamma}(0) = 8f_{\eta_c}/(3M_V^2)$, $M_V = 3.5$ GeV for $\langle PVV \rangle$, and $M_V = 4.0$ GeV for $\langle AVV \rangle$. (b) Comparison of the LD form factor predicted by the LD sum rule for the $\langle PVV \rangle$ correlator with recent BABAR measurements [5].

the exact threshold at a fixed value of β is found to be practically Q^2 independent in the region $Q^2 \geq 10$ – 15 GeV². We therefore believe that in this region the assumption of a Q^2 -independent threshold leads to trustworthy results.²

In order to get numerical estimates for the form factor, we adopt the charm-quark mass $\bar{m}_c(\bar{m}_c) = 1.29^{+0.05}_{-0.11}$ GeV [29] and the value $f_{\eta_c} = 0.3947 \pm 0.0024$ GeV of the η_c decay constant from lattice QCD [30]. The corresponding predictions for the form factors obtained from two different correlators are shown in Fig. 2(a). The assumption of Q^2 -independent thresholds for $\langle PVV \rangle$ and $\langle AVV \rangle$ leads to a spread of predictions for $F_{\eta_c \gamma}(Q^2)$ at finite Q^2 . Conservatively, this may be regarded as an indication of the expected accuracy of the LD model at the level of around 10%.

We stress once more that we cannot guarantee the applicability of the LD model at $Q^2 \leq 10$ – 15 GeV². Nevertheless, let us compare our LD predictions for small Q^2 with experiment. Using $\Gamma(\eta_c \rightarrow \gamma\gamma) = 7.20 \pm 2.12$ keV [29], one obtains $F_{\eta_c \gamma}(Q^2 = 0) = 0.08 \pm 0.01$ GeV⁻¹. The LD model using the $\langle AVV \rangle$ correlator yields $F_{\eta_c \gamma}(0) = 0.067$ GeV⁻¹; the LD form factor from the $\langle PVV \rangle$ correlator has $F_{\eta_c \gamma}(0) = 0.086$ GeV⁻¹. The latter value agrees very well with experiment, so optimistically one may expect the LD model for $\langle PVV \rangle$ to provide reliable predictions for the form factor for all Q^2 . Figure 2(b) compares these $\langle PVV \rangle$ results with the available experimental data from BABAR [5].

²A similar analysis [15] showed that, for light pseudoscalar mesons, the assumption of a Q^2 -independent threshold yields reliable results for the form factor in the region Q^2 larger than a few GeV². For smaller Q^2 , the threshold may differ sizably from the asymptotic threshold.

IV. CONCLUSIONS

We analyzed QCD sum rules for the $\gamma^* \gamma^* \rightarrow P$ transition form factor $F_{P\gamma\gamma}(Q_1^2, Q_2^2)$, utilizing two different three-point functions, $\langle AVV \rangle$ and $\langle PVV \rangle$, in the LD limit. We also revisited the decay constants f_P of massive $\bar{q}q$ pseudoscalar ground states from LD sum rules for the two-point functions $\langle AA \rangle$, $\langle AP \rangle$, and $\langle PP \rangle$, since f_P determines the asymptotics of the form factor $F_{P\gamma\gamma}(Q_1^2, Q_2^2)$ within the framework of pQCD factorization theorems. Our results are the following:

- (1) In the LD limit, the sum rules for the two-point functions $\langle AA \rangle$, $\langle AP \rangle$, and $\langle PP \rangle$ require different effective thresholds for the ground-state pseudoscalar meson. The sum rules and their thresholds coincide only in the nonrelativistic limit, i.e., for infinitely heavy quarkonia of finite radius.
- (2) Analyzing the form factors $F_{P\gamma\gamma}(Q_1^2, Q_2^2)$ obtained from LD QCD sum rules for the $\langle PVV \rangle$ and $\langle AVV \rangle$ three-point functions, we have determined the corresponding exact effective thresholds at large momentum transfer $Q^2 = Q_2^2$ and a fixed ratio $\beta \equiv Q_1^2/Q_2^2$ by matching the LD form factors to their pQCD asymptotic behavior for large Q^2 . These exact thresholds corresponding to $Q^2 \rightarrow \infty$ do depend on the ratio β . This perfectly confirms our previous findings that the effective thresholds in QCD sum rules depend, in general, on the external kinematical variables of the problem under consideration [31].
- (a) The chiral limit forms the sole exception: There the exact effective threshold for the $\langle AVV \rangle$ correlator does not depend on β and is equal to $s_{\text{eff}} = 4\pi^2 f_P^2$. Moreover, this effective threshold coincides with the effective threshold of the transverse part of $\langle AA \rangle$.

- (b) For massive quarks, the β -dependent effective thresholds for the three-point functions $\langle PVV \rangle$ and $\langle AVV \rangle$ turn out to differ from each other and from the thresholds of the two-point functions. Our results for the thresholds for these two three-point functions are given in Fig. 1.
- (3) The *LD model* for the form factor emerges if one assumes that the effective threshold $s_{\text{eff}}(\beta, Q^2)$ at finite Q^2 does not differ sizably from its asymptotic behavior $s_{\text{eff}}(\beta, Q^2 \rightarrow \infty)$. For light pseudoscalar mesons, this conjecture is found to be justified for Q^2 larger than a few GeV^2 , according to the results from quantum-mechanical potential models and to the experimental data on the $\pi\gamma$, $\eta\gamma$, and $\eta'\gamma$ form factors. For η_c , the nonrelativistic quantum-mechanical potential model reveals the exact effective threshold $s_{\text{eff}}(\beta, Q^2)$ to be close to $s_{\text{eff}}(\beta, Q^2 \rightarrow \infty)$ for $Q^2 \geq 10\text{--}15 \text{ GeV}^2$. Also in QCD, the LD approach is expected to yield reliable predictions for the $\eta_c\gamma$ transition form factor in this Q^2 region. Taking into account the results for the $F_{\eta_c\gamma}$ form factor derived from the $\langle PVV \rangle$ and $\langle AVV \rangle$ correlators, we conservatively estimate the accuracy of our predictions in this region of Q^2 to be around 10%; the accuracy improves rather fast with rising Q^2 . The numerical results for $F_{\eta_c\gamma}(Q^2)$ from QCD sum rules may be well described by a monopole parametrization. Combining the results from the $\langle PVV \rangle$ and $\langle AVV \rangle$ correlators, we obtain

$$F_{\eta_c\gamma}(Q^2) = \frac{F_{\eta_c\gamma}(0)}{1 + Q^2/M_V^2},$$

$$F_{\eta_c\gamma}(0) = \frac{2e_c^2 N_c f_{\eta_c}}{M_V^2},$$

$$M_V = 3.75 \pm 0.25 \text{ GeV}.$$

The Lepage–Brodsky approximate formula for the $\pi\gamma$ form factor [17], interpolating between the axial anomaly at $Q^2 = 0$ and the pQCD asymptotics at $Q^2 \rightarrow \infty$, too may be cast into this form, with $M_V = 2\pi f_\pi$ and the relevant charge factor $(e_u^2 - e_d^2)/\sqrt{2}$ replacing e_c^2 . For the pion, $M_V = 2\pi f_\pi = 0.81 \text{ GeV}$ is close to the ρ -meson mass. Thus, the predictions of LD QCD sum rules for both light and heavy pseudoscalars may be reasonably interpolated by the monopole formula

$$F_{P\gamma}(Q^2) = \frac{2e_P^2 N_c f_P}{Q^2 + M_V^2}, \quad (4.1)$$

with the mass parameter M_V not far from the mass of the ground-state vector meson with the relevant quantum numbers and e_P^2 the corresponding charge factor.

- (4) We investigated the onset of the pQCD behavior of $F_{\eta_c\gamma}(Q^2)$ and found that, at $Q^2 = 100 \text{ GeV}^2$, the form factor already reaches about 90% of its pQCD factorization value. This conclusion does not depend on the choice of the correlator and is thus a solid prediction of the LD QCD sum rules. The onset of the pQCD behavior of $F_{\eta_c\gamma}(Q^2)$ is delayed with Q^2 , compared to the case of the light pseudoscalars. Note, however, that we predict a much faster onset of the pQCD regime than a recent analysis [13], where the form factor at $Q^2 = 100 \text{ GeV}^2$ reaches only 65% of its asymptotic value.

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