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Four-point functions of the stress tensor and conserved currents in AdS_4/CFT_3

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We compute four-point functions of the stress tensor and conserved currents in AdS_4/CFT_3 using bulk perturbation theory. We work at tree-level in the bulk theory, which we take to be either pure gravity or Yang-Mills theory in AdS. We bypass the tedious evaluation of Witten diagrams using recently developed recursion relations for these correlators. In this approach, the four-point function is obtained as the sum of residues of a rational function at easily identifiable poles. We write down an explicit formula for the four-point correlator with arbitrary external helicities and momenta. We verify that, precisely as conjectured in a companion paper, the Maximally Helicity Violating (MHV) amplitude of gravitons or gluons appears as the coefficient of a specified singularity in the MHV stress-tensor or current correlator. We comment on the remarkably simple analytic structure of our answers in momentum space.

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I. INTRODUCTION

It is remarkable that even fifteen years after AdS/CFT was discovered [1] there are very few explicit computations of boundary correlation functions, at four points or higher, from the bulk point of view [2]. In principle, such computations are straightforward: we need to link interaction vertices with bulk-bulk and bulk-boundary propagators, and integrate over their positions [3]. However, in practice these computations are difficult for two reasons. First, it is hard to do the bulk integrals in closed form, although this can be sidestepped by using clever tricks [4] or by transforming to Mellin space [5]. However, in theories like gravity the interaction vertices themselves are very complicated. For example, the four-point vertex, even in four-dimensional flat space, contains 2850 terms [6] and is even more complicated in AdS. Consequently, the four-point function of the stress tensor has never before been computed explicitly.

In this paper, we point out that going to momentum space on the boundary in AdS_4 solves both these problems at once for correlators of conserved currents of the stress tensor. In AdS_4, the momentum-space bulk to boundary and bulk to bulk propagators for gluons and gravitons can be written in terms of elementary functions and so doing z-integrals is very simple.

Furthermore, by generalizing insights from flat space computations of gravity and Yang-Mills amplitudes [7,8], we are able to sidestep the tedious evaluation of interaction vertices. Instead, we compute explicit expressions for the four-point functions of the stress tensor by using the known three-point functions as input and combining this with a knowledge of the analytic properties of the correlators.

More specifically, we implement the recursion relations proposed in a companion paper [9]. These relations allow us to write down a formula for the four-point function in terms of residues of a rational function, at specific poles. This rational function is obtained through the product of two deformed three-point "transition amplitudes" as we explain in more detail in Sec. IV.

In [9], it was pointed out that momentum space also allows us to take an elegant "flat space limit." Here we take the flat space limit of our results for correlation functions of the stress tensor and obtain exactly the famous formulas for maximally helicity violating (MHV) amplitudes of gluons and gravitons in four-dimensional flat space.

To facilitate this comparison, we write our results for MHV correlators in the spinor helicity formalism that was originally developed for four-dimensional flat space amplitudes but, as pointed out in [10], is also useful for computations in AdS_4.

Apart from this flat space limit, our answers have some other interesting structural properties. For example, we can immediately see the contribution of the entire *conformal block* of the stress tensor itself in the correlator. Once again momentum space makes this simple. Here, we just have to multiply correlators to get the contribution of the primary and all its descendants rather than worrying about the complicated expressions for conformal blocks in position space.

Second, the transition amplitudes that appear in our computations are finite. So we avoid the divergences that appear in momentum space AdS integrals from the region near the boundary. An interesting consequence of this is that the correlators can be written as *rational functions* of the external spinors, and norms of partial sums of the external momenta. In particular we do not find any logarithms in our answers. We comment more on this interesting fact in the Discussion section.

A brief overview of this paper is as follows. We start by reviewing the spinor helicity formalism in Sec. II. The four-point computations require three-point transition amplitudes, which are very similar to correlators, but are obtained by replacing a bulk to boundary propagator with a normalizable mode. We compute these three-point functions for Yang-Mills theory and gravity in Sec. III. In Sec. IV, we use these results to write down an explicit formula for the four-point function in terms of residues of a rational function at prespecified poles. In Sec. V, we evaluate this formula for correlators of conserved currents, with two positive helicity and two negative helicity insertions, using the spinor helicity formalism and verify that, in the flat space limit, it reduces to the scattering amplitude of gluons, as conjectured in [9]. In Sec. VI, we evaluate this formula for stress tensor correlators, with the same combination of helicities, and, once again verify, that in the flat space limit it reduces to the maximally helicity violating amplitude of gravitons.

The main idea of this paper is presented in Sec. IV and the reader who is not interested in the fine details of four-point correlators can skip straight to this section. Moreover, we should warn the reader that some of the computations in Secs. V and VI are a little heavy on algebra. For this reason, we have provided a Mathematica program [11] that can be used to automate the formulas that are implemented there.

II. SPINOR HELICITY FORMALISM

We start by reviewing the spinor helicity formalism for correlation functions in three-dimensional conformal field theories that was introduced in [10].

We will use the mostly positive metric. So, for two vectors on the boundary

$$\boldsymbol{k}\cdot\boldsymbol{k} = (k_1)^2 + (k_2)^2 - (k_0)^2. \tag{2.1}$$

In this paper, just as in [9] we use bold-face for vectors but not their components. We use i, j etc. for boundary spacetime indices and μ, ν etc. for bulk spacetime indices. We use m, n etc. to index particle-numbers but one difference from [9] is that here it is convenient to place this index in subscripts rather than superscripts. Also, the components of a momentum vector come with a naturally lowered index.

Our σ matrix conventions are the following:

$$\sigma^0_{\alpha\dot\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1_{\alpha\dot\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2_{\alpha\dot\alpha} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3_{\alpha\dot\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

Given a three-momentum $\boldsymbol{k} = (k_0, k_1, k_2)$, we convert it into spinors using

$$k_{\alpha\dot\alpha} = k_0\sigma^0_{\alpha\dot\alpha} + k_1\sigma^1 + k_2\sigma^2 + i|\boldsymbol{k}|\sigma^3 = \lambda_\alpha\bar\lambda_{\dot\alpha}, \tag{2.3}$$

where

$$|\boldsymbol{k}| \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}} = \sqrt{k_1^2 + k_2^2 - k_0^2}. \tag{2.4}$$

If $\boldsymbol{k}$ is spacelike to start with, then the σ^3 component will be imaginary.

In components, we have the following expressions for the spinors

$$\lambda = \left(\sqrt{k_0 + i|\boldsymbol{k}|}, \frac{k_1 + ik_2}{\sqrt{k_0 + i|\boldsymbol{k}|}}\right),$$
$$\bar\lambda = \left(\sqrt{k_0 + i|\boldsymbol{k}|}, \frac{k_1 - ik_2}{\sqrt{k_0 + i|\boldsymbol{k}|}}\right). \tag{2.5}$$

We have the freedom to rescale the spinors by any complex number: $\lambda \to \alpha\lambda$, $\bar\lambda \to \frac{1}{\alpha}\bar\lambda$, without changing the momentum. If we do this with spinors corresponding to an external particle, then this rescales the polarization vectors and amplitudes pick up a simple phase. However, when we use the recursion relations, we also need spinors for an internal particle when we cut a propagator to form the product of two amplitudes. In such cases, λ_{int} and $\bar\lambda_{\text{int}}$ always come together and so we can rescale them without any physical effect at all. For example, we could choose:

$$\lambda_{\text{int}} = \left(1, \frac{k_1 + ik_2}{k_0 + i|\boldsymbol{k}|}\right), \qquad \bar\lambda_{\text{int}} = (k_0 + i|\boldsymbol{k}|, k_1 - ik_2). \tag{2.6}$$

We can raise and lower spinor indices using the ϵ tensor. We choose the ϵ tensor to be $i\sigma_2$ for both the dotted and the undotted indices. This means that

$$\epsilon^{01} = 1 = -\epsilon^{10}, \tag{2.7}$$

and spinor dot products are defined via

$$\langle\lambda_1, \lambda_2\rangle = \epsilon^{\alpha\beta}\lambda_{1\alpha}\lambda_{2\beta}, = \lambda_{1\alpha}\lambda_2^\alpha,$$
$$\langle\bar\lambda_1\bar\lambda_2\rangle = \epsilon^{\dot\alpha\dot\beta}\bar\lambda_{1\dot\alpha}\bar\lambda_{2\dot\beta} = \lambda_{1\dot\alpha}\bar\lambda_1^{\dot\alpha}. \tag{2.8}$$

However, we should expect our expressions for CFT_3 correlators to only have a manifest $SO(2,1)$ invariance. This means that we might have mixed products between dotted and undotted indices. Such a mixed product extracts the z-component of vector and is performed by contracting with σ^3

$$2i|\boldsymbol{k}| = (\sigma^3)^{\alpha\dot\alpha}k_{\alpha\dot\alpha} \equiv [\lambda, \bar\lambda], \tag{2.9}$$

The reader should note that we use square brackets only for this mixed product; products of both left- and right-handed spinors are denoted by angular brackets. Second, we note that this mixed dot product is symmetric:

$$[\lambda, \bar\lambda] = [\bar\lambda, \lambda]. \tag{2.10}$$

When we take the dot products of two 3-momenta, we have

$$\boldsymbol{k}\cdot\boldsymbol{q} \equiv (k_1q_1 + k_2q_2 - k_0q_0)$$
$$= -\tfrac{1}{2}\left(\langle\lambda_k, \lambda_q\rangle\langle\bar\lambda_k, \bar\lambda_q\rangle + \tfrac{1}{2}[\lambda_k, \bar\lambda_k][\lambda_q, \bar\lambda_q]\right). \tag{2.11}$$

Note that we have made a choice of the metric that is mostly positive.

Another fact to keep in mind is that

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 \Rightarrow \lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 = \lambda_3\bar{\lambda}_3 + \tfrac{1}{2}([\lambda_1, \bar{\lambda}_1] + [\lambda_2, \bar{\lambda}_2] - [\lambda_3, \bar{\lambda}_3])\sigma^3. \tag{2.12}$$

We also need a way to convert dotted to undotted indices. We write

$$\hat{\lambda}_{\dot{\alpha}} = \sigma^3_{\alpha\dot{\alpha}}\lambda^{\alpha}, \qquad \hat{\bar{\lambda}}_{\alpha} = \sigma^3_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}. \tag{2.13}$$

This has the property that

$$\langle \bar{\mu}, \hat{\lambda}\rangle = [\bar{\mu}, \lambda], \tag{2.14}$$

where the quantity on the right-hand side is defined in (2.9).

With all this, we can write down polarization vectors for conserved currents. The polarization vectors for a momentum vector $\boldsymbol{k}$ associated with spinors $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}}$ are given by

$$\epsilon^+_{\alpha\dot{\alpha}} = 2\frac{\hat{\bar{\lambda}}_\alpha \bar{\lambda}_{\dot{\alpha}}}{[\lambda, \bar{\lambda}]} = \frac{\hat{\bar{\lambda}}_\alpha \bar{\lambda}_{\dot{\alpha}}}{i|\boldsymbol{k}|}, \qquad \epsilon^-_{\alpha\dot{\alpha}} = 2\frac{\lambda_\alpha \hat{\lambda}_{\dot{\alpha}}}{[\lambda, \bar{\lambda}]} = \frac{\lambda_\alpha \hat{\lambda}_{\dot{\alpha}}}{i|\boldsymbol{k}|}. \tag{2.15}$$

These vectors are normalized so that

$$\boldsymbol{\epsilon}^+ \cdot \boldsymbol{\epsilon}^+ = \boldsymbol{\epsilon}^- \cdot \boldsymbol{\epsilon}^- = 0, \qquad \boldsymbol{\epsilon}^+ \cdot \boldsymbol{\epsilon}^- = 2. \tag{2.16}$$

Polarization tensors for the stress tensor are just outer-products of these vectors with themselves:

$$e^{\pm}_{ij} = \epsilon^{\pm}_i \epsilon^{\pm}_j. \tag{2.17}$$

In this paper, we will compute three and four-point functions of the stress tensor and conserved currents in momentum space:

$$T(\boldsymbol{e_1}, \boldsymbol{k_1}, \ldots, \boldsymbol{e_n}, \boldsymbol{k_n}) = e_{1 i_1 j_1} \ldots e_{n i_n j_n} \langle T^{i_1 j_1}(\boldsymbol{k}^1) \ldots T^{i_n j_n}(\boldsymbol{k}^n)\rangle, \tag{2.18}$$

where

$$\langle T^{i_1 j_1}(\boldsymbol{k}^1) \ldots T^{i_n j_n}(\boldsymbol{k}^n)\rangle \equiv \int \langle \mathcal{T}\{T^{i_1 j_1}(\boldsymbol{x_1}) \ldots T^{i_n j_n}(\boldsymbol{x_n})\}\rangle \times e^{i\sum_{m=1}^n \boldsymbol{k_m}\cdot \boldsymbol{x_m}} d^d x_m, \tag{2.19}$$

and $\mathcal{T}$ is the time-ordering symbol. Given the explicit formulas for polarization vectors, we can also label correlators using the helicity and momenta of the various insertions. For example:

$$T(+, \boldsymbol{k_1}, +, \boldsymbol{k_2}, -, \boldsymbol{k_3}) \equiv T(\boldsymbol{e}^+_{\boldsymbol{1}}, \boldsymbol{k_1}, \boldsymbol{e}^+_{\boldsymbol{2}}, \boldsymbol{k_2}, \boldsymbol{e}^-_{\boldsymbol{3}}, \boldsymbol{k_3}). \tag{2.20}$$

We will use the same notation to refer to correlators of conserved currents, and the meaning should be clear from the context.

III. THREE-POINT TRANSITION AMPLITUDES

In this section we compute three-point transition amplitudes that are an essential building block for the four-point computation. We start with Yang-Mills theory where the Feynman rules are quite easy to establish and then move on to gravity.

We remind the reader that transition amplitudes are computed by replacing one bulk to boundary propagator with a normalizable mode. Below, we use spinors $\lambda_1, \ldots \lambda_3$, and $\bar{\lambda}_1, \ldots \bar{\lambda}_3$ to specify the three momenta in the amplitude. We will use $p = -i|\boldsymbol{k_3}|$ to indicate that this leg is distinguished because it is the one that is associated with the normalizable mode.

A. Yang-Mills theory

Let us first review the form of the gauge-boson bulk to boundary propagator in AdS_4. As we will see below for both gauge-bosons and gravitons propagating in Ads_4, the bulk to boundary and bulk to bulk propagators are very simple in momentum space. This allows us to easily perform the z-integrals that appear in transition amplitudes.

It is convenient to work in "axial gauge" where we set the radial-component to 0. In this gauge, the non-normalizable free wave-functions in AdS (these are the same as the bulk to boundary propagators) are given by

$$A_i = \sqrt{\frac{2}{\pi}}\epsilon_i(|\boldsymbol{k}|z)^{1/2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} K_{1/2}(|\boldsymbol{k}|z) = \epsilon_i e^{-|\boldsymbol{k}|z} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}; \quad A_0 = 0; \qquad \boldsymbol{\epsilon}\cdot\boldsymbol{k} = 0, \tag{3.1}$$

for $\boldsymbol{k}^2 > 0$ i.e. for spacelike momenta. For timelike momenta, the modified Bessel function K should replaced by a Hankel function of the first kind—$H^{(1)}(|\boldsymbol{k}|z)$. However, it is more convenient to continue using the expressions above and simply interpret $|\boldsymbol{k}|$ as an imaginary quantity when $\boldsymbol{k}$ is timelike.[1]

This normalization of the bulk-boundary propagators is chosen so that the two-point function of the currents is normalized as:

$$T(\boldsymbol{\epsilon_1}, \boldsymbol{k_1}, \boldsymbol{\epsilon_2}, \boldsymbol{k_2}) = -(2\pi)^3 i\delta^3(\boldsymbol{k_1} - \boldsymbol{k_2})(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2})|\boldsymbol{k_1}|. \tag{3.2}$$

Below, we will also need the normalizable free wave-function of the gauge field, which is

$$A_i = \epsilon_i z^{1/2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} J_{1/2}(|\boldsymbol{k}|z); \qquad A_0 = 0; \qquad \boldsymbol{\epsilon}\cdot\boldsymbol{k} = 0, \tag{3.3}$$

and exists only for *timelike momenta*, $\boldsymbol{k}^2 > 0$. We caution the reader that we have normalized this solution differently from the bulk to boundary propagator.

[1]Strictly speaking, the identity, $H^{(1)}_\alpha(x) = \frac{2}{\pi}(-i)^{\alpha+1}K_\alpha(-ix)$ tells us that we should take $|\boldsymbol{k}|$ to have a *negative* imaginary part for timelike momenta.

Next, we need the Feynman rules for Yang-Mills theory in AdS. For simplicity, let us consider color-ordered correlators, which correspond to color-ordered amplitudes in the bulk. The Feynman rules for color-ordered diagrams, as generalized to AdS, have a three-point and a four-point vertex, which is given by:

$$V_3 = \frac{1}{\sqrt{2}}[(a_1^\mu \overleftrightarrow{\nabla}_\nu (a_2)_\mu) a_3^\nu + (a_3^\mu \overleftrightarrow{\nabla}_\nu (a_1)_\mu) a_2^\nu + (a_2^\mu \overleftrightarrow{\nabla}_\nu (a_3)_\mu) a_1^\nu],$$
$$V_4 = i\left[(\boldsymbol{a_1}\cdot\boldsymbol{a_3})(\boldsymbol{a_2}\cdot\boldsymbol{a_4}) - \frac{1}{2}(\boldsymbol{a_1}\cdot\boldsymbol{a_2})(\boldsymbol{a_3}\cdot\boldsymbol{a_4}) - \frac{1}{2}(\boldsymbol{a_1}\cdot\boldsymbol{a_4})(\boldsymbol{a_2}\cdot\boldsymbol{a_3})\right], \tag{3.4}$$

where the a_n represent the external lines that meet at the vertex and $\boldsymbol{A}\overleftrightarrow{\nabla}\boldsymbol{B} \equiv \boldsymbol{A}\nabla\boldsymbol{B} - \boldsymbol{B}\nabla\boldsymbol{A}$ for two vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$. The connection coefficients are given by

$$\Gamma^\rho_{\alpha\beta} = \tfrac{1}{2}g^{\rho\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) = \tfrac{1}{z}(\delta^\rho_0 \eta_{\alpha\beta} - \delta^0_\alpha \delta^\rho_\beta - \delta^0_\beta \delta^\rho_\alpha). \tag{3.5}$$

Here 0 represents the radial direction in AdS.

Below we work out the three-point transition amplitudes for the different combinations of helicities. We will use the symbol E_p below to mean:

$$E_p \equiv (|\boldsymbol{k_1}| + |\boldsymbol{k_2}| + |\boldsymbol{k_3}|) = |\boldsymbol{k_1}| + |\boldsymbol{k_2}| + ip. \tag{3.6}$$

First, we note that for every amplitude we have a leading factor that comes from the z integrals

$$R^{\rm YM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p) = \frac{2\sqrt{|\boldsymbol{k_1}||\boldsymbol{k_2}|}}{\pi}\int_0^\infty z^{3/2}K_{1/2}(|\boldsymbol{k_1}|z)K_{1/2}(|\boldsymbol{k_2}|z)J_{1/2}(pz)dz = \frac{\sqrt{\frac{2p}{\pi}}}{|\boldsymbol{k_1}|^2 + 2|\boldsymbol{k_2}||\boldsymbol{k_1}| + |\boldsymbol{k_2}|^2 + p^2}. \tag{3.7}$$

We should remind the reader that the answers for correlators worked out in [10] involve the radial integral:

$$R^{\rm YM}_{\rm PM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, |\boldsymbol{k_3}|) = \left(\frac{2}{\pi}\right)^{3/2}\sqrt{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}|} \times \int_0^\infty z^{3/2}K_{1/2}(|\boldsymbol{k_1}|z)K_{1/2}(|\boldsymbol{k_2}|z)K_{1/2}(|\boldsymbol{k_3}|z)dz = \frac{1}{|\boldsymbol{k_1}| + |\boldsymbol{k_2}| + |\boldsymbol{k_3}|}. \tag{3.8}$$

Apart from a normalization factor that arises because the normalizable mode in the transition amplitude is normalized differently compared to the bulk-boundary propagators, the answers for transition amplitudes and correlators are related by the following simple substitution in the term that comes from the radial integral:

$$R^{\rm YM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, |\boldsymbol{p}|) = \frac{1}{\sqrt{2\pi p}}(R^{\rm YM}_{\rm PM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, -i|\boldsymbol{p}|) - R^{\rm YM}_{\rm PM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, i|\boldsymbol{p}|)). \tag{3.9}$$

The tensor structures that we need to compute is given by

$$T_3^* = R^{\rm YM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)\frac{i}{\sqrt{2}}\{(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2})(\boldsymbol{k_2} - \boldsymbol{k_1})\cdot\boldsymbol{\epsilon_3} + (2\boldsymbol{\epsilon_2}\cdot\boldsymbol{k_1})(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_3}) - 2(\boldsymbol{\epsilon_1}\cdot\boldsymbol{k_2})(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_3})\}. \tag{3.10}$$

(Here the dependence of T on the momenta and polarizations is not shown explicitly although we are using the subscript 3 to remind the reader that this is a three-point function and the superscript $*$ to indicate that this is a transition amplitude.) Except for the radial part, which is modified as explained above, the spinor expressions that we obtain from polarization contractions are the same as the expressions for correlators in [10].

1. ++− Amplitude

We use the polarization vectors given in Sec. II. Notice that there is a leading factor of $\frac{1}{\sqrt{2}}$ from the interaction vertex that gets multiplied with the norm factors from the denominator of the polarizations. We get another factor of 2 in the denominator when we convert momentum dot products to spinor contractions. With these observations, we see that the $++-$ amplitude is given by

$$T_3^*(+,+,-) = \frac{iR^{\rm YM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p}(\langle\bar\lambda_1, \bar\lambda_2\rangle^2\langle\lambda_2, \lambda_3\rangle \times [\bar\lambda_2, \lambda_3] + \langle\bar\lambda_2, \bar\lambda_1\rangle[\bar\lambda_2, \lambda_1][\bar\lambda_1, \lambda_3]^2 - \langle\bar\lambda_1, \bar\lambda_2\rangle[\bar\lambda_1, \lambda_2][\bar\lambda_2, \lambda_3]^2). \tag{3.11}$$

Actually each term inside the brackets is proportional to the same quantity and we can write the whole amplitude in the flat space MHV form multiplied by a prefactor. To see this, we note the following relations:

$$\lambda_1\bar\lambda_1 + \lambda_2\bar\lambda_2 + \lambda_3\bar\lambda_3 = iE_p\sigma^3, \tag{3.12}$$

which leads to

$$\langle\lambda_1, \lambda_3\rangle\langle\bar\lambda_3, \bar\lambda_2\rangle = -iE_p[\lambda_1, \bar\lambda_2], \tag{3.13}$$

and similar identities for other pairs of spinors. Moreover, we also have the identity

$$\langle\lambda_1, \lambda_2\rangle\langle\bar\lambda_1, \bar\lambda_2\rangle = -2((\boldsymbol{k_1}\cdot\boldsymbol{k_2}) - |\boldsymbol{k_1}||\boldsymbol{k_2}|) = (|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)^2 - (\boldsymbol{k_1} + \boldsymbol{k_2})^2 = -iE_p(p + i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)). \tag{3.14}$$

Substituting this we find that

$$T_3^*(+,+,-) = \frac{-R^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p}(|\boldsymbol{k_2}|+ip-|\boldsymbol{k_1}|) \times (ip+|\boldsymbol{k_1}|-|\boldsymbol{k_2}|)(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|-ip) \times \frac{\langle\bar\lambda_1,\bar\lambda_2\rangle^4}{\langle\bar\lambda_1,\bar\lambda_2\rangle\langle\bar\lambda_2,\bar\lambda_3\rangle\langle\bar\lambda_3,\bar\lambda_1\rangle}. \quad (3.15)$$

2. +++ Amplitude

The + + + amplitude is given by

$$T_3^*(+,+,+) = \frac{iR^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p}(\langle\bar\lambda_1,\bar\lambda_2\rangle^2[\lambda_2,\bar\lambda_3]\langle\bar\lambda_2,\bar\lambda_3\rangle + \langle\bar\lambda_2,\bar\lambda_1\rangle[\bar\lambda_2,\lambda_1]\langle\bar\lambda_1,\bar\lambda_3\rangle^2 - 2\langle\bar\lambda_1,\bar\lambda_2\rangle[\bar\lambda_1,\lambda_2]\langle\bar\lambda_2,\bar\lambda_3\rangle^2). \quad (3.16)$$

After using the identities above, we find that

$$T_3^*(+,+,+) = \frac{-R^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p} \times E_p\langle\bar\lambda_1,\bar\lambda_2\rangle\langle\bar\lambda_2,\bar\lambda_3\rangle\langle\bar\lambda_3,\bar\lambda_1\rangle. \quad (3.17)$$

3. −−− Amplitude

The − − − amplitude is related to the + + + amplitude by parity and is given by

$$T_3^*(-,-,-) = \frac{-R^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p} \times E_p\langle\lambda_1,\lambda_2\rangle\langle\lambda_2,\lambda_3\rangle\langle\lambda_3,\lambda_1\rangle. \quad (3.18)$$

4. −−+ Amplitude

The − − + amplitude is related to the + + − amplitude by parity and is given by

$$T_3^*(-,-,+) = \frac{-R^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_2}|p}(|\boldsymbol{k_2}|+ip-|\boldsymbol{k_1}|) \times (ip+|\boldsymbol{k_1}|-|\boldsymbol{k_2}|)(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|-ip) \times \frac{\langle\lambda_1,\lambda_2\rangle^4}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_2,\lambda_3\rangle\langle\lambda_3,\lambda_1\rangle}. \quad (3.19)$$

The list above covers all possible three-point transition amplitudes. The amplitude for any other combination of helicities can be obtained by just cyclically permuting the spinor expressions, while keeping $R^{\rm YM}$ unchanged.

5. Flat space limit

When the three-point amplitudes are written in the forms above, it is manifest that the flat space limit described in [9] holds. Let us remind the reader that in [9], we conjectured that the n-point conserved-current correlator in 3 dimensions and the flat space gluon-scattering amplitude in 4 dimensions should be related, at tree-level, through

$$M(\boldsymbol{\epsilon_1},\tilde{\boldsymbol{k}}_1,\ldots\boldsymbol{\epsilon_n},\tilde{\boldsymbol{k}}_n) = \lim_{(\sum|\boldsymbol{k}_m|)\to 0}\Big(\sum|\boldsymbol{k}_m|\Big)T(\boldsymbol{\epsilon_1},\boldsymbol{k}_1,\ldots\boldsymbol{\epsilon_n},\boldsymbol{k}_n). \quad (3.20)$$

Here $\tilde{\boldsymbol{k}}_m$ are the on-shell four-dimensional vectors produced by taking the three-dimensional vector $\boldsymbol{k}_m$ and appending its norm to form the four-dimensional vector $\tilde{\boldsymbol{k}}_m = \{\boldsymbol{k}_m, i|\boldsymbol{k}_m|\}$

Now, as we mentioned above, to compute correlators rather than transition amplitudes all we need to do is to replace $R^{\rm YM}$ above with $R^{\rm YM}_{\rm PM}$ defined in (3.8).

For example, looking at the + + − correlator (the − − + case works in exactly the same way), we see that

$$T_3(+,+,-) \underset{E_p\to 0}{\longrightarrow} i\frac{2\sqrt{2}}{E_p}\frac{\langle\bar\lambda_1,\bar\lambda_2\rangle^4}{\langle\bar\lambda_1,\bar\lambda_2\rangle\langle\bar\lambda_2,\bar\lambda_3\rangle\langle\bar\lambda_3,\bar\lambda_1\rangle}. \quad (3.21)$$

This is because at $E_p = 0$, we can replace $\frac{|\boldsymbol{k_1}|+|\boldsymbol{k_2}|-ip}{p} = -2i$ and so the numerator in $R^{\rm YM}_{\rm PM}$ neatly cancels with the denominator leaving behind the factor of $\frac{1}{E_p}$. Of course this is multiplied with the famous 3-point gluon amplitude in four dimensions, which is precisely what we expect from our flat-space conjecture.[2]

B. Gravity

We now turn to the computation of three-point transition amplitudes in the pure gravity theory, using the Hilbert action in the bulk. Just as we found above, we find that the answers for transition amplitudes are very similar to the answers for correlators, except that the part of the answer that comes from the radial integral over the bulk-boundary propagators gets modified as in (3.9).

The bulk to boundary propagator for gravity is given by the expression:

$$h_{ij}(\boldsymbol{e},\boldsymbol{k},\boldsymbol{x},z) = \frac{e_{ij}}{z^2}(|\boldsymbol{k}|z)^{d/2}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\sqrt{\frac{2}{\pi}}K_{3/2}(|\boldsymbol{k}|z) = \frac{e_{ij}}{z^2}(1+|\boldsymbol{k}|z)e^{-|\boldsymbol{k}|z}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad (3.22)$$

It is important to note that in (3.22), both indices on h are lowered. If one index had been raised the leading factor of $\frac{1}{z^2}$ would be absent. As explained above, this form of the bulk to boundary propagator is correct for spacelike momentum $\boldsymbol{k}\cdot\boldsymbol{k} > 0$. For timelike momentum, we should analytically continue the expression above while taking $|\boldsymbol{k}|$ to have a negative imaginary part.

[2] The careful reader might note that we have an extra factor of $2\sqrt{2}$. This is present because our polarization vectors are unconventionally normalized as shown in (2.16) so that $\boldsymbol{\epsilon}^+(\boldsymbol{k})\cdot\boldsymbol{\epsilon}^-(\boldsymbol{k}) = 2$. This normalization is convenient because below we will extend momentum vectors by their polarizations and this helps remove factors of $\sqrt{2}$ there; however these factors sometimes reappear in final results as above.

When we refer to the "normalizable mode" that enters transition amplitudes, we are referring to the solution

$$h_{ij}(\boldsymbol{e}, \boldsymbol{k}, \boldsymbol{x}, z) = \frac{1}{z^2} e_{ij} z^{3/2} J_{3/2}(|\boldsymbol{k}|z) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.23}$$

This is because it is this term that naturally enters the bulk to bulk propagator. We will have to be careful about these different normalizations when we compare correlators to transition amplitudes below.

1. Interaction vertices

To obtain the three-point gravity transition amplitude, we first need to expand the Hilbert action out to third order in fluctuations.[3] The computations in this subsection were performed using the excellent program Xact [13] that allowed us to automate the tensor manipulations below.

We start with the cubic action given in [14]

$$S = \frac{-1}{6} \int \sqrt{|g|} \Big(V_{\mu\nu} - \frac{1}{2} g_{\mu\nu} V \Big) h^{\mu\nu}, \tag{3.24}$$

where

$$\begin{aligned} V^{\mu\nu} = &-\nabla_\rho (h^{\rho\sigma}(\nabla^\mu h^\nu_\sigma + \nabla^\nu h^\mu_\sigma - \nabla_\sigma h^{\mu\nu})) \\ &+ \nabla^\nu (h^{\rho\sigma}\nabla^\mu h_{\rho\sigma}) + \frac{1}{2}(\nabla^\mu h^\nu_\rho + \nabla^\nu h^\mu_\rho \\ &- \nabla_\rho h^{\mu\nu})\nabla^\rho h^{ab} g_{ab} - \frac{1}{2}\nabla^\mu h_{\rho\sigma}\nabla^\nu h^{\rho\sigma} \\ &+ \nabla_\rho h^\mu_\sigma \nabla^\rho h^{\sigma\nu} - \nabla_\sigma h^\mu_\rho \nabla^\rho h^{\sigma\nu}. \end{aligned} \tag{3.25}$$

This term still contains various terms with a double derivative. To remove these we need to add the two-derivative terms

$$\begin{aligned} B = &-\frac{1}{2} g^{cd} g_{\mu\nu} \nabla_d (h^{\mu\nu} h^{\rho\sigma} \nabla_c h_{\rho\sigma}) + \nabla_\nu (h^{\mu\nu} h^{\rho\sigma} \nabla_\mu h_{\rho\sigma}) \\ &+ \frac{1}{2} g^{cd} g_{\mu\nu} \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_c h_{d\sigma}) \\ &+ \frac{1}{2} g^{cd} g_{\mu\nu} \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_d h_{c\sigma}) \\ &- \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_\mu h_{\nu\sigma}) - \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_\nu h_{\mu\sigma}) \\ &- \frac{1}{2} g^{cd} g_{\mu\nu} \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_\sigma h_{cd}) + \nabla_\rho (h^{\mu\nu} h^{\rho\sigma} \nabla_\sigma h_{\mu\nu}). \end{aligned} \tag{3.26}$$

Here, we should note that although in position space these terms can only give a delta function contribution to boundary correlators, we could have been worried about them in momentum space. This is because the recursion relations involve multiplying two three-point functions in momentum space, or convoluting two three-point functions in position space; in this manner what was a delta function contribution may become important. However, fortuitously, when we evaluate this boundary term on linearized solutions to the equations of motion in the gauge

$$h^{0\mu} = 0, \qquad h^\mu_\mu = 0, \tag{3.27}$$

and *on-shell*, so that

$$\partial_\mu h^{\mu\nu} = 0, \tag{3.28}$$

the boundary terms in B genuinely vanish upon integration. We find then that

$$\begin{aligned} (V_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} V) h^{\mu\nu} - B = &\tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_c h_{\rho\sigma} \nabla_d h^{\mu\nu} + \tfrac{1}{4} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_c h_{\rho\sigma} \nabla_d h^{\rho\sigma} - h^{\rho\sigma} \nabla_\mu h_{\rho\sigma} \nabla_\nu h^{\mu\nu} \\ &- \tfrac{1}{2} h^{\mu\nu} \nabla_\mu h_{\rho\sigma} \nabla_\nu h^{\rho\sigma} - \tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_c h_{d\sigma} \nabla_\rho h^{\mu\nu} - \tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_d h_{c\sigma} \nabla_\rho h^{\mu\nu} \\ &+ h^{\rho\sigma} \nabla_\mu h_{\nu\sigma} \nabla_\rho h^{\mu\nu} - h^{\mu\nu} \nabla_\nu h_{\mu\sigma} \nabla_\rho h^{\rho\sigma} - \tfrac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_c h_{d\rho} \nabla^\rho h^{ab} \\ &- \tfrac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_d h_{c\rho} \nabla^\rho h^{ab} + \tfrac{1}{2} g_{ab} h^{\mu\nu} \nabla_\mu h_{\nu\rho} \nabla^\rho h^{ab} + \tfrac{1}{2} g_{ab} h^{\mu\nu} \nabla_\nu h_{\mu\rho} \nabla^\rho h^{ab} \\ &+ \tfrac{1}{4} g_{ab} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_\rho h_{cd} \nabla^\rho h^{ab} - \tfrac{1}{2} g_{ab} h^{\mu\nu} \nabla_\rho h_{\mu\nu} \nabla^\rho h^{ab} - \tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla_\rho h_{c\sigma} \nabla^\rho h^\sigma{}_d \\ &+ h^{\mu\nu} \nabla_\rho h_{\mu\sigma} \nabla^\rho h^\sigma{}_\nu + \tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\rho\sigma} \nabla_\rho h^{\mu\nu} \nabla_\sigma h_{cd} + \tfrac{1}{2} g^{cd} g_{\mu\nu} h^{\mu\nu} \nabla^\rho h^\sigma{}_d \nabla_\sigma h_{c\rho} \\ &- h^{\rho\sigma} \nabla_\rho h^{\mu\nu} \nabla_\sigma h_{\mu\nu} - h^{\mu\nu} \nabla^\rho h^\sigma{}_\nu \nabla_\sigma h_{\mu\rho}. \end{aligned} \tag{3.29}$$

We write this term out in detail because this is what we would have to use if we were to try and compute an exchange Feynman diagram. Fortunately, in our method we only need the *on-shell* three-point function. When we impose just the traceless conditions from (3.27), we find a remarkable simplification:

$$\begin{aligned} (V_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} V) h^{\mu\nu} - B = &-h^{\mu\nu} (\tfrac{3}{2} \nabla_\mu h^{\rho\sigma} \nabla_\nu h_{\rho\sigma} + + \nabla^\rho h_{\mu\nu} \nabla_\sigma h_\rho{}^\sigma - 2 \nabla_\nu h_{\rho\sigma} \nabla^\sigma h_\mu{}^\rho + \nabla_\rho h_{\nu\sigma} \nabla^\sigma h_\mu{}^\rho \\ &- \nabla_\sigma h_{\nu\rho} \nabla^\sigma h_\mu{}^\rho). \end{aligned} \tag{3.30}$$

[3]It is possible to obtain the three-point on-shell amplitude without going through this process, and by using the flat space result, as was done in [12]. Our approach is more direct. It also has the advantage that it helps us keep track of the boundary terms that we are adding to the action. It also lets us see how the on-shell computation is much simpler than using Feynman rules.

We now convert all covariant derivatives to partial derivatives using the connection coefficients (3.5), and again impose the tracelessness condition on h. This leads to

$$(V_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}V)h^{\mu\nu} - B = 4h_a{}^c h^{ab}h_{bc} - \tfrac{3}{2}h^{ab}\partial_a h^{cd}\partial_b h_{cd} - h^{ab}\partial^c h_{ab}\partial_d h_c^d \tag{3.31}$$

$$+ 2h^{ab}\partial_b h_{cd}\partial^d h_a{}^c - h^{ab}\partial_c h_{bd}\partial^d h_a{}^c + h^{ab}\partial_d h_{bc}\partial^d h_a{}^c + 2zh_a{}^c h^{ab}\partial_0 h_{bc}. \tag{3.32}$$

If we now also use the on-shell condition $\partial_a h^{ab} = 0$, then the third term in the first line above [Eq. (3.31)] drops out. This results in

$$(V_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}V)h^{\mu\nu} - B = 4h_a{}^c h^{ab}h_{bc} - \tfrac{3}{2}h^{ab}\partial_a h^{cd}\partial_b h_{cd} + 2h^{ab}\partial_b h_{cd}\partial^d h_a{}^c - h^{ab}\partial_c h_{bd}\partial^d h_a{}^c + h^{ab}\partial_d h_{bc}\partial^d h_a{}^c + 2zh_a{}^c h^{ab}\partial_0 h_{bc}. \tag{3.33}$$

By adding another total derivative term, which vanishes on-shell we find

$$(V_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}V)h^{\mu\nu} - B + \partial_c(h^{ab}h_{bd}\partial_d h_a^c) = 4h_a{}^c h^{ab}h_{bc} - \tfrac{3}{2}h^{ab}\partial_a h^{cd}\partial_b h_{cd} + 3h^{ab}\partial_b h_{cd}\partial^d h_a{}^c + h^{ab}\partial_d h_{bc}\partial^d h_a{}^c + 2zh_a{}^c h^{ab}\partial_0 h_{bc}. \tag{3.34}$$

We can make this even simpler and get rid of the z derivatives, if we remember that in (3.24), this term is multiplied by $\sqrt{-g} = \frac{1}{z^{d+1}}$. Now,

$$\begin{aligned}
&\frac{1}{z^{d+1}}(4h_a{}^c h^{ab}h_{bc} + h^{ab}\partial_d h_{bc}\partial^d h_a{}^c + 2zh_a{}^c h^{ab}\partial_0 h_{bc}) \\
&= \frac{1}{z^{d+1}}\left(4h_a^c h_b^a h_c^b + z^2\eta^{ij}h_b^a\partial_i(h_c^b)\partial_j h_a^c + z^4 h_a^b\partial_0\left(\frac{h_b^c}{z^2}\right)\partial_0 h_c^a + 2z^3 h_a^c h_b^a\partial_0\left(\frac{h_c^b}{z^2}\right)\right) \\
&= \frac{1}{z^{d+1}}\left(z^2\eta^{ij}h_b^a\partial_i(h_c^b)\partial_j h_a^c + \frac{1}{2}z^2\partial_0(h_a^b h_b^c\partial_0 h_c^a) - \frac{1}{2}z^2 h_a^b h_b^c\partial_0^2 h_c^a\right) \\
&= \frac{1}{2z^{d-1}}\eta^{ij}\partial_i(h_b^a h_c^b\partial_j h_a^c) + \frac{1}{2z^{d-1}}h_b^a h_c^b\eta^{ij}\partial_i\partial_j h_a^c + \frac{1}{2}\partial_0\left(\frac{1}{z^{d-1}}h_a^b h_b^c\partial_0 h_c^a\right) \\
&= -\frac{1}{2}h_a^b h_b^c\partial_0\left(\frac{1}{z^{d-1}}\partial_0 h_c^a\right).
\end{aligned} \tag{3.35}$$

However, on-shell, h precisely satisfies the equation (see, for example, the detailed review of perturbation theory in [15])

$$\frac{1}{z^{d-1}}h_b^a h_c^b\eta^{ij}\partial_i\partial_j h_a^c - \frac{1}{2}h_a^b h_b^c\partial_0\left(\frac{1}{z^{d-1}}\partial_0 h_c^a\right) = 0. \tag{3.36}$$

After recalling the factor of $\frac{-1}{6}$ in (3.24), this leads to the following expression for the three-point function.

$$\begin{aligned}
T(\boldsymbol{e_1},\boldsymbol{k_1},\boldsymbol{e_2},\boldsymbol{k_2},\boldsymbol{e_3},\boldsymbol{k_3}) &= \sum_\pi\int\frac{F_\pi}{z^{d-1}} \\
F_\pi &= \left(\frac{1}{4}(\boldsymbol{\epsilon_{\pi_1}}.\boldsymbol{k_{\pi_2}})(\boldsymbol{\epsilon_{\pi_1}}.\boldsymbol{k_{\pi_3}})(\boldsymbol{\epsilon_{\pi_2}}.\boldsymbol{\epsilon_{\pi_3}})^2 - \frac{1}{2}(\boldsymbol{\epsilon_{\pi_1}}.\boldsymbol{k_{\pi_2}})(\boldsymbol{\epsilon_{\pi_1}}.\boldsymbol{\epsilon_{\pi_3}})(\boldsymbol{\epsilon_{\pi_2}}.\boldsymbol{k_{\pi_3}})(\boldsymbol{\epsilon_{\pi_2}}.\boldsymbol{\epsilon_{\pi_3}})\right) \\
&\quad\times\phi(|\boldsymbol{k_1}|z)\phi(|\boldsymbol{k_2}|z)\phi(|\boldsymbol{k_3}|z),
\end{aligned} \tag{3.37}$$

where π runs over the permutation group of 3 elements and ϕ is the radial part of the wave-function defined by

$$h_j^i(\boldsymbol{e},\boldsymbol{k},\boldsymbol{x},z) \equiv z^2\eta^{il}\epsilon_l\epsilon_j e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\phi(|\boldsymbol{k}|z), \tag{3.38}$$

with the linearized solutions h_j^i defined in (3.22). Note that ϕ carries information about whether any of the wave functions we are using is normalizable.

2. Answers for three-point stress tensor transition amplitudes

Let us now compute the three-point transition amplitudes of the stress-tensor that we need to compute four-point correlators. First we need the radial integral; this gives a polarization-independent part of the answer, which is then multiplied by some function that depends on the polarizations.

The radial integral gives us

$$\begin{aligned} R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p) &= \int \phi(|\boldsymbol{k_1}|z)\phi(|\boldsymbol{k_2}|z)\phi(|\boldsymbol{k_3}|z)\frac{dz}{z^2} \\ &= \frac{2}{\pi}(|\boldsymbol{k_1}||\boldsymbol{k_2}|)^{3/2} \\ &\times \int \frac{dz}{z^2} z^{3/2}K_{3/2}(|\boldsymbol{k_1}|z)z^{3/2}K_{3/2}(|\boldsymbol{k_2}|z)z^{3/2}J_{3/2}(pz) \\ &= \frac{p^{3/2}(|\boldsymbol{k_1}|^2 + 4|\boldsymbol{k_2}||\boldsymbol{k_1}| + |\boldsymbol{k_2}|^2 + p^2)\sqrt{\frac{2}{\pi}}}{(|\boldsymbol{k_1}|^2 + 2|\boldsymbol{k_2}||\boldsymbol{k_1}| + |\boldsymbol{k_2}|^2 + p^2)^2}. \end{aligned} \tag{3.39}$$

The prefactor that enters the correlator was calculated in [10] and is given by:

$$\begin{aligned} R^{\rm gr}_{\rm PM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, |\boldsymbol{k_3}|) &= \left(\frac{2}{\pi}|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}|\right)^{3/2} \\ &\times \int_0^\infty \frac{1}{z^2} z^{3/2}K_{3/2}(|\boldsymbol{k_1}|z)z^{3/2}K_{3/2}(|\boldsymbol{k_2}|z)z^{3/2}K_{3/2}(|\boldsymbol{k_3}|z) \\ &= -\frac{|\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_1}|}{(|\boldsymbol{k_1}| + |\boldsymbol{k_2}| + |\boldsymbol{k_3}|)^2} + |\boldsymbol{k_1}| + |\boldsymbol{k_2}| + |\boldsymbol{k_3}| \\ &- \frac{|\boldsymbol{k_1}||\boldsymbol{k_2}| + |\boldsymbol{k_3}||\boldsymbol{k_2}| + |\boldsymbol{k_1}||\boldsymbol{k_3}|}{|\boldsymbol{k_1}| + |\boldsymbol{k_2}| + |\boldsymbol{k_3}|}. \end{aligned} \tag{3.40}$$

The radial integral for the transition amplitude is convergent but, in the case of the correlator, it is divergent; the value in (3.40) comes from cutting it off at $z = \epsilon$ and picking up the ϵ^0 piece. In this respect, transition amplitudes are nicer than correlators. We discuss this in some more detail below.

Apart from an overall normalization, which appears because the normalizable mode is normalized differently from the bulk-boundary propagator, note that the answer for the radial integral that enters the transition amplitude is closely related to the term that enters the correlator:

$$\begin{aligned} R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p) = -i\sqrt{\frac{2}{\pi}}p^{-(3/2)}(R^{\rm gr}_{\rm PM}(\boldsymbol{k_1}, \boldsymbol{k_2}, ip) \\ - R^{\rm gr}_{\rm PM}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, -ip)). \end{aligned} \tag{3.41}$$

We adopt the same notation as (3.6) for the variable E_p. With this definition, we find that, except for the part the comes from the radial integrals, the graviton transition amplitudes can be written as the "square" of the gauge boson amplitudes explored in the previous subsection.

$++-$ *Amplitude* Let us consider the first term in the sum over permutations of (3.37), and take particles 1 and 2 to have positive helicity, and particle 3 to have negative helicity. Then we find

$$\begin{aligned} F_1 = \frac{-R^{\rm gr}(k_1, k_2, p)}{16|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2}\langle\bar\lambda_1, \bar\lambda_2\rangle[\bar\lambda_1, \lambda_2][\bar\lambda_1, \lambda_3][\bar\lambda_2, \lambda_3]^3 \\ \times\left(\frac{-1}{4}\langle\bar\lambda_1, \bar\lambda_3\rangle[\bar\lambda_2, \lambda_3] - \frac{1}{2}[\bar\lambda_1, \lambda_3]\langle\bar\lambda_2, \bar\lambda_3\rangle\right). \end{aligned} \tag{3.42}$$

Then, after summing over permutations and using the identities of III A 1 we find that the $++-$ amplitude is given by:

$$\begin{aligned} T_3^*(+, +, -) = \frac{R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)}{32|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2}(|\boldsymbol{k_2}| + ip - |\boldsymbol{k_1}|)^2 \\ \times (ip + |\boldsymbol{k_1}| - |\boldsymbol{k_2}|)^2(|\boldsymbol{k_1}| + |\boldsymbol{k_2}| - ip)^2 \\ \times\left(\frac{\langle\bar\lambda_1, \bar\lambda_2\rangle^4}{\langle\langle\bar\lambda_1, \bar\lambda_2\rangle\langle\bar\lambda_2, \bar\lambda_3\rangle\langle\bar\lambda_3, \bar\lambda_1\rangle}\right)^2 \end{aligned} \tag{3.43}$$

$+++$ *Amplitude* The $+++$ amplitude is given by

$$\begin{aligned} T_3^*(+, +, +) = \frac{R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)}{32|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2} \\ \times E_p^2(\langle\bar\lambda_1, \bar\lambda_2\rangle\langle\bar\lambda_2, \bar\lambda_3\rangle\langle\bar\lambda_3, \bar\lambda_1\rangle)^2. \end{aligned} \tag{3.44}$$

$---$ *Amplitude* The $---$ amplitude is related to the $+++$ amplitude by parity and is given by

$$\begin{aligned} T_3^*(-, -, -) = \frac{R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)}{32|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2} \\ \times E_p^2(\langle\lambda_1, \lambda_2\rangle\langle\lambda_2, \lambda_3\rangle\langle\lambda_3, \lambda_1\rangle)^2. \end{aligned} \tag{3.45}$$

$--+$ *Amplitude* The $--+$ amplitude is related to the $++-$ amplitude by parity and is given by

$$\begin{aligned} T_3^*(-, -, +) = \frac{R^{\rm gr}(|\boldsymbol{k_1}|, |\boldsymbol{k_2}|, p)}{32|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2}(|\boldsymbol{k_2}| + ip - |\boldsymbol{k_1}|)^2 \\ \times (ip + |\boldsymbol{k_1}| - |\boldsymbol{k_2}|)^2(|\boldsymbol{k_1}| + |\boldsymbol{k_2}| - ip)^2 \\ \times\left(\frac{\langle\lambda_1, \lambda_2\rangle^4}{\langle\langle\lambda_1, \lambda_2\rangle\langle\lambda_2, \lambda_3\rangle\langle\lambda_3, \lambda_1\rangle}\right)^2 \end{aligned} \tag{3.46}$$

One point that might cause some confusion is the asymmetry between p and $-p$ in the formulae above. This asymmetry arises from the choice of polarization-vectors, which require us to choose a sign for the norm of the third momentum. However, recall that in obtaining the four-point function we always sum over the polarizations of the intermediate state, and in doing this, the asymmetry will disappear.

No divergences from the boundary

A common property of AdS/CFT correlators in momentum space is that they must be regulated to get rid of divergences from the boundary at $z = 0$. This is true of the correlation function computations of [10] and also of (3.40). However, it is remarkable that simple power counting tells us that the radial integrals that enter transition amplitudes are convergent.

This is clear from (3.37). A non-normalizable wavefunction behaves like z^0 near the boundary (this is true provided one index is raised and another is lowered), while the normalizable mode behaves like z^3. Although we would have obtained a $\frac{1}{z^4}$ from the $\sqrt{-g}$ factor, we also get one factors of z^2 that comes from the inverse metric required to contract the derivatives in the interaction vertex. Consequently, the integrand F_π in (3.37) goes like z near the boundary, and leads to a convergent integral.

This removes a possible complication in using the recursion relations. The computation of the four-point function involves the product of two transition amplitudes. So, naively one might have worried a $\frac{1}{\epsilon}$ term from one amplitude could have combined with a ϵ term from another amplitude to give a finite contribution. there are no $\frac{1}{\epsilon}$ terms at all.

3. Flat space limit

In [9], we conjectured that the correlation function of the stress tensor and the graviton amplitude should be related, at tree-level, through

$$M(\boldsymbol{e_1}, \tilde{\boldsymbol{k}}_1, \ldots \boldsymbol{e_n}, \tilde{\boldsymbol{k}}_n) = \lim_{\sum |\boldsymbol{k}_m| \to 0} \frac{(\sum |\boldsymbol{k}_m|)^{n-1}}{(\prod |\boldsymbol{k}_m|)\Gamma(n-1)} \times T(\boldsymbol{e_1}, \boldsymbol{k_1}, \ldots \boldsymbol{e_n}, \boldsymbol{k_n}). \tag{3.47}$$

Just as in the conserved-current case $\tilde{\boldsymbol{k}}_m$ are on-shell four-dimensional vectors produced related to the three-dimensional vectors through $\tilde{\boldsymbol{k}}_m = \{\boldsymbol{k}_m, i|\boldsymbol{k}_m|\}$

To compute correlators rather than transition amplitudes all we need to do is to replace $R^{\rm gr}$ by $R^{\rm gr}_{\rm PM}$ defined in (3.40). With this replacement, let us consider the $++-$ stress tensor correlator. We see that when we take $E_p \to 0$, the correlator becomes

$$T^{++-} = 2\frac{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}|}{E_p^2}\left(\frac{\langle \bar{\lambda}_1, \bar{\lambda}_2\rangle^4}{\langle \bar{\lambda}_1, \bar{\lambda}_2\rangle\langle \bar{\lambda}_2, \bar{\lambda}_3\rangle\langle \bar{\lambda}_3, \bar{\lambda}_1\rangle}\right)^2, \tag{3.48}$$

which is consistent with our conjecture.

On the other hand, if we consider the T^{+++} correlator, then we find that it has no pole at $E_p = 0$ at all, which reflects that fact that the all-plus graviton scattering amplitude vanishes for the Hilbert action.

Note that as we explained in [9], flat space S-matrix elements can be extracted even if we go beyond the Hilbert action. For example if we compute correlation functions using a W^3 action in flat space, this gives a nonzero all-plus scattering amplitude. Corresponding to this, the W^3 action gives a correlator in Ads$_4$ that has a singularity of order E_p^6. (See Eq. 2.18 of [10].)

IV. FORMULAS FOR FOUR-POINT FUNCTIONS

We will now use the recursion relations developed in [9] to write down formulas for the four-point functions of stress tensors and currents. This process proceeds in the following steps. First, we describe, in detail, a one-parameter deformation of each external momentum by a null vector with the property that it preserves the norm of the momentum. The analytic properties of the correlator under this extension can be used to obtain recursion relations for the four-point function in terms of deformed three-point functions; we describe this procedure next. In the next section we evaluate these formulas for correlators of both the stress tensor and conserved currents and verify that the answers have the correct flat space limit.

A. Extending the momenta

For the four-point function, we start by deforming all four momenta through

$$\boldsymbol{k}_m \to \boldsymbol{k}_m + \alpha_m \boldsymbol{\epsilon}_m w, \tag{4.1}$$

where there is *no sum* on the m in the second term. The four α's are fixed by the equation

$$\sum_{p=1}^{4} \alpha_p \boldsymbol{\epsilon}_p = 0. \tag{4.2}$$

This has a unique solution up to one complex multiplicative parameter that can be absorbed in the definition of w.

In fact the extension (4.1) can be conveniently rephrased in terms of spinors. For each momentum, only one of the spinors—either λ_m or $\bar{\lambda}_m$ is extended—as shown below, where we use the notation $\beta_m = \frac{\alpha_m}{i|\boldsymbol{k}_m|}$,

$$\begin{aligned} &\text{negative polarization: } \lambda_m(w) = \lambda_m; \\ &\bar{\lambda}_m(w) = \bar{\lambda}_m + \beta_m \hat{\lambda}_m w; \\ &\text{positive polarization: } \lambda_m(w) = \lambda_m + \beta_m \hat{\bar{\lambda}}_m w; \\ &\bar{\lambda}_m(w) = \bar{\lambda}_m. \end{aligned} \tag{4.3}$$

Explicit Expressions for β_m It is quite easy to find explicit expressions for the β_m given a set of external helicities. We enumerate these expressions for different possible external helicities.

(1) $\{\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}, \mathbf{h_4}\} = \{\mathbf{1}, \mathbf{-1}, \mathbf{1}, \mathbf{-1}\}$ In terms of the β_m variables we have the equations:

$$\beta_1 \hat{\bar{\lambda}}_1 \bar{\lambda}_1 + \beta_2 \lambda_2 \hat{\lambda}_2 + \beta_3 \hat{\bar{\lambda}}_3 \bar{\lambda}_3 + \beta_4 \lambda_4 \hat{\lambda}_4 = 0. \tag{4.4}$$

Dotting this equation with $\lambda_4 \bar{\lambda}_3$, and then $\lambda_2 \bar{\lambda}_3$ and then $\lambda_2 \hat{\lambda}_4$ we find that

$$\begin{aligned} \frac{\beta_2}{\beta_1} &= \frac{-[\bar{\lambda}_1, \lambda_4]\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle}{\langle \lambda_2, \lambda_4\rangle[\lambda_2, \bar{\lambda}_3]}; \\ \frac{\beta_3}{\beta_1} &= \frac{-[\lambda_2, \bar{\lambda}_1][\lambda_4, \bar{\lambda}_1]}{[\lambda_2, \bar{\lambda}_3][\lambda_4, \bar{\lambda}_3]}; \\ \frac{\beta_4}{\beta_1} &= \frac{-[\bar{\lambda}_1, \lambda_2]\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle}{\langle \lambda_4, \lambda_2\rangle[\lambda_4, \bar{\lambda}_3]}. \end{aligned} \tag{4.5}$$

It is this combination of helicities that we will use in Secs. V and VI, and all appearances of β_m in those sections refer to the quantities defined above.

(2) $\{\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}, \mathbf{h_4}\} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{-1}\}$ We now have the equations

$$\beta_1 \hat{\bar{\lambda}}_1 \bar{\lambda}_1 + \beta_2 \hat{\bar{\lambda}}_2 \bar{\lambda}_2 + \beta_3 \hat{\bar{\lambda}}_3 \bar{\lambda}_3 + \beta_4 \lambda_4 \hat{\lambda}_4 = 0. \tag{4.6}$$

This leads to

$$\frac{\beta_2}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_3 \rangle [\lambda_4, \bar{\lambda}_1]}{\langle \bar{\lambda}_2, \bar{\lambda}_3 \rangle [\lambda_4, \bar{\lambda}_2]}; \quad \frac{\beta_3}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle [\lambda_4, \bar{\lambda}_1]}{\langle \bar{\lambda}_3, \bar{\lambda}_2 \rangle [\lambda_4, \bar{\lambda}_3]}; \quad \frac{\beta_4}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_3 \rangle}{[\lambda_4, \bar{\lambda}_2][\lambda_4, \bar{\lambda}_3]}. \tag{4.7}$$

(3) $\{\mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3}, \mathbf{h_4}\} = \{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}\}$ This gives rise to

$$\beta_1 \hat{\bar{\lambda}}_1 \bar{\lambda}_1 + \beta_2 \hat{\bar{\lambda}}_2 \bar{\lambda}_2 + \beta_3 \hat{\bar{\lambda}}_3 \bar{\lambda}_3 + \beta_4 \hat{\bar{\lambda}}_4 \bar{\lambda}_4 = 0, \tag{4.8}$$

which has the solution

$$\frac{\beta_2}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_4 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_3 \rangle}{\langle \bar{\lambda}_2, \bar{\lambda}_4 \rangle \langle \bar{\lambda}_2, \bar{\lambda}_3 \rangle}; \quad \frac{\beta_3}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_4 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle}{\langle \bar{\lambda}_3, \bar{\lambda}_4 \rangle \langle \bar{\lambda}_3, \bar{\lambda}_2 \rangle}; \quad \frac{\beta_4}{\beta_1} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_3 \rangle}{\langle \bar{\lambda}_4, \bar{\lambda}_2 \rangle \langle \bar{\lambda}_4, \bar{\lambda}_3 \rangle}. \tag{4.9}$$

All other helicity combinations can be obtained through interchanges in the expressions above or using parity.

B. Recursion relations

With this extension, we can now write down the recursion relations derived in [9]. However, when the boundary dimension is odd, as it is in this case, we find an important simplification. To see this we rewrite the expression for the bulk to bulk propagator given in [15,16] and used in [9] as follows:

$$\begin{aligned} G_{ij}^{\text{axial, ab}}(\boldsymbol{k}, z, z') &= \int_0^\infty -ipdp\Big[\frac{(zz')^{1/2} J_{1/2}(pz) J_{1/2}(pz') \mathcal{T}_{ij} \delta^{ab}}{(\boldsymbol{k}^2 + p^2 - i\epsilon)}\Big] \\ &= \int_0^\infty \frac{-2idp}{\pi}\Big[\frac{\sin(pz)\sin(pz')\mathcal{T}_{ij}\delta^{ab}}{(\boldsymbol{k}^2 + p^2 - i\epsilon)}\Big] \\ &= \int_{-\infty}^\infty \frac{-idp}{\pi}\Big[\frac{\sin(pz)\sin(pz')\mathcal{T}_{ij}\delta^{ab}}{(\boldsymbol{k}^2 + p^2 - i\epsilon)}\Big] \\ &= \frac{1}{2}\int_{-\infty}^\infty -ipdp\Big[\frac{(zz')^{1/2} J_{1/2}(pz) J_{1/2}(pz') \mathcal{T}_{ij} \delta^{ab}}{(\boldsymbol{k}^2 + p^2 - i\epsilon)}\Big]. \end{aligned} \tag{4.10}$$

Here, we have used the fact that $J_{1/2}$ is just a sine function in disguise, then obsered that the integrand is manifestly even in p and used this to rewrite the propagator as an integral from $(-\infty, \infty)$. The graviton propagator can be similarly written as an integral over the entire real line,

$$G_{ij,kl}^{\text{grav}} = \int_{-\infty}^\infty \frac{-idp}{\pi}\Big[\frac{\sin(pz)\sin(pz')}{z^2(z')^2(\boldsymbol{k}^2 + p^2 - i\epsilon)} \times \frac{1}{2}\Big(\mathcal{T}_{ik}\mathcal{T}_{jl} + \mathcal{T}_{il}\mathcal{T}_{jk} - \frac{2\mathcal{T}_{ij}\mathcal{T}_{kl}}{d-1}\Big)\Big]. \tag{4.11}$$

The advantage of writing the propagator as the third line of (4.10) is that when we now obtain p-integrals in Witten diagrams these can be done just through an algebraic procedure of extracting residues. These simplifications happen for all odd boundary dimensions. With this observation the recursion relations of [9], specialized to $d = 3$, can be written as:

$$\begin{aligned} T(h_1, \boldsymbol{k_1}(w), \ldots h_4, \boldsymbol{k_4}(w)) &= \int_{-\infty}^\infty \Big[\sum_\pi \mathcal{I}_\pi(w, p) + \mathcal{B}(w, p)\Big] dp \\ \mathcal{I}_\pi(w, p) &= \frac{p}{2 \cdot 2^s} \sum_{h^{\text{int}}, \pm} \frac{-i\mathcal{T}^2}{p^2 + (\boldsymbol{k}_{\boldsymbol{\pi_1}}(w) + \boldsymbol{k}_{\boldsymbol{\pi_2}}(w))^2} \frac{w - w^{\mp}(p)}{w^{\pm}(p) - w^{\mp}(p)} \\ \mathcal{T}^2 &\equiv T^*(h_{\pi_1}, \boldsymbol{k}_{\boldsymbol{\pi_1}}(p), h_{\pi_2}, \boldsymbol{k}_{\boldsymbol{\pi_2}}(p), h_{\text{int}}, \boldsymbol{k}_{\text{int}}) T^*(-h_{\text{int}}, -\boldsymbol{k}_{\text{int}}, h_{\pi_3}, \boldsymbol{k}_{\boldsymbol{\pi_3}}(p), h_{\pi_4}, \boldsymbol{k}_{\boldsymbol{\pi_4}}(p)). \end{aligned} \tag{4.12}$$

Here $T(\boldsymbol{h}^1, \boldsymbol{k}^1(w), \ldots \boldsymbol{h}^4, \boldsymbol{k}^4(w))$ is the four-point correlator with momenta $\boldsymbol{k}^m(w)$ extended according to (4.1) and polarization vectors that are specified in terms of the helicity by (2.15). The T^* that appear on the right-hand sides are three-point transition amplitudes that were computed in Sec. III. As explained in [9], $\mathcal{B}$ is a polynomial in w with coefficients that are rational functions of p; this term ensures that the p-integral converges and also that the correlator to have the right behavior at large w. We show below that we do not need to evaluate this term explicitly for the four-point function.

The reader should also note that we have a leading factor of $\frac{1}{2 \cdot 2^s}$ in the definition of $\mathcal{I}_\pi$ compared to [9]. Here $s = 1$ for currents and $s = 2$ for the stress tensor. The factor of $\frac{1}{2}$ comes from the fact that our integral runs over $(-\infty, \infty)$ instead of $(0, \infty)$. The second factor of $\frac{1}{2^s}$ comes from the normalization of our polarization vectors in (2.16).

We are actually interested in the original undeformed correlator, which is recovered by setting $w = 0$ in (4.12). We have

$$\mathcal{I}_{\pi}(0, p) = \frac{p}{2 \cdot 2^s} \sum_{h_{\text{int}}, \pm} \frac{i\mathcal{T}^2}{p^2 + (\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2})^2} \times \frac{w^{\mp}(p)}{w^{\pm}(p) - w^{\mp}(p)}. \tag{4.13}$$

Partitions: The recursion relations (4.12) involve a sum over partitions. For a non-color-ordered amplitude, we need to sum over three partitions in the four-point function. These three partitions are

$$\pi \in \{\{1, 2, 3, 4\}, \{1, 3, 2, 4\}, \{1, 4, 3, 2\}\}. \tag{4.14}$$

We will also call these partitions the s, t, and u partitions respectively.

$w^{\pm}$ as a function of p: To use the recursion relations we need to specify $w^{\pm}$ as a function of p. Consider a partition of the four external legs, described by π. Then the pole under the extension (4.1) is at the value of $w = w^{\pm}$ where

$$(\boldsymbol{k}_{\pi_1} + \alpha_{\pi_1}\boldsymbol{\epsilon}_{\pi_1} w^{\pm} + \boldsymbol{k}_{\pi_2} + \alpha_{\pi_2}\boldsymbol{\epsilon}_{\pi_2} w^{\pm})^2 + p^2 = 0. \tag{4.15}$$

We can write $w^{\pm} = u \pm v$, where we have defined the auxiliary quantities

$$u = -\frac{\alpha_{\pi_1}(\boldsymbol{\epsilon}_{\pi_1}.\boldsymbol{k}_{\pi_2}) + \alpha_{\pi_2}(\boldsymbol{\epsilon}_{\pi_2}.\boldsymbol{k}_{\pi_1})}{2\alpha_{\pi_1}\alpha_{\pi_2}(\boldsymbol{\epsilon}_{\pi_1}.\boldsymbol{\epsilon}_{\pi_2})},$$
$$v = \frac{\sqrt{(\alpha_{\pi_1}(\boldsymbol{\epsilon}_{\pi_1}.\boldsymbol{k}_{\pi_2}) + \alpha_{\pi_2}(\boldsymbol{\epsilon}_{\pi_2}.\boldsymbol{k}_{\pi_1}))^2 - 4\alpha_{\pi_1}\alpha_{\pi_2}(\boldsymbol{\epsilon}_{\pi_1}.\boldsymbol{\epsilon}_{\pi_2})(p^2 + (\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2})^2)}}{2\alpha_{\pi_1}\alpha_{\pi_2}(\boldsymbol{\epsilon}_{\pi_1}.\boldsymbol{\epsilon}_{\pi_2})}. \tag{4.16}$$

As we will see below, we need to evaluate these expressions only at specific values of p and in those cases, they often simplify considerably.

We should also stress that what is important is that there are two solutions for w, given a value of p. Which solution we call w^+ and which we call w^- is of no relevance and we will be somewhat cavalier about this below.

Rational Integrands: The integrands $\mathcal{I}_{\pi}(w, p)$ in (4.12) might seem to have square-roots but, in fact, all these square-roots cancel. This is because of the sum in front, which takes $v \leftrightarrow -v$. As a consequence, all the integrands depend only on even powers of v, which means that there are no square-roots after accounting for both terms.

Intermediate Spinors: Now, let us consider the spinors for the intermediate leg. The intermediate momentum is just

$$\boldsymbol{k}_{\text{int}} = -\boldsymbol{k}_{\pi_1}(w^{\pm}) - \boldsymbol{k}_{\pi_2}(w^{\pm}) = \boldsymbol{k}_{\pi_3}(w^{\pm}) + \boldsymbol{k}_{\pi_4}(w^{\pm}). \tag{4.17}$$

The three-point amplitudes above are written in terms of spinors. However, the key point is that in choosing a decomposition of $\boldsymbol{k}_{\text{int}}$ into spinors, we can rescale $\lambda_{\text{int}} \to \alpha\lambda_{\text{int}}$ and $\bar{\lambda}_{\text{int}} \to \alpha^{-1}\bar{\lambda}_{\text{int}}$ by any complex number α without affecting the final answer. This is because in the recursion relations above when we have h_{int} on the left, we have $-h_{\text{int}}$ on the right and so λ_{int} and $\bar{\lambda}_{\text{int}}$ always come together. Consequently, we can choose the intermediate spinors using (2.6), and avoid any square-roots.

Other, more covariant looking, choices are possible. For example one could take

$$\lambda_{\text{int}} = \lambda_{\pi_2}(w^{\pm}) + \lambda_{\pi_1}(w^{\pm})\frac{i(|\boldsymbol{k}_{\pi_1}| - |\boldsymbol{k}_{\pi_2}| - ip)}{[\bar{\lambda}_{\pi_2}(w^{\pm}), \lambda_{\pi_1}(w^{\pm})]}$$
$$\bar{\lambda}_{\text{int}} = -\bar{\lambda}_{\pi_2}(w^{\pm}) - \frac{i(|\boldsymbol{k}_{\pi_1}| + |\boldsymbol{k}_{\pi_2}| - ip)}{\langle\lambda_{\pi_1}(w^{\pm}), \lambda_{\pi_2}(w^{\pm})\rangle}\hat{\lambda}_{\pi_1}(w^{\pm}). \tag{4.18}$$

In fact in the calculations of Secs. V and VI, we will never need the intermediate spinors explicitly. Instead we will use identities like (5.2) to rewrite expressions that involve λ_{int} like (5.1) as expressions that are free of these terms like (5.3).

Boundary Term $\mathcal{B}$: Now, we have argued above that the integrand term in (4.12) is rational and even in p. So, merely by polynomial division, we can write it in the following form:

$$\sum_{\pi} \mathcal{I}_{\pi}(w, p) = \frac{\mathcal{N}(p^2, w)}{\mathcal{D}(p^2, w)} + \mathcal{Q}(p^2, w), \tag{4.19}$$

where $\mathcal{N}$, $\mathcal{D}$, $\mathcal{Q}$ are polynomials and $\frac{\mathcal{N}}{\mathcal{D}}$ dies off at least as fast as $\frac{1}{p^2}$ for large p. The fact that $\mathcal{Q}$ is polynomial in w follows from the fact that the highest power of p in $\mathcal{D}$ is independent of w. (Note that $\mathcal{T}^2$ is purely a function of p and the only dependence on w in $\mathcal{I}_{\pi}(w, p)$ comes through the factors that are explicitly displayed.)

To ensure the convergence of the integral at large p and the correct behavior of the correlator at large w, we need to set:

$$\mathcal{B}(p, w) = -\mathcal{Q}(p^2, w) + \sum_m b_m(p) w^m, \tag{4.20}$$

where $b_m(p)$ are rational functions of p with a convergent integral over the real line. For conserved currents, we can take $b_m(p) = 0$ since the correlator vanishes at large w.

For stress tensor correlators, the behavior of the correlator at large w is completely fixed by the Ward identities as shown in [9] and we can take the $b_m(p)$ to be any functions that satisfy:

$$T(h_1, \boldsymbol{k_1}(w), \ldots h_4, \boldsymbol{k_4}(w)) \underset{w\to\infty}{\longrightarrow} \sum_m w^m \int_{-\infty}^{\infty} b_m(p) dp. \tag{4.21}$$

However, we know that the Ward identities contribute only *local terms* at large w; in momentum space, this corresponds to terms that are analytic in at least two momenta. These terms are not themselves of physical interest and so, at the level of the four-point function, we can just forget about the b_m functions.

Algebraic Evaluation of the p Integral: We now show that the entire p integral can be done just by picking out residues of the integrand at prespecified poles. Now that we have dealt with the behavior of the integrand at large w, we will specialize to $w = 0$ for simplicity.

Since, with the addition of $\mathcal{B}$ the integrand vanishes at large p by construction, we can close the contour through either the upper or the lower half plane. This leaves us just with the task of evaluating some residues. In fact we do not need to evaluate $\mathcal{Q}$ explicitly either:

$$\int_{-\infty}^{\infty} \frac{\mathcal{N}(p^2, w)}{\mathcal{D}(p^2, w)} dp = 2\pi i \sum_{\text{poles}} \text{Res}[I_\pi(w, p)], \tag{4.22}$$

where we sum over the poles of the integrand at finite p in the upper half plane.

Second, it is, in fact, quite easy to specify the locations of *all poles* in the integrand. There are two sources of poles: (a) the poles in the three-point function where $p = \pm i(|\boldsymbol{k}_{\pi_1}| + |\boldsymbol{k}_{\pi_2}|)$ (b) the pole where the propagator vanishes $p^2 + (\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2})^2 = 0$.

So, for each partition π, there are exactly *three poles* in the upper half plane. This set is given by:

$$\mathcal{P}_\pi = \{i(|\boldsymbol{k}_{\pi_1}| + |\boldsymbol{k}_{\pi_2}|), i(|\boldsymbol{k}_{\pi_3}| + |\boldsymbol{k}_{\pi_4}|), i\sqrt{(\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2})^2}\}. \tag{4.23}$$

This leads to our final formula for the four-point function:

$$T(h^1, \boldsymbol{k^1}, \ldots h^4, \boldsymbol{k^4}) = 2\pi i \sum_\pi \sum_{p_0 \in \mathcal{P}_\pi} \text{Res}_{p=p_0}[\mathcal{I}_\pi(0, p)], \tag{4.24}$$

Here $\mathcal{I}(0, p)$ is specified in (4.13). The three-point transition amplitudes that appear there are specified in Sec. III, with intermediate momenta and spinors given by (4.17) and (4.18). Moreover, $w^{\pm}(p)$ is specified by (4.15), the set of partitions π is specified by (4.14), and the set of poles $\mathcal{P}$ is specified by (4.23).

This leads to a straightforward algorithm that is implemented in the attached Mathematica code [11]. We also evaluate this formula in several cases below.

An Aside: Before we conclude this section, let us comment briefly on the various kinds of terms that appear in the formula above. First, note that the residue at $p = i(|\boldsymbol{k}_{\pi_1}| + |\boldsymbol{k}_{\pi_2}|)$ or $p = i(|\boldsymbol{k}_{\pi_3}| + |\boldsymbol{k}_{\pi_4}|)$ is a rational function of the external spinors. This is guaranteed since I_π is a rational function and so is the location of the pole. Furthermore, the analysis of [9] (and our explicit computations below) tells us that when we take the flat space limit, it is these two terms that give us the correct singularity in the final answer.

On the other hand the last entry in the set (4.23) is not important in the flat space limit. It also has a different analytic structure, and it can be written as a rational function of the external spinors and the norms of the sums of momenta. If we choose to write it purely as a function of the original spinors then we get square-roots in this term, which arise because the location of the pole involves a square root.

However, this term has an interesting relation to the operator product expansion that we should mention. First, it is easy to check that for this pole one of the possible solutions for $w^{\pm}(p)$ is just $w^- = 0$. (There is another solution to (4.15) but since one solution is 0 this does not contribute due to the $\frac{w^{\mp}}{w^{pm} - w^{+}}$ factor in front of the integrand.) The residue at this pole is merely:

$$2\pi i \sum_{h_{\text{int}}} T^*(h_1, \boldsymbol{k}_{\pi_1}, h_2, \boldsymbol{k}_{\pi_2}, h_{\text{int}}, \boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2}) \times T^*(h_1, \boldsymbol{k}_{\pi_1}, h_2, \boldsymbol{k}_{\pi_2}, -h_{\text{int}}, \boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2}) \frac{i}{4}$$

Now the correlator is obtained by contracting the bulk vertices with a bulk-boundary propagator. From the relations between Bessel functions:

$$z^\nu J_\nu(pz) = \frac{-2i}{\pi p^\nu}(-ipz)^\nu K_\nu(-ipz) - iz^\nu N_\nu(pz), \tag{4.25}$$

it is easy to check that one term in the transition amplitude is the correlator:

$$T^*(h_1, \boldsymbol{k}_{\pi_1}, h_2, \boldsymbol{k}_{\pi_2}, h_{\text{int}}, \boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2}) = -i\sqrt{\frac{2}{\pi}} \frac{T(h_1, \boldsymbol{k}_{\pi_1}, h_2, \boldsymbol{k}_{\pi_2}, h_{\text{int}}, \boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2})}{(i|\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2}|)^\nu} + \ldots \tag{4.26}$$

where $\nu = \frac{1}{2}$ for currents and $\nu = \frac{3}{2}$ for the stress tensor and where $\ldots$ is the term that comes from the Neumann function above. So we see that one term in our final answer is exactly the product of the three-point functions divided by the two point function as predicted by the operator product expansion.

We should emphasize that although this conformal block drops out in the flat space limit, the Witten diagram involving the exchange of a graviton does not. The flat space limit of [9] was derived diagram by diagram; so the exchange Witten diagram goes over to the flat space exchange diagram. This is consistent because the exchange Witten diagram involves more than just the conformal block of the stress-tensor (as is discussed, for example, in section 6.4 of [17]) and so it survives in the flat space limit.

Here should caution the reader that although the pole at $p = i|\boldsymbol{k}_{\pi_1} + \boldsymbol{k}_{\pi_2}|$ accounts for the contribution of the conformal block of the stress tensor or the conserved current itself, we have not shown that the remainder of the correlator including the ... in (4.26) and the contribution from the other poles can be exactly accounted for by the contribution of all double trace operators.[4] This deserves some further study.

V. MHV CORRELATORS FOR CONSERVED CURRENTS

In this section, we will expand the formula above in terms of spinors for the four-point function of currents. We start by analyzing the color-ordered MHV correlator $+-+-$, and then describe the full (i.e. non-color-ordered) MHV correlator. Although the calculations below might seem tedious, our final answer for the color-ordered MHV correlator is quite simple and is given in (5.16). The expressions below are also implemented in the attached Mathematica program [11], which may be useful while following this analysis. We show, explicitly, for both the color-ordered and full MHV amplitude that taking the flat space limit just leads to the Parke-Taylor formula for four-gluon scattering.

A. Color-Ordered MHV Correlator

To obtain the color-ordered amplitude we only need to sum over two partitions: the (12)(34) partition and the (41)(23) partition.

Let us start by analyzing the (12)(34) partition, which we will call the "s" partition. In fact the "t" partition: (14)(23) is just obtained by taking all the results here and interchanging $2 \leftrightarrow 4$. So that will not require a separate calculation.

Let us expand out the three-point functions that appear in the formula above. Doing this, we find:

$$\begin{aligned} I_s = \sum_{\pm}\bigg\{ & \frac{R^{\mathrm{YM}}(|\boldsymbol{k}_1|, |\boldsymbol{k}_2|, p)}{2\sqrt{2}|\boldsymbol{k}_1||\boldsymbol{k}_2|p}(|\boldsymbol{k}_2| + ip - |\boldsymbol{k}_1|)(ip + |\boldsymbol{k}_1| - |\boldsymbol{k}_2|)(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| - ip)\frac{R^{\mathrm{YM}}(|\boldsymbol{k}_3|, |\boldsymbol{k}_4|, -p)}{2\sqrt{2}|\boldsymbol{k}_3||\boldsymbol{k}_4|p}(|\boldsymbol{k}_4| - ip - |\boldsymbol{k}_3|) \\ & \times (-ip + |\boldsymbol{k}_3| - |\boldsymbol{k}_4|)(|\boldsymbol{k}_3| + |\boldsymbol{k}_4| + ip)\bigg[\bigg(\frac{\langle\bar{\lambda}_1, \bar{\lambda}_{\mathrm{int}}\rangle^3}{\langle\bar{\lambda}_1, \bar{\lambda}_2(w^\pm)\rangle\langle\bar{\lambda}_2(w^\pm), \bar{\lambda}_{\mathrm{int}}\rangle}\frac{\langle\lambda_4, \lambda_{\mathrm{int}}\rangle^3}{\langle\lambda_3(w^\pm), \lambda_4\rangle\langle\lambda_3(w^\pm), \lambda_{\mathrm{int}}\rangle}\bigg) \\ & + \bigg(\frac{\langle\lambda_2, \lambda_{\mathrm{int}}\rangle^3}{\langle\lambda_1(w^\pm)\rangle, \lambda_2\rangle\langle\lambda_1(w^\pm), \lambda_{\mathrm{int}}\rangle}\frac{\langle\bar{\lambda}_3, \bar{\lambda}_{\mathrm{int}}\rangle^3}{\langle\bar{\lambda}_3, \bar{\lambda}_4(w^\pm)\rangle\langle\bar{\lambda}_4(w^\pm), \bar{\lambda}_{\mathrm{int}}\rangle}\bigg)\bigg]\bigg\}\frac{1}{4}\frac{ip}{(p^2 + (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2)}\frac{w^\mp}{w^\pm - w^\mp} \end{aligned} \tag{5.1}$$

We can simplify by expanding out R^{YM} and also recognizing that

$$\lambda_1(w^\pm)\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2(w^\pm) + \lambda_{\mathrm{int}}\bar{\lambda}_{\mathrm{int}} = i(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + ip)\sigma^3. \tag{5.2}$$

This leads to

$$\begin{aligned} I_s = \sum_{\pm}\bigg\{ & \frac{(|\boldsymbol{k}_2| + ip - |\boldsymbol{k}_1|)(ip + |\boldsymbol{k}_1| - |\boldsymbol{k}_2|)(|\boldsymbol{k}_3| + ip - |\boldsymbol{k}_4|)(ip + |\boldsymbol{k}_4| - |\boldsymbol{k}_3|)}{16\pi|\boldsymbol{k}_1||\boldsymbol{k}_2||\boldsymbol{k}_3||\boldsymbol{k}_4|(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + ip)(|\boldsymbol{k}_3| + |\boldsymbol{k}_4| - ip)} \\ & \times\bigg[\frac{(\langle\bar{\lambda}_1, \bar{\lambda}_2(w^\pm)\rangle\langle\lambda_4, \lambda_2\rangle + iE_p^{12}[\bar{\lambda}_1, \lambda_4])^3}{\langle\bar{\lambda}_1, \bar{\lambda}_2(w^\pm)\rangle\langle\lambda_4, \lambda_3(w^\pm)\rangle\langle\bar{\lambda}_2(w^\pm), \bar{\lambda}_1\rangle\langle\lambda_1(w^\pm), \lambda_3(w^\pm)\rangle - iE_p^{12}[\lambda_3(w^\pm), \bar{\lambda}_2(w^\pm)])} \\ & + \frac{(\langle\lambda_2, \lambda_1(w^\pm)\rangle\langle\bar{\lambda}_3, \bar{\lambda}_1\rangle + iE_p^{12}[\lambda_2, \bar{\lambda}_3])^3}{\langle\lambda_1(w^\pm), \lambda_2\rangle(\langle\lambda_1(w^\pm), \lambda_2\rangle\langle\bar{\lambda}_4(w^\pm), \bar{\lambda}_2(w^\pm)\rangle + iE_p^{12}[\lambda_1(w^\pm), \bar{\lambda}_4(w^\pm)])\langle\bar{\lambda}_3, \bar{\lambda}_4(w^\pm)\rangle}\bigg] \\ & \times\frac{-i}{p^2 + (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2}\frac{w^\mp}{w^\pm - w^\mp}\bigg\}. \end{aligned} \tag{5.3}$$

where we have defined

$$E_p^{nm} \equiv ip + |\boldsymbol{k}_n| + |\boldsymbol{k}_m|. \tag{5.4}$$

[4]There is, of course, an infinite sequence of such double trace operators whose exact spectrum can be easily worked out using character decomposition [18]. For example, below conformal weight 8, we find the spectrum of double trace operators of the stress tensor [in the notation (weight, spin)]: (6, 0) ⊕ (6, 1) ⊕ (6, 2) ⊕ (6, 3) ⊕ (6, 4) ⊕ (7, 0) ⊕ (7, 1) ⊕ (7, 2) ⊕ (7, 3) ⊕ 2(7, 4) ⊕ (7, 5).

As we described in Sec. IV, it is clear that there are two kinds of poles that appear in I_s. One type is the pole that appears from the constituent three-point amplitudes: the existence of such a pole is required by the fact that the three-point amplitude must have the correct flat space limit i.e. the flat space three-point amplitude must appear as the coefficient of singularities at $ip + |\mathbf{k_1}| + |\mathbf{k_2}| = 0$ and $ip + |\mathbf{k_3}| + |\mathbf{k_4}| = 0$. The second kind of pole appears when the propagator factor above $p^2 + (\mathbf{k_1} + \mathbf{k_2})^2$ vanishes.

Poles from the three-point amplitude: We have written the expression above so that it is very easy to extract the pole at $E_p^{12} = 0$. First, let us note that when $E_p^{12} = 0$, the value of $w^{\pm}(p)$ simplifies. Denoting this value by $w^{\pm}(p = i(|\mathbf{k_1}| + |\mathbf{k_2}|)) \equiv w_{s_1}^{\pm}$, we have

$$((\lambda_1 + \beta_1 \hat{\bar{\lambda}}_1 w_{s_1}^{\pm})\bar{\lambda}_1 + \lambda_2(\bar{\lambda}_2 + \beta_2 \hat{\lambda}_2 w_{s_1}^{\pm}))^2 = 0. \quad (5.5)$$

This condition requires either

$$\langle \lambda_1 + \beta_1 \hat{\bar{\lambda}}_1 w_{s_1}^{\pm}, \lambda_2 \rangle = 0, \quad \text{or} \quad \langle \bar{\lambda}_1, \bar{\lambda}_2 + \beta_2 \hat{\lambda}_2 w_{s_1}^{\pm} \rangle = 0. \quad (5.6)$$

These equations are solved by

$$\beta_1 w_{s_1}^{+} = -\frac{\langle \lambda_2, \lambda_1 \rangle}{[\lambda_2, \bar{\lambda}_1]}; \quad (5.7)$$

or

$$\beta_2 w_{s_1}^{-} = -\frac{\langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle}{[\bar{\lambda}_1, \lambda_2]}. \quad (5.8)$$

We remind the reader that the β_m are given by (4.5).

To proceed further we recognize that at the pole $w_{s_1}^{+}$ the second line of the big square bracket in (5.3), which corresponds to $h_{\text{int}} = -1$, vanishes. So we only need to evaluate the first line in the big square bracket at the point where $E_p^{12} = 0$.

Some short calculations tell us that, the propagator factor simplifies

$$\frac{i}{-(|\mathbf{k_1}| + |\mathbf{k_2}|)^2 + (\mathbf{k_1} + \mathbf{k_2})} \frac{w_{s_1}^{\mp}}{w_{s_1}^{\pm} - w_{s_1}^{\mp}} = \frac{i}{\langle \lambda_1, \lambda_1 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle} \frac{w_{s_1}^{\mp}}{w_{s_1}^{\pm} - w_{s_1}^{\mp}} = \frac{1}{\langle \bar{\lambda}_1, \bar{\lambda}_2(w_{s_1}^{+}) \rangle \langle \lambda_1, \lambda_2 \rangle}. \quad (5.9)$$

Another short calculation tells us that

$$\lambda_1(w_{s_1}^{+}) = \lambda_2 \frac{[\lambda_1, \bar{\lambda}_1]}{[\lambda_2, \bar{\lambda}_1]},$$
$$\lambda_3(w_{s_1}^{+}) = \frac{[\lambda_1, \bar{\lambda}_3] + [\lambda_4, \bar{\lambda}_4] - iE^T}{[\lambda_2, \bar{\lambda}_3]} \lambda_2 - \lambda_4 \frac{[\lambda_4, \bar{\lambda}_4] - iE^T}{[\lambda_4, \bar{\lambda}_3]}. \quad (5.10)$$

Plugging these factors in, we find that

$$(2\pi i)\text{Res}_{p=i(|\mathbf{k_1}|+|\mathbf{k_2}|)}[I_s(p)] = \frac{E^{124,3}}{4|\mathbf{k_3}||\mathbf{k_4}||\mathbf{k_1}|} \frac{\langle \lambda_2, \lambda_4 \rangle [\lambda_4, \bar{\lambda}_3][\lambda_2, \bar{\lambda}_3][\lambda_2, \bar{\lambda}_1]}{E^{12,34} \langle \lambda_1, \lambda_2 \rangle E^T} \quad (5.11)$$

$$+ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leftrightarrow (\bar{\lambda}_2, \bar{\lambda}_1, \bar{\lambda}_4, \bar{\lambda}_3). \quad (5.12)$$

Here in (5.12) we have recognized that the contribution from (5.8) is just obtained by the interchange indicated.

Poles from the propagator: We now turn to the second kind of pole, which comes when the propagator vanishes. When $p^2 + (\mathbf{k_1} + \mathbf{k_2})^2 = 0$, one of the solutions—which we will denote by $w^{-}(\sqrt{(\mathbf{k_1} + \mathbf{k_2})^2})$—is just 0. The other solution is complicated, but it does not contribute to the answer at all; the factor $\frac{w^-}{w^+ - w^-}$ that appears in (5.3) vanishes since $w^- = 0$ at this value of p.

$$\mathcal{A}_s = (2\pi i)\text{Res}_{p=i\sqrt{(\mathbf{k_1}+\mathbf{k_2})^2}}[I_s(p)] = \frac{iE^{1,2s}E^{2,1s}E^{4,3s}E^{3,4s}}{16E^{12,s}E^{34s}|\mathbf{k_1} + \mathbf{k_2}|}. \quad (5.13)$$

$$\times \left[\frac{(\langle \lambda_2, \lambda_1 \rangle \langle \bar{\lambda}_3, \bar{\lambda}_1 \rangle + iE^{12,s}[\lambda_2, \bar{\lambda}_3])^3}{\langle \lambda_1, \lambda_2 \rangle [\bar{\lambda}_3, \bar{\lambda}_4] (\langle \lambda_1, \lambda_2 \rangle \langle \bar{\lambda}_4, \bar{\lambda}_2 \rangle + iE^{12,s} \langle \lambda_1, \bar{\lambda}_4 \rangle)} \right. \quad (5.14)$$

$$\left. + \frac{(\langle \lambda_4, \lambda_2 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle + iE^{12,s}[\lambda_4, \bar{\lambda}_1])^3}{\langle \lambda_4, \lambda_3 \rangle \langle \bar{\lambda}_1, \bar{\lambda}_2 \rangle (\langle \lambda_1, \lambda_3 \rangle \langle \bar{\lambda}_2, \bar{\lambda}_1 \rangle - iE^{12,s}[\lambda_3, \bar{\lambda}_2])} \right]. \quad (5.15)$$

As we mentioned above, this residue is quite interesting since it contributes exactly the product of the undeformed transition amplitude on the left and the right. This contains the contribution of the conformal block of the current itself and is consistent with what we would expect from the operator product expansion applied in momentum space. However, we repeat the caveat that it is necessary to also show that the remaining terms are consistent with the contribution of double trace operators.

1. Final answer for the color-ordered current correlator

The final answer for the color-ordered MHV current correlator can now just be obtained by interchanges from the answer above. The answer for the four-point amplitude is given by

$$T^{+-+-}(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}, \mathbf{k_4}) = \frac{\mathcal{F}}{E^T} + \mathcal{A}, \quad (5.16)$$

where $\mathcal{F}$ is the term with a pole that corresponds to the flat space limit and $\mathcal{A}$ is an intrinsically AdS term that would vanish in flat space. We have,

$$\mathcal{F} = \frac{E^{124,3}}{4|\boldsymbol{k_3}||\boldsymbol{k_4}||\boldsymbol{k_1}|}\frac{\langle\lambda_2,\lambda_4\rangle[\lambda_4,\bar\lambda_3][\lambda_2,\bar\lambda_3][\lambda_2,\bar\lambda_1]}{E^{12,34}\langle\lambda_1,\lambda_2\rangle} \tag{5.17}$$

$$+ (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \leftrightarrow (\bar\lambda_2,\bar\lambda_1,\bar\lambda_4,\bar\lambda_3) \tag{5.18}$$

$$+ (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\bar\lambda_1,\bar\lambda_2,\bar\lambda_3,\bar\lambda_4) \leftrightarrow (\lambda_3,\lambda_4,\lambda_1,\lambda_2,\bar\lambda_3,\bar\lambda_4,\bar\lambda_1,\bar\lambda_2) \tag{5.19}$$

$$+ (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\bar\lambda_1,\bar\lambda_2,\bar\lambda_3,\bar\lambda_4) \leftrightarrow (\bar\lambda_4,\bar\lambda_3,\bar\lambda_2,\bar\lambda_1,\lambda_4,\lambda_3,\lambda_2,\lambda_1)\Big] \tag{5.20}$$

$$+ (\lambda_2,\bar\lambda_2) \leftrightarrow (\lambda_4,\bar\lambda_4). \tag{5.21}$$

Note that within the big square bracket, all interchanges correspond to the expression in (5.17). So, for example to get (5.20) we take (5.17) and perform the interchanges indicated. However to get (5.21) we take the whole square bracket and perform the interchange indicated. For $\mathcal{A}$ we have

$$\mathcal{A} = \mathcal{A}_s + (\lambda_2,\bar\lambda_2) \leftrightarrow (\lambda_4,\bar\lambda_4), \tag{5.22}$$

where $\mathcal{A}_s$ is specified in the three lines (5.13), (5.14), and (5.15). We take all three lines and perform the substitution indicated.

2. *Flat space limit of the answer*

We can, quite easily, take the flat space limit of the answer above. We just need to look at the $\mathcal{F}$ term above. Second, at $E^T = 0$, various spinor identities can be used to simplify the function.

However, here we will take a different route that is somewhat more elegant, and also gives us a check on the final answer. Consider the functions

$$M^{\mathrm{MHV}}(w) = \frac{\langle\lambda_2,\lambda_4\rangle^4}{\langle\lambda_1(w),\lambda_2\rangle\langle\lambda_2,\lambda_3(w)\rangle\langle\lambda_3(w),\lambda_4\rangle\langle\lambda_4,\lambda_1(w)\rangle},$$
$$M^{\overline{\mathrm{MHV}}}(w) = \frac{\langle\bar\lambda_1,\bar\lambda_3\rangle^4}{\langle\bar\lambda_1,\bar\lambda_2(w)\rangle\langle\bar\lambda_2(w),\bar\lambda_3\rangle\langle\bar\lambda_3,\bar\lambda_4(w)\rangle\langle\bar\lambda_4(w),\bar\lambda_1\rangle}, \tag{5.23}$$

where the spinors are extended as above. Note that, as above, λ_2, λ_4, $\bar\lambda_1$, $\bar\lambda_4$ do not change in this extension.

Then (5.17) is actually just proportional to the residue of $M^{\mathrm{MHV}}(w)$ at $w^+_{s_1}$, while (5.18) is just proportional to the residue of $M^{\overline{\mathrm{MHV}}}(w)$ at $w^-_{s_1}$ In fact, we have

$$(5.17) = \frac{E^{123,4}E^{123,4}}{2|\boldsymbol{k_3}||\boldsymbol{k_4}|E^T}\lim_{w\to w^+_{s_1}}\frac{\langle\lambda_1(w),\lambda_2\rangle}{\langle\lambda_1(0),\lambda_2\rangle}M^{\mathrm{MHV}}(w),$$
$$(5.18) = \frac{E^{123,4}E^{123,4}}{2|\boldsymbol{k_3}||\boldsymbol{k_4}|E^T}\lim_{w\to w^-_{s_1}}\frac{\langle\bar\lambda_1,\bar\lambda_2(w)\rangle}{\langle\bar\lambda_1,\bar\lambda_2(0)\rangle}M^{\overline{\mathrm{MHV}}}(w), \tag{5.24}$$

These are true as *exact* statements without setting $E^T = 0$. However, when we set $E^T = 0$, the factor in front of the limit just becomes 1. Then we can see that by adding together the various terms in the expression for $\mathcal{F}$, we will just get the sum of the holomorphic and anti-holomorphic MHV amplitudes as the residue of the pole at $E^T = 0$. These amplitudes are, of course, equal in the flat space limit.

To prove the assertion (5.24), consider the expression for the integrand (5.1). At the pole $w^+_{s_1}$, we have $\lambda_1(w^+_{s_1}) \propto \lambda_2$. So, the spinor expressions that appear in (5.1) are

$$\frac{\langle\bar\lambda_1,\bar\lambda_{\mathrm{int}}\rangle^3}{\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle\langle\bar\lambda_2(w^+_{s_1}),\bar\lambda_{\mathrm{int}}\rangle}\frac{\langle\lambda_4,\lambda_{\mathrm{int}}\rangle^3}{\langle\lambda_3(w^+_{s_1}),\lambda_4\rangle\langle\lambda_{\mathrm{int}},\lambda_3(w^+_{s_1})\rangle}\times\frac{1}{\langle\lambda_1,\lambda_2\rangle\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle}, \tag{5.25}$$

where the third term comes from the propagator after using (5.9).

However, we can write

$$\langle\bar\lambda_1,\bar\lambda_{\mathrm{int}}\rangle\langle\lambda_4,\lambda_{\mathrm{int}}\rangle = -\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle\langle\lambda_4,\lambda_2\rangle, \tag{5.26}$$

where we have just used the fact that

$$-\lambda_{\mathrm{int}}\bar\lambda_{\mathrm{int}} = \lambda_1(w^+_{s_1})\bar\lambda_1 + \lambda_2\bar\lambda_2(w^+_{s_1}). \tag{5.27}$$

Using this to simplify both the numerator and the denominator we find that the spinor expression becomes

$$\begin{aligned}(5.25) &= \frac{\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle^3}{\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle\langle\bar\lambda_2(w^+_{s_1}),\bar\lambda_1\rangle}\\ &\quad\times\frac{\langle\lambda_4,\lambda_2\rangle^3}{\langle\lambda_3(w^+_{s_1}),\lambda_4\rangle\langle\lambda_1(w^+_{s_1}),\lambda_3(w^+_{s_1})\rangle}\\ &\quad\times\frac{1}{\langle\lambda_1,\lambda_2\rangle\langle\bar\lambda_1,\bar\lambda_2(w^+_{s_1})\rangle}\\ &= \frac{\langle\lambda_4,\lambda_2\rangle^3}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_1(w^+_{s_1}),\lambda_3(w^+_{s_1})\rangle\langle\lambda_3(w^+_{s_1}),\lambda_4\rangle}\\ &= \frac{\langle\lambda_4,\lambda_2\rangle^4}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_2,\lambda_3(w^+_{s_1})\rangle\langle\lambda_2,\lambda_3(w^+_{s_1})\rangle\langle\lambda_4,\lambda_1(w^+_{s_1})\rangle},\end{aligned} \tag{5.28}$$

where in the last step we have used the identity that

$$\langle\lambda_2,\lambda_4\rangle\langle\lambda_1(w^+_{s_1}),\lambda_3(w^+_{s_1})\rangle = \langle\lambda_1(w^+_{s_1}),\lambda_4\rangle\langle\lambda_2,\lambda_3(w^+_{s_1})\rangle. \tag{5.29}$$

Equation (5.24) now follows immediately when we recognize the prefactor that appears in the correlator.

We can now use (5.24) to independently reconstruct an expression for the amplitude. This is because given the function $M(w)$ defined in (5.23), we can force it have the correct residues at the four poles $w^+_{s_1}$, $w^+_{s_2}$, $w^+_{t_1}$, $w^+_{t_2}$. The simplest way to do this is by Lagrange interpolation.

We construct the rational function

$$M^{\rm AdS}(w)=M^{\rm MHV}(w)P^{+}(w)$$
$$M^{\overline{\rm AdS}}(w)=M^{\overline{\rm MHV}}(w)P^{-}(w)$$
$$\begin{aligned}P^{\pm}(w)\equiv&\frac{(w-w^{\pm}_{s_2})(w-w^{\pm}_{t_1})(w-w^{\pm}_{t_2})}{(w^{\pm}_{s_1}-w^{\pm}_{s_2})(w^{\pm}_{s_1}-w^{\pm}_{t_1})(w^{\pm}_{s_1}-w^{\pm}_{t_2})}\frac{E^{123,4}E^{124,3}}{2|\boldsymbol{k_3}||\boldsymbol{k_4}|}\\&+\frac{(w-w^{\pm}_{s_1})(w-w^{\pm}_{t_1})(w-w^{\pm}_{t_2})}{(w^{\pm}_{s_2}-w^{\pm}_{s_1})(w^{\pm}_{s_2}-w^{\pm}_{t_1})(w^{\pm}_{s_2}-w^{\pm}_{t_2})}\frac{E^{134,2}E^{234,1}}{2|\boldsymbol{k_1}||\boldsymbol{k_2}|}\\&+\frac{(w-w^{\pm}_{s_1})(w-w^{\pm}_{s_2})(w-w^{\pm}_{t_1})}{(w^{\pm}_{t_1}-w^{\pm}_{s_2})(w^{\pm}_{t_1}-w^{\pm}_{s_2})(w^{\pm}_{t_1}-w^{\pm}_{t_2})}\frac{E^{134,2}E^{124,3}}{2|\boldsymbol{k_3}||\boldsymbol{k_2}|}\\&+\frac{(w-w^{\pm}_{s_1})(w-w^{\pm}_{s_2})(w-w^{\pm}_{t_2})}{(w^{\pm}_{t_2}-w^{\pm}_{s_1})(w^{\pm}_{t_2}-w^{\pm}_{s_2})(w^{\pm}_{t_2}-w^{\pm}_{t_1})}\frac{E^{134,2}E^{234,1}}{2|\boldsymbol{k_1}||\boldsymbol{k_2}|}.\end{aligned} \tag{5.30}$$

$P^{+}(w)$ is a Lagrange polynomial with the property that modulates the residues of $M^{\rm MHV}(w)$ to produce the correct residues required in $M^{\rm AdS}(w)$. Note that $M^{\rm AdS}(w)$ still has the desired falloff at infinity because $M^{\rm MHV}(w)\longrightarrow\frac{1}{w^4}$ at large w and $P^{+}(w)\longrightarrow w^3$ at large w.

In terms of these functions, we have

$$\mathcal{F}=M^{\rm AdS}(0)+M^{\overline{\rm AdS}}(0). \tag{5.31}$$

In the flat space limit, we have the *identity*

$$P^{+}(w)=P^{-}(w)=2,\quad \text{at } E^{T}=0. \tag{5.32}$$

So, it is clear that in the flat space limit, we have

$$T^{+-+-}\underset{E^T\to 0}{\longrightarrow}\frac{4}{E^T}M^{\rm MHV}=\frac{4}{E^T}M^{\overline{\rm MHV}}, \tag{5.33}$$

which is exactly what we expect from the flat space conjecture.[5]

Finally we notice that this method provides a check on our answer in (5.17). We can now do some algebra given the explicit positions of the poles. We have already computed the values of $w^{\pm}_{s_1}$ in (5.7) and all other poles are given by obvious substitutions in those equations. For example, we have

$$\beta_3 w^{+}_{s2}=-\frac{\langle\lambda_4,\lambda_3\rangle}{[\lambda_4,\bar\lambda_3]}, \tag{5.34}$$

$$\beta_4 w^{-}_{s2}=-\frac{\langle\bar\lambda_3,\bar\lambda_4\rangle}{[\lambda_3,\lambda_4]}. \tag{5.35}$$

We see that

$$\begin{aligned}\frac{1}{\frac{w^{+}_{s_1}}{w^{+}_{s_2}}-1}&=\frac{-1}{\frac{\langle\lambda_2,\lambda_1\rangle[\lambda_4,\bar\lambda_1]}{\langle\lambda_4,\lambda_3\rangle\langle\lambda_2,\bar\lambda_3\rangle}+1}\\&=-\frac{\langle\lambda_4,\lambda_3\rangle[\lambda_2,\bar\lambda_3]}{\langle\lambda_2,\lambda_1\rangle[\lambda_4,\bar\lambda_1]+\langle\lambda_4,\lambda_3\rangle[\lambda_2,\bar\lambda_3]}\\&=-\frac{\langle\lambda_4,\lambda_3\rangle[\lambda_2,\bar\lambda_3]}{i\langle\lambda_2,\lambda_4\rangle E^{12,34}}.\end{aligned} \tag{5.36}$$

Similarly,

$$\frac{1}{\frac{w^{+}_{s_1}}{w^{+}_{t_1}}-1}=\frac{[\lambda_2,\bar\lambda_1]\langle\lambda_4,\lambda_1\rangle}{\langle\lambda_2,\lambda_4\rangle[\lambda_1,\bar\lambda_1]}, \tag{5.37}$$

and

$$\frac{1}{\frac{w^{+}_{s_1}}{w^{+}_{t_2}}-1}=-\frac{[\lambda_4,\bar\lambda_3]\langle\lambda_2,\lambda_3\rangle}{i\langle\lambda_2,\lambda_4\rangle E^{123,4}}. \tag{5.38}$$

We can now substitute (5.36), (5.37), and (5.38) in (5.31) and (5.30) to recover our expression for $\mathcal{F}$.

$$\mathcal{F}=\frac{E^{124,3}}{4|\boldsymbol{k_3}||\boldsymbol{k_4}||\boldsymbol{k_1}|}\frac{\langle\lambda_2,\lambda_4\rangle[\lambda_4,\bar\lambda_3][\lambda_2,\bar\lambda_3][\lambda_2,\bar\lambda_1]}{(E^{12,34})\langle\lambda_1,\lambda_2\rangle}+\ldots \tag{5.39}$$

where the ... indicate the various interchanges indicated in (5.18), (5.19), (5.20), and (5.21). This matches precisely with our previous answer.

B. Full MHV amplitude

To evaluate the full MHV amplitude we also need to consider the (13)(24) partition, which we will call the "u" partition. With the value of w being given by (4.15), we can write down an expression for the integrand.

$$\begin{aligned}I_u=\sum_{\pm}\Big\{&\frac{R^{\rm YM}(|\boldsymbol{k_1}|,|\boldsymbol{k_3}|,p)}{2\sqrt{2}|\boldsymbol{k_1}||\boldsymbol{k_3}|p}\frac{R^{\rm YM}(|\boldsymbol{k_2}|,|\boldsymbol{k_4}|,p)}{2\sqrt{2}|\boldsymbol{k_2}||\boldsymbol{k_4}|p}\Big[(|\boldsymbol{k_1}|+ip-|\boldsymbol{k_3}|)^2(ip+|\boldsymbol{k_3}|-|\boldsymbol{k_1}|)(|\boldsymbol{k_1}|+|\boldsymbol{k_3}|-ip)(|\boldsymbol{k_4}|+ip-|\boldsymbol{k_2}|)\\&\times(ip+|\boldsymbol{k_2}|-|\boldsymbol{k_4}|)(|\boldsymbol{k_2}|+|\boldsymbol{k_4}|+ip)\frac{\langle\bar\lambda_1,\bar\lambda_3\rangle^3}{\langle\bar\lambda_1,\bar\lambda_{\rm int}\rangle\langle\bar\lambda_3,\bar\lambda_{\rm int}\rangle}\frac{\langle\lambda_4,\lambda_2\rangle^3}{\langle\lambda_2,\lambda_{\rm int}\rangle\langle\lambda_4,\lambda_{\rm int}\rangle}\\&+(E^{13}_p)(E^{24}_{-p})(\langle\bar\lambda_1,\bar\lambda_3\rangle\langle\bar\lambda_1,\bar\lambda_{\rm int}\rangle\langle\bar\lambda_3,\bar\lambda_{\rm int}\rangle\langle\lambda_2,\lambda_4\rangle\langle\lambda_2,\lambda_{\rm int}\rangle\langle\lambda_4,\lambda_{\rm int}\rangle)\Big]\Big\}\frac{1}{4}\frac{ip}{(p^2+(\boldsymbol{k_1}+\boldsymbol{k_3})^2)}\frac{w^{\mp}}{w^{\pm}-w^{\mp}}.\end{aligned} \tag{5.40}$$

[5]The extra factor of 4 comes, once again, from the unconventional normalization of our polarization vectors.

We can now simplify this, as above by using the identities

$$\lambda_1\bar{\lambda}_1 + \lambda_3\bar{\lambda}_3 + \lambda_{\text{int}}\bar{\lambda}_{\text{int}} = iE_p^{13}\sigma^3, \tag{5.41}$$

$$\lambda_2\bar{\lambda}_2 + \lambda_4\bar{\lambda}_4 - \lambda_{\text{int}}\bar{\lambda}_{\text{int}} = iE_{-p}^{24}\sigma^3. \tag{5.42}$$

Rewriting terms that involve $|\lambda_{\text{int}}\rangle\langle\bar{\lambda}_{\text{int}}|$ using this, we find:

$$I_u = \sum_{\pm}\bigg\{\frac{-i}{16\pi|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k_3}||\mathbf{k_4}|}\bigg[\frac{(|\mathbf{k_1}| + ip - |\mathbf{k_3}|)(ip + |\mathbf{k_3}| - |\mathbf{k_1}|)(|\mathbf{k_4}| + ip - |\mathbf{k_2}|)}{(E_p^{13})(E_{-p}^{24})} \tag{5.43}$$

$$\times\frac{(ip + |\mathbf{k_2}| - |\mathbf{k_4}|)\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle^3\langle\lambda_2, \lambda_4\rangle^3}{(\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle\langle\lambda_2, \lambda_3(w^{\pm}(p))\rangle + iE_p^{13}[\bar{\lambda}_1, \lambda_2])(\langle\bar{\lambda}_3, \bar{\lambda}_1\rangle\langle\lambda_4, \lambda_1(w^{\pm}(p))\rangle + iE_p^{13}[\bar{\lambda}_3, \lambda_4])} \tag{5.44}$$

$$+\frac{\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle\langle\lambda_2, \lambda_4\rangle(\langle\bar{\lambda}_3, \bar{\lambda}_2(w^{\pm})\rangle\langle\lambda_4, \lambda_2\rangle + iE_{-p}^{24}[\bar{\lambda}_3, \lambda_4])}{(|\mathbf{k_1}| + |\mathbf{k_3}| - ip)(|\mathbf{k_2}| + |\mathbf{k_4}| + ip)} \tag{5.45}$$

$$\times(\langle\bar{\lambda}_1, \bar{\lambda}_4(w^{\pm})\rangle\langle\lambda_2, \lambda_4\rangle + iE_{-p}^{24}[\bar{\lambda}_1, \lambda_2])\bigg]\times\frac{1}{p^2 + (\mathbf{k_1} + \mathbf{k_3})^2}\frac{w^{\mp}}{w^{\pm} - w^{\mp}}\bigg\}. \tag{5.46}$$

We now see that at the pole $E_p^{13} = 0$, only the term with $h_{\text{int}} = -1$ contributes with

$$T_{u_1} = (2\pi i)\text{Res}_{p=i(|\mathbf{k_1}|+|\mathbf{k_3}|)}[I_u] = \frac{-i}{2}\frac{E^{123,4}E^{134,2}}{E^T|\mathbf{k_4}||\mathbf{k_2}|}\sum_{\pm}\frac{\langle\lambda_2, \lambda_4\rangle^3}{\langle\lambda_2, \lambda_3(w_{u_1}^{\pm})\rangle\langle\lambda_4, \lambda_1(w_{u_1}^{\pm})\rangle\langle\lambda_1, \lambda_3\rangle}\frac{w_{u_1}^{\mp}}{w_{u_1}^{\pm} - w_{u_1}^{\mp}}. \tag{5.47}$$

Here $w_{u_1}^{\pm}$ are the two solutions to the quadratic equation

$$\langle\lambda_1(w_{u_1}^{\pm}), \lambda_3(w_{u_1}^{\pm})\rangle = 0. \tag{5.48}$$

Although these individual solutions involve square-roots as we have pointed out above, we are always summing over both solutions, which gets rid of all the roots and leaves us with a rational function of the spinors. The reader may, if she prefers, easily use this to rewrite (5.47) as a rational function of the unextended spinors.

From the formula for the integrand above, it is easier to extract the residue at $p = -i(|\mathbf{k_2}| + |\mathbf{k_4}|)$, which is, in any case, the same as the residue at $p = i(|\mathbf{k_2}| + |\mathbf{k_4}|)$ since the integrand is even in p. To do this, we write (5.44) using (5.42). [We can also extract the residue at E_p^{24} directly using (5.45) and (5.46) but that is less convenient.] When we do this we find that

$$T_{u_2} = (2\pi i)\text{Res}_{p=i(|\mathbf{k_2}|+|\mathbf{k_4}|)}[I_u] = [T_{u_1}]_{1\leftrightarrow\bar{2},3\leftrightarrow\bar{4}} = \frac{-i}{2}\frac{E^{124,3}E^{234,1}}{E^T|\mathbf{k_3}||\mathbf{k_1}|}\sum_{\pm}\frac{\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle^3}{\langle\bar{\lambda}_1, \bar{\lambda}_4(w_{u_2}^{\pm})\rangle\langle\bar{\lambda}_3, \bar{\lambda}_2(w_{u_2}^{\pm})\rangle\langle\lambda_2, \lambda_4\rangle}\frac{w_{u_2}^{\mp}}{w_{u_2}^{\pm} - w_{u_2}^{\mp}}. \tag{5.49}$$

where $w_{u_2}^{\pm}$ are defined by the quadratic equation $\langle\bar{\lambda}_2(w_{u_2}^{\pm}), \bar{\lambda}_4(w_{u_4}^{\pm})\rangle = 0$.

The full amplitude also involves the pole at $p = i|\mathbf{k_1} + \mathbf{k_3}|$. This is given by

$$\begin{aligned} T_{u_3} &= (2\pi i)\text{Res}_{p=i|\mathbf{k_1}+\mathbf{k_3}|}[I_u] \\ &= \frac{i}{16|\mathbf{k_1}||\mathbf{k_2}||\mathbf{k_3}||\mathbf{k_4}||\mathbf{k_1} + \mathbf{k_3}|}\bigg[\frac{E^{3s,1}E^{1s,3}E^{4,s2}E^{2,s4}\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle^3\langle\lambda_2, \lambda_4\rangle^3}{E^{13,s}E^{24s}(\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle\langle\lambda_2, \lambda_3\rangle + iE^{13,s}[\bar{\lambda}_1, \lambda_2])(\langle\bar{\lambda}_3, \bar{\lambda}_1\rangle\langle\lambda_4, \lambda_1\rangle + iE^{13,s}[\bar{\lambda}_3, \lambda_4])} \\ &\quad + \frac{\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle\langle\lambda_2, \lambda_4\rangle(\langle\bar{\lambda}_3, \bar{\lambda}_2\rangle\langle\lambda_4, \lambda_2\rangle + iE^{24s}[\lambda_3, \lambda_4])(\langle\bar{\lambda}_1, \bar{\lambda}_4\rangle\langle\lambda_2, \lambda_4\rangle + iE^{24s}[\bar{\lambda}_1, \lambda_2])}{E^{13s}E^{24,s}}\bigg]. \end{aligned} \tag{5.50}$$

The full contribution of this partition is given by $T_{u_1} + T_{u_2} + T_{u_3}$.

To get the full MHV current correlator we just need to add the contributions from the s and t partitions that we have already computed above. One note of caution is that each of these terms now comes with the appropriate color-factor. For example, the s-channel partition is multiplied by the color-factor $f^{12e}f^{e34}$ and similarly for the t and u channels.

We see now that the flat space limit is manifest. In analogy to the color-ordered correlators, let us define:

$$\mathcal{M}^{\text{AdS}}(w) = (M^{\text{MHV}}(w)P^{+}(w) + M^{\overline{\text{MHV}}}(w)P^{-}(w))f^{12e}f^{e34} + (\tilde{M}^{\text{MHV}}(w)\tilde{P}^{+}(w) + \tilde{M}^{\overline{\text{MHV}}}\tilde{P}^{-}(w))f^{14e}f^{e23}. \quad (5.51)$$

Here

$$\tilde{M}^{\text{MHV}}(w) = \frac{\langle\lambda_2, \lambda_4\rangle^3}{\langle\lambda_1(w), \lambda_4\rangle\langle\lambda_2, \lambda_3(w)\rangle\langle\lambda_3(w), \lambda_1(w)\rangle}, \qquad \tilde{M}^{\overline{\text{MHV}}}(w) = \frac{\langle\bar{\lambda}_1, \bar{\lambda}_3\rangle^3}{\langle\bar{\lambda}_1, \bar{\lambda}_4(w)\rangle\langle\bar{\lambda}_4(w), \bar{\lambda}_2(w)\rangle\langle\bar{\lambda}_2(w), \bar{\lambda}_3\rangle}, \quad (5.52)$$

If we adopt the notation:

$$w^{+}_{y_1} = w^{+}_{u_1}; \qquad w^{+}_{y_2} = w^{-}_{u_1}; \qquad w^{-}_{y_1} = w^{+}_{u_2}; \qquad w^{-}_{y_2} = w^{-}_{u_2}, \quad (5.53)$$

then we can define the $\tilde{P}$ functions using

$$\begin{aligned}\tilde{P}^{+}(w) \equiv{}& \frac{(w-w^{+}_{y_2})(w-w^{+}_{t_1})(w-w^{+}_{t_2})}{(w^{+}_{y_1}-w^{+}_{y_2})(w^{+}_{y_1}-w^{+}_{t_1})(w^{+}_{y_1}-w^{+}_{t_2})}\frac{E^{123,4}E^{134,2}}{2|\boldsymbol{k_2}||\boldsymbol{k_4}|} + \frac{(w-w^{+}_{y_1})(w-w^{+}_{t_1})(w-w^{+}_{t_2})}{(w^{+}_{y_2}-w^{+}_{y_1})(w^{+}_{y_2}-w^{+}_{t_1})(w^{+}_{y_2}-w^{+}_{t_2})}\frac{E^{123,4}E^{134,2}}{2|\boldsymbol{k_2}||\boldsymbol{k_4}|}\\ &+ \frac{(w-w^{+}_{y_1})(w-w^{+}_{y_2})(w-w^{+}_{t_1})}{(w^{+}_{t_1}-w^{+}_{y_2})(w^{+}_{t_1}-w^{+}_{y_2})(w^{+}_{t_1}-w^{+}_{t_2})}\frac{E^{124,3}E^{134,2}}{2|\boldsymbol{k_3}||\boldsymbol{k_2}|} + \frac{(w-w^{+}_{y_1})(w-w^{+}_{y_2})(w-w^{+}_{t_1})}{(w^{+}_{t_2}-w^{+}_{y_1})(w^{+}_{t_2}-w^{+}_{y_2})(w^{+}_{t_2}-w^{+}_{t_1})}\frac{E^{234,1}E^{123,4}}{2|\boldsymbol{k_1}||\boldsymbol{k_4}|}.\end{aligned} \quad (5.54)$$

The other interpolating function $\tilde{P}^{-}$ is defined similarly:

$$\begin{aligned}\tilde{P}^{-}(w) \equiv{}& \frac{(w-w^{-}_{y_2})(w-w^{-}_{t_1})(w-w^{-}_{t_2})}{(w^{-}_{y_1}-w^{-}_{y_2})(w^{-}_{y_1}-w^{-}_{t_1})(w^{-}_{y_1}-w^{-}_{t_2})}\frac{E^{124,3}E^{234,1}}{2|\boldsymbol{k_1}||\boldsymbol{k_3}|} + \frac{(w-w^{-}_{y_1})(w-w^{-}_{t_1})(w-w^{-}_{t_2})}{(w^{-}_{y_2}-w^{-}_{y_1})(w^{-}_{y_2}-w^{-}_{t_1})(w^{-}_{y_2}-w^{-}_{t_2})}\frac{E^{124,3}E^{234,1}}{2|\boldsymbol{k_1}||\boldsymbol{k_3}|}\\ &+ \frac{(w-w^{-}_{y_1})(w-w^{-}_{y_2})(w-w^{-}_{t_1})}{(w^{-}_{t_1}-w^{-}_{y_2})(w^{-}_{t_1}-w^{-}_{y_2})(w^{-}_{t_1}-w^{-}_{t_2})}\frac{E^{124,3}E^{134,2}}{2|\boldsymbol{k_3}||\boldsymbol{k_2}|} + \frac{(w-w^{-}_{y_1})(w-w^{-}_{y_2})(w-w^{-}_{t_1})}{(w^{-}_{t_2}-w^{-}_{y_1})(w^{-}_{t_2}-w^{-}_{y_2})(w^{-}_{t_2}-w^{-}_{t_1})}\frac{E^{234,1}E^{123,4}}{2|\boldsymbol{k_1}||\boldsymbol{k_4}|}.\end{aligned} \quad (5.55)$$

It is this term evaluated at $w = 0$—$\mathcal{M}^{\text{AdS}}(0)$—that plays the role that $\mathcal{F}$ played in (5.16). Just as in the case above, near $E^T = 0$, we find that $\tilde{P}^{\pm} = 2$. So the full MHV amplitude (including contributions from other partitions) goes like:

$$\frac{4}{E^T}(M^{\text{MHV}}(0)f^{12e}f^{e34} + \tilde{M}^{\text{MHV}}(0)f^{14e}f^{e23}) + \ldots$$

where $\ldots$ are terms that are nonsingular at $E^T = 0$. This is exactly what we need.

VI. MHV CORRELATORS FOR THE STRESS TENSOR

We now turn to correlation functions of the stress tensor. Our objective in this section is to write out the formula (4.24) explicitly in terms of spinors and check that the MHV graviton amplitude appears in the flat space limit. We will achieve this in two steps. First we expand out the integrand that appears in (4.24). Then we expand out the residues that appear in that formula. The procedure for writing down the integrand is almost identical to the case of conserved currents. However, there is one important difference in the final result; this is the fact that the poles that appear from the three-point amplitudes are now double poles. Consequently, to extract the residue we need to take a derivative. This complicates our final formulas.

We will carry out this procedure with the same configuration of external polarizations that we used for conserved currents. Namely, we will take $\{h_1, h_2, h_3, h_4\} = \{1, -1, 1, -1\}$. The generalization to other helicity configurations involves a straightforward procedure that is very similar to the one that we present in detail below. The reader may also use the computer program that accompanies this paper to generate answers for any combination of external helicities either analytically or numerically.

We will first write down the integrands for the three different partitions that appear in (4.24). These are the (12)(34) partition, the (14)(23) partition, and the (13)(24) partition. In each case, we then extract the residue at the poles specified in (4.23).

A. (12)(34) Partition:

Let us start by considering the (12)(34) partition. We can write down an expression for the p-integrand corresponding to this partition, using the three-point amplitudes above. This expression is given by

$$\mathcal{I}_s=\sum_{\pm}\bigg\{\frac{R^{\rm gr}(|\boldsymbol{k_1}|,|\boldsymbol{k_2}|,p)}{32|\boldsymbol{k_1}|^2|\boldsymbol{k_2}|^2p^2}(|\boldsymbol{k_2}|+ip-|\boldsymbol{k_1}|)^2(ip+|\boldsymbol{k_1}|-|\boldsymbol{k_2}|)^2(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|-ip)^2\frac{R^{\rm gr}(|\boldsymbol{k_3}|,|\boldsymbol{k_4}|,-p)}{32|\boldsymbol{k_3}|^2|\boldsymbol{k_4}|^2p^2}$$
$$\times(|\boldsymbol{k_4}|-ip-|\boldsymbol{k_3}|)^2(-ip+|\boldsymbol{k_3}|-|\boldsymbol{k_4}|)^2(|\boldsymbol{k_3}|+|\boldsymbol{k_4}|+ip)^2\bigg[\bigg(\frac{\langle\bar\lambda_1,\bar\lambda_{\rm int}\rangle^3}{\langle\bar\lambda_1,\bar\lambda_2(w_s^{\pm})\rangle\langle\bar\lambda_2(w_s^{\pm}),\bar\lambda_{\rm int}\rangle}\frac{\langle\lambda_4,\lambda_{\rm int}\rangle^3}{\langle\lambda_3,(w_s^{\mp}),\lambda_4\rangle\langle\lambda_3w_s^{\mp}),\lambda_{\rm int}\rangle}\bigg)^2$$
$$+\bigg(\frac{\langle\lambda_2,\lambda_{\rm int}\rangle^3}{\langle\lambda_1(w_s^{\pm}),\lambda_2\rangle\langle\lambda_1(w_s^{\pm}),\lambda_{\rm int}\rangle}\frac{\langle\bar\lambda_3,\bar\lambda_{\rm int}\rangle^3}{\langle\bar\lambda_3,\bar\lambda_4(w_s^{\mp})\rangle\langle\bar\lambda_4(w_s^{\mp}),\bar\lambda_{\rm int}\rangle}\bigg)^2\bigg]\frac{ip}{8(p^2+(\boldsymbol{k_1}+\boldsymbol{k_2})^2)}\frac{w_s^{\mp}}{w_s^{\pm}-w_s^{\mp}}\bigg\}. \tag{6.1}$$

Note that if, on the left-hand side, we use the norm $|\boldsymbol{k}_{\rm int}|=ip$, on the right-hand side we need to use $(-ip)$. We also need to flip the sign of $\lambda_{\rm int}$, but this does not matter because of the fact that the integrand involves only the "square" of this term. By expanding out $R^{\rm gr}$ and recalling the identity (5.2), we find

$$\mathcal{I}_s=\sum_{\pm}\frac{1}{2^{12}\pi}\bigg\{\frac{(|\boldsymbol{k_1}|^2+4|\boldsymbol{k_2}||\boldsymbol{k_1}|+|\boldsymbol{k_2}|^2+p^2)(|\boldsymbol{k_3}|^2+4|\boldsymbol{k_4}||\boldsymbol{k_3}|+|\boldsymbol{k_4}|^2+p^2)}{(|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|)^2(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|+ip)^2(|\boldsymbol{k_3}|+|\boldsymbol{k_4}|-ip)^2}$$
$$\times(|\boldsymbol{k_2}|+ip-|\boldsymbol{k_1}|)^2(ip+|\boldsymbol{k_1}|-|\boldsymbol{k_2}|)^2(|\boldsymbol{k_3}|+ip-|\boldsymbol{k_4}|)^2(ip+|\boldsymbol{k_4}|-|\boldsymbol{k_3}|)^2$$
$$\times\bigg[\frac{(\bar\lambda_1,\bar\lambda_2(w_s^{\pm}))\langle\lambda_4,\lambda_2\rangle+iE_p^{12}[\bar\lambda_1,\lambda_4])^6}{\langle\bar\lambda_1,\bar\lambda_2(w_s^{\pm})\rangle^2(\langle\bar\lambda_2(w_s^{\pm}),\bar\lambda_1\rangle\langle\lambda_1,(w_s^{\mp}),\lambda_3(w_s^{\mp})\rangle-iE_p^{12}[\lambda_3(w_s^{\mp}),\bar\lambda_2(w_s^{\pm})])^2\langle\lambda_4,\lambda_3(w_s^{\mp})\rangle^2}$$
$$+\frac{(\langle\lambda_2,\lambda_1(w_s^{\pm})\rangle\langle\bar\lambda_3,\bar\lambda_1\rangle+iE_p^{12}[\lambda_2,\bar\lambda_3])^6}{\langle\lambda_1(w_s^{\pm}),\lambda_2\rangle^2(\langle\lambda_1(w_s^{\pm}),\lambda_2\rangle\langle\bar\lambda_4(w_s^{\mp}),\bar\lambda_2(w_s^{\mp})\rangle+iE_p^{12}[\lambda_1(w_s^{\pm}),\bar\lambda_4(w_s^{\mp})])^2\langle\bar\lambda_3,\bar\lambda_4(w_s^{\mp})\rangle^2}\bigg]$$
$$\times\frac{i}{p^2+(\boldsymbol{k_1}+\boldsymbol{k_2})^2}\frac{w_s^{\mp}}{w_s^{\pm}-w_s^{\mp}}\bigg\}, \tag{6.2}$$

where we have remind the reader that $E_p^{nm}\equiv ip+|\boldsymbol{k_n}|+|\boldsymbol{k_m}|$.

Extracting the Residues: We now proceed to implement (4.24) and extract the residue from the integrand above. An important difference from the conserved-current computation is that, as we mentioned above, the poles that appear from three-point amplitudes are double poles. So extracting the residue involves taking a derivative of the integrand with respect to p at the pole.

It seems more convenient to perform this procedure through "logarithmic differentiation": first we write down an expression for the value of the integrand, with the singular term stripped off, at the pole. Next we write down an expression for the *ratio* of the derivative of this term to the term itself.

The procedure of extracting the value of the integrand is almost identical to the one we followed to obtain (5.11). The difference is that various terms are squared. When $E_p^{12}=0$ the value of $w^{\pm}(p)$ for this partition is still defined by (5.7) and (5.8). To simplify the integrand we throw away all terms proportional to E_p^{12}. Apart from (5.10), we need one more explicit expression for an extended spinor:

$$\bar\lambda_2(w_{s1}^+)=\bar\lambda_3\frac{[\lambda_2,\bar\lambda_2]+[\lambda_1,\bar\lambda_1]-iE^T}{[\lambda_2,\bar\lambda_3]}-\frac{[\lambda_1,\bar\lambda_1]}{[\lambda_2,\bar\lambda_1]}\bar\lambda_1+\frac{iE^T}{\langle\lambda_2,\lambda_4\rangle}\hat\lambda_4. \tag{6.3}$$

From these expressions, we find

$$\mathcal{V}_{s_1}^+=(2\pi i)\lim_{E_p^{12}\to0}(E_p^{12})^2\mathcal{I}_s$$
$$=\bigg[\frac{-i|\boldsymbol{k_2}|(E^{34,12}(E^T)+2|\boldsymbol{k_3}||\boldsymbol{k_4}|)(E^{3,124})^2}{256|\boldsymbol{k_1}||\boldsymbol{k_3}|^2|\boldsymbol{k_4}|^2(E^T)^2(E^{34,12})^2\langle\lambda_1,\lambda_2\rangle}\bigg]\langle\lambda_2,\lambda_4\rangle$$
$$\times[\lambda_2,\bar\lambda_1]^2(\langle\bar\lambda_1,\bar\lambda_3\rangle\langle\lambda_2,\lambda_4\rangle E^{12,34}+E^T[\bar\lambda_1,\lambda_4]$$
$$\times[\lambda_2,\bar\lambda_3])[\lambda_4\bar\lambda_3]^2[\lambda_2,\bar\lambda_3]. \tag{6.4}$$

We now turn to an evaluation of the derivative. As we mentioned above it is convenient to work with the quantity:

$$\mathcal{D}_{s_1}^+=\lim_{E_p^{12}\to0}\frac{d}{dp}\log[(E_p^{12})^2\mathcal{I}_s], \tag{6.5}$$

where it is understood that the limit is taken at the value of $w^{\pm}(p)$ in (6.2) corresponding to $w_{s_1}^+$.

Before we evaluate this expression, note that we can also define $w^{\pm}$ through

$$\langle\lambda_1(w_s),\lambda_2\rangle\langle\bar\lambda_1,\bar\lambda_2(w_s)\rangle=-(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|-ip)E_p^{12}. \tag{6.6}$$

At a pole in the p-integral, where $E_p^{12}=0$, one of the terms on the left-hand side must vanish. We have defined $w_{s_1}^+$ to be the pole where the first dot product vanishes and $w_{s_1}^-$ to be the pole where the second dot product vanishes. With a slight abuse of notation, defining $\frac{dw^{\pm}}{dp}\equiv\frac{dw_{s_1}^{\pm}}{dp}$ at this point, we have

$$\beta_1 \frac{dw_{s_1}^+}{dp}[\bar{\lambda}_1, \lambda_2]\langle\bar{\lambda}_1, \bar{\lambda}_2(w_s^+)\rangle = -2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|) \quad (6.7)$$

$$\Rightarrow \gamma_1^+ \equiv \frac{dw_{s_1}^+}{dp} = \frac{-2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)}{\beta_1[\bar{\lambda}_1, \lambda_2]\langle\bar{\lambda}_1, \bar{\lambda}_2(w_{s_1}^+)\rangle} = \frac{-2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)}{[\bar{\lambda}_1, \lambda_2](\beta_1\langle\bar{\lambda}_1, \bar{\lambda}_2\rangle + \beta_2\langle\lambda_1, \lambda_2\rangle)}. \quad (6.8)$$

Furthermore

$$\beta_2 \frac{dw_{s_1}^-}{dp}\langle\lambda_1(w_s), \lambda_2\rangle[\bar{\lambda}_1, \lambda_2] = -2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|) \quad (6.9)$$

$$\Rightarrow \gamma_1^- \equiv \frac{dw_{s_1}^-}{dp} = \frac{-2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)}{\beta_2\langle\lambda_1(w_{s_1}^-), \lambda_2\rangle[\bar{\lambda}_1, \lambda_2]} = -\gamma_1^+. \quad (6.10)$$

With this observation and notation, $\mathcal{D}_{s_1}^+$ is given by

$$\mathcal{D}_{s_1}^+ = 2i(|\boldsymbol{k_1}| + |\boldsymbol{k_2}|)\left(\frac{1}{E^{34,12}E^T + 2|\boldsymbol{k_3}||\boldsymbol{k_4}|} + \frac{1}{\langle\lambda_1, \lambda_2\rangle\langle\bar{\lambda}_1, \bar{\lambda}_2\rangle}\right) + i\left(\frac{2}{E^{3,124}} + \frac{2}{E^{4,123}} + \frac{2}{E^T}\right) - \frac{1}{\langle\bar{\lambda}_1, \bar{\lambda}_2(w_{s_1}^+)\rangle}\left(\frac{6[\bar{\lambda}_1, \lambda_4]}{\langle\lambda_4, \lambda_2\rangle} - \frac{2[\lambda_3(w_{s_1}^+), \bar{\lambda}_2(w_{s_1}^+)]}{\langle\lambda_1(w_{s_1}^+), \lambda_3(w_{s_1}^+)\rangle}\right) + 2\gamma_1^+\left\{\frac{\beta_2[\bar{\lambda}_1, \lambda_2]}{\langle\bar{\lambda}_1, \bar{\lambda}_2(w_{s_1}^+)\rangle} - \frac{\beta_3[\bar{\lambda}_3, \lambda_4]}{\langle\lambda_4, \lambda_3(w_{s_1}^+)\rangle} + \frac{\beta_1[\bar{\lambda}_1, \lambda_3(w_{s_1}^+)] - \beta_3[\lambda_1(w_{s_1}^+), \bar{\lambda}_3]}{\langle\lambda_1(w_{s_1}^+), \lambda_3(w_{s_1}^+)\rangle}\right\} - \gamma_1^+\frac{w_{s_1}^+ + w_{s_1}^-}{w_{s_1}^-(w_{s_1}^+ - w_{s_1}^-)}. \quad (6.11)$$

If the reader wishes to expand this expression out in terms of unextended spinors, the reader can do so using (5.10) and (6.3) and the additional identity:

$$\lambda_3(w_{s1}^+) = \frac{[\lambda_3, \bar{\lambda}_3] + [\lambda_4, \bar{\lambda}_4] - iE^T}{[\lambda_2, \bar{\lambda}_3]}\lambda_2 - \lambda_4\frac{[\lambda_4, \bar{\lambda}_4] - iE^T}{[\lambda_4, \bar{\lambda}_3]}. \quad (6.12)$$

However, this does not provide much additional insight so we have left the expression above as is.

Note that the contribution of $\mathcal{V}_{s_1}^-\mathcal{D}_{s_1}^-$, which is just the residue at $E_p^{12} = 0$ but with $w^\pm(p) = w_{s_1}^-$ in (6.2) can be easily incorporated, just through some substitutions. We have

$$T_{s_1} = (2\pi i)\mathrm{Res}_{p=i(|\boldsymbol{k_1}|+|\boldsymbol{k_2}|)}[\mathcal{I}_s] = \mathcal{V}_{s_1}^+\mathcal{D}_{s_1}^+ + (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leftrightarrow (\bar{\lambda}_2, \bar{\lambda}_1, \bar{\lambda}_4, \bar{\lambda}_3). \quad (6.13)$$

There is another pole in $\mathcal{I}_s$ at $E_p^{34} = 0$.[6] The contribution of this pole is just obtained by interchanging (12) with (34):

$$T_{s_2} = [T_{s_1}]_{1\leftrightarrow 3, 2\leftrightarrow 4}. \quad (6.14)$$

We now turn to the second kind of contribution to the correlator from this partition, which comes from the pole where the denominator of the propagator vanishes. Fortunately, this is a simple pole. This pole occurs at

$$p = i|\boldsymbol{k_1} + \boldsymbol{k_2}|. \quad (6.15)$$

Note that the factor $\frac{w^+}{w^+ - w^-}$ becomes the identity, since $w^- = 0$. Also, we do not need the other pole w^+ at all, since that contribution vanishes because of this factor (which, in that case, would be $\frac{w^-}{w^+ - w^-}$). We have already seen this in the calculation for current-correlators. Picking up the pole at $w = 0$, we have

$$T_{s_3} = (2\pi i)\mathrm{Res}_{p=i|\boldsymbol{k_1}+\boldsymbol{k_2}|}\mathcal{I}_s = \frac{-i(E^{12,s}E^{12s} + 2|\boldsymbol{k_2}||\boldsymbol{k_1}|)(E^{34,s}E^{34s} + 2|\boldsymbol{k_4}||\boldsymbol{k_3}|)(E^{1s,2})^2(E^{2s,1})^2(E^{3s,4})^2(E^{4s,3})^2}{2^{12}(|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|)^2(E^{12,s})^2(E^{34s})^2|\boldsymbol{k_1} + \boldsymbol{k_2}|} \times\left[\frac{(\langle\bar{\lambda}_1, \bar{\lambda}_2\rangle\langle\lambda_4, \lambda_2\rangle + iE^{12,s}[\bar{\lambda}_1, \lambda_4])^6}{\langle\bar{\lambda}_1, \bar{\lambda}_2\rangle^2(\langle\bar{\lambda}_2, \bar{\lambda}_1\rangle\langle\lambda_1, \lambda_3\rangle - iE^{12,s}[\lambda_3, \bar{\lambda}_2])^2\langle\lambda_3, \lambda_4\rangle^2} + \frac{(\langle\lambda_2, \lambda_1\rangle\langle\bar{\lambda}_3, \bar{\lambda}_1\rangle + iE^{12,s}[\lambda_2, \bar{\lambda}_3])^6}{\langle\lambda_1, \lambda_2\rangle^2(\langle\lambda_1, \lambda_2\rangle\langle\bar{\lambda}_4, \bar{\lambda}_2\rangle + iE^{12,s}[\lambda_1, \bar{\lambda}_4])^2\langle\bar{\lambda}_3, \bar{\lambda}_4\rangle^2}\right], \quad (6.16)$$

where we remind the reader that s stands for sum and so, for example,

$$E^{12,s} \equiv |\boldsymbol{k_1}| + |\boldsymbol{k_2}| - |\boldsymbol{k_1} + \boldsymbol{k_2}|. \quad (6.17)$$

[6]Although the pole that is manifest in (6.2) is actually at $E_{-p}^{34} = 0$, we should remember that the integrand is even in p and we can rewrite it to make the pole at $E_p^{34} = 0$ manifest instead.

To sum up the contribution of the (12)(34) partition is given by

$$T_s = T_{s_1} + T_{s_2} + T_{s_3}, \tag{6.18}$$

where the three terms on the right are given by (6.13), (6.14), and (6.16).

B. (14)(23) Partition

This leads to the same expression as above with the substitution $2 \leftrightarrow 4$. The computation of the residue is exactly the same as the one for the (12)(34) partition with $4 \leftrightarrow 2$:

$$T_t = [T_s]_{\lambda_4 \leftrightarrow \lambda_2, \bar{\lambda}_4 \leftrightarrow \bar{\lambda}_2}. \tag{6.19}$$

C. (13)(24) Partition

This partition has a slightly different structure. We can write the integrand for this partition as $I_u = \sum_\pm I_u^\pm$, corresponding to the two different values of w where

$$I_u^\pm = \frac{R^{\mathrm{gr}}(|\mathbf{k_1}|, |\mathbf{k_3}|, p)}{32|\mathbf{k_1}|^2|\mathbf{k_3}|^2 p^2} \frac{R^{\mathrm{gr}}(|\mathbf{k_2}|, |\mathbf{k_4}|, p)}{32|\mathbf{k_2}|^2|\mathbf{k_4}|^2 p^2} \times \frac{ip}{8(p^2 + (\mathbf{k_1}+\mathbf{k_3})^2)} \frac{w_u^\mp}{w_u^\pm - w_u^\mp}\Big[(|\mathbf{k_1}| + ip - |\mathbf{k_3}|)^2 (ip + |\mathbf{k_3}| - |\mathbf{k_1}|)^2$$
$$\times (|\mathbf{k_1}| + |\mathbf{k_3}| - ip)^2 (|\mathbf{k_4}| + ip - |\mathbf{k_2}|)^2 (ip + |\mathbf{k_2}| - |\mathbf{k_4}|)^2 (|\mathbf{k_2}| + |\mathbf{k_4}| + ip)^2 \Big(\frac{\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle^3}{\langle \bar{\lambda}_1, \bar{\lambda}_{\mathrm{int}}\rangle \langle \bar{\lambda}_3, \bar{\lambda}_{\mathrm{int}}\rangle} \frac{\langle \lambda_4, \lambda_2\rangle^3}{\langle \lambda_2, \lambda_{\mathrm{int}}\rangle\langle \lambda_4, \lambda_{\mathrm{int}}\rangle}\Big)^2$$
$$+ (E_p^{13})^2 (E_{-p}^{24})^2 (\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle \langle \bar{\lambda}_1, \bar{\lambda}_{\mathrm{int}}\rangle \langle \bar{\lambda}_3, \bar{\lambda}_{\mathrm{int}}\rangle \langle \lambda_2, \lambda_4\rangle \langle \lambda_2, \lambda_{\mathrm{int}}\rangle \langle \lambda_4, \lambda_{\mathrm{int}}\rangle)^2 \Big]. \tag{6.20}$$

We can write this expression as

$$I_u^\pm = \frac{(|\mathbf{k_1}|^2 + 4|\mathbf{k_3}||\mathbf{k_1}| + |\mathbf{k_3}|^2 + p^2)(|\mathbf{k_2}|^2 + 4|\mathbf{k_2}||\mathbf{k_4}| + |\mathbf{k_4}|^2 + p^2)}{2^{12}\pi(|\mathbf{k_1}||\mathbf{k_4}||\mathbf{k_3}||\mathbf{k_2}|)^2}$$
$$\times \Big[\frac{(|\mathbf{k_1}| + ip - |\mathbf{k_3}|)^2 (ip + |\mathbf{k_3}| - |\mathbf{k_1}|)^2 (|\mathbf{k_4}| + ip - |\mathbf{k_2}|)^2 (ip + |\mathbf{k_2}| - |\mathbf{k_4}|)^2}{(E_p^{13})^2 (E_{-p}^{24})^2}$$
$$\times \frac{\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle^6 \langle \lambda_4, \lambda_2\rangle^6}{(\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle \langle \lambda_2, \lambda_3(w_u^\pm)\rangle + iE_p^{13}[\bar{\lambda}_1, \lambda_2])^2 (\langle \bar{\lambda}_3, \bar{\lambda}_1\rangle \langle \lambda_4, \lambda_1(w_u^\pm)\rangle + iE_p^{13}[\bar{\lambda}_3, \lambda_4])^2}$$
$$+ \frac{\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle^2 \langle \lambda_2, \lambda_4\rangle^2}{(|\mathbf{k_1}| + |\mathbf{k_3}| - ip)^2 (|\mathbf{k_2}| + |\mathbf{k_4}| + ip)^2} (\langle \bar{\lambda}_3, \bar{\lambda}_2(w_u^\pm\rangle \langle \lambda_4, \lambda_2\rangle + iE_{-p}^{24}[\bar{\lambda}_3, \lambda_4])^2 (\langle \bar{\lambda}_1, \bar{\lambda}_4(w_u^\pm)\rangle \langle \lambda_2, \lambda_4\rangle$$
$$+ iE_{-p}^{24}[\bar{\lambda}_1, \lambda_2])^2 \Big] \frac{i}{p^2 + (\mathbf{k_1} + \mathbf{k_3})^2} \frac{w_u^\mp}{w_u^\pm - w_u^\mp}. \tag{6.21}$$

Extracting the Residues: Let us start by picking up the residue at $p = i(|\mathbf{k_1}| + |\mathbf{k_3}|)$. We see from the expression above that only the term with $h_{\mathrm{int}} = -1$ contributes to this residue; however, both values of w are important. We have

$$\mathcal{V}_{u_1}^\pm = (2\pi i) \lim_{E_p^{13} \to 0} (E_p^{12})^2 I_u^\pm = \frac{|\mathbf{k_1}||\mathbf{k_3}|(E^{24,13}E^T + 2|\mathbf{k_2}||\mathbf{k_4}|)(E^{4,123})^2 (E^{2,134})^2 \langle \bar{\lambda}_1, \bar{\lambda}_3\rangle \langle \lambda_4, \lambda_2\rangle^6}{64(|\mathbf{k_4}||\mathbf{k_2}|)^2 (E^T)^2 \langle \lambda_2, \lambda_3(w_u^\pm)\rangle^2 \langle \lambda_4, \lambda_1(w_u^\pm)\rangle^2 \langle \lambda_1, \lambda_3\rangle} \frac{w_{u_1}^\mp}{w_{u_1}^\pm - w_{u_1}^\mp}. \tag{6.22}$$

Now we need the derivative of the log of the integrand. Recall that w can be defined through

$$\langle \lambda_1 + \hat{\bar{\lambda}}_1 \beta_1 w^\pm, \lambda_3 + \hat{\bar{\lambda}}_3 \beta_3 w^\pm \rangle \langle \bar{\lambda}_1, \bar{\lambda}_3\rangle = -((|\mathbf{k_1}| + |\mathbf{k_3}| - ip)E_p^{13}, \tag{6.23}$$

and this also leads to an expression for the derivative when $E_p^{13} = 0$:

$$\gamma_3^\pm \equiv \frac{dw_{u_1}^\pm}{dp} = \frac{2i(|\mathbf{k_1}| + |\mathbf{k_3}|)}{\langle \bar{\lambda}_1, \bar{\lambda}_3\rangle (\beta_3[\lambda_1(w^\pm), \bar{\lambda}_3] - \beta_1[\lambda_3(w^\pm), \bar{\lambda}_1])}. \tag{6.24}$$

We have $\gamma_3^+ = -\gamma_3^-$.

We can now evaluate the derivative that we need

$$\begin{aligned}\mathcal{D}^{+}_{u_1} &= \lim_{E^{13}_p \to 0} \frac{d}{dp}\log[(E^{13}_p)^2 \mathcal{I}^{+}_u] \\ &= \frac{2i(|\boldsymbol{k_1}|+|\boldsymbol{k_3}|)}{E^{24,13}E^T + 2|\boldsymbol{k_2}||\boldsymbol{k_4}|} + \frac{2i(|\boldsymbol{k_2}|+|\boldsymbol{k_3}|)}{\langle\lambda_1,\lambda_3\rangle\langle\bar\lambda_1,\bar\lambda_3\rangle} \\ &\quad + \frac{2i}{E^{4,123}} + \frac{2i}{E^{2,134}} + \frac{2i}{E^T} - \gamma_3^{+}\bigg(\frac{2\beta_3[\lambda_2,\bar\lambda_3]}{\langle\lambda_2,\lambda_3(w_u^{\pm})\rangle} \\ &\quad + \frac{2\beta_1[\lambda_4,\bar\lambda_1]}{\langle\lambda_4,\lambda_1(w_u^{\pm})\rangle} + \frac{w_{u_1}^{+}+w_{u_1}^{-}}{w_{u_1}^{-}(w_{u_1}^{+}-w_{u_1}^{-})}\bigg) \\ &\quad + \frac{2}{\langle\bar\lambda_2,\bar\lambda_3\rangle}\bigg(\frac{[\bar\lambda_1,\lambda_2]}{\langle\lambda_2,\lambda_3(w_u^{\pm})\rangle} - \frac{[\bar\lambda_3,\lambda_4]}{\langle\lambda_4,\lambda_1(w_u^{\pm})\rangle}\bigg). \end{aligned} \tag{6.25}$$

We can now write

$$T_{u_1} = \sum_{\pm} \mathcal{V}^{\pm}_{u_1}\mathcal{D}^{\pm}_{u_1}. \tag{6.26}$$

The residue at $p = i(|\boldsymbol{k_2}| + |\boldsymbol{k_4}|)$ can be obtained by interchanges in the expression above:

$$T_{u_2} = [T_{u_1}]_{1\leftrightarrow\bar2, 3\leftrightarrow\bar4}. \tag{6.27}$$

(It is understood that alongside we also take $\bar1 \leftrightarrow 2$, $\bar3 \leftrightarrow 4$.)

Finally, we turn to the contribution from the pole at $p^2 = -(\boldsymbol{k_2}+\boldsymbol{k_4})^2 = -(\boldsymbol{k_1}+\boldsymbol{k_3})^2$, which occurs at $w = 0$. This is a first-order pole and we can evaluate it as above.

$$\begin{aligned} T_{u_3} &= (2\pi i)\mathrm{Res}_{p=i|\boldsymbol{k_1}+\boldsymbol{k_3}|}\mathcal{I}_u \\ &= \frac{-i(E^{13,s}E^{13s} + 2|\boldsymbol{k_1}||\boldsymbol{k_3}|)(E^{24,s}E^{24s} + 2|\boldsymbol{k_2}||\boldsymbol{k_4}|)}{2^{12}(|\boldsymbol{k_1}||\boldsymbol{k_4}||\boldsymbol{k_3}||\boldsymbol{k_2}|)^2|\boldsymbol{k_1}+\boldsymbol{k_3}|}\Bigg[\frac{(E^{1s,3})^2(E^{3s,1})^2(E^{2s,4})^2(E^{24s})^2\langle\bar\lambda_1,\bar\lambda_3\rangle^2\langle\lambda_4,\lambda_2\rangle^6}{(E^{13,s})^2(E^{24,s})^2(\langle\lambda_2,\lambda_3\rangle + iE^{13,s}\frac{[\bar\lambda_1,\lambda_2]}{\langle\bar\lambda_1,\bar\lambda_3\rangle})^2(\langle\lambda_4,\lambda_1\rangle + iE^{13,s}\frac{[\bar\lambda_3,\lambda_4]}{\langle\bar\lambda_3,\bar\lambda_1\rangle})^2} \\ &\quad + \frac{\langle\bar\lambda_1,\bar\lambda_3\rangle^2\langle\lambda_2,\lambda_4\rangle^6}{(E^{13s})^2(E^{24,s})^2}\bigg(\langle\bar\lambda_3,\bar\lambda_2\rangle + iE^{24s}\frac{[\bar\lambda_3,\lambda_4]}{\langle\lambda_4,\lambda_2\rangle}\bigg)^2\bigg(\langle\bar\lambda_1,\bar\lambda_4\rangle + iE^{24s}\frac{[\bar\lambda_1,\lambda_2]}{\langle\lambda_2,\lambda_4\rangle}\bigg)^2\Bigg]. \end{aligned} \tag{6.28}$$

The contribution from this partition can be written as

$$T_u = T_{u_1} + T_{u_2} + T_{u_3}, \tag{6.29}$$

summing the contributions of the three terms on the right-hand side, which are computed above.

D. Final answer

The final answer for the stress tensor correlator can now be written in terms of all the contributions above:

$$T^{+-+-}(\boldsymbol{k_1}, \ldots \boldsymbol{k_4}) = T_s + T_t + T_u, \tag{6.30}$$

where the contributions from the three partitions are given in (6.18), (6.19), and (6.29). The formula above involves only four structurally distinct expressions. These are the expressions for the residue at $w_{s_1}^{+}$, at w_{s_3}, at $w_{u_1}^{+}$ and w_{u_3}. The formula above tells us that all other expressions are given by simple interchanges in these expressions.

E. Flat space limit

Although the final formulas above are more complicated than the corresponding formulas for currents, it is not difficult to extract the MHV graviton amplitude from them. Our analysis is very similar to subsection VA 2, so we will be brief here.

First let us recall some facts about the flat space MHV graviton amplitude. Consider the rational functions of w:

$$\begin{aligned} \mathcal{M}^{\mathrm{MHV}}(w) &= i\frac{\langle\lambda_4,\lambda_2\rangle^6\langle\bar\lambda_1,\bar\lambda_3\rangle}{\langle\lambda_3(w),\lambda_4\rangle\langle\lambda_1(w),\lambda_2\rangle\langle\lambda_2,\lambda_3(w)\rangle\langle\lambda_1(w),\lambda_4\rangle\langle\lambda_1(w),\lambda_3(w)\rangle}, \\ \mathcal{M}^{\overline{\mathrm{MHV}}}(w) &= i\frac{\langle\bar\lambda_3,\bar\lambda_1\rangle^6\langle\lambda_2,\lambda_4\rangle}{\langle\bar\lambda_3,\bar\lambda_4(w)\rangle\langle\bar\lambda_1,\bar\lambda_2(w)\rangle\langle\bar\lambda_1,\bar\lambda_4(w)\rangle\langle\bar\lambda_2(w),\bar\lambda_3\rangle\langle\bar\lambda_2(w),\bar\lambda_4(w)\rangle}, \end{aligned} \tag{6.31}$$

with w extended according to (4.1). The usual MHV graviton amplitude is given by $M^{\mathrm{MHV}}(0) = M^{\overline{\mathrm{MHV}}}(0)$ [8,19].

The trick is to break $M^{\mathrm{MHV}}(0)$ into partial fractions. This can be achieved by using the Cauchy theorem and writing $M^{\mathrm{MHV}}(0)$ as the sum of the residues of $M^{\mathrm{MHV}}(w)$ at its poles. However, these poles are precisely at the values of w considered in our previous subsection. In particular, we have

$$M^{\rm MHV}(0) = \Big[\lim_{w\to w_{s_1}^+}\frac{\langle\lambda_1(w),\lambda_2\rangle}{\langle\lambda_1,\lambda_2\rangle} + \lim_{w\to w_{t_1}^+}\frac{\langle\lambda_1(w),\lambda_4\rangle}{\langle\lambda_1,\lambda_4\rangle} + \lim_{w\to w_{s_2}^+}\frac{\langle\lambda_3(w),\lambda_4\rangle}{\langle\lambda_3,\lambda_4\rangle} + \lim_{w\to w_{t_2}^+}\frac{\langle\lambda_3(w),\lambda_2\rangle}{\langle\lambda_3,\lambda_2\rangle} + \sum_{\pm}\frac{w_{u_1}^{\mp}}{w_{u_1}^{\mp}-w_{u_1}^{\pm}}\lim_{w\to w_{u_1}^{\pm}}\frac{\langle\lambda_3(w),\lambda_1(w)\rangle}{\langle\lambda_3,\lambda_1\rangle}\Big]M^{\rm MHV}(w), \tag{6.32}$$

which is just the sum of the residues of $M^{\rm MHV}(w)$ at all its poles. We can write down an analogous expression for $M^{\overline{\rm MHV}}(w)$.

Now, let us turn to our computations of the stress tensor correlator. To check the flat space limit, we need to focus on the coefficient of $(E^T)^{-3}$ and check that it matches the expressions above. However, a term of this kind only comes from the $\frac{1}{E^T}$ term in the derivatives. We can see a term of this form in (6.11) and (6.25). So, to check the coefficient we must only look at the *values of the integrands* (with the poles stripped off) at $E^T = 0$.

This is easy to do. For example, consider (6.2). Near $E^T = 0$, we can see that

$$\mathcal{V}_{s_1}^{+} = \frac{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|}{2(E^T)^2}\frac{\langle\bar\lambda_1,\bar\lambda_2(w_{s_1}^+)\rangle\langle\lambda_4,\lambda_2\rangle^6}{\langle\lambda_1(w_{s_1}^+),\lambda_3(w_{s_1}^+)\rangle^2\langle\lambda_4,\lambda_3(w_{s_1}^+)\rangle^2\langle\lambda_1,\lambda_2\rangle} + \ldots \tag{6.33}$$

where the ... are terms less singular in E^T. Here, we have just dropped the E_p^{12} terms in (6.2) and simplified other factors using $E^T = 0$. Now, using the fact that $\lambda(w_{s_1}^+) \propto \lambda_2$, we can write

$$\mathcal{V}_{s_1}^{+} = \frac{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|\langle\bar\lambda_1,\bar\lambda_2(w_{s_1}^+)\rangle\langle\lambda_4,\lambda_2\rangle^6\langle\lambda_2,\lambda_4\rangle}{2(E^T)^2\langle\lambda_1(w_{s_1}^+),\lambda_3(w_{s_1}^+)\rangle\langle\lambda_2,\lambda_3(w_{s_1}^+)\rangle\langle\lambda_1(w_{s_1}^+),\lambda_4\rangle\langle\lambda_4,\lambda_3(w_{s_1}^+)\rangle^2\langle\lambda_1,\lambda_2\rangle} + \ldots \tag{6.34}$$

Dropping terms that are less singular at $E^T = 0$, this becomes:

$$\mathcal{V}_{s_1}^{+} = \frac{1}{2(E^T)^2}\frac{\langle\bar\lambda_1,\bar\lambda_3\rangle\langle\lambda_4,\lambda_2\rangle^6}{\langle\lambda_1(w_{s_1}^+),\lambda_3(w_{s_1}^+)\rangle\langle\lambda_2,\lambda_3(w_{s_1}^+)\rangle\langle\lambda_1(w_{s_1}^+),\lambda_4\rangle\langle\lambda_3(w_{s_1}^+),\lambda_4\rangle\langle\lambda_1,\lambda_2\rangle} + \ldots \tag{6.35}$$

Combining this with the $\frac{2i}{E^T}$ from $\mathcal{D}_{s_1}^{+}$ in (6.11), we see that

$$T_{s_1} = \frac{i}{(E^T)^3}\frac{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|\langle\bar\lambda_1,\bar\lambda_3\rangle\langle\lambda_4,\lambda_2\rangle^6}{\langle\lambda_1(w_{s_1}^+),\lambda_3(w_{s_1}^+)\rangle\langle\lambda_2,\lambda_3(w_{s_1}^+)\rangle\langle\lambda_1(w_{s_1}^+),\lambda_4\rangle\langle\lambda_3(w_{s_1}^+),\lambda_4\rangle\langle\lambda_1,\lambda_2\rangle} + \ldots \tag{6.36}$$

However, this is exactly the first term in (6.32). Working through the other terms we see that the full gravitational correlator can be written

$$T^{+-+-}(\boldsymbol{k_1},\ldots\boldsymbol{k_4}) = \frac{|\boldsymbol{k_1}||\boldsymbol{k_2}||\boldsymbol{k_3}||\boldsymbol{k_4}|}{(E^T)^3}(M^{\rm MHV}(0) + M^{\overline{\rm MHV}}(0)) + \ldots \tag{6.37}$$

where ... are terms less singular in E^T. This is *precisely* consistent with the flat space limit conjectured in [9], which we remind the reader was

$$M^{+-+-}(\boldsymbol{k_1},\ldots,\boldsymbol{k_4}) = \lim_{E^T\to 0}\frac{(E^T)^3}{2\prod_{m=1}^4|\boldsymbol{k_m}|}T^{+-+-}(\boldsymbol{k_1},\ldots\boldsymbol{k_4}). \tag{6.38}$$

This concludes our discussion of the flat space limit of the MHV graviton correlator. It is not difficult to see that for NMHV and N^2MHV configurations, the graviton correlator (with a pure Hilbert action) has no $\frac{1}{(E^T)^3}$ singularities and so its flat space limit vanishes just as one would expect.

VII. DISCUSSION

In this paper, we have obtained four-point functions of the stress-tensor in AdS_4/CFT_3 from a bulk gravity computation. Although the evaluation of Witten diagrams in AdS is very complicated, we utilized the technique devised in a companion paper to directly obtain the final answer using three-point functions as an input. To our knowledge, this is the first time explicit expressions for the four-point function of the stress-tensor have been written down.

To summarize briefly, we wrote down a general formula for the four-point function for arbitrary external helicities and momenta in terms of the residues of a rational integrand at prespecified poles. This formula is given in (4.24). In the case of the MHV current and stress tensor correlators, we evaluated this formula in terms of the external spinors. The answer for the color-ordered MHV correlator is given in (5.16) and for the full MHV correlator is given in (5.51). The answer for the full MHV stress tensor correlator is given in (6.30). We also verified that, in the flat space limit, these answers give exactly the flat space MHV gluon and graviton amplitudes.

From a structural perspective, it is interesting that our recursion relations also remove the divergences that usually appear in momentum space computations from the region near the boundary. What is striking, and related to this, is that our final answers are purely rational functions of the momenta, their norms, and the norms of the sum of the momenta i.e. $\boldsymbol{k}_m$ $|\boldsymbol{k}_m|$ and $|\boldsymbol{k}_{m_1} + \boldsymbol{k}_{m_2}|$. In particular, our answers are free of logarithms in momentum space.

The fact that stress tensor correlators have such an analytic structure is also supported by the observation that their correlators in AdS_4 can be obtained by doing a flat space computation (on half of flat space) using conformal gravity [20]. Similarly, current correlators in AdS_4 with pure Yang-Mills in the bulk can also be obtained by doing a computation on 4-dimensional flat space, cut off at $z = 0$. These computations do not lead to any logarithms in momentum space.

Now logarithms in position space come from the fact that when we expand a four-point correlator of some operator ϕ in a large-N theory, $\langle\phi(0)\phi(\boldsymbol{x})\phi(\boldsymbol{y}_1)\phi(\boldsymbol{y}_2)\rangle$, in terms of the contribution of various operators in the OPE when x is close to 0, we get terms like $|\boldsymbol{x}|^{\Delta_0 - 2\Delta_\phi + \delta}$ where Δ_ϕ is the dimension of ϕ, Δ_O is the dimension of the operator in the OPE-channel under consideration, and δ is a small "anomalous dimension" proportional to a negative power of N. Expanding this term in a $\frac{1}{N}$ expansion we get logarithms. From the fact that the Fourier transform of $|\boldsymbol{x}|^{\Delta_O - 2\Delta_\phi + \delta}$ is proportional to $|\boldsymbol{k}|^{-d-\Delta_O+2\Delta_\phi-\delta}$, one might naively suspect that the absence of logarithms in momentum space is indicative of the absence of anomalous dimensions for double trace operators of the stress tensor.

However, this logic is not quite correct.[7] In the case where $\Delta_O - 2\Delta_\phi$ is a positive *even integer*, if we carefully consider this Fourier transform we find that we can get logarithms in position space without corresponding logarithms in momentum space. This is related to the fact that the Fourier transform of $\frac{1}{|\boldsymbol{k}|^{2m+3}}$ in 3 dimensions, where m is a non-negative integer, is proportional to $|\boldsymbol{x}|^{2m}\log(|\boldsymbol{x}|)$ plus some terms that are analytic in $\boldsymbol{x}$ and depend on the precise $i\epsilon$ prescription that we use while performing the Fourier transform. In fact, the double trace operators of the stress tensor that appear in the four-point correlator, which we have computed, give a contribution that is of this form. Their contribution in the OPE can be written in the form $Q(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2)|\boldsymbol{x}|^{2m+\delta}$ where Q is a polynomial in $\boldsymbol{x}$. Consequently, they are subject to the subtlety above. It would be nice to perform this Fourier transform in full detail and extract the anomalous dimensions and OPE coefficients of double trace operators from our results.

As we mentioned above, the OPE provides another check on our final answer. The residue from the third pole in the list (4.23) automatically contains the product of three-point functions of the stress tensor multiplied with the appropriate power of the two point function. This accounts for the entire contribution of the conformal block of the stress tensor itself.[8] It would be nice to show explicitly that the rest of the correlator is consistent with the expectation that it comes from the contribution of double trace operators.

Turning now to finer details in the four-point correlator, we notice that our answers for are relatively simple but our expressions for stress-tensor correlators are still somewhat unwieldy. It would be nice to put these answers in their "simplest possible" form. One complication is that in momentum space it is only the Lorentz subgroup $SO(2, 1)$, of the conformal group on the boundary, that is manifest. For scattering amplitudes in four-dimensional flat space, the Lorentz group $SO(3, 1)$ and a knowledge of how amplitudes scale under dilatations at tree-level is sufficient to ensure that four-point function essentially depends only on one variable—the scattering angle. This simplification cannot be obtained here just by using the Lorentz group of the boundary.

Of course the full conformal group is far more powerful but the constraints of special conformal invariance are differential equations in momentum space. This makes these constraints rather tedious to implement.

One could also ask whether a knowledge of the flat space limit of the correlator, and some further assumptions about its analytic structure are enough to determine it completely. This deserves further attention. It is also possible that the simplest form of the four-point correlators will be obtained by using another formalism, such as twistor space or a simpler set of recursion relations. We hope that the results presented here will be useful as inputs and checks on any such new method. So we have included a Mathematica program [11] that allows for the automated evaluation of the formulas in this paper.

We should mention that it is quite easy to generalize this formalism to supersymmetric theories. While graviton tree-amplitudes are the same in supergravity and pure

[7] I would like to thank Liam Fitzpatrick, Jared Kaplan and Joao Penedones for a discussion on this question.

[8] As we mentioned above, an advantage of momentum space is that we do not have to worry about conformal blocks: the contribution of all descendants in a channel is just obtained by multiplying correlators, on the left and the right, involving the primary.

gravity, it is possible that expressing the amplitude in a manifestly supersymmetric form will make it more compact and also reveal links between correlators for different possible external helicities. We leave these investigations to future work.

Finally, we should point out that in this paper we have observed several advantages of going to momentum space in AdS_4 for computations involving massless fields with spin. The wave-functions are very simple, and so z-integrals are easy to do; complicated interactions can be analyzed by generalizing flat space techniques in momentum space; conformal blocks are trivial and it is also easy to take a flat space limit. This suggests that it would be very useful to analyze the Vasiliev theory in AdS_4—which contain massless higher spin fields—in momentum space. We leave this to future work.

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