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Generalized self-dual Maxwell-Chern-Simons-Higgs model

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We present a consistent Bogomol'nyi-Prasad-Sommerfield (BPS) framework for a generalized Maxwell-Chern-Simons-Higgs model. The overall model, including its self-dual potential, depends on three different functions, $h(|\phi|, N)$, $w(|\phi|)$, and $G(|\phi|)$, which are functions of the scalar fields only. The BPS energy is proportional to the magnetic flux when $w(|\phi|)$ and $G(|\phi|)$ are related to each other by a differential constraint. We present an explicit nonstandard model and its topologically nontrivial static configurations, which are described by the usual radially symmetric profile. Finally, we note that the nonstandard results behave in a similar way as their standard counterparts, as expected, reinforcing the consistence of the overall construction.

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I. INTRODUCTION

Topologically nontrivial structures have been intensively studied in many areas of physics [1]. In particular, in the context of classical field theories, these structures are described as finite-energy solutions to some nonlinear models. In this case, such models must be endowed by a spontaneous symmetry breaking potential for the matter self-interaction, since topological solutions are formed during symmetry breaking phase transitions.

In this context, the most common topological defects are kinks [2], vortices [3] and magnetic monopoles [4]: the first ones are one-dimensional structures described by a single real scalar field, while vortices and monopoles are two- and three-dimensional configurations arising as static solutions of some Abelian-and a non-Abelian-Higgs models.

In particular, vortices are stable configurations described by a radially symmetric profile. The simplest version of such structures arises as electrically noncharged solution of a planar Maxwell-Higgs model endowed by a fourth-order Higgs potential. However, under special circumstances, vortices also arise as electrically charged configurations of some Chern-Simons- and Maxwell-Chern-Simons-Higgs (MCS-Higgs) models [5]. In all these cases, such structures can be found as numerical solutions to a set of first-order differential equations, named Bogomol'nyi-Prasad-Sommerfield (BPS) ones [6]. In this case, as finite-energy solutions, they have interesting applications, mainly concerning the superconductivity phenomena [3].

Moreover, during the last years, beyond the standard models cited above, nonusual ones have been intensively studied. These theories, generically named *k-field models*, are endowed by nonusual kinetic terms, which change the dynamics of the overall model in a nonstandard way. Here, it is important to point out that the motivation regarding such generalization arises in a rather natural way, in the context of string theories.

In fact, *k*-field theories have been used as effective models mainly in Cosmology, as an attempt to explain the actual accelerated inflationary phase of the universe [7] via the so-called *k*-essence models [8]. Furthermore, they have been applied in the study of strong gravitational waves [9], dark-[10] and tachyon-matter [11], and others [12].

In such a nonstandard scenario, topologically nontrivial configurations, named *topological k-solutions*, can exist even in the absence of a symmetry breaking potential [13], from which one notes that the existence of such structures are quite sensible to the presence of nonstandard kinetic terms. On the other hand, as an attempt to study topological *k*-solutions via the comparison between them and their canonical counterparts, some of us have already considered the existence of such solutions in the context of symmetry breaking *k*-field models [14]. Moreover, interesting results concerning these models and the corresponding solutions can be found in Ref. [15].

In general, given nontrivial kinetic terms, *k*-field models can be highly nonlinear, and the corresponding *k*-solutions can be quite hard to find. In this case, the development of a consistent self-dual theoretical framework is quite useful and desirable, since it helps to find topological *k*-structures as solutions to some nonstandard BPS equations. Here, as in the usual approach, such equations can be obtained via the minimization of the energy functional related to the nonstandard model, the resulting self-dual *k*-solutions being the minimal energy configurations possible ever.

In this sense, in a recent work, some of us have presented a BPS theoretical framework consistent with a generalized self-dual Maxwell-Higgs model endowed by nonusual kinetic terms to both gauge and scalar fields [16]. Now, we introduce an extension of that work. Here, we develop a general first-order approach consistent with a nonstandard self-dual MCS-Higgs model. The overall model, including its self-dual potential, depends on three different functions,

$h(|\phi|, N)$, $w(|\phi|)$ and $G(|\phi|)$, which are functions of the scalar sector of the model. In this context, we look for topologically nontrivial configurations via the usual radially symmetric static *Ansatz*, and the finite-energy generalized numerical solutions we found behave in the same general way as their standard counterparts, as expected.

This letter is outlined as follows: in the next Sec. II, we introduce the generalized model and develop the BPS framework which allows us to get to its nonstandard BPS equations. Also, we verify the consistence of the overall construction by using it to develop a generalization of the usual MCS-Higgs case, the general model being controlled by two real parameters, α and b. Then, in Section III, we perform the numerical analysis concerning the nonstandard BPS equations previously presented by means of the relaxation technique. Also, we depict the corresponding self-dual k-solutions for the electromagnetic sector and comment on the main features they engender. Finally, in Sec. IV, we present our ending comments and perspectives.

From now on, we use the natural units system and a plus-minus signature for the planar Minkowski metric.

II. THE MODEL

The planar Lagrangian density describing the generalized Maxwell-Chern-Simons-Higgs model is given by

$$\mathcal{L}_d = -\frac{h(|\phi|, N)}{4} F_{\mu\nu}F^{\mu\nu} - \frac{k}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + w(|\phi|)|D_\mu\phi|^2 + \frac{h(|\phi|, N)}{2}\partial_\mu N\partial^\mu N - U(|\phi|, N), \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field strength tensor, $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$ is the covariant derivative of the Higgs field and $\epsilon^{\mu\nu\rho}$ is the $(1+2)$— dimensional Levi-Civita's tensor (with $\epsilon^{012} = +1$). The additional real scalar field N provides the stabilization of the self-dual solutions arising in the presence of both Maxwell and Chern-Simons terms. Here, $h(|\phi|, N)$ and $w(|\phi|)$ are positive-definite dimensionless functions of the scalar fields of the model. The Higgs potential $U(|\phi|, N)$ supporting spontaneous symmetry breaking is supposed to have the following structure:

$$U(|\phi|, N) = \frac{k^2(N + G(|\phi|))^2}{2h(|\phi|, N)} + e^2N^2|\phi|^2 w(|\phi|), \tag{2}$$

from which one notes that $U(|\phi|, N)$ is defined in terms of $h(|\phi|, N)$, $w(|\phi|)$, and $G(|\phi|)$, with $\dim G = 1/2$. It is worthwhile to point out that the specific form of the potential in (2) assures the self-duality of the generalized model (1), i.e., it is a consequence of the BPS construction and can be derived from the energy density (10).

We introduce the mass scale M of the model, and use it to perform the scale transformations: $x^\mu \to M^{-1}x^\mu$, $\phi \to M^{1/2}\phi$, $N \to M^{1/2}N$, $A^\mu \to M^{1/2}A^\mu$, $k \to Mk$, $e \to M^{1/2}e$, and $v \to M^{1/2}v$, where v is the vacuum expectation value of the Higgs field. Then, we get that $G \to M^{1/2}G$ and $\mathcal{L}_d \to M^3\ \mathcal{L}$, and the modified model (1) is now described by the dimensionless Lagrange density $\mathcal{L}$, which has the same form as $\mathcal{L}_d$. Also, for simplicity, we choose $e = v = k = 1$.

It is well known that Chern-Simons theories only exhibit electrically charged static solutions. In fact, the static Gauss law related to (1) is (j runs over spatial indices only)

$$\partial_j(h\partial^j A^0) + 2|\phi|^2A^0w = F_{12}, \tag{3}$$

from which one observes that, even in the presence of nontrivial $h(|\phi|, N)$, $w(|\phi|)$, and $G(|\phi|)$, the absence of electrically uncharged static solutions still holds, since the temporal gauge ($A^0 = 0$) does not solve (3).

Now, since the gauge $A^0 = 0$ cannot be implemented and supposing that $h(|\phi|, N)$, $w(|\phi|)$, and $G(|\phi|)$ are nontrivial functions, the looking for static solutions for the second-order Euler-Lagrange equations related to (1) can be a quite difficult task, even in the presence of suitable boundary conditions. In this context, the adequate implementation of the BPS formalism is quite useful, since it helps one to find finite-energy configurations as solutions of a given set of first-order differential equations.

So, from now on, we focus our attention on the development of a BPS framework consistent with (1) and (2). Specifically, we look for radially symmetric solutions using the standard static *Ansatz*

$$\phi(r, \theta) = g(r)e^{in\theta} \quad \text{and} \quad N(r, \theta) = \mp A^0(r), \tag{4}$$

$$\mathbf{A}(r, \theta) = -\frac{\hat{\theta}}{r}(a(r) - n), \tag{5}$$

where (r, θ) are polar coordinates and $n = \pm1, \pm2, \pm3\ldots$ is the winding number (vorticity) of the configuration. The fields $g(r)$, $a(r)$, and $A^0(r)$ must obey the usual boundary conditions

$$g(0) = 0, \qquad a(0) = n, \qquad A_0'(0) = 0, \tag{6}$$

$$g(\infty) = 1, \qquad a(\infty) = 0, \qquad A_0(\infty) = 0, \tag{7}$$

where prime denotes the derivative with respect to r.

Now, we implement the Bogomol'nyi approach on the energy functional related to (1). Given the generalized potential (2), the radially symmetric nonstandard energy density can be written as

$$\varepsilon = \frac{h}{2}\left(\frac{1}{r}\frac{da}{dr}\right)^2 + h\left(\frac{dA^0}{dr}\right)^2 + \frac{(G \mp A^0)^2}{2h} + w\left(\left(\frac{dg}{dr}\right)^2 + \frac{g^2a^2}{r^2}\right) + 2g^2A_0^2w, \tag{8}$$

where $h = h(g, A_0)$, $w = w(g)$, and $G = G(g)$. It is clear that, in order to guarantee a positive-definite energy, both h

and w must be positive-definite and finite. Now, by using the static Gauss law (3) together with the constraint

$$\frac{dG}{dg} = 2wg, \tag{9}$$

the energy density (8) can be rewritten in the form

$$\varepsilon = \frac{h}{2}\left(\frac{1}{r}\frac{da}{dr} \mp \frac{1}{h}(G \mp A^0)\right)^2 + w\left(\frac{dg}{dr} \mp \frac{ag}{r}\right)^2 \pm \frac{1}{r}\frac{d}{dr}(aG) + \frac{1}{r}\frac{d}{dr}\left(rhA^0\frac{dA^0}{dr}\right). \tag{10}$$

At this point, we see that the resulting total energy is minimized by the nonstandard first-order equations

$$\frac{dg}{dr} = \pm\frac{ag}{r}, \tag{11}$$

$$\frac{1}{r}\frac{da}{dr} = \pm\frac{1}{h}(G \mp A^0). \tag{12}$$

Equations (11) and (12) are the BPS equations related to the generalized MCS-Higgs model (1). In the present case, the BPS total energy $E_{\rm bps}$ related to the solutions of (11) and (12) can be explicitly evaluated

$$E_{\rm bps} = 2\pi\int r\varepsilon_{\rm bps}dr = \pm 2\pi n, \tag{13}$$

where the minimal energy density $\varepsilon_{\rm bps}$ is defined as

$$\varepsilon_{\rm bps} = \pm\frac{1}{r}\frac{d}{dr}(aG) + \frac{1}{r}\frac{d}{dr}\left(rhA^0\frac{dA^0}{dr}\right). \tag{14}$$

Also, to compute the BPS energy (13), we have used the boundary conditions (6) and (7) together with

$$G(r=0) = -1, \qquad G(r=\infty) = 0. \tag{15}$$

Finally, we assume that $h(|\phi|, N)$ and $w(|\phi|)$ behave as

$$h(r=0) = H_0, \qquad h(r=\infty) = H_\infty, \tag{16}$$

$$w(r=0) = W_0, \qquad w(r=\infty) = W_\infty, \tag{17}$$

where H_0, W_0, and H_∞ are real non-negative constants, and W_∞ is a real positive constant.

From (13), one notes that the total energy $E_{\rm bps}$ is quantized according to the winding number n. Furthermore, it can be related to the magnetic flux Φ_B in the standard way. Also, we point out that the first-order framework developed here is implemented for any function $h(|\phi|, N)$ finite and positive-definite, i.e., given some function $h(|\phi|, N)$, the set formed by (11) and (12) gives the BPS configurations of the generalized MCS-Higgs model (1). On the other hand, in the absence of (2) or (9), the energy functional (8) cannot be written as (10), and the development of a consistent nonstandard first-order formalism cannot be performed.

The prescription for the development of generalized self-dual Maxwell-Chern-Simons-Higgs models is as follows: given any function $w(|\phi|)$, positive and finite, the corresponding $G(|\phi|)$ is obtained by means of Eq. (9). Then, given any function $h(|\phi|, N)$, also positive and finite, the resulting self-dual potential $U(|\phi|, N)$ is determined via (2). In this scenario, the BPS states of the generalized model (1) are the solutions of the Eqs. (3), (11), and (12). The total energy is given by (13), and the energy density by (14).

In the standard MCS-Higgs case, BPS solutions with nontrivial topology only exist in the asymmetric vacuum of the potential

$$U_s(|\phi|, N) = \tfrac{1}{2}(|\phi|^2 + N - 1)^2 + N^2|\phi|^2, \tag{18}$$

which is defined by $N = 0$ and $|\phi| = 1$. So, for simplicity, we assume that also the generalized potential (2) achieves its asymmetric vacuum when $N = 0$ and $|\phi| = 1$. As a consequence, topologically nontrivial self-dual solutions of (1) exist in the same asymmetric phase as their standard counterparts.

The standard MCS-Higgs model is trivially recovered starting from our generalized framework by setting $w(|\phi|) = h(|\phi|, N) = 1$. So, instead of recovering the usual model, we introduce a generalization of such theory. The generalized model is defined by

$$w(|\phi|) = b(|\phi|^2 - 1)^{b-1}, \tag{19}$$

where b is a positive odd number. The corresponding $G(|\phi|)$ is

$$G(|\phi|) = (|\phi|^2 - 1)^b. \tag{20}$$

For simplicity, we choose $h(|\phi|, N)$ as follows:

$$h(N) = \alpha N^2 + 1, \tag{21}$$

where α is a real non-negative number. Note that $\alpha = 0$ and $b = 1$ leads us back to the standard case. The resulting generalized self-dual potential $U(|\phi|, N)$ (2) is

$$U(|\phi|, N) = \frac{(N + (|\phi|^2 - 1)^b)^2}{2(\alpha N^2 + 1)} + bN^2|\phi|^2(|\phi|^2 - 1)^{b-1}. \tag{22}$$

In the present case, (3), (11), and (12) can be written, respectively, as

$$(\alpha A_0^2 + 1)\left(\frac{d^2A^0}{dr^2} + \frac{1}{r}\frac{dA^0}{dr}\right) + 2\alpha A^0\left(\frac{dA^0}{dr}\right)^2 = 2bg^2(g^2 - 1)^{b-1}A^0 \mp \frac{1}{r}\frac{da}{dr}, \tag{23}$$

$$\frac{dg}{dr} = \pm\frac{ag}{r}, \tag{24}$$

$$\frac{1}{r}\frac{da}{dr} = \frac{\pm(g^2 - 1)^b - A^0}{\alpha A_0^2 + 1}, \tag{25}$$

which must be solved according to the finite-energy boundary conditions (6) and (7).

In the next section, we solve (23)–(25) via (6) and (7) for different values of α and b. Then, we use the numerical solutions we found for $g(r)$, $a(r)$, and $A^0(r)$ to depict the corresponding profiles for the electric field

$$E(r) = -\frac{dA^0}{dr}, \tag{26}$$

the magnetic field

$$B(r) = -\frac{1}{r}\frac{da}{dr}, \tag{27}$$

and the minimal energy density (14). Also, we comment on the main features that the nonstandard numerical solutions engender.

III. NUMERICAL SOLUTIONS

In this section, we focus our attention on the nonstandard numerical solutions themselves. The equations to be studied are (23)–(25), and the fields $g(r)$, $a(r)$, and $A^0(r)$ must behave according to the finite-energy boundary conditions (6) and (7).

Here, for simplicity, we choose $n = 1$. Then, we solve the first-order system numerically by means of the relaxation technique, for different values of the real parameters α and b. The profiles of the physical fields and energy density are depicted in the figures below. The system was solved for $\alpha = 0$ and $b = 1$ (usual case, dashed-dotted black line), $\alpha = 0$ and $b = 3$ (dotted blue line), $\alpha = 5$ and $b = 1$ (dashed red line), and $\alpha = 5$ and $b = 3$ (solid green line).

In Fig. 1, we depict the numerical solutions for the electric field (26), and we see that the nonstandard ones behave, in general, as their usual counterpart: starting from 0 (zero), the solutions reach their maximum values at some finite distance R from the origin, and vanish as r goes to infinity. However, we point out that different solutions exhibit not only different amplitudes, but also slightly different characteristic lengths: in general, increments on α and/or b lead to decrements on amplitudes and/or characteristic length. Even in this case, such variations are expected to occur in the context of a nonstandard theory as (1); see, for instance, [14].

In Fig. 2, we present the profile of the magnetic field (27). The same manner as for the electric field, the generalized solutions behave as the standard one: the magnetic fields are lumps centered at the origin, and they decrease monotonically as r goes to infinity. In this case, for a fixed α and an increasing b, the amplitudes experience a small increase and the characteristic length decreases. On the other hand, for a fixed b and an increasing α, the amplitude will decrease significantly, but the characteristic lengths remain approximately fixed by a compensatory effect relating α and b.

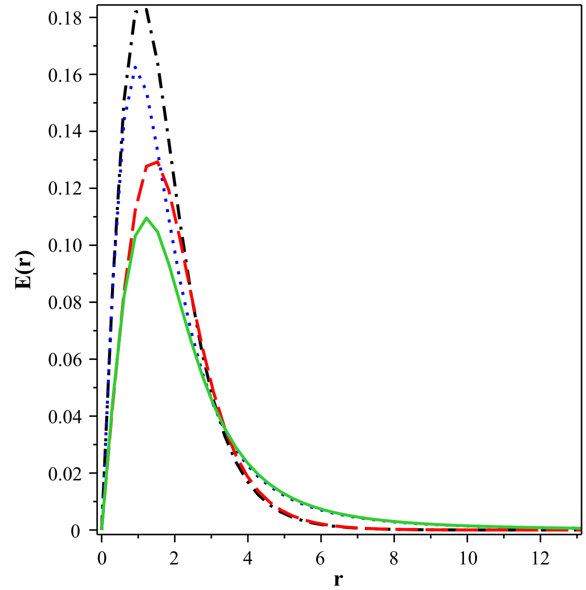

FIG. 1 (color online). Solutions to $E(r)$ for $\alpha = 0$ and $b = 1$ (dashed-dotted black line), $\alpha = 0$ and $b = 3$ (dotted blue line), $\alpha = 5$ and $b = 1$ (dashed red line), and $\alpha = 5$ and $b = 3$ (solid green line).

Numerical solutions for the minimal energy density $\varepsilon_{\rm bps}$ (14) are depicted in Fig. 3, and we see that also such solutions are lumps centered at $r = 0$ that decrease monotonically to $r \to 0$. The lumps' profiles are similar to those in the magnetic field. However, in this case, for a fixed α and an increasing b, the amplitudes experienced a large increase and the characteristic lengths decreased slightly. On the other hand, for a fixed b and an increasing α, the amplitude will decrease significantly but the characteristic

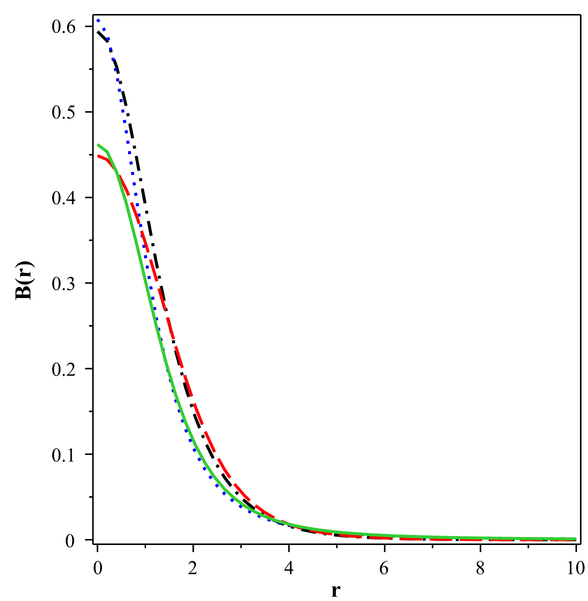

FIG. 2 (color online). Solutions to $B(r)$. Conventions as in Fig. 1.

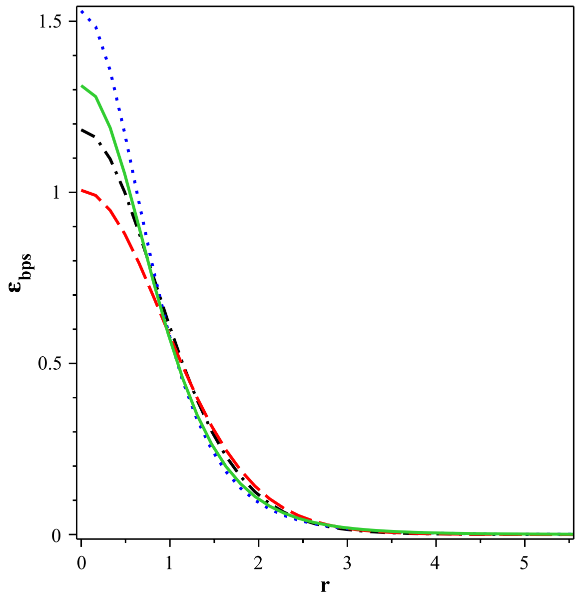

FIG. 3 (color online). Solutions to $\varepsilon_{\rm bps}$. Conventions as in Fig. 1.

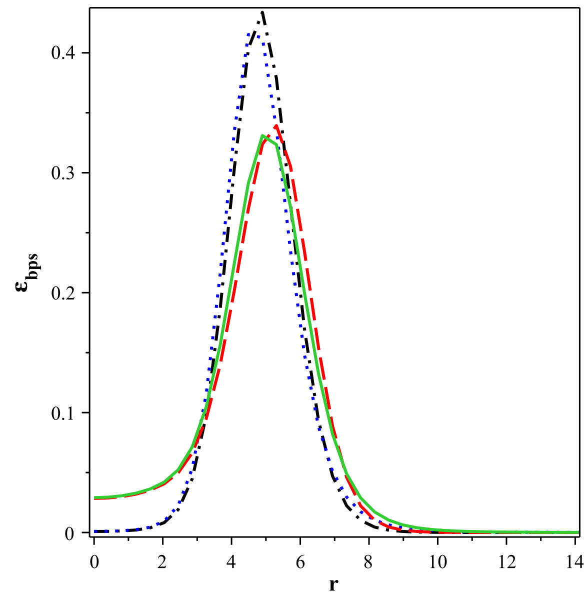

FIG. 5 (color online). Solutions to $\varepsilon_{\rm bps}$ for $n = 5$. Conventions as in Fig. 1.

lengths will remain almost fixed by a compensatory effect relating α and b.

To end this section, we have depicted the numerical profiles for $B(r)$ and $\varepsilon_{\rm bps}$ for $n = 5$ (see Figs. 4 and 5, respectively). It is well known that in the standard model, for a large winding number, the profiles of both the magnetic field and the energy density go from a lump ($n = 1$) to a ring ($n \gg 1$). From Figs. 4 and 5, one notes that also in this limit (increasing vorticity), the generalized solutions mimic the usual ones, as expected. For large n with fixed α and a increasing b, the profiles experience a slight change. However, for b fixed and an increment in α, the profiles' amplitudes experience a considerable decrease.

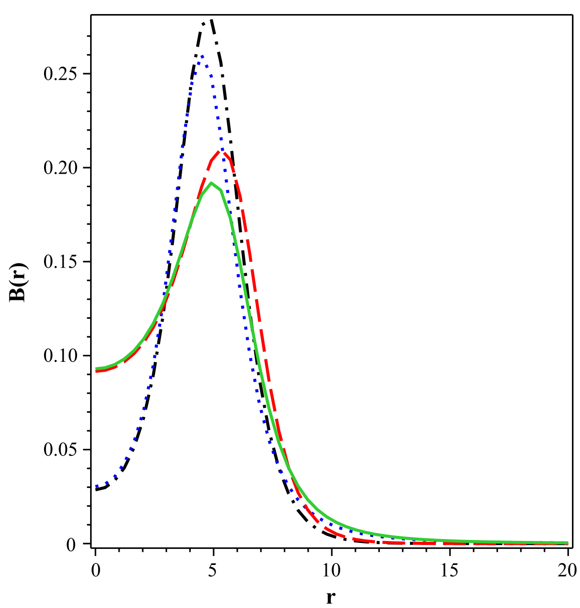

FIG. 4 (color online). Solutions to $B(r)$ for $n = 5$. Conventions as in Fig. 1.

IV. ENDING COMMENTS

In the present paper, we have performed the development of a first-order theoretical framework in the context of a generalized MCS-Higgs model given by (1). In the present case, the nonstandard model, including its self-dual potential (2), is given in terms of three different functions, $h(|\phi|, N)$, $w(|\phi|)$, and $G(|\phi|)$, which are functions of the scalar fields only. Here, in order to avoid problems with the energy of the model, $h(|\phi|, N)$ and $w(|\phi|)$ must be positive-definite. Then, given the general structure for the self-dual potential (2), the consistence of the nonstandard first-order approach only holds when $w(|\phi|)$ and $G(|\phi|)$ are related by the differential constraint (9). On the other hand, one notes that there is no additional constraint to be imposed on $h(|\phi|, N)$.

After performing the construction of the generalized MCS-Higss model, we illustrate such a realization providing an explicit nonstandard model specified by (19)–(21). In this case, the nonstandard model is controlled by two real parameters, α and b. Immediately, we have investigated radially symmetric self-dual configurations given by the usual static *Ansatz* (4) and (5), and considering that the fields behave according to the usual finite-energy boundary conditions (6) and (7).

We have integrated the generalized BPS equations by means of the relaxation technique, for different values of α and b. The numerical profiles we found are depicted in Figs. 1–5. In general, we have seen that the nonstandard solutions mimic their usual counterparts. Also, as expected, we have noted variations on the amplitudes and on the characteristic lengths of the modified solutions. Furthermore, we have identified the rules controlling such variations.

This work is an extension of a recent investigation performed by some of us [16]. In this sense, a very interesting issue to be considered concerns the development of a nonstandard self-dual MCS-Higgs model which exhibits the very same numerical solutions engendered by the standard theory. In this case, such an investigation would be a generalization of the one presented in [17], regarding a nonstandard Maxwell-Higgs model. Another issue concerns the supersymmetric extension of the self-dual model (1) and (2). These issues are in advance, and we hope report interesting results in the near future.

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