Renormalization group procedure for effective particles: Elementary example of an exact solution with finite mass corrections and no involvement of vacuum

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The renormalization group procedure for effective particles in the front form of Hamiltonian dynamics is applied to an elementary quantum field theory for two species of particles mixed through a masslike interaction term. The model interaction generates only finite terms and the procedure yields a whole family of equivalent effective theories. The exact solution for the family is found without involvement of the vacuum state in the dynamics. A physical spectrum is obtained at the end of the procedure in the form of free particles with definite masses. Since the procedure is designed in general terms, it could be used for the purpose of constructing effective dynamics also in other theories than the elementary model.

I. INTRODUCTION

The renormalization group procedure for effective particles (RGPEP) [1] in the front form (FF) of Hamiltonian dynamics [2] is designed for application in solving relativistic quantum field theories. In realistic theories, where nonperturbative solutions of the RGPEP equations are hard to find, the equations can be initially solved only order by order in a perturbative expansion [3] or after making other drastic simplifications of unknown accuracy. This article describes instead an application of the RGPEP to an elementary but instructive model of a quantum field theory in four dimensions which is soluble exactly. The model exhibits a nonperturbative scale evolution of a mass matrix for effective particles and shows how the RGPEP can deal with the quantum vacuum problem [4]. The final result of the RGPEP in the model is a free theory of the particles whose masses appear in the exact eigenvalues of the initial Hamiltonian.

The article begins with a brief summary of the RGPEP. The procedure defines a whole family of equivalent Hamiltonians that are labeled by a renormalization group parameter s. This means that a Hamiltonian is written in terms of different creation and annihilation operators for every value of s, but the coefficients of products of these operators also evolve with s so that the Hamiltonian as an operator is not changed. A need for such design can be seen in the context of QCD, where a canonical formulation starts with the bare quanta of quark and gluon fields whose local interactions are capable of producing violent energy changes among virtual, intermediate states during the time development of quantum systems. On the other hand, the same theory is expected to provide a precise relativistic description of the hadrons whose spectrum is classified using the constituent quarks, i.e., effective particles meant to be soft in their interactions. The design of RGPEP is tuned for application to solving quantum field theories so that different degrees of freedom, such as the high-energy partonlike quanta for very small s and low-energy constituentlike quanta for s approaching hadronic size in QCD, can be continuously transformed into each other as alternatives for conveniently describing various physical phenomena in one theory. In addition to quoting earlier literature, several details of how the RGPEP can serve the general purpose are illustrated in the article using an exact solution of an elementary model.

The model is defined starting from a classical Lagrangian density for two real scalar fields ϕ and χ with masses μ and ν, respectively. The only interaction term added is of the form $m^2 \phi \chi$. Canonical FF quantization leads to the Hamiltonian $\mathcal{P}^-$ that properly counts the FF free energy, $p^- = p^0 - p^3$, for particles with masses μ and ν and changes them into each other at the rate implied by the parameter m in the mass-mixing term. The bare vacuum state $|0\rangle$ is an eigenstate of $\mathcal{P}^-$ with an eigenvalue 0. The RGPEP transforms creation and annihilation operators for bare particles of masses μ and ν into operators for effective particles of masses that depend on the parameter s. The dependence is found in a simple analytic form. In the limit $s \rightarrow \infty$, the masses of effective particles become equal to the eigenvalues of the mass matrix in the model. These eigenvalues appear in the spectrum of the Hamiltonian. The vacuum state $|0\rangle$ is not involved in the RGPEP evolution of operators as functions of s.

The elementary example thus demonstrates that the RGPEP approach considerably differs from the standard canonical approach. In particular, the RGPEP does not encounter the vacuum problem that was pointed out by Dirac as a general drawback of standard formulations of quantum field theory [4]. In the elementary model, the standard formulation could be defined by first quantizing the free fields ϕ and χ and then adding the interaction term due to the mass mixing. Such approach would suffer from vacuum divergences because the quantum mass-mixing interaction term creates pairs of bare particles from a bare vacuum. Particles in the pairs may have arbitrary large relative momentum. Therefore, the states resulting from action of the Hamiltonian could not belong to the Hilbert

space; they have infinite norms. One would have to introduce a cutoff on the momenta to make the norm finite. Any such cutoff would violate the Lorentz symmetry. Similar features lead to questions concerning existence of the Schrödinger picture in all physically important relativistic quantum field theories [4]. One would have to adjust the standard quantization procedure to the size of the mass-mixing interaction term in order to avoid the vacuum problem. Unfortunately, when interactions get more complicated, it is not known exactly what to do with interactions that produce a complex vacuum. In contrast, the RGPEP is capable of solving the elementary model entirely in the Schrödinger picture. This result is obtained in a quantum theory in which the strength of the mass-mixing term is a free parameter, i.e., there is no need for adjusting the quantization procedure to the interaction term. The elementary example thus supports the idea that the RGPEP is a tool worth trying also in application to more complex theories, where the vacuum problem is so difficult that it has so far not been resolved by available methods.

Since the RGPEP involves basic elements of the canonical field quantization [5,6] and renormalization of Hamiltonians by techniques other than integrating out high-energy degrees of freedom [7,8], the elementary model application is described including all details needed to make the presentation self-contained. Besides the FF of Hamiltonian dynamics, the description often refers to the commonly used form of dynamics, designated the instant form (IF) by Dirac [2].

Section II briefly introduces the RGPEP. The elementary example is defined in Sec. III. Section IV describes solution of the RGPEP equations. The vacuum problem is discussed in Sec. V. Section VI concludes the article. The Appendix describes derivation of the same physical solution but obtained using an alternative RGPEP generator to the one used in the main text.

II. SUMMARY OF RGPEP

The concept of effective particles as degrees of freedom in a relativistic quantum field theory is introduced through a transformation [3]

$$\psi_s = \mathcal{U}_s \psi_0 \mathcal{U}_s^\dagger. \tag{1}$$

ψ_s is a quantum field operator built from creation and annihilation operators for effective particles of size s. These creation and annihilation operators are commonly denoted by q_s. The operator ψ_0 is the field operator built from the particle operators q_0 that correspond to the bare, pointlike particles, and $s = 0$. By definition, all kinematical quantum numbers that label operators q on both sides of Eq. (1), such as a three-momentum, charge, spin, isospin, flavor, color, and the like, are not altered by $\mathcal{U}_s$.

The intuitive interpretation of parameter s in terms of a size of the effective particles in the elementary model will be explained later. It is based on the general RGPEP feature

that effective interactions contain the form factors that limit how far off energy shell the interactions can extend. The corresponding energy width of the form factors is determined by $1/s$ (see below). The value $s = 0$ corresponds to absence of form factors. For a finite s, the effective Hamiltonian is band diagonal on the energy scale and the bandwidth is $\sim 1/s$. The principle of using the band-diagonal structure for the purpose of renormalization is formulated in [9]. It is convenient to use the parameter $t = s^4$ and label operators with t rather than s itself.

A canonical Hamiltonian density is built from products of fields ψ_0 and their derivatives. A corresponding Hamiltonian is obtained by integrating the density over a space-time hyper-surface. The result is a polynomial $\mathcal{H}_0(q_0)$ with coefficients c_0. If a term in $\mathcal{H}_0(q_0)$ contains a product of n operators q_0, the coefficient has n arguments. Each argument is a set of quantum numbers carried by a corresponding particle. Similarly, an effective-particle Hamiltonian $\mathcal{H}_t(q_t)$ is defined through its coefficients c_t.

The RGPEP employs the equality

$$\mathcal{H}_t(q_t) = \mathcal{H}_0(q_0), \tag{2}$$

which means that the same dynamics is expressed in terms of different operators. The change of q_0 to q_t is accompanied with the change of coefficients c_0 to c_t so that the physics is not changed. For example, the expansion of eigenstates of the Hamiltonian into the t-dependent Fock components involves the wave functions that depend on t, but the states as a whole do not depend on t at all.

Variation of the coefficients c_t with t is described by the equation obtained by differentiating both sides of

$$\mathcal{H}_t(q_0) = \mathcal{U}_t^\dagger \mathcal{H}_0(q_0) \mathcal{U}_t, \tag{3}$$

with respect to t. One obtains

$$\mathcal{H}_t'(q_0) = [\mathcal{G}_t(q_0), \mathcal{H}_t(q_0)], \tag{4}$$

where $\mathcal{G}_t = -\mathcal{U}_t^\dagger \mathcal{U}_t'$ is called a generator. Correspondingly,

$$\mathcal{U}_t = T \exp\left(-\int_0^t d\tau \mathcal{G}_\tau\right), \tag{5}$$

where T orders operators from left to right in the order from a smallest to largest t.

In the RGPEP, the generator is defined by

$$\mathcal{G}_t = [\mathcal{H}_f, \mathcal{H}_{Pt}]. \tag{6}$$

The operator $\mathcal{H}_f$, called the free Hamiltonian, is the part of $\mathcal{H}_0(q_0)$ that does not depend on the coupling constants,

$$\mathcal{H}_f = \sum_i p_i^- q_{0i}^\dagger q_{0i}. \tag{7}$$

The sum over subscript i extends over all particle species and their quantum numbers, including integration over momenta, and

$$p_i^- = \frac{p_i^{\perp 2} + m_i^2}{p_i^+}. \qquad (8)$$

This is the FF free energy of a particle with mass m_i and kinematical momentum components p_i^+ and $p_i^\perp$. The operator $\mathcal{H}_{Pt}$ is defined using the Hamiltonian $\mathcal{H}_t$. Namely, if $\mathcal{H}_t(q_0)$ is of the form

$$\mathcal{H}_t(q_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n) q_{0i_1}^{\dagger} \cdots q_{0i_n}, \quad (9)$$

where the coefficients $c_t(i_1, \ldots, i_n)$ are to be found using RGPEP, the operator $\mathcal{H}_{Pt}(q_0)$ is defined by

$$\mathcal{H}_{Pt}(q_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \ldots, i_n} c_t(i_1, \ldots, i_n) \left(\frac{1}{2} \sum_{k=1}^{n} p_{i_k}^+ \right)^2 q_{0i_1}^{\dagger} \cdots q_{0i_n}. \qquad (10)$$

This means that $\mathcal{H}_{Pt}$ differs from $\mathcal{H}_t$ by multiplication of each and every term by a square of a total $+$ momentum involved in a term. The motivation for this definition of $\mathcal{H}_{Pt}$ is explained in Refs. [1,3] and exemplified below in the elementary model in Sec. IV B. Generally, the reason for multiplication by a square of a total $+$ momentum involved in a term is to recover 7 kinematical symmetries of the FF of Hamiltonian dynamics. Without the multiplication, the generator would depend on the frame of reference and consequently the effective-particle dynamics would also depend on the frame of reference.

In summary, the coefficients c_t of products of operators q_t in the effective Hamiltonians $\mathcal{H}_t(q_t)$ are solutions of the equation

$$\mathcal{H}_t' = [[\mathcal{H}_f, \mathcal{H}_{Pt}], \mathcal{H}_t], \qquad (11)$$

where all operators are written as polynomials in q_0 and the initial condition is provided by a regulated canonical Hamiltonian with counterterms.

The counterterms are calculated in the RGPEP using a condition that for finite t the coefficients c_t with finite arguments do not depend on the regularization parameters used in the canonical Hamiltonian [9]. The difficulty of satisfying the cutoff-independence condition for c_t originates in the fact that the coefficients appear in the solution for $\mathcal{H}_t(q_t)$ while the counterterms are inserted in the initial condition $\mathcal{H}_0(q_0)$ and in-between there is a solution of the RGPEP that spans the range from 0 to t. However, there is no special difficulty associated with the counterterms in the example discussed below because the coefficients c_t with finite arguments do not develop any dependence on regularization in the model and solutions for them are known exactly. Therefore, the adjustment of counterterms in the model only amounts to specifying their finite parts. These parts form the initial mass matrix. The only regularization dependence in the example appears in one overall constant term in $\mathcal{H}_t$, which is a pure number and drops out from Eq. (11).

The generic feature of narrowness of $\mathcal{H}_t$ as t increases can be seen by introducing a projector R on a subspace in the Fock space. Let $\mathcal{H}_R = R\mathcal{H}_t R$. The corresponding projected equation reads (for details, see Appendix C in [1])

$$\mathcal{H}_R' = [[\mathcal{H}_f, \mathcal{H}_{PR}], \mathcal{H}_R]. \qquad (12)$$

The free Hamiltonian $\mathcal{H}_f$ commutes with R. The matrix version of Eq. (12) resembles the Wegner flow equation introduced in the IF of dynamics for Hamiltonians in condensed matter physics [10–12]. Similar equations are also successfully used in nuclear physics [13–15]. In relativistic quantum field theories, a narrow matrix must be obtained from Eq. (12) for large t because the trace of $\mathcal{H}_R^2$ does not depend on t and thus [1]

$$\left(\sum_{mn} |\mathcal{H}_{Imn}|^2 \right)' = -2 \sum_{km} (\mathcal{M}_{km}^2 - \mathcal{M}_{mk}^2)^2 |\mathcal{H}_{Ikm}|^2 \leq 0, \quad (13)$$

where $\mathcal{M}_{km}$ denotes an invariant mass of the particles in a state labeled with k that are connected through the interaction $\mathcal{H}_I$ to the particles in a state labeled by m. The interaction Hamiltonian is defined by $\mathcal{H}_I = \mathcal{H} - \mathcal{H}_f$ and the matrix elements $\mathcal{H}_{mn} = \langle m|\mathcal{H}|n \rangle$ are evaluated in the basis built from eigenstates $|m\rangle$ of $\mathcal{H}_f$. Equation (13) means that the sum of moduli squared of all matrix elements of the interaction Hamiltonian decreases as t increases until all off-diagonal matrix elements of the interaction Hamiltonian between states with different free invariant masses vanish. For a sizable value of s, the width of the narrow invariant-mass band in $\mathcal{H}_R$ is s^{-1}.

III. MODEL HAMILTONIAN

Let a theory of two real scalar fields ϕ and χ have a classical Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}[(\partial \phi)^2 - \mu^2 \phi^2] + \tfrac{1}{2}[(\partial \chi)^2 - \nu^2 \chi^2] - m^2 \phi \chi. \qquad (14)$$

The last term is called the mass-mixing term.

A. Classical Hamiltonian

In terms of the variables $x^{\pm} = x^0 \pm x^3$ and $x^\perp = (x^1, x^2)$ used to label points in space-time, so that $\partial^{\pm} = 2\partial/\partial x^{\mp}$, the Lagrangian density reads

$$\mathcal{L} = \tfrac{1}{2}[\partial^+ \phi \partial^- \phi - (\partial^\perp \phi)^2 - \mu^2 \phi^2]$$
$$+ \tfrac{1}{2}[\partial^+ \chi \partial^- \chi - (\partial^\perp \chi)^2 - \nu^2 \chi^2] - m^2 \phi \chi. \qquad (15)$$

The FF of dynamics involves the four-momentum [16,17]

$$P^\mu = \frac{1}{2} \int dx^- d^2 x^\perp \mathcal{T}^{+\mu}(x), \qquad (16)$$

where the energy-momentum tensor density component relevant for constructing the model Hamiltonian is

$$\mathcal{T}^{+-}(x) = \partial^+ \phi \partial^- \phi + \partial^+ \chi \partial^- \chi - 2\mathcal{L}. \qquad (17)$$

Hence,

$$P^- = \frac{1}{2} \int dx^- d^2x^\perp [(\partial^\perp \phi)^2 + \mu^2 \phi^2 + (\partial^\perp \chi)^2$$
$$+ \nu^2 \chi^2 + 2m^2 \phi \chi]. \tag{18}$$

B. Quantization

Let the fields ϕ and χ at $x^+ = 0$ have the Fourier decompositions [18]

$$\phi(x^-, x^\perp) = \int [p] a_p e^{-ipx}, \tag{19}$$

$$\chi(x^-, x^\perp) = \int [p] b_p e^{-ipx}, \tag{20}$$

where $[p]$ denotes the measure $dp^+ d^2 p^\perp / [2|p^+|(2\pi)^3]$ of integration over momentum variables $p^+ = p^0 + p^3$ and $p^\perp = (p^1, p^2)$. In this notation, the integration over momentum variables extends from $-\infty$ to $+\infty$ for all three components of p (a need for a cutoff on the range of p is still ignored at this point). Quantum theory is obtained by imposing commutation relations

$$[a_p, a_q] = [b_p, b_q] = 2p^+ (2\pi)^3 \delta^3(p + q). \tag{21}$$

The absence of † in the commutation relations is intended, since it is the sign of p^+ that distinguishes the operators that create field quanta from operators that annihilate them. Such kinematical distinction between creation and annihilation processes is not available in the standard, IF approaches. The "annihilation" operators with negative p^+ correspond to creation operators and one has

$$a_{-p} = a_p^\dagger, \tag{22}$$

$$b_{-p} = b_p^\dagger. \tag{23}$$

Note that these relations involve the change of sign of $p^\perp$.

After quantization, the classical fields ϕ and χ are turned into operators that create and annihilate quanta on the front hyperplane, $\hat{\phi}$ and $\hat{\chi}$. The commutation relations of Eq. (21) correspond to the spatial commutation relations

$$[\hat{\phi}(x), \partial^+ \hat{\phi}(y)] = [\hat{\chi}(x), \partial^+ \hat{\chi}(y)] = i\delta^3(x - y). \tag{24}$$

The inverse relations are

$$a_p = |p^+| \int d^3x e^{+ipx} \hat{\phi}(x), \tag{25}$$

$$b_p = |p^+| \int d^3x e^{+ipx} \hat{\chi}(x), \tag{26}$$

where $d^3x = dx^- d^2x^\perp$ and integrals extend from $-\infty$ to $+\infty$ on the $x^+ = 0$ hyperplane (the behavior of fields in spatial infinity remains unspecified at this point).

C. Quantum Hamiltonian

The quantum Hamiltonian is obtained from Eq. (18) by inserting operator versions of Eqs. (19) and (20) for $\hat{\phi}$ and $\hat{\chi}$, respectively, and by normal ordering,

$$\mathcal{P}^- = \frac{1}{2} \int dx^- d^2x^\perp :[(\partial^\perp \hat{\phi})^2 + \mu^2 \hat{\phi}^2 + (\partial^\perp \hat{\chi})^2$$
$$+ \nu^2 \hat{\chi}^2 + 2m^2 \hat{\phi} \hat{\chi}]:. \tag{27}$$

The normal ordering is defined using Feynman's convention [19] with the ordering parameter set equal to p^+. In this convention, it is understood that operators a_p are ordered in products according to the value of p^+ so that the greater p^+ the further to the right the operator.

All terms in the Hamiltonian are bilinear in fields and all of them contain one and the same integral

$$\int dx^- d^2x^\perp \int [qp] e^{-iqx-ipx} = \int [qp] 2(2\pi)^3 \delta^3(q + p). \tag{28}$$

According to Eq. (27),

$$\mathcal{P}^- = \frac{1}{2} \int [p] \frac{1}{|p^+|} :[(p^{\perp 2} + \mu^2) a_p a_{-p}$$
$$+ (p^{\perp 2} + \nu^2) b_p b_{-p} + 2m^2 a_p b_{-p}]:. \tag{29}$$

The normal ordering produces the operator that properly counts the FF energy of field quanta,

$$\mathcal{P}^- = \int [p] \theta(p^+) \left[\frac{p^{\perp 2} + \mu^2}{p^+} a_p^\dagger a_p + \frac{p^{\perp 2} + \nu^2}{p^+} b_p^\dagger b_p \right.$$
$$+ \left. \frac{m^2}{p^+} (a_p^\dagger b_p + b_p^\dagger a_p) \right]. \tag{30}$$

The last term describes the mixing of bare particles of type a associated with field ϕ and of type b associated with field χ. From now on, the function $\theta(p^+)$ is included in the integration measure $[p]$.

The diverging number that is removed by the FF normal ordering in $\mathcal{P}^-$,

$$\Omega^- = 2(2\pi)^3 \delta^3(0) \int [p] (p^{\perp 2} + \mu^2/2 + \nu^2/2), \tag{31}$$

involves factors $V_F = 2(2\pi)^3 \delta^3(0)$ and $\rho_\Omega = \int [p] (p^{\perp 2} + \mu^2/2 + \nu^2/2)$. Factor V_F has an interpretation of a volume of the front that a Hamiltonian density is integrated over. Factor ρ_Ω is associated with a ground-state energy, cf. [20–24]. As a number, Ω^- does not contribute to the commutators in Eq. (11) and it is not included in the RGPEP discussion in the next section. However, regarding application of the RGPEP to more complex theories, one should remember that the vacuum issue is not limited in them to a constant such as Ω^-, cf. [25,26].

IV. SOLUTION OF THE RGPEP EQUATION

According to Sec. II, Eq. (11) has the form

$$\mathcal{P}_t^{-\,'} = [[\mathcal{P}_f^-, \mathcal{P}_{Pt}^-], \mathcal{P}_t^-], \tag{32}$$

and should be solved using Eq. (30) as the initial condition,

$$\mathcal{P}_0^-(a_0, b_0) = \int [p]\Big[\frac{p^{\perp 2} + \mu^2}{p^+} a_{0p}^\dagger a_{0p} + \frac{p^{\perp 2} + \nu^2}{p^+} b_{0p}^\dagger b_{0p}$$
$$+ \frac{m^2}{p^+}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p}) \Big]. \tag{33}$$

A. Equations for coefficients c_t

On the basis of direct inspection of how the RGPEP equation works in the model and thus gained knowledge of the operator structures that it is capable of generating, the relevant operators can be written as

$$\mathcal{P}_t^-(a_0, b_0) = \int [p][A_{tp} a_{0p}^\dagger a_{0p} + B_{tp} b_{0p}^\dagger b_{0p}$$
$$+ C_{tp}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p})], \tag{34}$$

$$\mathcal{P}_f^-(a_0, b_0) = \int [p](A_{0p} a_{0p}^\dagger a_{0p} + B_{0p} b_{0p}^\dagger b_{0p}), \tag{35}$$

$$\mathcal{P}_{Pt}^-(a_0, b_0) = \int [p]p^{+2}[A_{tp} a_{0p}^\dagger a_{0p} + B_{tp} b_{0p}^\dagger b_{0p}$$
$$+ C_{tp}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p})], \tag{36}$$

where the coefficients generically denoted by c_t in Sec. II read

$$A_{tp} = \frac{p^{\perp 2} + \mu_t^2}{p^+}, \tag{37}$$

$$B_{tp} = \frac{p^{\perp 2} + \nu_t^2}{p^+}, \tag{38}$$

$$C_{tp} = \frac{m_t^2}{p^+}. \tag{39}$$

The initial conditions for these coefficients, denoted by c_0 in Sec. II, are set by fixing the mass-squared parameters at $t = 0$,

$$\mu_0 = \mu, \tag{40}$$

$$\nu_0 = \nu, \tag{41}$$

$$m_0 = m, \tag{42}$$

with constants μ, ν, and m, taken from $\mathcal{P}_0^-$ in Eq. (33). These parameters include the finite parts of mass-squared counterterms as discussed in Sec. II.

In this notation, the generator has the form

$$[\mathcal{P}_f^-, \mathcal{P}_{Pt}^-] = \int [p](A_{0p} - B_{0p})p^{+2}C_{tp}(a_{0p}^\dagger b_{0p} - b_{0p}^\dagger a_{0p}). \tag{43}$$

Equation (32) reads

$$\mathcal{P}_t^{-\,'}(a_0, b_0) = \int [p][A_{tp}' a_{0p}^\dagger a_{0p} + B_{tp}' b_{0p}^\dagger b_{0p} + C_{tp}'(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p})] \tag{44}$$

$$= \int [p](-p^{+2})(A_{0p} - B_{0p})(A_{tp} - B_{tp})C_{tp}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p}) \tag{45}$$

$$+ \int [p]2p^{+2}(A_{0p} - B_{0p})C_{tp}^2(a_{0p}^\dagger a_{0p} - b_{0p}^\dagger b_{0p}). \tag{46}$$

By equating coefficients in front of the same bare particle operators (or evaluating matrix elements between bare one-particle states of types a and b), one obtains a set of equations for the coefficients A_{tp}, B_{tp}, and C_{tp} in $\mathcal{P}_t^-(a_0, b_0)$. Namely,

$$A_{tp}' = 2p^{+2}(A_{0p} - B_{0p})C_{tp}^2, \tag{47}$$

$$B_{tp}' = -2p^{+2}(A_{0p} - B_{0p})C_{tp}^2, \tag{48}$$

$$C_{tp}' = (-p^{+2})(A_{0p} - B_{0p})(A_{tp} - B_{tp})C_{tp}. \tag{49}$$

This set contains as many triplets of equations as there are different triplets of momentum labels p, which a priori is an infinite number when one does not regulate the field ex-

pansions into their Fourier components by imposing cutoffs on some discretized set of variables p^+ and $p^\perp$. However, it is clear that the modes with different values of p are decoupled. They evolve in t independently of each other. This simplification is a consequence of the bilinear nature of the initial Lagrangian. In addition, new generic simplifications occur thanks to the FF boost invariance of Eq. (11).

B. Generic simplification due to boost invariance

In full detail, Eqs. (47)–(49) read

$$\Big(\frac{p^{\perp 2} + \mu_t^2}{p^+} \Big)' = 2p^{+2}\Big(\frac{p^{\perp 2} + \mu^2}{p^+} - \frac{p^{\perp 2} + \nu^2}{p^+} \Big)\Big(\frac{m_t^2}{p^+} \Big)^2, \tag{50}$$

$$\left(\frac{p^{\perp 2} + \nu_t^2}{p^+}\right)' = -2p^{+2}\left(\frac{p^{\perp 2} + \mu^2}{p^+} - \frac{p^{\perp 2} + \nu^2}{p^+}\right)\left(\frac{m_t^2}{p^+}\right)^2,$$ (51)

$$\left(\frac{m_t^2}{p^+}\right)' = (-p^{+2})\left(\frac{p^{\perp 2} + \mu^2}{p^+} - \frac{p^{\perp 2} + \nu^2}{p^+}\right)$$
$$\times \left(\frac{p^{\perp 2} + \mu_t^2}{p^+} - \frac{p^{\perp 2} + \nu_t^2}{p^+}\right)\frac{m_t^2}{p^+}.$$ (52)

It is visible that the kinematical variables p^+ and $p^\perp$ drop out. This feature is special to the FF of dynamics. Thus, the *a priori* infinite set of different equations for infinitely many coefficients with different kinematical variables p actually reduces to a single set of just 3 equations for 3 mass parameters that are independent of p,

$$(\mu_t^2)' = 2\delta\mu^2(m_t^2)^2,$$ (53)

$$(\nu_t^2)' = -2\delta\mu^2(m_t^2)^2,$$ (54)

$$(m_t^2)' = -\delta\mu^2(\mu_t^2 - \nu_t^2)m_t^2,$$ (55)

where

$$\delta\mu^2 = \mu^2 - \nu^2.$$ (56)

This set can be written as a differential matrix equation,

$$\begin{bmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{bmatrix}' = \left[\left[\begin{bmatrix} \mu^2 & 0 \\ 0 & \nu^2 \end{bmatrix}, \begin{bmatrix} 0 & m_t^2 \\ m_t^2 & 0 \end{bmatrix}\right], \begin{bmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{bmatrix}\right],$$ (57)

for a 2×2 matrix that will be called mass-squared matrix below.

Note that Eq. (57) would be a Wegner-like equation if the first matrix on the right-hand side contained μ_t and ν_t instead of the initial mass parameters μ and ν. Such change corresponds to inserting μ_t^2 and ν_t^2 in place of μ^2 and ν^2, respectively, in $\mathcal{P}_f$ of Eq. (35). The resulting Wegner-like equation for the mass-squared matrix can be solved by proceeding in a way analogous to the one described below. This is shown in the Appendix. The explicit solution described in next sections is for constant masses in $\mathcal{H}_f$.

C. Analytic solution for masses

One can introduce a dimensionless variable

$$u = \delta\mu^4 t,$$ (58)

and, denoting differentiation with respect to u with a prime, one obtains Eqs. (53)–(55), in the form

$$\alpha' = 2\gamma^2,$$ (59)

$$\beta' = -2\gamma^2,$$ (60)

$$\gamma' = -(\alpha - \beta)\gamma,$$ (61)

where the dimensionless functions of u are

$$\alpha = \mu_t^2/\delta\mu^2,$$ (62)

$$\beta = \nu_t^2/\delta\mu^2,$$ (63)

$$\gamma = m_t^2/\delta\mu^2.$$ (64)

If $\mu^2 = \nu^2$, so that $\delta\mu^2 = 0$, the mass parameters do not evolve with t irrespective of the initial value of mass-mixing parameter m. It is assumed from now on that $\mu^2 > \nu^2$, so that $\delta\mu^2 > 0$.

Regarding the mass degeneracy in the initial theories, one should observe that in order to trigger an RGPEP evolution towards a solution when initially $\mu = \nu$, one has to introduce an artificial splitting of masses in $\mathcal{H}_f$. For example, such splitting is needed in the case of local theories with massless bare particles and chiral symmetry. Two other physically important cases in which the mass degeneracy and its minimal lifting may play important roles as far as an application of RGPEP is concerned, are neutrinos in electroweak interactions and u and d quarks in QCD.

Equations (59) and (60) imply that the sum $\alpha + \beta$ as a function of u is a constant. This constant, denoted by $\mathcal{T} = T/\delta\mu^2$, results from the constancy of a trace of the mass-squared matrix, $T = m_1^2 + m_2^2$, where m_1^2 and m_2^2 denote its eigenvalues. The remaining coupled set of equations reads

$$\delta' = 4\gamma^2,$$ (65)

$$\gamma' = -\delta\gamma,$$ (66)

where $\delta = \alpha - \beta$. Multiplying the first of these two equations by 2δ and the second by 2γ, one arrives at

$$\delta^{2'} = 8\delta\gamma^2,$$ (67)

$$\gamma^{2'} = -2\delta\gamma^2,$$ (68)

and concludes that

$$\epsilon^2 = \delta^2 + 4\gamma^2$$ (69)

does not depend on u. In fact,

$$\epsilon^2 = \mathcal{T}^2 - 4\mathcal{D},$$ (70)

where $\mathcal{D} = D/\delta\mu^4$ and D is the determinant of the mass-squared matrix. Hence, $\epsilon^2 = (m_1^2 - m_2^2)^2/\delta\mu^4$. Using the constant ϵ, one can eliminate γ^2 from Eq. (67) to obtain

$$\delta' = \epsilon^2 - \delta^2,$$ (71)

which is an ordinary differential equation. Since the difference between eigenvalues of a Hermitian 2×2 matrix is never smaller than the difference between its diagonal

matrix elements, one always has $\delta' > 0$ except when $\delta = \epsilon$ and the mass-squared matrix is diagonalized. Without any loss of generality, one can assume $\epsilon > 0$.

Integrations of Eqs. (71) and then (66) produce solutions for the elements of mass-squared matrix as functions of t,

$$\mu_t^2 = \tfrac{1}{2}(\mu^2 + \nu^2) + \tfrac{1}{2}\delta\mu_t^2, \tag{72}$$

$$\nu_t^2 = \tfrac{1}{2}(\mu^2 + \nu^2) - \tfrac{1}{2}\delta\mu_t^2, \tag{73}$$

$$\delta\mu_t^2 = \delta\mu^2 \frac{\cosh x_t + \epsilon \sinh x_t}{\cosh x_t + \epsilon^{-1} \sinh x_t}, \tag{74}$$

$$m_t^2 = m^2 \frac{1}{\cosh x_t + \epsilon^{-1} \sinh x_t}, \tag{75}$$

where $x_t = \delta\mu^2 \delta m^2 t$. Note that $\epsilon = \sqrt{1 + (2m^2/\delta\mu^2)^2}$. For $t \to \infty$, one obtains

$$\mu_\infty^2 = m_1^2, \tag{76}$$

$$\nu_\infty^2 = m_2^2, \tag{77}$$

$$m_\infty^2 = 0. \tag{78}$$

These results mean that the RGPEP eventually produces a Hamiltonian for the two new species of particles of types 1 and 2 that are free, i.e., they no longer mix due to interactions, and their masses squared are given by the eigenvalues m_1^2 and m_2^2 of the initial mass-squared matrix.

D. Effective particles

The result of RGPEP is a family of Hamiltonians $P_t = \mathcal{P}_t(a_t, b_t)$ for $t \geq 0$, which is obtained from $\mathcal{P}_t(a_0, b_0)$ in Eq. (34) by replacement of a_{0p} and b_{0p} by a_{tp} and b_{tp}, respectively. The effective-particle operators are obtained from Eq. (1). Namely,

$$a_{tp} = \mathcal{U}_t a_{0p} \mathcal{U}_t^\dagger, \tag{79}$$

$$b_{tp} = \mathcal{U}_t b_{0p} \mathcal{U}_t^\dagger, \tag{80}$$

where $\mathcal{U}_t$ is given in Eq. (5) as a solution of

$$\mathcal{U}_t' = -\mathcal{U}_t [\mathcal{P}_f^-, \mathcal{P}_{Pt}^-]. \tag{81}$$

The generator, i.e., the commutator on the right-hand side of Eq. (81), is given in Eq. (43). Using results of the previous section, the generator can be written as

$$[\mathcal{P}_f^-, \mathcal{P}_{Pt}^-] = \delta\mu^2 m_t^2 \int [p](a_{0p}^\dagger b_{0p} - b_{0p}^\dagger a_{0p}). \tag{82}$$

Boost invariance of the RGPEP thus yields the generator that is a product of a function of t times a constant operator. The t-ordered exponential in Eq. (5) is

$$\mathcal{U}_t = \exp(\varphi_t A), \tag{83}$$

where

$$\varphi_t = -\delta\mu^2 \int_0^t m_\tau^2 d\tau \tag{84}$$

$$= \arctan\sqrt{\frac{\epsilon + 1}{\epsilon - 1}} - \arctan e^{x_t}\sqrt{\frac{\epsilon + 1}{\epsilon - 1}}, \tag{85}$$

$$A = \int [k](a_{0k}^\dagger b_{0k} - b_{0k}^\dagger a_{0k}). \tag{86}$$

The effective-particle operators a_{tp} and b_{tp} are obtained from the formula

$$q_{tp} = e^{\varphi_t A} q_{0p} e^{-\varphi_t A} \tag{87}$$

with $q = a$ and $q = b$, respectively, using

$$[A, a_{0p}] = -b_{0p}, \tag{88}$$

$$[A, b_{0p}] = a_{0p}. \tag{89}$$

Suppose there exists a combination

$$q_{0p} = a_{0p} + z_1 b_{0p}, \tag{90}$$

for which one has

$$[A, q_{0p}] = z_2 q_{0p}, \tag{91}$$

where z_1 and z_2 are some complex numbers. This is possible when $z_1 = z_2 = \pm i$ and

$$e^{\varphi_t A} q_{0p\pm} e^{-\varphi_t A} = e^{\pm i\varphi_t} q_{0p\pm}. \tag{92}$$

where $q_{0p\pm} = a_{0p} \pm i b_{0p}$. Knowing that

$$a_{0p} = \tfrac{1}{2}(q_{0p+} + q_{0p-}), \tag{93}$$

$$b_{0p} = \frac{-i}{2}(q_{0p+} - q_{0p-}), \tag{94}$$

one obtains

$$a_{tp} = \cos\varphi_t a_{0p} - \sin\varphi_t b_{0p}, \tag{95}$$

$$b_{tp} = \sin\varphi_t a_{0p} + \cos\varphi_t b_{0p}, \tag{96}$$

and the inverse relations

$$a_{0p} = \cos\varphi_t a_{tp} + \sin\varphi_t b_{tp}, \tag{97}$$

$$b_{0p} = -\sin\varphi_t a_{tp} + \cos\varphi_t b_{tp}. \tag{98}$$

Equations (95) and (96) provide explicit definitions of annihilation operators for effective particles corresponding to the RGPEP parameter $t = s^4$. The corresponding relations for creation operators are obtained by Hermitian conjugation.

E. Interpretation of s as the effective-particle size

The interpretation of parameter s as a size of effective particles requires explanation in the context of our mass-mixing model because the mass-mixing interaction does not change any three-momentum that could be an argument of a form factor whose width might be related to a concept of a particle size. However, in more advanced

theories, interactions change an invariant mass of the interacting particles when their relative momenta change. To be specific, consider a fermion of mass m_f that emits a boson of mass m_b. The associated change of invariant-mass squared is

$$\mathcal{M}_{fb,f}^2 = \left(\sqrt{m_f^2 + k^2} + \sqrt{m_b^2 + k^2} \right)^2 - m_f^2. \quad (99)$$

The exponential factor of the type $\exp(-s^4 \mathcal{M}_{fb,f}^4)$ becomes $\exp[-(2sk)^4]$ for large k. This is the origin of interpreting the parameter s as a size of effective particles in complex theories. Namely, only particles with small size s can interact producing a large momentum k, cf. [18].

In the mass-mixing model, there is no change of relative three-momentum involved. Instead, the interaction strength m_t^2 in Eq. (75) is limited in strength roughly by $\exp(-\delta\mu^2 \delta m^2 s^4)$. The change of interaction strength comes solely from the change of a particle mass. Therefore, the role of the effective-particle size parameter s is reduced to taming changes in the mass. The pointlike, bare particles at $s = 0$ can change mass through a mass-mixing interaction by arbitrary amounts that are introduced in the initial $\mathcal{P}^-$. But the effective particles of large s can change mass only by amounts not exceeding $1/s$, as if the motion of their constituents could not involve a large excitation without breaking them apart. Thus, when s is large, the effective particles can only change their masses by small amounts. Eventually, when $s \to \infty$, they cannot change mass at all, which means that they do not interact through a mass-mixing term at all (see Sec. IV G below). In any case, the RGPEP suggests that mass mixing in low-energy effective theories should be small. Realistic effective theories appear to share this feature.

F. Constancy of the Hamiltonian

The effective Hamiltonian, $\mathcal{P}_t^- = \mathcal{P}_t^-(a_t, b_t)$, is obtained from $\mathcal{P}_t^-(a_0, b_0)$ in Eq. (34) by replacing a_{0p} and b_{0p} in the latter by a_{tp} and b_{tp}. The result is

$$\mathcal{P}_t^- = \int [p][A_{tp} a_{tp}^\dagger a_{tp} + B_{tp} b_{tp}^\dagger b_{tp} + C_{tp}(a_{tp}^\dagger b_{tp} + b_{tp}^\dagger a_{tp})], \quad (100)$$

where the coefficients A_{tp}, B_{tp}, and C_{tp} are given in Eqs. (37)–(39), respectively, and the mass parameters in them are given in Eqs. (72), (73), and (75). Using Eqs. (95) and (96), one obtains

$$\mathcal{P}_t^- = \mathcal{P}_0^- + \int [p]\left[\frac{\Delta\mu^2}{p^+} a_{0p}^\dagger a_{0p} + \frac{\Delta\nu^2}{p^+} b_{0p}^\dagger b_{0p} \right.$$
$$\left. + \frac{\Delta m^2}{p^+}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p}) \right], \quad (101)$$

where

$$\Delta\mu^2 = \mu_t^2 c^2 + \nu_t^2 s^2 + 2m_t^2 cs - \mu^2, \quad (102)$$

$$\Delta\nu^2 = \mu_t^2 s^2 + \nu_t^2 c^2 - 2m_t^2 cs - \nu^2, \quad (103)$$

$$\Delta m^2 = -(\mu_t^2 - \nu_t^2)cs + m_t^2(c^2 - s^2) - m^2, \quad (104)$$

$s = \sin\varphi_t$, and $c = \cos\varphi_t$. Direct inspection demonstrates that $\Delta\mu^2 = \Delta\nu^2 = \Delta m^2 = 0$ for all values of t in the range from 0 to ∞, which means that the operators $\mathcal{P}_t^- = \mathcal{P}_t^-(a_t, b_t)$ and $\mathcal{P}_0^- = \mathcal{P}_0^-(a_0, b_0)$ are the same for all values of t.

G. Spectrum of the theory

The initial Hamiltonian, $\mathcal{P}_0^-$ in Eq. (33), is transformed as a result of the RGPEP to $\mathcal{P}_t^-$ in Eq. (100). At the same time, the RGPEP secures equality $\mathcal{P}_t^- = \mathcal{P}_0^-$, as shown in Sec. IV F. Since the eigenvalues and eigenstates of $\mathcal{P}_0^-$ and $\mathcal{P}_t^-$ are identical, one can derive them using any value of t one wishes. The simplest to discuss is the case of $t \to \infty$, because in this case there is no mass mixing, $m_\infty = 0$. The mixing vanishes in the limit $t \to \infty$ provided that initially $\mu \neq \nu$. This is assumed in what follows. The case of $\mu = \nu$ is addressed near the end of this section.

The effective theory with $t = \infty$ is a free theory, with a correspondingly simple spectrum. Details of the spectrum are described below for two reasons. One reason is the completeness of the article. The other reason is a preparation for the discussion in Sec. V concerning the ground state, or vacuum. Simplicity of the RGPEP illustrated here is contrasted with complexity of other approaches there.

In the limit of $t \to \infty$,

$$\mathcal{P}_\infty^-(a_\infty, b_\infty) = \int [p]\left[\frac{p^{\perp 2} + \mu_\infty^2}{p^+} a_{\infty p}^\dagger a_{\infty p} \right.$$
$$\left. + \frac{p^{\perp 2} + \nu_\infty^2}{p^+} b_{\infty p}^\dagger b_{\infty p} \right], \quad (105)$$

where

$$a_{\infty p} = \cos\varphi_\infty a_{0p} - \sin\varphi_\infty b_{0p}, \quad (106)$$

$$b_{\infty p} = \sin\varphi_\infty a_{0p} + \cos\varphi_\infty b_{0p}, \quad (107)$$

and the angle φ_∞ is

$$\varphi_\infty = -\arctan\sqrt{\frac{\epsilon - 1}{\epsilon + 1}}. \quad (108)$$

Note that this angle is the same as the one in Eq. (A10) that results from solving RGPEP equations with a different generator in the Appendix.

Eigenvalues of the Hamiltonian in Eq. (105) are free energies of $n_{\infty 1}$ particles of mass m_1 and $n_{\infty 2}$ particles of mass m_2, each with some momentum components p^+ and $p^\perp$,

$$P^-_{\{p_{1i}, i=1,\ldots,n_{\infty 1}\},\{p_{2j}, j=1,\ldots,n_{\infty 2}\}} = \sum_{i=1}^{n_{\infty 1}} \frac{p_{1i}^{\perp 2} + m_1^2}{p_{1i}^+}$$
$$+ \sum_{j=1}^{n_{\infty 2}} \frac{p_{2j}^{\perp 2} + m_2^2}{p_{2j}^+}. \quad (109)$$

The spectrum is degenerate. The eigenstates can be closely identified because the RGPEP provides expressions for the operators a_∞ and b_∞. A complete set of eigenstates (not normalized) is defined by writing

$$|\{p_{1i}, i = 1, \ldots, n_{\infty 1}\}, \{p_{2j}, j = 1, \ldots, n_{\infty 2}\}\rangle$$
$$= \prod_{i=1}^{n_{\infty 1}} a^\dagger_{\infty p_{1i}} \prod_{j=1}^{n_{\infty 2}} b^\dagger_{\infty p_{2j}} |0\rangle, \quad (110)$$

where $|0\rangle$ denotes the vacuum state. The vacuum state is annihilated by all annihilation operators of all particles for all values of t and one can treat $|0\rangle$ as one and the same state for all values of the parameter $t = s^4$.

Since the creation operators $a^\dagger_{\infty p}$ and $b^\dagger_{\infty p}$ are given by linear combinations of $a^\dagger_{0p}$ and $b^\dagger_{0p}$ implied by Eqs. (106) and (107) through Hermitian conjugation, the eigenstates defined in Eq. (110) can also be written as combinations of states created from the same vacuum state by products of the operators $a^\dagger_{0p}$ and $b^\dagger_{0p}$ with the corresponding momenta. The total number of particles in every resulting component of an eigenstate is the same. However, an eigenstate with definite numbers $n_{\infty 1}$ and $n_{\infty 2}$ of effective particles with $t = \infty$ corresponds to a combination of states with varying numbers of initial particles, n_{01} and n_{02}, that satisfy the condition $n_{01} + n_{02} = n_{\infty 1} + n_{\infty 2}$. If the total number of particles is large, a simple state of effective particles with $t = \infty$ is a complex mixture of many states made of bare particles corresponding to $t = 0$.

When $\mu = \nu$, the RGPEP does not change the particle operators, since the generator is zero. On the other hand, it is clear that a nonzero mixing term m^2 causes the eigenvectors of mass-squared matrix in a classical Lagrangian to be definite combinations of the initial basis vectors. In the quantum theory, in order to generate a solution using the RGPEP, one may introduce a small artificial difference between the initial masses. When the initial mass degeneracy corresponds to symmetry, the small artificial difference that breaks the degeneracy breaks also the symmetry. The RGPEP can be said to use consequences of such small breaking to finesse quantum symmetry-breaking solutions.

In summary, the RGPEP produces the spectrum in a simple way. However, the simplicity is to some extent deceptive because the RGPEP allows one to ignore questions concerning the vacuum state $|0\rangle$. The next section discusses this issue.

V. THE VACUUM PROBLEM

The vacuum problem appears in the quantization of fields [4–6]. One starts with quantizing a free classical theory. This renders a quantum theory of noninteracting particles in terms of a free Hamiltonian H_0. Interaction terms are added to H_0 in the form of H_I. The latter can be constructed by starting from local products of classical fields multiplied by coupling constants and replacing the classical fields with the quantized ones. The vacuum problem becomes apparent when one attempts to solve the eigenvalue problem for $H = H_0 + H_I$. The problem is that H_I takes eigenstates of H_0 out of the Hilbert space. In particular, the ground state of the free theory, denoted by $|0\rangle$, is changed by H_I to a state with an infinite norm. The situation is further discussed below using the mass-mixing example, in which the vacuum problem appears in a similar way as in the model used by Dirac to discuss the vacuum problem [4].

A. Vacuum problem due to mass mixing

The parameter m^2 in the mixing term in the Lagrangian of Eq. (14) is treated as a coupling constant. Setting $m = 0$, one obtains a Lagrangian density of a free theory,

$$\mathcal{L}_0 = \tfrac{1}{2}[(\partial\phi)^2 - \mu^2\phi^2] + \tfrac{1}{2}[(\partial\chi)^2 - \nu^2\phi^2]. \quad (111)$$

The IF quantization of a free theory is well known and nothing new is said here about it except for stressing one aspect that concerns the vacuum. Namely, when one evaluates $H_0 = \int d^3x \mathcal{H}_0$, where the Hamiltonian density $\mathcal{H}_0$ is canonically obtained from $\mathcal{L}_0$, the terms that involve products of two creation or two annihilation operators all cancel out, as desired. This happens because of the free-energy formulas, $E_a^2(p) = \mu^2 + p^2$ and $E_b^2(p) = \nu^2 + p^2$, that are used in defining the time derivatives, or canonical momenta for the field variables. These energy formulas produce the desired cancellations in the sum of terms involving π_ϕ^2, $\vec{\nabla}\phi^2$, and $\mu^2\phi^2$, and similarly for π_χ^2, $\vec{\nabla}\chi^2$, and $\nu^2\chi^2$. The resulting H_0 in the IF of dynamics has the form

$$H_0(a_0, b_0) = \int [p]_a \sqrt{\mu^2 + p^2}\, a^\dagger_{0p} a_{0p}$$
$$+ \int [p]_b \sqrt{\nu^2 + p^2}\, b^\dagger_{0p} b_{0p}. \quad (112)$$

The integration measures obtain the subscripts a and b because of the energies in their denominators. The non-zero commutation relations are $[a_{0p}, a^\dagger_{0q}] = 2E_a(p) \times (2\pi)^3\delta^3(\vec{p} - \vec{q})$ and $[b_{0p}, b^\dagger_{0q}] = 2E_b(p)(2\pi)^3\delta^3(\vec{p} - \vec{q})$. Possible additional quantum numbers can be ignored here.

An infinite constant Ω^0 has been removed by the IF normal ordering, analogous to the constant Ω^- removed from the FF Hamiltonian $\mathcal{P}^-$, see Eqs. (30) and (31). The constant Ω^0 can be subtracted this way [4,27], or it can also

be included in variational estimates of the ground-state energy when interaction terms are taken into account [20,21].

In a theory set up this way, the IF vacuum problem emerges in the model due to the interaction term,

$$H_I = \int d^3x m^2 \phi \chi \qquad (113)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{4E_a E_b} (a_{0p}^\dagger b_{0-p}^\dagger + a_{0p}^\dagger b_{0p}$$
$$+ b_{0p}^\dagger a_{0p} + a_{0p} b_{0-p}). \qquad (114)$$

The result of action of H_I on the vacuum state is

$$H_I|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{4E_a E_b} a_{0p}^\dagger b_{0-p}^\dagger |0\rangle. \qquad (115)$$

This state has an infinite norm. The infinity occurs through two factors. One factor is the volume of space in which the states with definite three-momentum are normalized. In the case of a state in Eq. (115), the three-momentum is zero. Another source of infinity is the integral over all momentum labels p. This divergence results from the infinite number of momentum scales in the theory.

Acting on the state in Eq. (115) with H_I again also generates infinity. Multiple action of H_I creates further infinities. For example, the infinities appear in action of the evolution operator $U = \exp(-iHt)$ on $|0\rangle$, since U involves all powers of H_I [4].

If states of the theory are built starting from $|0\rangle$, the mixing operator H_I creates infinities in all of them. Removal of the infinities requires a cutoff on the range of momentum p in the Fourier expansions of fields ϕ and χ. However, every cutoff on the momentum range in a theory violates the Lorentz symmetry [4]. One has to redesign the quantization procedure in the example in order to recover the quantum theory that was straightforwardly found as a solution using the RGPEP in previous sections.

On the one hand, it is known in the elementary model what needs to be done to solve it. On the other hand, one can look at the model as sharing some basic features with theories in which a more complex H_I is added to H_0 and it is not known how to deal with the vacuum problem in them beyond perturbation theory. Therefore, the model is of interest as a potential source of ideas about how to use the RGPEP to try to work around the vacuum problem in complex theories and attempt to break through the barriers that this problem poses in general.

B. General scope of vacuum problems

Dirac pointed out that problems with vacuum may require a reinterpretation of quantum field theory [4]. He argued for such reinterpretation in the case of QED. Similar divergences occur in the vacuum problem of QCD but they cannot be as easily worked around as Dirac suggested for QED [26].

One reason is that the coupling constant in QCD is much larger than in QED. The QED coupling constant is so small that one can use very large cutoffs in diverging terms in perturbation theory and still does not need to worry about the Lorentz-symmetry violation in practice. In QCD, where the coupling constant is much larger than in QED, the cutoffs would have to be much smaller than in QED in order to exclude large terms in perturbation theory. But much smaller cutoffs on $|\vec{p}|$ could lead to effects that violate the Lorentz symmetry much stronger. Asymptotic freedom enables perturbative calculations in QCD but does not solve the vacuum problem. The other reason is the need for explaining spontaneous chiral symmetry breaking [28,29] for which a nontrivial vacuum structure is seen as the origin. The third reason is the desire to explain confinement. Confinement is often associated in the literature with a concept of a complex ground state. In any case, the ground state of QCD still awaits a construction. More generally, questions concerning a ground state, spontaneous symmetry breaking, and mass generation are of concern in the present standard model and theories trying to explain its origin. A famous ambiguity involved in the vacuum concept is the vacuum energy density, which can be seen as relevant to cosmology [30,31].

In the FF of Hamiltonian dynamics, the vacuum problem does not appear in the same way as in the IF. For example, the vacuum problem in the FF version of QCD can be formulated as a renormalization group problem for Hamiltonians [26]. Using the RGPEP, one can also envision a scenario for solving the canonical FF of QCD in which the effects commonly associated with a gluon condensate in vacuum [32] may actually originate in an analogous expectation value but merely in the gluon medium that exists only inside the volume of a hadron, rather than in the entire space [1]. Discussions of the idea that condensate parameters may actually correspond to expectation values of operators in the medium present inside hadrons, instead of the entire space, are available in [33–36], including implications for cosmology.

The scope of vacuum problems is broad enough to suggest that the features that enable RGPEP to work around the vacuum problem and produce an exact quantum solution in the elementary example should be identified. This is done in the next section.

C. RGPEP path around the vacuum

The general features that enable RGPEP to circumvent the vacuum problem and still produce a relativistic solution in the model stem from the FF of Hamiltonian dynamics. The key properties of the FF are the positivity of p^+ and boost invariance. The RGPEP takes advantage of these properties in the design of its generator.

The positivity of p^+ results from the assumption that for a free particle of an arbitrary mass $\mu > 0$ one can write for arbitrary three-momentum $\vec{p}$ that

$$p^+ = \sqrt{\mu^2 + \vec{p}^2} + p^z \geq 0. \qquad (116)$$

Thus, one assumes in the FF of quantum dynamics that a creation operator for a particle may only carry positive p^+ as a label. This feature is summarized in Eqs. (22) and (23) in Sec. III B.

Positivity of p^+ in Eq. (116) implies that the classical, translation-invariant mass-mixing interaction term in Eq. (18),

$$P_I^- = \int dx^- d^2x^\perp m^2 \phi\chi, \qquad (117)$$

results in the quantum interaction operator in Eq. (33),

$$\mathcal{P}_I^- = \int [p]\theta(p^+)\frac{m^2}{p^+}(a_{0p}^\dagger b_{0p} + b_{0p}^\dagger a_{0p}), \quad (118)$$

which does not contain any terms of the type $a_p^\dagger b_{-p}^\dagger$ and $a_p b_{-p}$ that appear in Eq. (114) for H_I in the IF of quantum dynamics. Such terms are excluded because both p^+ and $-p^+$ in them are required to be positive. This is not possible for particles of a finite mass in the presence of a cutoff on $p = |\vec{p}|$, as is visible in Eq. (116), no matter how large such cutoff is. Note also that the FF integration measure $[p]$ does not depend on the mass μ used in the condition (116).

When the cutoff on $|\vec{p}|$ can be made arbitrarily large, one can have boost invariance in practice in an arbitrarily large range of momenta provided that the theory respects the symmetry [26]. This is the case at $s = 0$ in the RGPEP. In order to maintain the Lorentz symmetry in an effective theory, the sliding cutoff parameter $\lambda = 1/s$ emerges in the RGPEP through its equations. They are so designed that the sliding cutoff is not limiting $|\vec{p}|$ of individual particles. Instead, the effective-theory cutoff limits only the changes of invariant mass caused by interactions. The mass is invariant with respect to all 7 FF kinematical symmetries, including boost invariance.

The boost invariance is secured by design of the RGPEP generator in Eq. (6). The commutator guarantees that only connected interactions are generated. The total transverse momenta of interacting particles before and after an interaction cancel each other in the arguments of resulting vertex form factors. Spectators do not contribute to these arguments. The multiplication by a total $+$ momentum squared of interacting particles in the definition of $\mathcal{H}_{Pt}$, Eq. (10), results in the factor p^{+2} in Eq. (36). In the absence of sensitivity to cutoffs on p^+, this factor removes p^+ from the RGPEP evolution equation entirely. Therefore, the arguments of resulting vertex form factors depend only on the change of invariant mass squared among the particles that are involved in the interaction.

These features, combined with the absence of divergences due to separation of momentum modes, reduce the RGPEP in the elementary mass-mixing model to solving an evolution equation for particle masses as functions of

$t = s^4$. Quite generally, renormalized equations for coefficients c_t in $\mathcal{H}_t$ may involve only masses, relative momenta, coupling constants, and the parameter t. Thus, in the elementary model, the equations involve only μ_t^2, ν_t^2, mass-mixing parameter m_t^2, and t itself. These equations are independent of the particle momentum p. As a result, the RGPEP equations render a different representation of the same relativistic quantum theory for every value of t.

Each and every one of the effective theories derived using the RGPEP is defined in terms of a different basis in the space of operators acting in the Fock space. In the mass-mixing example, the effective representations tend in the limit of $t \to \infty$ to a relativistic theory of free particles with masses m_1 and m_2. No variation of the ground state $|0\rangle$ with t is required in the procedure.

D. Standard, IF approach versus RGPEP

The comparison relies on a change of field variables in the classical Lagrangian of Eq. (14). The new variables are determined by diagonalization of the mass-squared matrix. The mass terms,

$$-2\mathcal{L}_{\text{mass}} = \mu^2\phi^2 + \nu^2\chi^2 + 2m^2\phi\chi, \qquad (119)$$

can be written in the form of a 2×2 matrix sandwiched with a doublet of fields $\Psi = [\phi, \chi]$. Namely,

$$-2\mathcal{L}_{\text{mass}} = \Psi^\dagger M^2 \Psi \qquad (120)$$

$$= [\phi, \chi]\begin{bmatrix} \mu^2 & m^2 \\ m^2 & \nu^2 \end{bmatrix}\begin{bmatrix} \phi \\ \chi \end{bmatrix}. \qquad (121)$$

The eigenvalues of matrix M^2, denoted by m_1^2 and m_2^2 above Eq. (65) in Sec. IV C, and the corresponding eigenvectors, are

$$m_{1,2}^2 = (\mu^2 + \nu^2)/2 \pm \sqrt{(\mu^2 - \nu^2)^2/4 + m^4}, \qquad (122)$$

$$v_1 = \begin{bmatrix} \cos\varphi_\infty \\ -\sin\varphi_\infty \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \sin\varphi_\infty \\ -\cos\varphi_\infty \end{bmatrix}, \qquad (123)$$

where φ_∞ is given in Eq. (108). Inverting the relation

$$\Psi = \xi v_1 + \zeta v_2, \qquad (124)$$

one can define the fields

$$\xi = \cos\varphi_\infty\phi - \sin\varphi_\infty\chi, \qquad (125)$$

$$\zeta = \sin\varphi_\infty\phi + \cos\varphi_\infty\chi. \qquad (126)$$

This is a unitary change of field variables. Since the terms that involve derivatives of the fields ϕ and χ have equal coefficients in the classical Lagrangian density of Eq. (14), the density can be written as

$$\mathcal{L} = \tfrac{1}{2}[(\partial\xi)^2 - m_1^2\xi^2] + \tfrac{1}{2}[(\partial\zeta)^2 - m_2^2\zeta^2]. \quad (127)$$

This classical expression can now be quantized from scratch in the IF of dynamics.

The IF quantization involves definitions of the fields ξ and ζ and their conjugated momenta. The quantization leads to a theory of particles with masses m_1 and m_2 when one defines the quantum fields ξ and ζ and their conjugated momenta π_ξ and π_ζ using energy expressions $E_\xi^2(p) = m_1^2 + p^2$ and $E_\zeta^2(p) = m_2^2 + p^2$ in defining the time derivatives of the fields, respectively. The new energy expressions guarantee that all terms of the type $a_{\xi p}^\dagger a_{\xi -p}^\dagger$ or $a_{\zeta p}^\dagger a_{\zeta -p}^\dagger$ cancel out in the Hamiltonian. Equations (125) and (126) imply that

$$a_{\xi p} = \cos\varphi_\infty a_{0p} - \sin\varphi_\infty b_{0p}, \qquad (128)$$

$$a_{\zeta p} = \sin\varphi_\infty a_{0p} + \cos\varphi_\infty b_{0p}. \qquad (129)$$

These relations match Eqs. (106) and (107). The matching shows that the IF quantization of fields ξ and ζ produces the same result as the solution obtained entirely in one quantum theory using the RGPEP, in which there is no need to requantize the theory due to inclusion of the mass-mixing interaction term.

On the basis of knowing the full quantum implications of the mass-mixing interaction term in the FF of Hamiltonian dynamics, one can also write expressions for the initial IF quantum fields $\hat{\phi}$, $\hat{\chi}$, $\hat{\pi}_\phi$, and $\hat{\pi}_\chi$, in terms of the fields $\hat{\xi}$, $\hat{\zeta}$, $\hat{\pi}_\xi$, and $\hat{\pi}_\zeta$, using Eq. (124) and right energies for the time derivatives needed in $\hat{\pi}_\xi$ and $\hat{\pi}_\zeta$. Substituting these expressions into a classical IF Hamiltonian that canonically corresponds to the Lagrangian density of Eq. (14), one obtains the IF quantum Hamiltonian that explicitly describes the same physics as the FF quantum Hamiltonian obtained from the RGPEP at $t = \infty$. The vacuum-altering terms cancel out for all modes with a finite momentum.

However, when in a more complex theory than the elementary example some additional interaction terms cause divergences and other effects that are difficult to see through, the IF quantization approach may get stuck due to lack of a right guess for the time derivatives. In contrast, the RGPEP still indicates a direction for further studies in realistic cases. Namely, while the free theory that results from diagonalization of a bilinear part in a Lagrangian density is certainly not sufficient for establishing how to deal with the IF vacuum problem, the RGPEP promises some capability to work around the vacuum problem using the FF.

The FF Hamiltonian at $t = s^4 = 0$, $\mathcal{P}_0^-$, involves fields ϕ and χ. Their conjugate "momenta," $\pi_\phi = \partial^+ \phi$ and $\pi_\chi = \partial^+ \chi$, do not involve FF time derivatives, i.e., they do not involve derivatives with respect to x^+. Instead, the "momenta" are expressed through gradients of the fields in the front hyperplane. Rotation of the quantum fields automatically rotates the quantum "momenta."

By the way, the fields ξ and ζ can be used as initial variables also in the FF. The RGPEP provides no additional value in such setup, since there is no interaction between the free fields ξ and ζ. However, when more interactions are added, nothing prevents the RGPEP from application to the whole quantum theory using the effective-particle operators associated with the fields ξ and ζ, instead of ϕ and χ.

It should be mentioned that an interesting example of the IF application of a similarity renormalization group procedure in a fixed source model has been recently considered by Jones and Perry [37]. In the fixed source model, the interaction term is only linear in the quantum field variables, different momentum modes evolve separately, and one obtains the well-known solution in an elegant way. The fixed source model does not appear to suggest how to proceed in the IF when interaction terms involve more than one field and create a genuine vacuum problem.

VI. CONCLUSION

The case of a theory with two free fields with a mass-mixing interaction term can be generalized to theories with an arbitrary number $n > 2$ of fields and mass-mixing terms. In such theories, the RGPEP equation describes the evolution of a mass matrix of dimension $n \times n$ with s. The solution tends for $s \to \infty$ to a diagonal matrix, whose eigenvalues provide physical masses for n species of free particles.

Degeneracy of the mass matrix, which may correspond to a symmetry in a theory, prevents its full diagonalization via the RGPEP equation. In this case, an artificial infinitesimal breaking of the degeneracy can be introduced in order to enable RGPEP to identify a solution in the limit $s \to \infty$, as the artificial breaking is being removed.

The diagonalization of the mass matrix does not correspond to a minimization of a classical potential in the IF. Instead, it corresponds to identification of the eigenmodes in classical field oscillations. One has to use the eigenmodes in the IF quantization procedure in order to solve a vacuum problem in the absence of interactions other than the mass mixing. However, when such additional interaction terms, involving products of more than two fields, are included in a theory, the IF vacuum problem can no longer be solved using the field combinations that correspond to eigenvectors of the mass matrix. The additional interactions typically contribute to particle masses, bound states may develop, and, as it would have to happen in the case of confinement, the full theory eigenmodes do not even correspond to the fields present in an initial Lagrangian.

The intriguing feature of the RGPEP, illustrated here in the elementary model with a mass-mixing interaction term, is that it applies to quantum theories via steps that are essentially independent of the type of interaction one grapples with, while the vacuum problem is treated in a new way. Namely, the vacuum stays simple while the

interaction terms evolve towards expressions in terms of effective degrees of freedom. This feature makes the RGPEP a deserving candidate for application to more realistic theories than the elementary model discussed here. It is evident from the works referenced in this article that the RGPEP can be applied to realistic quantum field theories. The elementary example described here is thus of interest not only as an illustration of an exact nonperturbative solution of the RGPEP equations but also as the indicator of a difference between the options one has left for treating vacuum problems in the IF and FF of Hamiltonian dynamics in relativistic quantum field theories.

APPENDIX: SOLUTION FOR $\mathcal{H}_f$ DEPENDENT ON t

The discussion of Eq. (57) in Sec. IV B included the case of $\mathcal{H}_f$ containing masses dependent on t, which yields a 2×2 mass-squared matrix equation of the form

$$\begin{bmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{bmatrix}' = \left[\left[\begin{bmatrix} \mu_t^2 & 0 \\ 0 & \nu_t^2 \end{bmatrix}, \begin{bmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{bmatrix} \right], \begin{bmatrix} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{bmatrix} \right].$$

(A1)

This equation matches the Wegner equation for a Hamiltonian matrix [10] of a two-level system. Its analytic solution is well known but as far as the author knows it was never considered before in the context of particle masses in an exactly soluble quantum field theory in the FF of dynamics.

Proceeding as in Sec. IV C, one obtains

$$\delta' = 4\delta\gamma^2, \tag{A2}$$

$$\gamma' = -\delta^2\gamma. \tag{A3}$$

Multiplying the first of these two equations by 2δ and the second by 2γ, one arrives at

$$\delta^{2'} = 8\delta^2\gamma^2, \tag{A4}$$

$$\gamma^{2'} = -2\delta^2\gamma^2, \tag{A5}$$

which implies the same constant $\epsilon^2 = \delta^2 + 4\gamma^2$ as in Sec. IV C. After eliminating γ^2 from Eq. (A4),

$$\delta^{2'} = 2\delta^2(\epsilon^2 - \delta^2). \tag{A6}$$

The solutions corresponding to Eqs. (74) and (75), are

$$\delta\mu_t^2 = \delta\mu^2 \frac{\epsilon e^{x_t}}{\sqrt{\epsilon^2 - 1 + e^{2x_t}}}, \tag{A7}$$

$$m_t^2 = m^2 \frac{\epsilon}{\sqrt{\epsilon^2 - 1 + e^{2x_t}}}, \tag{A8}$$

where $x_t = (\delta m^2)^2 t$.

Since the generator given in Eqs. (43) and (82) is now altered to contain the varying $\delta\mu_t^2$ instead of the constant $\delta\mu^2$, the angle φ_t given by Eq. (84) is replaced by

$$\varphi_t = -\int_0^t \delta\mu_\tau^2 m_\tau^2 d\tau = \frac{1}{2}\left(\arctan\frac{1}{\sqrt{\epsilon^2 - 1}} - \arctan\frac{e^{(\delta m^2)^2 t}}{\sqrt{\epsilon^2 - 1}} \right). \tag{A9}$$

This result deviates from the result in Eq. (85) for finite values of t. The difference in the angles of rotation, φ_t, implies different combinations of operators a_{0p} and b_{0p} in Eqs. (95) and (96) for the same t. This means that the effective-particle operators at any finite $t > 0$ depend on the choice of the generator, although the Hamiltonians as operators are just one and the same operator for all values of t and both choices of the generator. When $t \rightarrow \infty$, Eqs. (84) and (A9) produce the same result for φ_∞ for arbitrary values of $\epsilon > 1$,

$$\varphi_\infty = \arctan\sqrt{\frac{\epsilon + 1}{\epsilon - 1}} - \pi/2 = \frac{1}{2}\arctan\frac{1}{\sqrt{\epsilon^2 - 1}} - \pi/4. \tag{A10}$$

Thus, the change in the generator from a constant $\mathcal{H}_f$ to a t-dependent full free part of $\mathcal{H}_t$ does not lead to any change in the effective particles that one obtains for $t \rightarrow \infty$ as a solution of the theory.

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