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# Hyperbolic vortices with large magnetic flux

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There has been some recent interest in the study of non-Abelian Bogomolny-Prasad-Sommerfield (BPS) monopoles in the limit of large magnetic charge. Most investigations have used a magnetic bag approximation, in which spherical symmetry is assumed within an Abelian description. In particular, this approach has been used to suggest the existence of two types of magnetic bags, with differing distributions of the zeros of the Higgs field, together with multilayer structures, containing several magnetic bags. This paper is concerned with the analogous situation of Abelian BPS vortices in the hyperbolic plane, in the limit of large magnetic flux. This system has the advantage that explicit exact solutions can be obtained and compared with a magnetic bag approximation. Exact BPS vortex solutions are presented that are analogous to the two types of magnetic bags predicted for BPS monopoles and it is shown that these structures can be combined to produce exact multilayer solutions.

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## I. INTRODUCTION

Vortices and monopoles are objects that arise in a range of theories in several areas, including particle physics, cosmology and condensed matter physics. A rather natural question concerns the possible structures that can arise when many of these objects are in close proximity, and this has led to the study of magnetic bags. The concept of a magnetic bag was first introduced by Bolognesi [1] in the context of vortices in the Abelian Higgs model in the Euclidean plane, where it is an approximation to a vortex solution with a large magnetic flux. In the simplest case the magnetic bag has circular symmetry and the approximation assumes that inside the bag the magnetic field is constant and the Higgs field vanishes, whereas outside the bag the magnetic field vanishes and the Higgs field takes its vacuum expectation value. Neither the magnetic field nor the Higgs field are continuous at the surface of the bag, but the surface contribution to the energy can be neglected because it is subleading in the limit of a large vortex number $N \gg 1$. A comparison with numerical solutions of circularly symmetric vortices provides support for the validity of this approximation [2]. Magnetic bags are particularly interesting in the Bogomolny-Prasad-Sommerfield (BPS) limit of critically coupled vortices, where it is expected that the shape of the bag can be varied because of the existence of the $2N$-dimensional moduli space of BPS vortex solutions.

Bolognesi [3] extended the magnetic bag idea to monopoles in non-Abelian Yang-Mills-Higgs theories in three-dimensional Euclidean space, where they describe configurations with large magnetic charge. Again the most interesting situation is the BPS limit, in this case associated with a massless Higgs field. In this context the approximation assumes that inside the bag both the Higgs field and the magnetic field vanish, whereas outside the bag the fields are taken to be abelian, consisting of a magnetic field and a scalar field that represents the magnitude of the non-Abelian Higgs field. These Abelian fields satisfy the Abelianized version of the Bogomolny equation. The simplest situation is to assume that the bag is spherically symmetric, though again it is expected that the shape of the bag can be varied because of the $4N$-dimensional moduli space associated with an $N$-monopole solution.

The magnetic bag description has been applied to several different aspects of monopoles and used to predict a number of interesting phenomena. Lee and Weinberg [4] have proposed that there are a range of magnetic bags, all of which have approximate spherical symmetry, but are distinguished by the distribution of the zeros of the Higgs field associated with the exact monopole solution. In particular, they propose that there is an extreme case, in which all the Higgs zeros are coincident at the center of the bag, and a second extreme case in which the Higgs zeros are distinct and (almost all) are located near the surface of the bag. Manton [5] has investigated the properties of multilayer structures, created by patching together nested sequences of magnetic bags, and Harland [6] has shown how magnetic bags may be described using a large $N$ limit of the Nahm transform. The magnetic bag approximation has also proved useful in the study of monopoles in anti-de Sitter spacetime [7], where it compares well with numerical solutions [8] even for rather small values of $N$.

The purpose of the present paper is to investigate some of the above issues in a system where exact explicit solutions are available for comparison with the magnetic bag approximation. The chosen system is the Abelian Higgs model in the hyperbolic plane and its associated BPS vortices. In particular, solutions with large magnetic flux are studied that include analogues of the two extreme types of magnetic bags predicted for BPS monopoles and it is shown that these structures can be combined to produce exact multilayer solutions.

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## II. HYPERBOLIC VORTICES

This section contains a brief review of BPS vortices in the hyperbolic plane. A more detailed discussion of this material can be found in Chapter 7 of [9].

The system of interest is the Abelian Higgs model, with complex scalar Higgs field $\phi$ and real gauge potential $a_i$. The theory is defined on the hyperbolic plane $\mathcal{H}^2$ of curvature $-1/2$, with metric

$$ds^2 = \Omega(dx^2 + dy^2), \quad \text{where} \quad \Omega = \frac{8}{(1 - x^2 - y^2)^2} \tag{2.1}$$

and $(x, y)$ are coordinates in the Poincaré disc model of $\mathcal{H}^2$, with $x^2 + y^2 < 1$. The energy of the model is given by the standard Ginzburg-Landau energy at critical coupling

$$E = \int_{\mathcal{H}^2} \left( \frac{1}{2}\Omega^{-1}B^2 + \frac{1}{2}D_i\phi\overline{D_i\phi} + \frac{\Omega}{8}(1 - |\phi|^2)^2 \right) d^2x, \tag{2.2}$$

where $B = \partial_1 a_2 - \partial_2 a_1$ is the magnetic field and $D_i\phi = \partial_i\phi - ia_i\phi$ the covariant derivative of the Higgs field. The vacuum expectation value of the Higgs field has been set to unity for convenience.

The magnetic flux through the hyperbolic plane is quantized,

$$\int_{\mathcal{H}^2} B d^2x = 2\pi N, \tag{2.3}$$

where the integer $N$ is the vortex number and is equal to the winding number of the phase of the Higgs field on the circle at infinity. $N$ is the first Chern number of the gauge field and it is also equal to the number of zeros of the Higgs field, counted with multiplicity. Without loss of generality, the vortex number $N$ will be taken to be positive in this paper.

A standard Bogomolny argument yields the energy bound $E \geq \pi N$, with equality attained by the BPS vortex solutions that solve the first order Bogomolny equations

$$D_1\phi + iD_2\phi = 0, \quad \text{and} \quad B = \frac{\Omega}{2}(1 - |\phi|^2). \tag{2.4}$$

As first observed by Witten [10], these Bogomolny equations are integrable for a particular value of the curvature of the hyperbolic plane ($-1/2$ in the conventions of this paper, hence its adoption from the outset).

Explicitly, the general $N$-vortex solution can be obtained in closed form by introducing the complex coordinate $z = x + iy$, in the unit disc, and setting

$$\phi = \left( \frac{1 - |z|^2}{1 - |f|^2} \right) \frac{df}{dz} \quad \text{and} \quad a_z = -i\frac{\partial}{\partial\bar{z}} \log\left( \frac{1 - |z|^2}{1 - |f|^2} \right), \tag{2.5}$$

where $f(z)$ is a rational map of the form

$$f = z\prod_{i=1}^{N} \left( \frac{z - \beta_i}{1 - \bar{\beta}_i z} \right), \tag{2.6}$$

with complex constants $\beta_i$, all inside the unit disc $|\beta_i| < 1$. The gauge freedom can be used to write any $N$-vortex solution in this form.

The $N$ complex parameters $\beta_i$ provide coordinates for the $N$-vortex moduli space $\mathcal{M}_N$, which has real dimension $2N$. These parameters determine the positions of the $N$ vortices, which are given by the points in the unit disc where $\frac{df}{dz}$ vanishes, since it follows from (2.5) that $\phi$ is zero at these points. In general, the relation between the solution parameters $\beta_i$ and the vortex positions is only known implicitly, since the relation requires knowledge of the zeros of the derivative of $f(z)$.

The simplest example of an $N$-vortex solution represents $N$ coincident vortices at the origin and is given by the map $f = z^{N+1}$, corresponding to the choice $\beta_i = 0 \ \forall \ i$. This solution has circular symmetry, since under an arbitrary spatial rotation through an angle $\chi$, given by $z \mapsto ze^{i\chi}$, the map transforms as $f \mapsto fe^{i(N+1)\chi}$, and a change in the phase of $f$ is simply a gauge transformation. In particular, the modulus of the Higgs field depends only on the distance from the origin and is given by

$$|\phi| = \frac{(N + 1)|z|^N}{|z|^{2N} + |z|^{2N-2} + \cdots + 1}. \tag{2.7}$$

## III. VORTICES AS MAGNETIC BAGS

A magnetic bag description of hyperbolic vortices in the large $N$ limit can be obtained by making the approximation that the Higgs field vanishes identically throughout some region $\mathcal{R} \subset \mathcal{H}^2$, with area $\mathcal{A}$. In this region the Bogomolny Eqs. (2.4) then imply that $B = \Omega/2$, that is, there is a constant magnetic flux per unit area. Outside the region $\mathcal{R}$ the Higgs field is taken to have its vacuum value $|\phi| = 1$, and the Bogomolny equations then determine that $B = 0$ in this region of space. If the gauge is chosen so that $\phi$ is real outside $\mathcal{R}$ then $\phi = 1$ and the energy density vanishes identically in this region. Generally the gauge in which $\phi$ is real is not a good one since an $N$-vortex configuration has the phase of $\phi$ winding $N$ times on the circle at infinity, so a gauge in which $\phi$ is real must be singular. However, it is assumed that the singularities occur inside the region $\mathcal{R}$, or on its boundary $\partial\mathcal{R}$, where this approximation for the Higgs field does not apply.

In this magnetic bag approximation the Higgs field and the magnetic field are not continuous on the boundary $\partial\mathcal{R}$, but neglecting the boundary contribution to the energy, which should be a reasonable approximation in the large $N$ limit, the energy (2.2) becomes

$$E = \int_{\mathcal{R}} \frac{1}{4}\Omega d^2x = \frac{1}{4}\mathcal{A}. \tag{3.1}$$

Within this approximation the vortex number is given by

$$N = \frac{1}{2\pi}\int_{\mathcal{R}} B d^2x = \frac{1}{4\pi}\int_{\mathcal{R}} \Omega d^2x = \frac{1}{4\pi}\mathcal{A} \tag{3.2}$$

and combining these two expressions yields the BPS energy formula $E = \pi N$.

The validity of this magnetic bag approximation can be tested in the simple case of a circular bag by comparing with the exact $N$-vortex solution (2.7). The Poincaré disc coordinate $z$ is not very convenient for analysing vortices in the large $N$ limit, because its modulus has a finite range. Therefore a coordinate transformation is first made by writing $z = e^{i\theta}\tanh(r/2^{3/2})$ so that the metric on the hyperbolic plane becomes

$$ds^2 = dr^2 + 2\sinh^2(r/\sqrt{2})d\theta^2, \tag{3.3}$$

where the radius $r \in [0, \infty)$ is the geodesic distance to the origin. In terms of this coordinate, the circular solution (2.7) becomes

$$|\phi| = \frac{2(N+1)(e^{\sqrt{2}r}-1)^N}{\sum_{j=0}^{N}\binom{2N+2}{2j+1}e^{\sqrt{2}jr}}. \tag{3.4}$$

In particular, the 1-vortex takes the simple kink form $|\phi| = \tanh(r/\sqrt{2})$.

The vortex number of a circular magnetic bag of radius $r = R$ is given by

$$N = \frac{1}{4\pi}\mathcal{A} = \cosh(R/\sqrt{2}) - 1, \tag{3.5}$$

so in the large $N$ limit, and neglecting terms that decay with $N$, the expression for the radius is

$$R = \sqrt{2}\log(2N). \tag{3.6}$$

In Fig. 1 the solid curves represent the exact solution (3.4) with $N = 10^2, 10^4, 10^6, 10^8$ (curves move to the right with increasing $N$). The dashed vertical lines correspond to the associated magnetic bag radius (3.6), revealing a reasonable approximation to the size of the vortex. However, the surface of the bag is assumed to have zero thickness, when in fact it is determined by the Higgs mass and has a finite value that is independent of $N$. Therefore a better approximation is to assume the same thickness as in the exact 1-vortex solution and take the continuous function

$$|\phi| = \tanh((r - R_0)/\sqrt{2})\Theta(r - R_0), \tag{3.7}$$

where $\Theta$ is the Heaviside step function and $R_0$ is a parameter that may be viewed as the radius of the magnetic bag with a finite thickness. The relation between the radius $R_0$, and the vortex number $N$ is determined by substituting the expression (3.7) into the formula

$$N = \frac{1}{2\pi}\int B d^2x = \frac{1}{4\pi}\int (1 - |\phi|^2)d^2x. \tag{3.8}$$

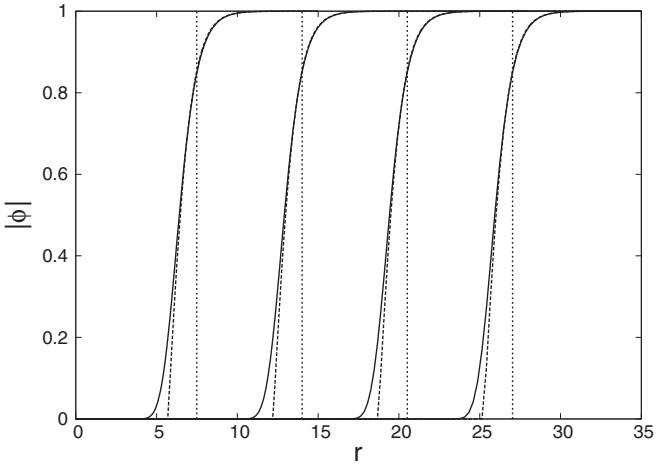


FIG. 1. The modulus of the Higgs field $|\phi|$ as a function of the radius $r$, for circularly symmetric vortices with vortex numbers $N = 10^2, 10^4, 10^6, 10^8$ (curves move to the right with increasing $N$). Solid curves represent the exact solution (3.4). Vertical dashed lines denote the value of the bag radius $R$ given by (3.6). Dashed curves correspond to the improved magnetic bag approximation (3.7).

Neglecting terms that tend to zero as $N \to \infty$, this formula may be written as

$$R_0 = R - \sqrt{2}\log\left(\frac{16 - \pi^2}{8 - 2\pi}\right), \tag{3.9}$$

which confirms that $R_0$ differs from the naive radius $R$ only by a term that is $O(1)$, associated with the finite thickness of the magnetic bag surface. The dashed curves in Fig. 1 correspond to the improved magnetic bag approximation (3.7) with the radius given by (3.9). It is clear that this yields an excellent approximation to the exact $N$-vortex solution.

The kink that appears as the large $N$ boundary bag profile is the hyperbolic vortex analogue of the monopole wall [11]. Its form can be obtained[1] by making a connection to the work of Manton and Rink [12] on a single vortex on a hyperbolic trumpet. Identify the angle $\theta$ with its shift by $2\pi/N$, to give a hyperbolic cone containing a single vortex. Introduce the angle on the cone $\chi_1 = N\theta$, which has period $2\pi$, and make a change of variable from $r$ to $\chi_2$ by writing $r = R - \sqrt{2}\log\chi_2$, where $R$ is given by (3.6). In terms of the coordinates $\chi_1$, $\chi_2$ the metric (3.3) becomes

$$ds^2 = \frac{2}{\chi_2^2}(d\chi_1^2 + d\chi_2^2), \tag{3.10}$$

where the large $N$ limit has been used to replace the hyperbolic function in (3.3) by its exponential approximation for large argument. The metric (3.10) is the upper half plane model of hyperbolic space, but as $\chi_1$ is periodic this surface is a hyperbolic trumpet. The vortex on this surface

[1]I thank Nick Manton for this observation.

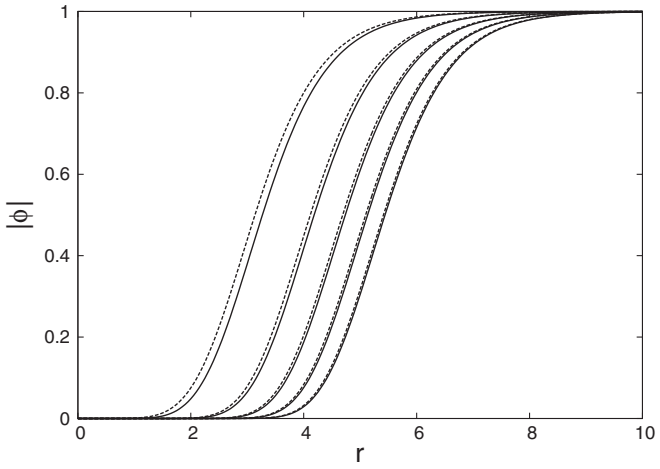


FIG. 2. The modulus of the Higgs field $|\phi|$ as a function of the radius $r$, for circularly symmetric vortices with vortex numbers $N = 10, 20, 30, 40, 50$ (curves move to the right with increasing $N$). Solid curves represent the exact solution (3.4) and dashed curves correspond to the boundary bag profile (3.11).

has been obtained in [12] and yields $|\phi| = \chi_2/\sinh\chi_2$. Converting back to the radial variable $r$ the boundary bag profile is

$$|\phi| = \frac{e^{(R-r)/\sqrt{2}}}{\sinh(e^{(R-r)/\sqrt{2}})} = \frac{2Ne^{-r/\sqrt{2}}}{\sinh(2Ne^{-r/\sqrt{2}})}, \tag{3.11}$$

which has a novel double exponential form.

Although the boundary bag profile (3.11) has been derived in the large $N$ limit, it provides an excellent approximation to the exact $N$-vortex solution (3.4) even for reasonably small values of $N$. This is demonstrated in Fig. 2, where solid curves represent the exact solution and dashed curves the boundary bag profile for $N = 10, 20, 30, 40, 50$.

Previous studies of magnetic bags, for both vortices and monopoles, have often used length units that scale with the bag radius, so that the bag position remains fixed as $N$ increases, to facilitate the comparison of bags with different values of $N$. Applied to the current situation an appropriate scaled radial variable is $r/R$, in terms of which the Higgs field of the exact $N$-vortex solution approaches the magnetic bag step function in the large $N$ limit, because of the logarithmic growth of $R$ with $N$ and the finite thickness of the bag.

## IV. DEFORMING THE MAGNETIC BAG

Magnetic bags assume that the Higgs field vanishes identically in a large region of space, but for a finite number of either monopoles or vortices there are only a finite number of points in space where the Higgs field is zero. In the case of monopoles, Lee and Weinberg [4] have proposed that there are two extreme types of magnetic bags, characterized by different distributions of the Higgs zeros. Both types of magnetic bags are approximately spherical, even though multimonopoles with exact spherical symmetry do not exist in the $SU(2)$ Yang-Mills-Higgs gauge theory. In the first type of magnetic bag, termed a non-Abelian bag in [4], the Higgs zeros are coincident at the center of the bag. The hyperbolic vortex analogue of this magnetic bag is clearly the circularly symmetric solution discussed in the previous section, where all $N$ zeros of the Higgs field are located at the origin. The term non-Abelian is not appropriate in the vortex context, so this type of solution will be referred to as a core magnetic bag, to denote that the zeros of the Higgs field lie deep within the core of the vortex. In the second type of magnetic bag, called an Abelian bag in the monopole context [4], the zeros of the Higgs field are distinct and most are located near the surface of the bag. The idea is that the zeros of the Higgs field may be viewed as moduli for the magnetic bag and varying the Higgs zeros inside the bag has little impact on the surface of the bag, but does change the bags character, and in particular the nature of the fields inside the bag. The evidence in support of this view [4] consists of an analysis of the topological features of the Higgs zeros, together with an extrapolation based on explicit knowledge of the Higgs zeros for some low charge monopoles with Platonic symmetry.

In this section an analysis will be presented of exact BPS vortex solutions that have approximate circular symmetry and are obtained as deformations of core magnetic bags, in which the Higgs zeros move out from the center towards the surface of the bag. This provides an explicit realization for hyperbolic vortices of the predicted behavior for monopoles and yields vortex analogues of both Abelian and non-Abelian monopole magnetic bags.

Consider the exact BPS $N$-vortex solution generated from the function

$$f(z) = \frac{z(z^N - \alpha)}{1 - \alpha z^N} \tag{4.1}$$

using the formula (2.5). Here $\alpha \in [0, 1)$ is a real parameter that controls the separation of the Higgs zeros. If $\alpha = 0$ then this solution reverts to the circularly symmetric solution of the previous section, that is, a core magnetic bag with $N$ coincident Higgs zeros at the origin. If $\alpha > 0$ then the solution has a cyclic $C_N$ symmetry and the $N$ zeros of the Higgs field are given by $z = \rho e^{2\pi ik/N}$, where $k = 0, 1, \ldots, N-1$ and $\rho \in (0, 1)$ is the real root of the equation

$$\rho^{2N} - \rho^N(N(\alpha^{-1} - \alpha) + \alpha^{-1} + \alpha) + 1 = 0. \tag{4.2}$$

In addition to determining the separation of the Higgs zeros, the parameter $\alpha$ also controls the nature of the Higgs field in the core of the bag, since at the origin ($z = 0$) the formula (2.5) gives that $|\phi| = \alpha$. If $\alpha \approx 0$ then the Higgs zeros are close to coincidence and the solution is only a small perturbation of the core magnetic

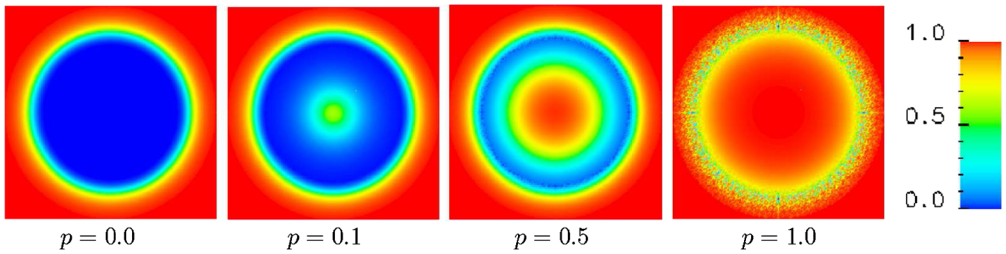


FIG. 3 (color online). The modulus of the Higgs field, $|\phi|$, for a vortex solution with $N = 10^4$ and increasing bag parameter $p = 0.0$, 0.1, 0.5, 1.0.

bag. To obtain a bag in which the Higgs zeros are close to the surface of the bag requires the opposite limit, that is $\alpha \approx 1$. However, this regime also includes a range of $\alpha$ for which the solution (4.1) describes $N$ well-separated single vortices located on the vertices of a regular $N$-gon and this is not a magnetic bag.

To determine an appropriate range of $\alpha$ it is useful to exchange $\alpha$ for the magnetic bag deformation parameter $p \in [0, \infty)$ defined by

$$\alpha = 1 - N^{-p}. \tag{4.3}$$

If $p = 0$ then the circularly symmetric solution is recovered, so the core magnetic bag is undeformed. Now consider the case $p > 0$, so that $\alpha \to 1$ as $N \to \infty$. The crucial issue is whether the Higgs zeros remain inside the magnetic bag as $p$ is increased from zero. Recall that the relation between $|z|$ and the radial coordinate $r$ is given by $|z| = \tanh(r/2^{3/2}) \approx 1 - 2e^{-r/\sqrt{2}}$, in the large $N$ limit. The circle associated with the surface of the bag has a radius $r = R = \sqrt{2}\log(2N)$ and corresponds to a value of $|z| = \rho_\star$ given by $\rho_\star = 1 - (1/N)$, which means that $\rho_\star^N \to 1/e$ as $N \to \infty$. The important point about this result is that as $N \to \infty$ then $\rho_\star^N$ has a nonzero limit that is strictly less than one. This is the property required by $\rho$, the solution of Eq. (4.2), to make sure that the $C_N$ symmetric solution does not describe $N$ well-separated vortices.

Note that if there are $N$ equally spaced vortices on a circle with a radius equal to that of the corresponding bag $r = R$, then the distance between neighboring vortices is $O(1)$, which is the same order as the thickness of the bag surface. Although the vortices are not well-separated in this regime, they are dilute, in the sense that a circularly symmetric magnetic bag approximation is only valid as an averaged description, since there is a large angular variation in $|\phi|$. This will be demonstrated below by considering the angular average

$$\langle|\phi|\rangle = \frac{1}{2\pi}\int_0^{2\pi} |\phi| d\theta \tag{4.4}$$

as a function of the radius $r$.

Substituting the form (4.3) into the root Eq. (4.2) gives

$$\rho^{2N} - 2(1 + N^{1-p})\rho^N + 1 = 0, \tag{4.5}$$

so clearly the critical value is $p = 1$. For $p = 0$ a core magnetic bag is obtained and as $p \to 1$ the dilute regime emerges. The bag deformation parameter $p \in [0, 1]$ describes the transition between these two extreme regimes.

To illustrate this behavior, Fig. 3 displays $|\phi|$ for a solution with vortex number $N = 10^4$ and increasing values of the bag deformation parameter $p$ from 0 to 1. The plotted region corresponds to $-1.2R \le X,\ Y \le 1.2R$, where $X$, $Y$ are the Cartesian coordinates given by $X + iY = re^{i\theta}$, and $R = \sqrt{2}\log(2N)$ is the bag radius. These plots confirm the expected behavior, with little variation of the surface of the bag, until the dilute regime is obtained, but a significant change in the character of the field in the interior of the bag, as it changes from a region of unbroken symmetry to a region of broken symmetry where the Higgs field attains its vacuum expectation value.

In Fig. 4 the angular average (4.4) is plotted as a function of the radius $r$ for the four solutions displayed in Fig. 3. Recall that the modulus of the Higgs field at the origin increases with $p$ since it is given by $|\phi| = 1 - N^{-p}$. It is clear from these graphs that the surface of the bag is virtually identical for the first three values $p = 0.0$, 0.1, 0.5, even though the interior of the bag changes

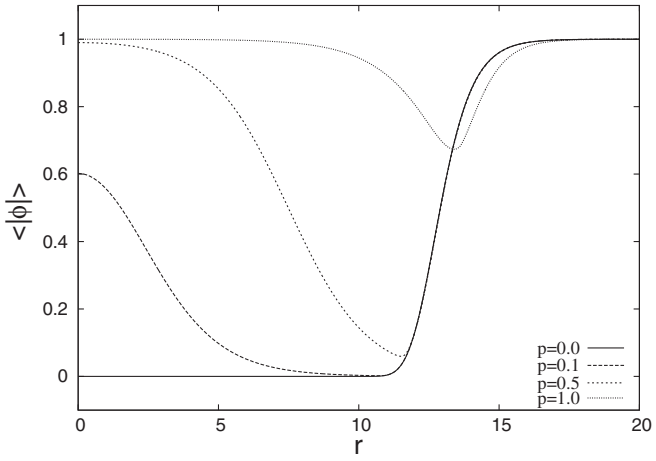


FIG. 4. The angular average of the modulus of the Higgs field, $\langle|\phi|\rangle$, as a function of the radius $r$, for a vortex solution with $N = 10^4$ and increasing bag parameter $p = 0.0$, 0.1, 0.5, 1.0.

dramatically. For $p = 0.1$ and $p = 0.5$ the minimum value of the angular average $\langle|\phi|\rangle$ is close to zero and is attained near the surface of the bag. For this range of $p$ there are angular directions along which the Higgs field exactly vanishes at a radius near to the surface of the bag, so the fact that the angular average is also close to zero confirms the approximate circular symmetry of the solution that arises when the vortices are in a dense regime. However, in the dilute regime $p = 1.0$ the angular average does not deviate far from the vacuum expectation value, despite the fact that there are still angular directions along which the Higgs field exactly vanishes at a radius near to the surface of the bag. In this sense the solution is far from being circularly symmetric and there is a large angular variation of the fields. In this dilute regime it is clear that any attempted magnetic bag approximation can only be valid in terms of a description of angularly averaged fields.

## V. MULTI-LAYER MAGNETIC BAGS

Manton [5] has applied a generalization of the magnetic bag approximation to study monopoles with a multilayer structure, consisting of a sequence of magnetic bags with increasing magnetic charges. Multilayer magnetic bags for hyperbolic vortices can be obtained as exact BPS solutions by taking $f(z)$ to have the obvious product form

$$f(z) = z\prod_{j=1}^{M}\frac{z^{N_j} - 1 + N_j^{-p_j}}{1 - (1 - N_j^{-p_j})z^{N_j}}. \tag{5.1}$$

For appropriate values of $N_j$ and $p_j$, with $j = 1, \ldots, M$, this solution describes a multilayer configuration with $M$ layers. Only the innermost layer can be a core magnetic bag, because the zeros of the Higgs field must be at the origin in this case. If two layers are core magnetic bags, say $p_k = p_{k+1} = 0$, then clearly the two layers degenerate to a single core magnetic bag with vortex number $N_k + N_{k+1}$.

As illustration, $|\phi|$ is displayed in Fig. 5 for a two-layer solution with $N_1 = 10^2$, $p_1 = 0$ and $N_2 = 10^8$, $p_2 = 0.3$. The two-layer structure is clearly visible in this plot, as is the behavior of the Higgs field, which is close to zero in each bag but returns to approximately its vacuum expectation value between the layers. Also plotted in Fig. 5 is the angular average $\langle|\phi|\rangle$ as a function of the radius $r$, displaying transitions between regions where the Higgs field is close to zero and regions where it is close to its vacuum expectation value.

The angular average confirms that this type of solution has approximate circular symmetry and can therefore be well-approximated by a multilayer magnetic bag description that assumes this symmetry. The appropriate form of this type of multilayer magnetic bag description is clearly a set of concentric nonoverlapping annuli, inside which the Higgs field is assumed to vanish and outside which it is taken to have its vacuum expectation value. Within this approximation, the vortex number in each layer is equal to the area of the annulus divided by $4\pi$. Obviously, the vortex number is higher in layers that are further out and, in particular, to have equally spaced layers requires an exponential growth of the vortex number. The innermost annulus can degenerate to a disc, to describe a core magnetic bag at the center, but this is the only annulus that is allowed to degenerate because of the nonoverlapping requirement.

The description presented above for hyperbolic vortices contrasts with the behavior of the Higgs field predicted in [5] for multilayer monopoles using the magnetic bag approximation. In the monopole case, the modulus of the assumed Abelianized Higgs field is not close to zero in each layer but rather approaches an effective vacuum

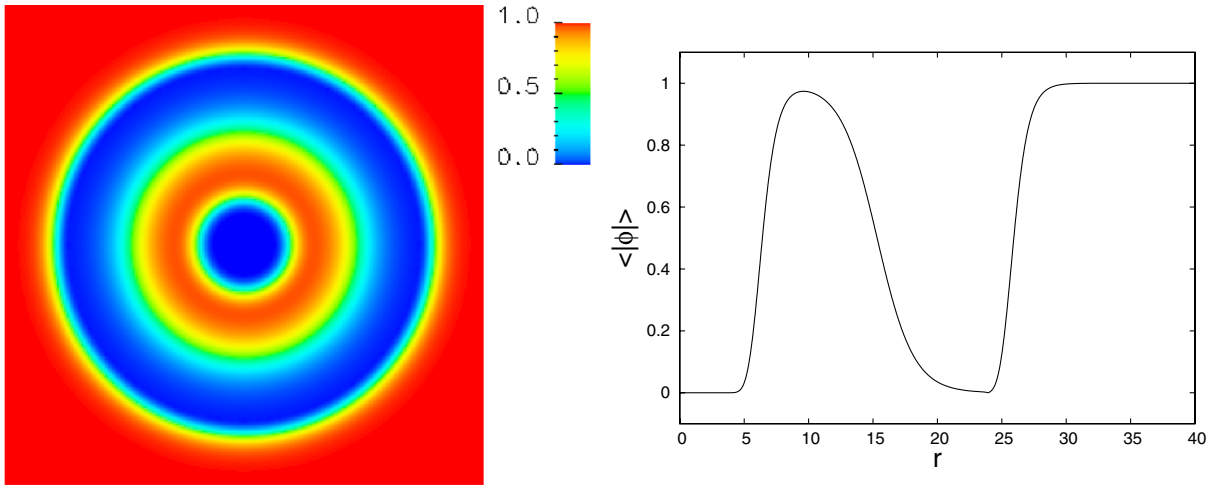


FIG. 5 (color online). A two-layer solution in which the inner layer has vortex number $N_1 = 10^2$ with bag parameter $p_1 = 0$ and the outer layer has vortex number $N_2 = 10^8$ with bag parameter $p_2 = 0.3$. The left-hand image displays the modulus of the Higgs field $|\phi|$. The right-hand plot is the angular average $\langle|\phi|\rangle$ as a function of the radius $r$.

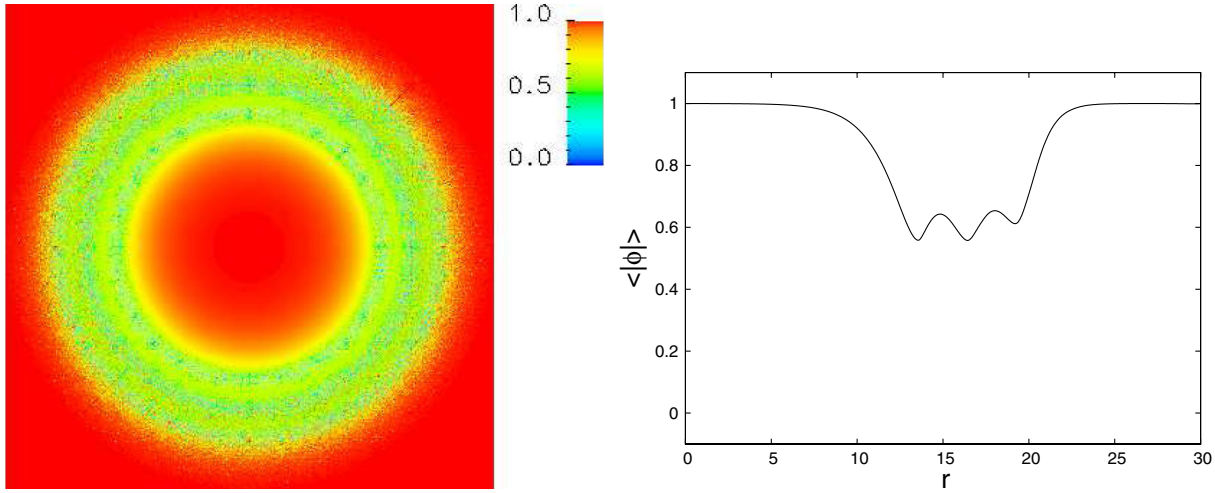


FIG. 6 (color online). A three-layer solution in which the inner layer has vortex number $N_1 = 10^4$ with bag parameter $p_1 = 1.0$, the middle layer has $N_2 = 10^5$ with $p_2 = 0.9$ and the outer layer has $N_3 = 10^6$ with $p_3 = 0.8$. The left-hand image displays the modulus of the Higgs field $|\phi|$. The right-hand plot is the angular average $\langle|\phi|\rangle$ as a function of the radius $r$.

expectation value determined by the magnetic charges and radii of the multilayer system. Given the above discussion, it is clear that this description can only apply to the angular average of the Higgs field in a dilute regime. Indeed the description in [5] is of multilayers consisting of dilute spherical clusters of large numbers of monopoles. This is consistent with the analysis in [4], where it is observed that $N$ monopoles located at the bag surface (which has radius $O(N)$) have an intermonopole separation that is $O(\sqrt{N})$.

To produce a hyperbolic vortex analogue of the kind of multilayer structure envisaged for monopoles, requires moving to the dilute regime, with bag deformation parameters closer to unity than zero. An example is presented in Fig. 6, for a three-layer solution where the vortex numbers are $N_1 = 10^4, N_2 = 10^5, N_3 = 10^6$ and the bag parameters are given by $p_1 = 1.0$, $p_2 = 0.9$, $p_3 = 0.8$. The dilute nature of the layers is clearly visible in this plot and contrasts with the dense regime displayed previously in Fig. 5. The vortex numbers and bag parameters have been chosen so that the layers are close together and produce a large region where the angular average of the modulus of Higgs field is reasonably constant but at a value that is only about half of the Higgs expectation value. A plot of the angular average $\langle|\phi|\rangle$ as a function of the radius is included in Fig. 6 to demonstrate this feature.

To provide a magnetic bag description of a dilute solution requires working with average values, as follows. Consider a region with area $\mathcal{A}$ and let $\varphi^2$ be the average value of $|\phi|^2$ throughout this region. The appropriate magnetic bag approximation is to assume that $|\phi|$ takes its vacuum expectation value outside this region and to average the second Bogomolny Eq. (2.4) inside the region to provide the approximate vortex number as $N = \mathcal{A}(1 - \varphi^2)/(4\pi)$. This formula shows that the previous dense magnetic bag approximation, which corresponds to the choice $\varphi^2 = 0$, yields minimal area for fixed vortex number. This mimics the interpretation of the non-Abelian monopole magnetic bag as the most compact arrangement of $N$ monopoles.

The two examples displayed in Figs. 5 and 6 are simply representative solutions that highlight the type of behavior that arises. The solution presented in Fig. 6 does not contain a core magnetic bag at its interior, but this could be added by including an additional layer with a vanishing bag parameter. Indeed it should be clear that there are many possible variations on multilayer magnetic bags that may or may not have core magnetic bags at their center and can include combinations of both dense and dilute layers. In the continuum limit this allows the construction of hyperbolic vortex analogues of Manton's monopole planets and galaxies [5].

## VI. CONCLUSION

Exact BPS vortex solutions of the Abelian Higgs model in the hyperbolic plane have been used to provide an explicit realization of magnetic bags. This has allowed an investigation of some of their properties and associated phenomena predicted in the context of BPS monopoles, including different extreme types of bags and multilayer bags.

Witten's ansatz [10] provides a direct mapping between the solutions considered in this paper and $SU(2)$ Yang-Mills instantons in four-dimensional Euclidean space with an $SO(3)$ rotational symmetry. The instanton number is identified with the vortex number $N$, so magnetic bags for vortices map to instantons with a large instanton number $N \gg 1$. It might be interesting to follow through the details of this correspondence and, in particular, to see how these

instantons are described within a large $N$ limit of the Atiyah-Drinfeld-Hitchin-Manin construction [13], in a manner similar to the recent study [6] of the large $N$ limit of the Nahm transform and its relation to monopole magnetic bags.

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