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Noncommutative Complex Scalar Field and Casimir Effect

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Using the noncommutative deformed canonical commutation relations proposed by Carmona *et al.* [J. M. Carmona, J. L. Cortés, J. Gamboa, and F. Mendez, J. High Energy Phys. 03 (2003) 058.][J. Gamboa, J. Lopéz-Sarrion, and A. P. Polychronakos, Phys. Lett. B **634**, 471 (2006).][J. M. Carmona, J. L. Cortés, Ashok Das, J. Gamboa, and F. Mendez, Mod. Phys. Lett. A **21**, 883 (2006).], a model describing the dynamics of the noncommutative complex scalar field is proposed. The noncommutative field equations are solved, and the vacuum energy is calculated to the second order in the parameter of noncommutativity. As an application to this model, the Casimir effect, due to the zero-point fluctuations of the noncommutative complex scalar field, is considered. It turns out that in spite of its smallness, the noncommutativity gives rise to a repulsive force at the microscopic level, leading to a modified Casimir potential with a minimum at the point $a_{\min} = \sqrt{\frac{5}{84}}\pi\theta$.

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I. INTRODUCTION

Since the birth of quantum mechanics and general relativity, many efforts have been made to understand the nature of spacetime at very short distances of the order of the Planck length, or at very high energies. There is now a common belief that the usual picture of spacetime as a smooth pseudo-Riemannian manifold should break down at very short distances of the order of the Planck length, due to the quantum gravity effects. Several physical arguments are used to motivate a deviation from the flat-space concept at very short distances of the order of the Planck length. Among the new concepts are quantum groups, quantum loop gravity, deformation theories, noncommutative geometry, noncommutative spacetime etc. The concept of noncommutative spacetime was suggested very early on by the founding fathers of quantum mechanics and quantum field theory. This was motivated by the need to remove the divergences that arise in quantum electrodynamics. However, this suggestion was ignored [1]. The concept of noncommutative spacetime was discovered in string theory and in the matrix model of M theory, where noncommutative gauge theory appears as a certain limit in the presence of a background field B [2–4].

In recent years, the idea of noncommutative spacetime has attracted considerable interest, and has penetrated into various fields in physics and mathematical physics, starting from the standard model of particle physics, strings, renormalization, to the quantum Hall effect, two-dimensional noncommutative harmonic oscillators, and noncommutative field theory [1,5–37]. One of the new features of noncommutative field theories is the UV/IR mixing phenomenon, in which the physics at high energies affects the physics at low energies, which does not occur in quantum field theories in which the coordinates commute [1,15,16]. The study of noncommutative spacetime and its implications to gauge and gravity theories, quantum field theories and other areas of theoretical physics, is motivated by the fact that the effects of noncommutativity of space may appear at very short distances of the order of the Planck length, or at very high energies. This may shed a light on the real microscopic geometry and structure of our universe.

In the last few years, there has been a great interest in the Casimir effect, for both the theoretical and the experimental sides; it finds applications in various physical phenomena, such as quantum field theory, condensed matter physics, elementary particle physics, quantum reflection of atoms on different surfaces and Bose-Einstein condensation [38,39].

In gravitation and cosmology, the Casimir effect can drive the inflation process and leads to interesting effects in brane models of the universe [39–43]. More practical reasons for the recent interest in Casimir effects are their implications for nanotechnology [38,39,43–45], where the attractive forces could lead to restrictions in the construction of nanodevices [39,46]. Casimir forces are usually attractive, but repulsive Casimir forces can be achieved in special circumstances; repulsive Casimir forces might prove useful in the construction of nanodevices, and other systems in which material components are in close proximity, including quantum levitation, quantum friction etc. [39,46].

The Casimir effect provides an effective mechanism for spontaneous compactification of extra spatial dimensions in multidimensional physics; indeed the vacuum fluctuations of higher-dimensional gravitational field may contribute an attractive Casimir force to push the size of the extra spaces in Kaluza-Klein unified theory and string theories to the Planck scale. Near the Planck scale, it is generally believed that the nonperturbative quantum gravity can stabilize the size of the extra spaces [39,41,47–49]. It has been shown that the Casimir energy could give repulsive force if some of the extra dimensions are noncommutative. This suggests that the noncommutativity of spatial dimensions provides a possible mechanism to stabilize the extra radius in high temperature [48–52].

In this paper, we present a model describing a noncommutative complex scalar field theory with commutative base space and noncommutative target space, and we explore possible implications that the noncommutativity in the target space might have on the Casimir force. It turns out that in spite of its smallness, the noncommutativity gives rise to a repulsive force at the microscopic level, leading to a modified Casimir potential with a minimum; this result is important, as mentioned above, in nanotechnology and in the stabilization of the size of the extra spatial dimensions.

By generalizing the noncommutative harmonic oscillator construction, an extension of quantum field theory based on the concept of noncommutative target space has been proposed in [30–32], where the properties and phenomenological implications of the noncommutative field have been studied and applied to different problems, including scalar, gauge and fermionic fields [30–34]. The idea of noncommutative target space has also been developed in the work of Balachandran *et al.* [53].

Our paper is organized as follows: In Sec. II, we consider a noncommutative action for a complex scalar field with self-interaction; in Sec. III, we derive and solve the free noncommutative field equations; in Sec. IV, we consider the noncommutative Casimir effect. Finally, in Sec. V, we draw our conclusions.

II. NONCOMMUTATIVE ACTION

Consider a complex scalar field $\Phi(x)$ with Lagrangian density given by [54–58]

$$\mathfrak{L} = -(\partial_\mu \Phi)^*(\partial^\mu \Phi) - m^2\Phi^*\Phi - g(\Phi^*\Phi)^2, \quad (1)$$

where m is the mass of the charged particles, and g is a positive parameter. The metric signature will be assumed to be $- + + \cdots$; in what follows, we take $\hbar = c = 1$.

The complex scalar field can be quantized using the canonical quantization rules. For this, we express it in terms of its real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$, where φ_1, φ_2 are real scalar fields; in terms of these real scalar fields the Lagrangian density reads

$$\mathfrak{L} = -\tfrac{1}{2}(\partial_\mu \varphi_a)^2 - \tfrac{1}{2}\mu^2[\varphi](\varphi_a)^2, \quad (2)$$

where $\mu^2[\varphi] = m^2 + \frac{1}{2}g(\varphi_a)^2$.

Let π_a be the canonical conjugate to φ_a

$$\pi_a = \frac{\partial \mathfrak{L}}{\partial \dot{\varphi}_a} = \dot{\varphi}_a \quad (3)$$

The Hamiltonian density then reads

$$\mathcal{H} = \pi_a\dot{\varphi}_a - \mathfrak{L} = \tfrac{1}{2}(\pi_a)^2 + \tfrac{1}{2}(\vec{\nabla}\varphi_a)^2 + \tfrac{1}{2}\mu^2[\varphi](\varphi_a)^2, \quad (4)$$

where the summation convention over repeated indices is assumed throughout this paper.

To quantize the system, we split the Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int}$ into free and interaction terms [54]

$$\mathcal{H}_0 = \tfrac{1}{2}(\pi_a)^2 + \tfrac{1}{2}(\vec{\nabla}\varphi_a)^2 + \tfrac{1}{2}m^2(\varphi_a)^2 \quad (5)$$

$$\mathcal{H}_{\rm int} = \tfrac{1}{2}g(\varphi_a\varphi_a)^2 \quad (6)$$

and then we pass to the interaction picture. In the interaction picture the equations of motion are given by

$$\dot{\varphi}_a(x) = \frac{\delta H_0}{\delta \pi_a(x)}, \qquad \dot{\pi}_a(x) = -\frac{\delta H_0}{\delta \varphi_a(x)}, \quad (7)$$

where $H_0 = \int d^3\vec{x}\mathcal{H}_0$ is the free Hamiltonian.

In the canonical quantization, the canonical variables φ_a and the canonical conjugates π_a are assumed to be operators satisfying the canonical commutation relations

$$\begin{aligned} &[\varphi_a(t,\vec{x}), \pi_b(t,\vec{y})] = i\delta_{ab}\delta^3(\vec{x}-\vec{y}) \\ &[\varphi_a(t,\vec{x}), \varphi_b(t,\vec{y})] = 0 \\ &[\pi_a(t,\vec{x}), \pi_b(t,\vec{y})] = 0. \end{aligned} \quad (8)$$

It is well-known, since the birth of quantum field theory in the papers of Born, Dirac, Fermi, Heisenberg, Jordan, and Pauli, that the free field behaves like an infinite number of coupled harmonic oscillators [54]. Using this analogy between free fields and an infinite number of coupled harmonic oscillators, one can impose noncommutativity on the configuration space of dynamical fields φ_a; to do this we recall that the two-dimensional harmonic oscillator noncommutative configuration space can be realized as a space where the coordinates $\hat{x}_a$, and the corresponding noncommutative momentum $\hat{p}_a$, are operators satisfying the commutation relations

$$[\hat{x}_a, \hat{x}_b] = i\theta^2\varepsilon_{ab} \quad [\hat{p}_a, \hat{p}_b] = 0 \quad [\hat{x}_a, \hat{p}_b] = i\delta_{ab}, \quad (9)$$

where θ is a parameter with dimension of length, and ε_{ab} is an antisymmetric constant matrix.

It is well-known that this noncommutative algebra can be mapped to the commutative Heisenberg-Weyl algebra [35–37]

$$[x_a, x_b] = 0 \qquad [p_a, p_b] = 0 \qquad [x_a, p_b] = i\delta_{ab} \quad (10)$$

through the relations

$$\hat{x}_a = x_a - \frac{1}{2}\theta^2\varepsilon_{ab}p_b \qquad \hat{p}_a = p_a. \quad (11)$$

To impose noncommutativity on the configuration space of dynamical fields φ_a, the noncommutative canonical variables $\hat{\varphi}_a$ and the noncommutative canonical conjugates $\hat{\pi}_a$ are assumed to be operators satisfying the noncommutative commutation relations [30–32]

$$[\hat{\varphi}_a(t,\vec{x}),\hat{\pi}_b(t,\vec{y})]=i\delta^3(\vec{x}-\vec{y})\delta_{ab}$$
$$[\hat{\varphi}_a(t,\vec{x}),\hat{\varphi}_b(t,\vec{y})]=i\theta\varepsilon_{ab}\delta^3(\vec{x}-\vec{y}) \tag{12}$$
$$[\hat{\pi}_a(t,\vec{x}),\hat{\pi}_b(t,\vec{y})]=0,$$

where θ is the parameter of noncommutativity, with dimension of length, which is assumed to be a constant, and ε_{ab} is a 2×2 real antisymmetric matrix

$$\varepsilon_{12}=-\varepsilon_{21}=1. \tag{13}$$

By generalizing the noncommutative harmonic oscillator construction an extension of quantum field theory, based on the concept of noncommutative fields satisfying the noncommutative commutation relations (12), has been proposed in [30–32], where the properties and phenomenological implications of the noncommutative field have been studied and applied to different problems including scalar, gauge and fermionic fields [30–34]. Another approach, based on the relations (15) between the noncommutative variables $\hat{\varphi}_a$ and $\hat{\pi}_a$ and the canonical variables φ_a and π_a, has been used and developed in the work of Balachandran *et al.* [53], where some periodic boundary conditions are used to study a free massless scalar field in the noncommutative target space $\mathbb{R}^2$, the theory has been quantized via the Hamiltonian formalism and applied to the study of the deformed black-body radiation spectrum.

Our approach is different from that proposed in [30–32], but it bears some similarities with the approach used in the work of Balachandran *et al.* [53]; both approaches are based on the relations (15) between the noncommutative variables $\hat{\varphi}_a$ and $\hat{\pi}_a$ and the canonical variables φ_a and π_a, but in our approach the equations of motions of the deformed theory will be used to quantize the deformed theory via the Peierls bracket.

The noncommutative Hamiltonian density is assumed to have the form

$$\hat{\mathcal{H}}=\tfrac{1}{2}(\hat{\pi}_a)^2+\tfrac{1}{2}(\vec{\nabla}\hat{\varphi}_a)^2+\tfrac{1}{2}\mu^2[\hat{\varphi}](\hat{\varphi}_a)^2. \tag{14}$$

It is easy to see that the noncommutative commutation relations (12) can be mapped to the canonical commutation relations (8) if the noncommutative variables $\hat{\varphi}_a$ and $\hat{\pi}_a$ are related to the canonical variables φ_a and π_a by the relations

$$\hat{\varphi}_a=\varphi_a-\frac{1}{2}\theta\varepsilon_{ab}\pi_b \qquad \hat{\pi}_a=\pi_a. \tag{15}$$

Using these transformations, the noncommutative Hamiltonian density (14) can be rewritten, up to a total derivative term and up to second order in the parameter θ, as

$$\hat{\mathcal{H}}=\tfrac{1}{2}\pi^{\sim}\Big(\mathbb{I}+\tfrac{1}{4}\theta^2(m^2-g\varepsilon\hat{\sigma}\varepsilon)\Big)\pi-\tfrac{1}{8}\theta^2\pi^{\sim}\vec{\nabla}^2\pi$$
$$+\tfrac{1}{2}\theta\pi^{\sim}(m^2-\vec{\nabla}^2+g(\varphi_a)^2)\varepsilon\varphi$$
$$+\tfrac{1}{2}\varphi^{\sim}\Big(m^2-\vec{\nabla}^2+\tfrac{1}{2}g(\varphi_a)^2\Big)\varphi+O(\theta^3),$$

where

$$\hat{\sigma}_{ab}=\frac{\delta^2}{\delta\varphi_a\delta\varphi_b}\Big[\frac{1}{4}(\varphi^{\sim}\varphi)^2\Big]=\varphi^{\sim}\varphi\delta_{ab}+2\varphi_a\varphi_b \tag{16}$$

with $\mathbb{I}$ denoting the 2×2 unit matrix, and $\varphi^{\sim}$ denoting the transpose of φ.

From now on we keep only the modifications due to the noncommutativity up to second order in the parameter θ.

The relation between π_a and $\dot{\varphi}_a$ is given by

$$\dot{\varphi}_a(x)=\frac{\delta\hat{H}}{\delta\pi_a(x)}, \tag{17}$$

where $\hat{H}=\int d^3x\hat{\mathcal{H}}$. Using the expression of $\hat{\mathcal{H}}$, one gets

$$\dot{\varphi}_a(x)=\Big(\mathbb{I}+\tfrac{1}{4}\theta^2(m^2-g\varepsilon\hat{\sigma}\varepsilon)\Big)_{ab}\pi_b(x)-\tfrac{1}{4}\theta^2\vec{\nabla}^2\pi_a(x)$$
$$+\tfrac{1}{2}\theta(m^2-\vec{\nabla}^2+g(\varphi_a)^2)\varepsilon_{ab}\varphi_b(x). \tag{18}$$

From this relation we get, by iteration, the following expression of π_a

$$\pi_a=\dot{\varphi}_a+\tfrac{1}{4}\theta^2(\vec{\nabla}^2-(m^2-g\varepsilon\hat{\sigma}\varepsilon))_{ab}\dot{\varphi}_b$$
$$-\tfrac{1}{2}\theta(m^2-\vec{\nabla}^2+g(\varphi_a)^2)\varepsilon_{ab}\varphi_b. \tag{19}$$

We note that the noncommutative Hamiltonian density can be derived from the following noncommutative Lagrangian density

$$\hat{\mathfrak{L}}=\tfrac{1}{2}\dot{\varphi}^{\sim}\Big(\mathbb{I}+\tfrac{1}{4}\theta^2(\vec{\nabla}^2-(m^2-g\varepsilon\hat{\sigma}\varepsilon))\Big)\dot{\varphi}$$
$$+\tfrac{1}{2}\theta\varphi^{\sim}(m^2-\vec{\nabla}^2+g(\varphi_a)^2)\varepsilon\dot{\varphi}$$
$$-\tfrac{1}{2}\varphi^{\sim}\Big(m^2-\vec{\nabla}^2+\tfrac{1}{2}g(\varphi_a)^2-\tfrac{1}{4}\theta^2(m^2-\vec{\nabla}^2+g(\varphi_a)^2)^2\Big)\varphi \tag{20}$$

via the usual Legendre transformation $\hat{\mathfrak{L}}=\pi_a\dot{\varphi}_a-\hat{\mathcal{H}}$.

III. NONCOMMUTATIVE FIELD EQUATIONS

Let us now consider the free theory, $g=0$; the noncommutative free Hamiltonian density reads

$$\hat{\mathcal{H}}=\tfrac{1}{2}\Big(1+\tfrac{1}{4}\theta^2m^2\Big)\pi^{\sim}\pi-\tfrac{1}{8}\theta^2\pi^{\sim}\vec{\nabla}^2\pi$$
$$+\tfrac{1}{2}\theta\pi^{\sim}(m^2-\vec{\nabla}^2)\varepsilon\varphi+\tfrac{1}{2}\varphi^{\sim}(m^2-\vec{\nabla}^2)\varphi. \tag{21}$$

The noncommutative field equations are given by

$$\dot{\varphi}_a(x) = \frac{\delta \hat{H}}{\delta \pi_a(x)}, \tag{22}$$

$$\dot{\pi}_a(x) = -\frac{\delta \hat{H}}{\delta \varphi_a(x)}. \tag{23}$$

From the first equation we get

$$\pi_a = \dot{\varphi}_a + \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\dot{\varphi}_a - \tfrac{1}{2}\theta(m^2 - \vec{\nabla}^2)\varepsilon_{ab}\varphi_b \tag{24}$$

The second equation gives

$$\dot{\pi}_a = -(m^2 - \vec{\nabla}^2)\varphi_a + \tfrac{1}{2}\theta(m^2 - \vec{\nabla}^2)\varepsilon_{ab}\pi_b. \tag{25}$$

The noncommutative field equations Eq. (24) and (25) may be written in the form

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]\varphi(x) = 0, \tag{26}$$

where

$$\begin{aligned} \mathcal{A} &= \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right]\mathbb{I} = \mathcal{A}^\sim \\ \mathcal{C} &= \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right](m^2 - \vec{\nabla}^2)\mathbb{I} = \mathcal{C}^\sim \\ \mathcal{B} &= \theta(m^2 - \vec{\nabla}^2)\varepsilon = -\mathcal{B}^\sim \end{aligned} \tag{27}$$

and $\mathbb{A}^\sim$ denotes the transpose of the operator $\mathbb{A}$.

It is easy to see that the field equations Eq. (26) may be derived from the Lagrangian Eq. (20)

$$\hat{\mathbb{L}} = \int d^3\vec{x}\left[\tfrac{1}{2}\dot{\varphi}^\sim \mathcal{A}\dot{\varphi} + \tfrac{1}{2}\varphi^\sim \mathcal{B}\dot{\varphi} - \tfrac{1}{2}\varphi^\sim \mathcal{C}\varphi\right]. \tag{28}$$

The general solution of Eq. (26) may be written as (see Appendix A):

$$\begin{aligned} \varphi(x) &= \sum_A [u_A^{(+)}(x)a_A + \bar{u}_A^{(+)}(x)\bar{a}_A] \\ &\quad + \sum_A [u_A^{(-)}(x)b_A + \bar{u}_A^{(-)}(x)\bar{b}_A] \end{aligned} \tag{29}$$

for some time-independent complex numbers a_A, b_A and their complex conjugates $\bar{a}_A$, $\bar{b}_A$, where $\bar{u}_A^{(\pm)}$ are the complex conjugates of the mode functions $u_A^{(\pm)}$

$$u_A^{(\pm)}(t, \vec{x}) = \chi_A(\vec{x})e^{-i\omega_A^{(\pm)}t}\zeta_A^{(\pm)}. \tag{30}$$

The frequencies $\omega_A^{(\pm)}$ are given by Eq. (A9)

$$\begin{aligned} \omega_A^{(\pm)} &= \tfrac{1}{2}[\mp\theta\bar{\sigma}_A + \sqrt{4\bar{\sigma}_A + \theta^2\bar{\sigma}_A^2}] \\ &\simeq \sqrt{\bar{\sigma}_A} \mp \tfrac{1}{2}\theta\bar{\sigma}_A + \tfrac{1}{8}\theta^2\bar{\sigma}_A^{3/2}, \end{aligned} \tag{31}$$

where χ_A are the eigenvectors of the operator $-\vec{\nabla}^2$ with eigenvalues σ_A, and $\bar{\sigma}_A = m^2 + \sigma_A$.

Quantization of the noncommutative complex scalar field theory is straightforward via the Peierls bracket (see [55] for more details). In the quantum theory, the field φ becomes a Hermitian operator, and the operator version of Eq. (29)

$$\begin{aligned} \varphi(x) &= \sum_A [u_A^{(+)}(x)a_A + \bar{u}_A^{(+)}(x)a_A^*] \\ &\quad + \sum_A [u_A^{(-)}(x)b_A + \bar{u}_A^{(-)}(x)b_A^*] \end{aligned} \tag{32}$$

holds for some constant operators a_A, b_A and their Hermitian conjugates a_A^*, b_A^*. By using the Wronskian relations Eq. (A13)–(A16) we get

$$\begin{aligned} a_A &= -i\int d^3\vec{x}u_A^{(+)*}(x)\overleftrightarrow{\mathbb{W}}\varphi(x) \\ a_A^* &= +i\int d^3\vec{x}u_A^{(+)\sim}(x)\overleftrightarrow{\mathbb{W}}\varphi(x) \\ b_A &= -i\int d^3\vec{x}u_A^{(-)*}(x)\overleftrightarrow{\mathbb{W}}\varphi(x) \\ b_A^* &= +i\int d^3\vec{x}u_A^{(-)\sim}(x)\overleftrightarrow{\mathbb{W}}\varphi(x). \end{aligned} \tag{33}$$

The quantum theory is obtained by setting

$$[\varphi_a(x), \varphi_b(y)] = i\tilde{G}_{ab}(x, y), \tag{34}$$

where $\tilde{G}$ is the commutator matrix

$$\begin{aligned} \tilde{G}(x, y) &= -i\sum_A u_A^{(+)}(x)u_A^{(+)*}(y) + i\sum_A \bar{u}_A^{(+)}(x)u_A^{(+)\sim}(y) \\ &\quad - i\sum_A u_A^{(-)}(x)u_A^{(-)*}(y) + i\sum_A \bar{u}_A^{(-)}(x)u_A^{(-)\sim}(y). \end{aligned}$$

Using the Wronskian relations Eq. (A13)–(A16) one can see that the commutator matrix $\tilde{G}$ is the unique function that solves the Cauchy problem

$$\varphi(x) = \int d^3\vec{y}\,\tilde{G}(x, y)\overleftrightarrow{\mathbb{W}}(y)\varphi(y) \quad \text{at the same time } t = x^0 = y^0. \tag{35}$$

Moreover, the commutator matrix $\tilde{G}$ satisfies the equation

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]\tilde{G}(x, y) = 0. \tag{36}$$

Using Eq. (34) and the Wronskian relations Eq. (A13)–(A16) we get the commutation relations

$$\begin{aligned} &[a_A, a_B^*] = \delta_{AB}, \qquad [a_A, a_B] = [a_A^*, a_B^*] = 0 \\ &[b_A, b_B^*] = \delta_{AB}, \qquad [b_A, b_B] = [b_A^*, b_B^*] = 0 \\ &[a_A, b_B^*] = [a_A, b_B] = [a_A^*, b_B^*] = [a_A^*, b_B] = 0. \end{aligned} \tag{37}$$

It is easy to show, by substituting the expression of π Eq. (24) into Eq. (21), that the noncommutative Hamiltonian operator can be written as

$$\hat{H} = \tfrac{1}{2}\int d^3\vec{x}\varphi^\sim(x)\overleftrightarrow{\mathbb{W}}(x)\dot{\varphi}(x) \tag{38}$$

using the Wronskian relations Eq. (A13)–(A16), and the expression of φ Eq. (32), the noncommutative Hamiltonian operator $\hat{H}$ of the system can be expressed as

$$\hat{H} = \sum_A (\omega_A^{(+)} a_A^* a_A + \omega_A^{(-)} b_A^* b_A) + \tfrac{1}{2}\sum_A (\omega_A^{(+)} + \omega_A^{(-)}), \tag{39}$$

where the commutation relations Eq. (37) have been used to get this form.

The noncommutative vacuum energy $E_{\rm vac}$ reads

$$E_{\rm vac} = \langle {\rm vac}|\hat{H}|{\rm vac}\rangle = \tfrac{1}{2}\sum_A (\omega_A^{(+)} + \omega_A^{(-)}) = \sum_A \left(\sqrt{m^2 + \sigma_A} + \frac{1}{8}\theta^2 (m^2 + \sigma_A)^{(3/2)}\right), \tag{40}$$

where the summation over A is constrained by the condition Eq. (A21). The noncommutative vacuum energy $E_{\rm vac}$, in the case where the free scalar field is confined in a D-dimensional rectangular box of volume $V = L^D$ with periodic boundary conditions on the walls of the box, can be written as

$$E_{\rm vac} = \sum_{n_1, n_2, \ldots, n_D} \left(\left[m^2 + \sum_{k=1}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{1/2} + \frac{1}{8}\theta^2 \left[m^2 + \sum_{k=1}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{3/2} \right), \tag{41}$$

where the summation over $n_1, n_2, \ldots, n_D$ is constrained by the condition Eq. (A23).

In the limit $L \to \infty$ we can approximate the sums that occur in Eq. (41) with (divergent) integrals

$$E_{\rm vac} = V \int \frac{d^D \vec{p}}{(2\pi)^D} \left([\vec{p}^2 + m^2]^{1/2} + \frac{1}{8}\theta^2 [\vec{p}^2 + m^2]^{3/2} \right). \tag{42}$$

Although these integrals are mathematically meaningless, one can use some sort of regularization technique that makes the integrals finite. Using the ζ- function regularization (see the definitions and intermediate stages of the calculation in Appendix B) [59], we get the following expression for the vacuum energy $E_{\rm vac}$

$$E_{\rm vac} = \left[\frac{V[m^2]^{(D+1)/2}}{(4\pi)^{D/2}} [l^2 m^2]^{-(s/2)} \frac{\Gamma(\frac{s-D-1}{2})}{\Gamma(\frac{s-1}{2})} + \frac{1}{8}\theta^2 \frac{V[m^2]^{(D+3)/2}}{(4\pi)^{D/2}} [l^2 m^2]^{-(3s/2)} \frac{\Gamma(\frac{3s-D-3}{2})}{\Gamma(\frac{3s-3}{2})} \right]_{s \to 0}. \tag{43}$$

If D is even, the right-hand side of Eq. (43) is analytic at $s = 0$ with the result

$$E_{\rm vac} = \frac{V[m^2]^{(D+1)/2}}{(4\pi)^{D/2}} \frac{\Gamma(-\frac{D+1}{2})}{\Gamma(-\frac{1}{2})} \left[1 + \frac{1}{8}\theta^2 \frac{3m^2}{D+1} \right] = \frac{V[m^2]^{(D+1)/2}}{(4\pi)^{D/2}} \frac{(-2)^{D/2}}{1.3.5\ldots(D+1)} \left[1 + \frac{1}{8}\theta^2 \frac{3m^2}{D+1} \right],$$

where we have used the following properties of the Γ-function [56,59]

$$z\Gamma(z) = \Gamma(z+1)$$

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-2)^n \sqrt{\pi}}{1.3.5\ldots(2n-1)}, \qquad n = 1, 2, 3, \ldots$$

When D is odd, the right-hand side of Eq. (43) is not analytic at $s = 0$; it has simple poles at $s = 0$, one simple pole from $\Gamma(\frac{s-D-1}{2})$, and another simple pole from $\Gamma(\frac{3s-D-3}{2})$. If we expand Eq. (43) about the pole, in the case where $D = 3$, we find

$$E_{\rm vac} = -\frac{V}{2}\left(\frac{m^2}{4\pi}\right)^2 \left\{ \left[1 + \frac{m^2}{48}\theta^2 \right] \frac{2}{s} - \frac{1}{2}\left[1 + \frac{5m^2}{24}\theta^2 \right] - \left[1 + \frac{m^2}{16}\theta^2 \right] \ln \frac{l^2 m^2}{4} \right\}. \tag{44}$$

To get this expression, the following formula has been used [56,59]

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) + O(\epsilon), \tag{45}$$

where n is a positive integer or zero, and γ is the Euler constant.

The vacuum energy, when D is odd, is divergent; this is just one example of a variety of ultraviolet divergences that are encountered in quantum field theory. They arise in a continuum theory due to the infinite number of degrees of freedom that exist even in a finite volume, they can be reabsorbed into a rescaling of the fields and into a rescaling of coupling constants. These ultraviolet divergences can be eliminated by hand since only energy differences can be observed; they are only important if we worry about gravitational phenomena, since in general relativity any form of energy contributes to the gravitational interaction [54,57].

IV. NONCOMMUTATIVE CASIMIR EFFECT

The Casimir effect is a nonclassical, electromagnetic, attractive force which is concerned with vacuum fluctuations in the electromagnetic field between two uncharged parallel conducting plates [38]. The size of this force was first predicted and calculated in 1948 by Casimir, who found that there is an attractive force per unit area between two parallel, uncharged, perfectly conducting plates separated by a distance a

$$F_{\text{Casimir}} = -\frac{\hbar c \pi^2}{240 a^4}.$$

This was first looked for by Sparnaay (1958), and recently has been confirmed by Lamoreaux, Mohideen and Roy, and recently by Chan, Aksyuk, Kleiman, Bishop, and Capasso [38,60–66].

In this section, we will consider the complex scalar field analogue of the Casimir effect, for this we consider a massive complex scalar field in a D-dimensional rectangular box, satisfying Dirichlet boundary conditions at $x_1 = 0$ and $x_1 = a$, but is unconfined in the remaining directions, let $L_1 = a$, $L_2 = L_3 = \ldots = L_D = L$ be the sides of the box, and $V = L_1 L_2 L_3 \ldots\ L_D$ its volume, ultimately we will let L becomes infinitely large. The normalized eigenvectors χ_A and the eigenvalues σ_A of $-\vec{\nabla}^2$ with Dirichlet boundary conditions Eq. (A25) on the walls of the box are given by Eq. (A26) and (A27) [59].

The noncommutative vacuum energy E_{vac} is given by Eq. (40)

$$\begin{aligned} E_{\text{vac}} &= E_{\text{vac}}^{(\text{C})} + E_{\text{vac}}^{(\text{NC})} \\ &= \sum_A \left(\sqrt{m^2 + \sigma_A} + \tfrac{1}{8}\theta^2 (m^2 + \sigma_A)^{3/2} \right), \end{aligned} \tag{46}$$

where $E_{\text{vac}}^{(\text{C})}$ is the classical vacuum energy

$$\begin{aligned} E_{\text{vac}}^{(\text{C})} &= \sum_A \sqrt{m^2 + \sigma_A} \\ &= \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_D=-\infty}^{\infty} \sqrt{m^2 + \left(\frac{\pi n_1}{a}\right)^2 + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2} \end{aligned} \tag{47}$$

and $E_{\text{vac}}^{(\text{NC})}$ is the pure noncommutative vacuum energy

$$\begin{aligned} E_{\text{vac}}^{(\text{NC})} &= \frac{1}{8}\theta^2 \sum_A (m^2 + \sigma_A)^{3/2} \\ &= \frac{1}{8}\theta^2 \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{\pi n_1}{a}\right)^2 \right. \\ &\quad \left. + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{3/2}. \end{aligned} \tag{48}$$

The classical vacuum energy can be written as

$$E_{\text{vac}}^{(\text{C})} = \lim_{s \to 0} E(s) = E(0), \tag{49}$$

where the energy ζ-function $E(s)$ is given by

$$\begin{aligned} E(s) &= l^{-s} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{n_1 \pi}{a}\right)^2 \right. \\ &\quad \left. + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{(1-s)/2}. \end{aligned} \tag{50}$$

In the limit $L \to \infty$, we can replace the sums over $n_2, n_3, \ldots, n_D$ with integrals, so the energy ζ- function becomes

$$E(s) = l^{-s} \frac{V}{a} \sum_{n_1=1}^{\infty} \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} \left[\left(\frac{n_1 \pi}{a}\right)^2 + \vec{p}^2 + m^2 \right]^{(1-s)/2}. \tag{51}$$

Using the relations Eq. (B7)–(B13)) in Appendix B, the energy ζ-function, when $m \to 0$, can be written as

$$E(s) = l^{-s} \frac{V}{(4\pi)^{(D-1)/2} a} \left(\frac{\pi}{a}\right)^{D-s} \frac{\Gamma(\frac{s-D}{2})}{\Gamma(\frac{s-1}{2})} \zeta(s - D), \tag{52}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann ζ-function. Let us now consider the interesting case where $D = 3$, in this case the vacuum energy takes the form

$$E_{\text{vac}}^{(\text{C})} = E(0) = -\frac{\pi^2 V}{6a^4} \zeta(-3) = -\frac{\pi^2 A}{720 a^3}, \tag{53}$$

where $A = L_1 L_2 = L^2$ is the area of the parallel (uncharged conducting) plates. The noncommutative vacuum energy $E_{\text{vac}}^{(\text{NC})}$

$$\begin{aligned} E_{\text{vac}}^{(\text{NC})} &= \frac{1}{8}\theta^2 l^{-3s} \sum_{n_1=1}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_D=-\infty}^{\infty} \left[m^2 + \left(\frac{n_1 \pi}{a}\right)^2 \right. \\ &\quad \left. \left. + \sum_{k=2}^{D} \left(\frac{2\pi n_k}{L}\right)^2 \right]^{(3(1-s))/2} \right|_{s \to 0} \end{aligned} \tag{54}$$

can be written as (see Eq. (B14)–(B18) in Appendix B)

$$\begin{aligned} E_{\text{vac}}^{(\text{NC})} &= \frac{1}{8}\theta^2 l^{-3s} \frac{V}{a(4\pi)^{(D-1)/2}} \frac{\Gamma(\frac{(3s-D-2)}{2})}{\Gamma(\frac{3s-3}{2})} \\ &\quad \times \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{a}\right)^2 + m^2 \right]^{-((3s-D-2)/2)}. \end{aligned} \tag{55}$$

When $m \to 0$, the noncommutative vacuum energy $E_{\text{vac}}^{(\text{NC})}$ becomes

$$\begin{aligned} E_{\text{vac}}^{(\text{NC})} &= \frac{1}{8}\theta^2 \frac{V}{a(4\pi)^{(D-1)/2}} \left(\frac{\pi}{a}\right)^{D+2} \frac{\Gamma(\frac{(3s-D-2)}{2})}{\Gamma(\frac{3s-3}{2})} \\ &\quad \times \left(\frac{l\pi}{a}\right)^{-3s} \zeta(3s - D - 2). \end{aligned} \tag{56}$$

In the case where $D = 3$, $E_{\text{vac}}^{(\text{NC})}$ takes the form

$$E_{\text{vac}}^{(\text{NC})} = -\frac{1}{8}\theta^2 \frac{\pi^4 V}{4a^6} \frac{2}{5} \zeta(-5) = +\frac{1}{8}\theta^2 \frac{\pi^4 A}{2520 a^5}. \tag{57}$$

The total vacuum energy $E_{\text{vac}} = E_{\text{vac}}^{(\text{C})} + E_{\text{vac}}^{(\text{NC})}$, is given by

$$E_{\text{vac}} = E_{\text{vac}}^{(\text{C})} + E_{\text{vac}}^{(\text{NC})} = -\frac{\hbar c \pi^2 A}{720 a^3} \left(1 - \frac{\pi^2 \theta^2}{28 a^2}\right). \tag{58}$$

The Casimir force reads

$$F_{\mathrm{Casimir}} = -\frac{\partial E_{\mathrm{vac}}}{\partial a} = -\frac{\hbar c \pi^2 A}{240 a^6}\left(a^2 - \frac{5}{84}\pi^2\theta^2\right), \quad (59)$$

where the first term represents the classical attractive Casimir force, while the second term represents the noncommutative Casimir force, which is repulsive. From Eq. (59) we see that the total vacuum energy E_{vac} has a minimum at

$$a_{\min} = \sqrt{\frac{5}{84}}\pi\theta, \qquad \theta \neq 0. \quad (60)$$

At the equilibrium point $a_{\min}$, the total vacuum energy E_{vac} takes the value

$$E_{\mathrm{vac}}^{\min} = E_{\mathrm{vac}}(a_{\min}) = -\frac{\hbar c\pi^2 A}{720 a_{\min}^3}\left(1 - \frac{1}{28}\frac{\pi^2\theta^2}{a_{\min}^2}\right) = -(3.8497 \times 10^{-28}\ \mathrm{J\,m})\frac{A}{\theta^3}. \quad (61)$$

It is well-known that the motion near the equilibrium may be approximately described as harmonic oscillations, indeed near the equilibrium we may write $a = a_{\min} + \delta$, expanding the total vacuum energy E_{vac} in a Taylor series

$$E_{\mathrm{vac}}(a) = E_{\mathrm{vac}}(a_{\min}) + E'_{\mathrm{vac}}(a_{\min})\delta + \tfrac{1}{2}E'_{\mathrm{vac}}(a_{\min})\delta^2 + \cdots \quad (62)$$

we get

$$E_{\mathrm{vac}}(a) = -\frac{\hbar c\pi^2 A}{720 a^3}\left(1 - \frac{1}{28}\frac{\pi^2\theta^2}{a^2}\right) \simeq -\frac{\hbar c\pi^2 A}{1800 a_{\min}^3} + \frac{1}{2}\left(\frac{\hbar c\pi^2 A}{120 a_{\min}^5}\right)\delta^2. \quad (63)$$

Hence the equation of motion near the equilibrium may be derived from the following (harmonic oscillator) Lagrangian

$$L = \frac{1}{2}\rho A\dot{\delta}^2 - \frac{1}{2}\rho A\omega^2\delta^2, \quad (64)$$

where ρ is the density of the parallel plate, and ω is the angular frequency of vibration

$$\omega = \sqrt{\frac{\hbar c\pi^2}{120\rho a_{\min}^5}} = \frac{3.9499 \times 10^{-13}}{\sqrt{\rho}}\frac{1}{\theta^{5/2}}. \quad (65)$$

V. CONCLUSION

Throughout this work we have considered a noncommutative complex scalar field theory with self-interaction, by imposing noncommutativity to the canonical commutation relations. The noncommutative field equations are derived and solved, the vacuum energy is calculated to the second order in the parameter of noncommutativity. As an example, we have considered the Casimir effect, due to the zero-point fluctuations of the noncommutative complex scalar field. It turns out that in spite of its smallness, the noncommutativity gives rise to a repulsive force at the microscopic level, leading to an effective Casimir potential with a minimum at the point $a_{\min} = \sqrt{\frac{5}{84}}\pi\theta$.

APPENDIX A: MODE FUNCTIONS AND WRONSKIAN RELATIONS

The noncommutative field equations Eq. (24) and (25) may be written in the form

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]\varphi(x) = 0 \quad (A1)$$

where

$$\mathcal{A} = \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right]\mathbb{I} = \mathcal{A}^{\sim}$$
$$\mathcal{C} = \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right](m^2 - \vec{\nabla}^2)\mathbb{I} = \mathcal{C}^{\sim} \quad (A2)$$
$$\mathcal{B} = \theta(m^2 - \vec{\nabla}^2)\varepsilon = -\mathcal{B}^{\sim}$$

and $\mathbb{A}^{\sim}$ denotes the transpose of the operator $\mathbb{A}$.

To get the general solution of Eq. (A1) one begins by looking for solutions of the form [55]

$$u_A(t, \vec{x}) = \chi_A(\vec{x})e^{-i\omega_A t}\zeta_A \quad (A3)$$

known as mode functions, where χ_A are the eigenvectors of the operator $-\vec{\nabla}^2$ with eigenvalues σ_A

$$-\vec{\nabla}^2\chi_A(\vec{x}) = \sigma_A\chi_A(\vec{x}) \quad (A4)$$

and ζ_A are 2×1 constant columns.

Insertion of Eq. (A3) into Eq. (A1) leads to the eigenvector-eigenvalue problem

$$\left[\left(1 - \tfrac{1}{4}\theta^2\bar{\sigma}_A\right)(\bar{\sigma}_A - \omega_A^2) + i\theta\bar{\sigma}_A\varepsilon\omega_A\right]\zeta_A = 0, \quad (A5)$$

where we have used the abbreviation $\bar{\sigma}_A = m^2 + \sigma_A$.

This eigenvector-eigenvalue problem has a nontrivial solution if and only if the frequencies ω_A are roots of the equation

$$\det\left[\left(1 - \tfrac{1}{4}\theta^2\bar{\sigma}_A\right)(\bar{\sigma}_A - \omega_A^2) + i\theta\bar{\sigma}_A\varepsilon\omega_A\right] = 0, \quad (A6)$$

which can be written in the equivalent form

$$\left(1 - \tfrac{1}{4}\theta^2\bar{\sigma}_A\right)^2(\bar{\sigma}_A - \omega_A^2)^2 - \theta^2\bar{\sigma}_A^2\omega_A^2 = 0. \quad (A7)$$

Hence, the frequencies ω_A are the positive roots of the equations

$$\omega_A^2 \pm \theta\bar{\sigma}_A\omega_A - \bar{\sigma}_A = 0. \quad (A8)$$

The solutions are given by

$$\begin{aligned}\omega_A^{(+)} &= \tfrac{1}{2}[-\theta\bar{\sigma}_A + \sqrt{4\bar{\sigma}_A + \theta^2\bar{\sigma}_A^2}]\\ &\simeq \sqrt{\bar{\sigma}_A} - \tfrac{1}{2}\theta\bar{\sigma}_A + \tfrac{1}{8}\theta^2\bar{\sigma}_A^{3/2}\\ \omega_A^{(-)} &= \tfrac{1}{2}[+\theta\bar{\sigma}_A + \sqrt{4\bar{\sigma}_A + \theta^2\bar{\sigma}_A^2}]\\ &\simeq \sqrt{\bar{\sigma}_A} + \tfrac{1}{2}\theta\bar{\sigma}_A + \tfrac{1}{8}\theta^2\bar{\sigma}_A^{3/2}.\end{aligned} \tag{A9}$$

Because the mode functions

$$u_A^{(\pm)}(t,\vec{x}) = \chi_A(\vec{x})e^{-i\omega_A^{(\pm)}t}\zeta_A^{(\pm)} \tag{A10}$$

form a complete set the general solution of Eq. (A1) may be expanded in terms of them

$$\varphi(x) = \sum_A[u_A^{(+)}(x)a_A + \bar{u}_A^{(+)}(x)\bar{a}_A] + \sum_A[u_A^{(-)}(x)b_A + \bar{u}_A^{(-)}(x)\bar{b}_A] \tag{A11}$$

for some time-independent complex numbers a_A, b_A and their complex conjugates $\bar{a}_A$, $\bar{b}_A$, where $\bar{u}_A^{(\pm)}$ are the complex conjugates of the mode functions $u_A^{(\pm)}$. Starting from the equations satisfied by the mode functions $u_A^{(\pm)}$

$$\left[-\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}\right]u_A^{(\pm)}(x) = 0 \tag{A12}$$

one can see, after some algebraic operations [55], that these mode functions satisfy the Wronskian relations

$$\begin{aligned}-i\int d^3\vec{x}u_A^{(+)*}\overleftrightarrow{\mathbb{W}}u_B^{(+)} &= \delta_{AB},\\ +i\int d^3\vec{x}u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(+)} &= \delta_{AB},\\ -i\int d^3\vec{x}u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(+)} &= 0,\\ +i\int d^3\vec{x}u_A^{(+)*}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(+)} &= 0,\end{aligned} \tag{A13}$$

$$\begin{aligned}-i\int d^3\vec{x}u_A^{(-)*}\overleftrightarrow{\mathbb{W}}u_B^{(-)} &= \delta_{AB},\\ +i\int d^3\vec{x}u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(-)} &= \delta_{AB},\\ -i\int d^3\vec{x}u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(-)} &= 0,\\ +i\int d^3\vec{x}u_A^{(-)*}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(-)} &= 0,\end{aligned} \tag{A14}$$

$$\begin{aligned}-i\int d^3\vec{x}u_A^{(+)*}\overleftrightarrow{\mathbb{W}}u_B^{(-)} &= 0,\\ +i\int d^3\vec{x}u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(-)} &= 0,\\ -i\int d^3\vec{x}u_A^{(-)*}\overleftrightarrow{\mathbb{W}}u_B^{(+)} &= 0,\\ +i\int d^3\vec{x}u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(+)} &= 0,\end{aligned} \tag{A15}$$

$$\begin{aligned}-i\int d^3\vec{x}u_A^{(+)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(-)} &= 0,\\ +i\int d^3\vec{x}u_A^{(+)*}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(-)} &= 0,\\ -i\int d^3\vec{x}u_A^{(-)\sim}\overleftrightarrow{\mathbb{W}}u_B^{(+)} &= 0,\\ +i\int d^3\vec{x}u_A^{(-)*}\overleftrightarrow{\mathbb{W}}\bar{u}_B^{(+)} &= 0,\end{aligned} \tag{A16}$$

where

$$\overleftrightarrow{\mathbb{W}}(x) = -\mathcal{A}(x)\frac{\vec{\partial}}{\partial t} + \mathcal{A}(x)\frac{\overleftarrow{\partial}}{\partial t} + \mathcal{B}(x) \tag{A17}$$

is the Wronskian operator corresponding to the differential operator [55]

$$\mathbb{F} = -\mathcal{A}\frac{\partial^2}{\partial t^2} + \mathcal{B}\frac{\partial}{\partial t} - \mathcal{C}. \tag{A18}$$

The Wronskian operator $\overleftrightarrow{\mathbb{W}}$ has the following symmetry and reality properties:

$$\overleftrightarrow{\mathbb{W}}^{\sim} = -\overleftrightarrow{\mathbb{W}}, \qquad \overleftrightarrow{\mathbb{W}}^{*} = -\overleftrightarrow{\mathbb{W}}. \tag{A19}$$

Here $\bar{\mathcal{O}}$, $\mathcal{O}^*$ and $\mathcal{O}^{\sim}$ denote the complex conjugate, the Hermitian conjugate and the transpose of the matrix (or the operator) $\mathcal{O}$, respectively.

In order that these Wronskian relations must hold, the operators $\mathcal{A}$ and $\mathcal{C}$ must be positive definite operators, but the eigenvalues of the operators $\mathcal{A}$ and $\mathcal{C}$ are given by

$$\begin{aligned}\mathcal{A}u_A^{(\pm)}(x) &= \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right]u_A^{(\pm)}(x)\\ &= \left(1 - \tfrac{1}{4}\theta^2\bar{\sigma}_A\right)u_A^{(\pm)}(x)\\ \mathcal{C}u_A^{(\pm)}(x) &= \left[1 - \tfrac{1}{4}\theta^2(m^2 - \vec{\nabla}^2)\right](m^2 - \vec{\nabla}^2)u_A^{(\pm)}(x)\\ &= \left(1 - \tfrac{1}{4}\theta^2\bar{\sigma}_A\right)\bar{\sigma}_A u_A^{(\pm)}(x)\end{aligned} \tag{A20}$$

so these eigenvalues are not positive for all indices A; to solve this problem, we use the fact that $\theta \sim 10^{-13}$ m [35–37,67], so $(1 - \frac{1}{4}\theta^2\bar{\sigma}_A) > 0$ for all indices A such that $\bar{\sigma}_A < \frac{4}{\theta^2} \sim 10^{26}$. To make the spectrum of the operators $\mathcal{A}$ and $\mathcal{C}$ bounded, we impose the following boundary conditions on the eigenfunctions $\chi_A(\vec{x})$ of the operator $-\vec{\nabla}^2$

$$\left|\frac{\partial}{\partial x_j}\chi_A(x_1,\ldots,x_j\ldots,x_D)\right|_{\vec{x}=\vec{a}}\leq\frac{\alpha}{\theta}\qquad j=1,2,\ldots,D \tag{A21}$$

at some arbitrary point $\vec{x}=\vec{a}$, and α is some constant with dimension $(\text{length})^{-(3/2)}$. Note that in the classical limit where $\theta\longrightarrow 0$ this condition is trivially satisfied.

As an example, we consider the free scalar field confined in a D-dimensional rectangular box of volume $V=L^D$ and impose periodic boundary conditions on the walls of the box, the normalized eigenfunctions $\chi_A(\vec{x})$ of the operator $-\vec{\nabla}^2$, are [59]

$$\sqrt{\frac{1}{V}}\exp\left[\sum_{k=1}^{D}\frac{2\pi i n_k}{L}x_k\right]\quad\text{with}\quad n_k=0,\pm1,\pm2,\ldots,$$

for each $k=1,2,\ldots,D$. (A22)

In this case the boundary conditions Eq. (A21) read

$$\left|\frac{2\pi n_j}{L}\right|\leq\frac{\alpha\sqrt{V}}{\theta}\qquad j=1,2,\ldots,D. \tag{A23}$$

If we choose $\alpha=\frac{1}{\sqrt{DV}}$ we get

$$\begin{aligned}\frac{1}{4}\theta^2\bar{\sigma}_A&=\frac{1}{4}\theta^2\left[m^2+\sum_{k=1}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right]\\&\leq\frac{1}{4}\theta^2m^2+\frac{\alpha^2DV}{4}<1,\end{aligned} \tag{A24}$$

where we have used the fact that θ is an infinitesimal parameter such that $\theta^2m^2<1$. Hence $\mathcal{A}$ and $\mathcal{C}$ are positive definite operators.

As a second example, we consider the free scalar field confined in a D-dimensional rectangular box of volume $V=L^D$ and impose Dirichlet boundary conditions on the walls of the box, the normalized eigenvectors χ_A of $-\vec{\nabla}^2$ with Dirichlet boundary conditions on the walls of the box

$$\begin{aligned}&\chi_A(0,x_2,x_3,\ldots,x_D)=\chi_A(L,x_2,x_3,\ldots,x_D)=0\\&\chi_A(x_1,\ldots,x_{k-1},0,x_{k+1},\ldots,x_D)\\&\quad=\chi_A(x_1,\ldots,x_{k-1},L,x_{k+1},\ldots,x_D),\qquad k=2,\ldots,D\end{aligned} \tag{A25}$$

are given by [59]

$$\begin{aligned}&-\vec{\nabla}^2\chi_A(\vec{x})=\sigma_A\chi_A(\vec{x})\\&\chi_A(\vec{x})=\sqrt{\frac{2}{V}}\sin\left(\frac{\pi n_1}{L}x_1\right)\exp\left[\sum_{k=2}^{D}\frac{2\pi i n_k}{L}x_k\right]\end{aligned} \tag{A26}$$

with $n_1=1,2,\ldots$ and $n_k=0,\pm1,\pm2,\ldots$ for $k=2,3,\ldots,D$.

The eigenvalues are given by

$$\sigma_A\equiv\sigma_{n_1n_2\ldots n_D}=\left(\frac{\pi n_1}{L}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2. \tag{A27}$$

In this case, the boundary conditions Eq. (A21) read

$$\begin{aligned}&\left|\frac{\partial}{\partial x_j}\chi_A(x_1,\ldots,x_j\ldots,x_D)\right|_{\vec{x}=\vec{a}}\\&\quad=\sqrt{\frac{2}{V}}\left|\sin\left(\frac{\pi n_1}{L}a_1\right)\right|\left|\frac{2\pi n_j}{L}\right|\leq\frac{\alpha}{\theta},\qquad j=2,\ldots,D\end{aligned} \tag{A28}$$

$$\begin{aligned}&\left|\frac{\partial}{\partial x_1}\chi_A(x_1,\ldots,x_j\ldots,x_D)\right|_{\vec{x}=\vec{a}}\\&\quad=\sqrt{\frac{2}{V}}\left|\frac{n_1\pi}{L}\cos\left(\frac{\pi n_1}{L}a_1\right)\right|\leq\frac{\alpha}{\theta},\end{aligned} \tag{A29}$$

leading to the constraints

$$\left|\frac{2\pi n_j}{L}\right|\leq\frac{1}{|\sin(\frac{\pi n_1}{L}a_1)|}\frac{\alpha\sqrt{V}}{\sqrt{2}\theta},\quad\frac{L}{a_1}\notin\mathbb{N},\quad j=1,2,\ldots,D \tag{A30}$$

$$\left|\frac{n_1\pi}{L}\right|\leq\frac{1}{|\cos(\frac{\pi n_1}{L}a_1)|}\frac{\alpha\sqrt{V}}{\sqrt{2}\theta},\qquad\frac{L}{a_1}\notin\mathbb{N}. \tag{A31}$$

If we choose $\alpha=\sqrt{\frac{a_1}{L(D-1)V}}$ and $a_1\approx 0$, we get

$$\begin{aligned}\frac{1}{4}\theta^2\bar{\sigma}_A&=\frac{1}{4}\theta^2\left[m^2+\left(\frac{\pi n_1}{L}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right]\\&\leq\frac{1}{4}\theta^2m^2+\frac{(D-1)L}{a_1}\frac{\alpha^2V}{8\pi}<1,\end{aligned} \tag{A32}$$

where we have used the fact that θ is an infinitesimal parameter such that $\theta^2m^2<1$. Hence $\mathcal{A}$ and $\mathcal{C}$ are positive definite operators.

APPENDIX B: ZETA FUNCTION REGULARIZATION

The noncommutative vacuum energy E_{vac} is given by

$$\begin{aligned}E_{\text{vac}}&=E_{\text{vac}}^{(\text{C})}+E_{\text{vac}}^{(\text{NC})}\\&=\sum_A\left(\sqrt{m^2+\sigma_A}+\frac{1}{8}\theta^2(m^2+\sigma_A)^{3/2}\right),\end{aligned} \tag{B1}$$

where $E_{\text{vac}}^{(\text{C})}$ is the classical vacuum energy

$$E_{\rm vac}^{\rm (C)} = \sum_A \sqrt{m^2+\sigma_A} = \sum_{n_1=1}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_D=-\infty}^{\infty}\sqrt{m^2+\left(\frac{\pi n_1}{a}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2} \tag{B2}$$

and $E_{\rm vac}^{\rm (NC)}$ is the pure noncommutative vacuum energy

$$E_{\rm vac}^{\rm (NC)} = \frac{1}{8}\theta^2\sum_{n_1=1}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_D=-\infty}^{\infty}\left[m^2+\left(\frac{\pi n_1}{a}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right]^{3/2}. \tag{B3}$$

To deal with the infinite sum of zero-point energies in Eq. (B2) and (B3), we must introduce a regularization method to extract finite expression [56,59,68,69]. One elegant way of doing this is to use ζ- function regularization [59]; the idea of the method is to define the divergent sum $\sum_A E_A$ over zero-point energies in Eq. (B2) and (B3) by the analytic continuation of a convergent sum. First, we consider the infinite sum in Eq. (B2), we define the energy ζ- function by [59]

$$E(s) = \sum_A E_A (lE_A)^{-s}, \tag{B4}$$

where $E_A = \sqrt{m^2+\sigma_A}$, s is a complex variable and l is a constant with units of length, introduced to keep (lE_A) dimensionless. This ensures that $E(s)$ has dimensions of energy for all values of s.

The classical vacuum energy can be written as

$$E_{\rm vac}^{\rm (C)} = \lim_{s\to 0} E(s) = E(0), \tag{B5}$$

where the energy ζ- function $E(s)$ is given by

$$E(s) = l^{-s}\sum_{n_1=1}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_D=-\infty}^{\infty}\left[m^2+\left(\frac{n_1\pi}{a}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right]^{(1-s)/2}. \tag{B6}$$

In the limit $L\to\infty$, we can replace the sums over $n_2, n_3, \ldots, n_D$ with integrals, so the energy ζ- function becomes

$$E(s) = l^{-s}\frac{V}{a}\sum_{n_1=1}^{\infty}\int\frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\left[\left(\frac{n_1\pi}{a}\right)^2+\vec{p}^2+m^2\right]^{(1-s)/2}. \tag{B7}$$

Using the identity

$$a^{-z} = \frac{1}{\Gamma(z)}\int_0^{\infty} dt t^{z-1}e^{-at}, \tag{B8}$$

which holds for $\mathrm{Re}(z)>0$ and $\mathrm{Re}(a)>0$, where $\Gamma(z)$ is Gamma function

$$\Gamma(z) = \int_0^{\infty} dt t^{z-1}e^{-t} \tag{B9}$$

defined for $\mathrm{Re}(z)>0$, we obtain the following expression for the energy ζ- function

$$E(s) = l^{-s}\frac{V}{a}\sum_{n_1=1}^{\infty}\frac{1}{\Gamma(\frac{s-1}{2})}\int_0^{\infty} dt t^{(s-3)/2} \times\exp\left(-\left[\left(\frac{n_1\pi}{a}\right)^2+m^2\right]t\right)\int\frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}}\exp(-\vec{p}^2 t).$$

The integration over the $(D-1)$- dimensional momentum integral on the right-hand side can be performed with the help of the relations [56,59,70,71]

$$\int d^n q f(q^2) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^{+\infty} dk k^{n-1} f(q^2) \tag{B10}$$

and

$$\int_0^{+\infty} dt t^{2s-1}e^{-\alpha t^2} = \frac{\alpha^{-s}}{2}\Gamma(s) \tag{B11}$$

with the results

$$E(s) = l^{-s}\frac{V}{(4\pi)^{(D-1)/2}a}\frac{\Gamma(\frac{s-D}{2})}{\Gamma(\frac{s-1}{2})}\sum_{n_1=1}^{\infty}\left[\left(\frac{n_1\pi}{a}\right)^2+m^2\right]^{(D-s)/2}. \tag{B12}$$

When $m\to 0$, the energy ζ- function becomes

$$E(s) = l^{-s}\frac{V}{(4\pi)^{(D-1)/2}a}\left(\frac{\pi}{a}\right)^{D-s}\frac{\Gamma(\frac{s-D}{2})}{\Gamma(\frac{s-1}{2})}\zeta(s-D), \tag{B13}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann ζ- function.

By the same steps we will now calculate the noncommutative vacuum energy $E_{\rm vac}^{\rm (NC)}$, let $\mathcal{E}(s)$ be the energy ζ- function

$$\mathcal{E}(s) = l^{-3s}\sum_{n_1=1}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_D=-\infty}^{\infty}\left[m^2+\left(\frac{n_1\pi}{a}\right)^2+\sum_{k=2}^{D}\left(\frac{2\pi n_k}{L}\right)^2\right]^{(3(1-s))/2}. \tag{B14}$$

Then

$$E_{\rm vac}^{\rm (NC)} = \lim_{s\to 0}\mathcal{E}(s) = \mathcal{E}(0). \tag{B15}$$

In the limit $L \to \infty$, we can replace the sums over $n_2, n_3, \ldots, n_D$ with integrals, so the energy ζ- function becomes

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a} \sum_{n=1}^{\infty} \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} \left[\left(\frac{n\pi}{a} \right)^2 + \vec{p}^2 + m^2 \right]^{(3(1-s))/2}. \tag{B16}$$

Using the relation (B8), one gets

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a} \sum_{n=1}^{\infty} \frac{1}{\Gamma(\frac{3}{2}(s-1))} \times \int_0^{\infty} dt t^{(3s-5)/2} \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}} e^{-[(n\pi/a)^2 + \vec{p}^2 + m^2]t}. \tag{B17}$$

The integration over the $(D-1)$- dimensional momentum integral on the right-hand side can be performed with the help of the relations (B10) and (B11), one finds

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a(4\pi)^{(D-1)/2}} \frac{1}{\Gamma(\frac{3}{2}(s-1))} \times \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{a} \right)^2 + m^2 \right]^{-((3s-D-2)/2)} \times \int_0^{\infty} dt t^{((3s-D-2)/2)-1} e^{-t}.$$

Using Eq. (B9) to perform the integration over t, we get

$$\mathcal{E}(s) = l^{-3s} \frac{V}{a(4\pi)^{(D-1)/2}} \frac{\Gamma(\frac{(3s-D-2)}{2})}{\Gamma(\frac{3s-3}{2})} \times \sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{a} \right)^2 + m^2 \right]^{-((3s-D-2)/2)}. \tag{B18}$$

When $m \to 0$, the energy ζ- function becomes

$$\mathcal{E}(s) = \frac{V}{a(4\pi)^{(D-1)/2}} \left(\frac{\pi}{a} \right)^{D+2} \frac{\Gamma(\frac{(3s-D-2)}{2})}{\Gamma(\frac{3s-3}{2})} \times \left(\frac{l\pi}{a} \right)^{-3s} \zeta(3s-D-2). \tag{B19}$$

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