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Supersymmetric fluid dynamics

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Recently, Navier-Stokes equations have been derived from the duality between the black branes and a conformal fluid on the boundary of anti-de Sitter$_5$. Nevertheless, the full correspondence has to be established between solutions of supergravity in anti-de Sitter$_5$ and supersymmetric field theories on the boundary. That prompts the construction of Navier-Stokes equations for a supersymmetric fluid. In the framework of rigid supersymmetry, there are several possibilities and we propose one candidate. We deduce the equations of motion in two ways: both from the divergenceless condition on the energy-momentum tensor and by a suitable parametrization of the auxiliary fields. We give the complete component expansion and a very preliminary analysis of the physics of this supersymmetric fluid.

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I. INTRODUCTION

In the recent literature, the duality between gravity on manifolds with boundary and conformal fluids on the boundary has been intensively studied [1–10]. It has been shown that given a black hole/brane solution with the corresponding set of zero modes one can derive the Navier-Stokes equations for the conformal fluid. That relation also permits the computation of the fluid dynamic coefficients beyond the perturbation theory in the case of strongly interacting theory. Several examples have been carefully studied and analyzed confirming different ideas in the AdS/CFT correspondence and using the membrane paradigm of black holes.

Nonetheless, the complete duality can be only rigorously established between supergravity in the bulk and superconformal fluid on the boundary. With this idea in mind, the authors of the present paper [11] have considered the fermionic zero modes of a supersymmetric black hole in $d = 5$, $N = 2$ supergravity in anti-de Sitter$_5$ (AdS$_5$) background and computed the corrections to the fluid dynamics due to fermionic bilinears (currents built in terms of a fermion field) on the boundary. Those corrections modify the Navier-Stokes equations governing the dynamics of the boundary theory and the complete set of these corrections can be parametrized by an effective action. For that purpose, here we propose a supersymmetry generalization of the usual Navier-Stokes equations to take into account the corrections due to the fermionic degrees of freedom.

The construction of supersymmetric Lagrangian leading to Navier-Stokes equations has been discussed in the literature. We have to recall works [12,13] where a possible supersymmetric action has been proposed. There, the bosonic degrees of freedom are parametrized by a conserved current j^μ, the dynamics is encoded into a function $f(j^2)$, and the corresponding equations of motion are obtained with the help of an auxiliary field a_μ coupled to the current. In such supersymmetric generalization, both the current j^μ and the auxiliary field a_μ are embedded into two distinct real superfields, V and A, whose lowest components are two scalar fields. The function $f(j^2)$ is replaced by a function $F(V)$ of the real superfield V. Expanding the action, we find that it cannot describe a generic fluid whose dynamics is described by the function $f(j^2)$, namely, it does not reduce to any generic bosonic models, but only to specific ones. On the other side, works [14–18] lead to generic supersymmetric models in lower dimensions and we have not been able to adapt them to our scopes. That noncovariant approach in lower dimensions seems to be suitable to study the AdS/condensed matter correspondence.

In contrast to [12,13], we observed that the conserved current can be better viewed as the middle component of a real linear superfield J. The linearity of that superfield implies the conservation of the current and it does not contain auxiliary fields [19–22]. To overcome the problem of describing a generic model reducing to any bosonic Navier-Strokes system, we constructed a derived superfield $\mathcal{J}_\mu$ which is a linear, real vector superfield and a linear function of J.

As mentioned above, the equations of motion of the fluid, namely, the Navier-Stokes equations, are derived with the help of an auxiliary field a_μ. However, in [12,13], a_μ has been replaced with a Kähler potential implementing the so-called Clebsch parametrization (see also [18,23,24] for a complete discussion on the Clebsch parametrization) in a very convenient way for supersymmetric generalizations. We show in detail that the two choices, namely, the conventional Clebsch parametrization and the use of the Kähler potential, are indeed equivalent

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locally. The origin of that potential has to be traced out into supergravity models as advocated in [25], and for a forthcoming analysis in a generic supergravity background, we adopt it in the present work. It is worth mentioning also the discussion in [26]. Finally, in terms of J, of the derived superfield $\mathcal{J}_\mu$ we are able to provide a general action whose bosonic truncation leads to any generic bosonic fluid.

One important issue is the dependence of the Kähler potential. We provide an argument showing that the choice of the Kähler potential does not affect the physics, but we are convinced that the implementation of local supersymmetry invariance coupling it to supergravity might clarify this issue.

We provide the complete Lagrangian by expanding the superfields in components and integrating over the θ's. Because of this expansion, the number of possible terms increases and the Lagrangian is really cumbersome. In order to grasp the meaning of it, we derive the superfield equations of motion and we compute their bosonic sector. The energy-momentum tensor for the Lagrangian restricted to the physical field C (the lowest component of the superfield J) is computed and some considerations are proposed.

The plan of the paper is the following: in Sec. II, we review the derivation of Navier-Stokes (NS) equations for the purely bosonic model. In particular, in II A two different methods to compute them are compared: the divergenceless condition on the energy-momentum tensor, and the invariance of the action under certain isometries; in II B Clebsch, parametrization of the vector field a_μ is considered. In Sec. III, the supersymmetric completion of the previous model is taken into account, the action is constructed, and explicit results are given for the bosonic sector. In III B, the supersymmetric generalization of the Clebsch parametrization is built, and the coupling to the linear multiplet J is written. In Sec. III E, we consider the limit $C = 0$ and we discuss the first two terms of the expansion of the action in order to compare it with the result of the fluid/gravity correspondence [11] where the contributions of the fermionic bilinears to the NS equations are computed. In III F, the issue of Kähler potential and its appearance in the equations of motion is discussed. Finally, in Appendix B the complete supersymmetric Lagrangian is presented.

II. BOSONIC LAGRANGIAN

A. Action and equations of motion

We first discuss the bosonic Lagrangian and we derive the equations of motion. The model is characterized by a divergenceless current j^μ and an auxiliary field a_μ coupled to a world-volume metric $g_{\mu\nu}$. The gauge invariance under $a_\mu \to a_\mu + \partial_\mu \lambda$ is guaranteed by the conservation of j^μ. The model is considered in four dimensions. There are two ways to get the equations of motion: the first one is by computing the energy-momentum tensor $T^{\mu\nu}$ and requiring the vanishing of its divergence. The second method is requiring the invariance of the action under certain isometries.

Let the action be

$$\mathcal{L} = \sqrt{-g}(j^\mu a_\mu + f(j^2)), \qquad j^2 = j^\mu j^\nu g_{\mu\nu}. \tag{2.1}$$

Note that the equation of motion obtained by taking the functional derivative with respect to (w.r.t.) an unconstrained a_μ yields $j^\mu = 0$. The function f is completely generic. Therefore, the correct equations of motion are obtained as follows: varying w.r.t. j^μ and $g_{\mu\nu}$ leads to

$$\begin{aligned} a_\mu &= -2f'(j^2)j_\mu, \\ T^{\mu\nu} &= f'(j^2)(j^\mu j^\nu - g^{\mu\nu}j^2) + \tfrac{1}{2}f(j^2)g^{\mu\nu}, \end{aligned} \tag{2.2}$$

and the vanishing of the divergence of energy-momentum tensor implies

$$\begin{aligned} \partial^\mu T_{\mu\nu} = 0 \to j^\mu[&f''(j^2)(j_\mu \partial_\nu j^2 - j_\nu \partial_\mu j^2) \\ &+ f'(j^2)(\partial_\nu j_\mu - \partial_\mu j_\nu)] \\ = 0.& \end{aligned} \tag{2.3}$$

These are the usual NS equations which, together with the conservation of the current j^μ, yield the complete information on the fluid dynamics.

Since we are primarily interested in AdS/CFT correspondence, we recall that the fluid on the dual side must be a conformal one. That forces $f(j^2)$ to be equal to $C(j^2)^{2/3}$, where C is a constant. This can be obtained by imposing the tracelessness of $T^{\mu\nu}$ or by studying the dilatation properties of the action, assuming that j^μ has dimension 3 in four dimensions.

Notice that Eq. (2.3) can also be obtained in the following way: consider the field-strength associated to the Abelian vector a_μ, $F_{\mu\nu} = (\partial_\mu a_\nu - \partial_\nu a_\mu)$; using the first of (2.2) into F and upon contraction with j^μ, we get

$$j^\mu F_{\mu\nu} = \partial^\mu T_{\mu\nu} = 0. \tag{2.4}$$

It should be noticed that, in both ways, the auxiliary field a_μ drops off the equations.

Equation (2.4) calls for an explanation. First of all, we observe that, j^μ being a divergenceless current, action (2.1) is invariant under the gauge symmetry $\delta a_\mu = \partial_\mu \lambda$. Let us perform an isometry transformation leaving the current j^μ invariant. In the form language, given $\mathcal{A} = a_\mu dx^\mu$, $\mathcal{J} = j^\mu \partial_\mu$, and $\mathcal{X} = X^\mu \partial_\mu$, we have

$$\begin{aligned} \mathcal{L}_{\mathcal{X}}(\mathcal{A}) &= \iota_{\mathcal{X}} d\mathcal{A} + d(\iota_{\mathcal{X}}\mathcal{A}), \\ \mathcal{L}_{\mathcal{X}}(\mathcal{J}) &= [\mathcal{X}, \mathcal{J}] = 0, \\ \mathcal{L}_{\mathcal{X}}(g) &= (\nabla_\mu X_\nu + \nabla_\nu X_\mu)dx^m \otimes dx^\nu, \end{aligned} \tag{2.5}$$

and in components

$$\delta a_\mu = -F_{\mu\nu}X^\nu + \partial_\mu(a_\nu X^\nu), \qquad \delta j_\mu = 0,$$
$$\delta g_{\mu\nu} = g_{\mu\rho}\partial_\nu X^\rho + g_{\nu\rho}\partial_\nu X^\rho + X^\rho\partial_\rho g_{\mu\nu} = 0, \tag{2.6}$$

where X^μ are the components of the Killing vector generating the isometry commuting with the current $\mathcal{J}$. Requiring the invariance of the action under such an isometry, one gets Eq. (2.4).

The condition $\delta j^\mu = 0$ (if $g_{\mu\nu} = \eta_{\mu\nu}$) can be reformulated as follows: given the vector field $X = X^\mu\partial_\mu$, the infinitesimal variation of j^μ can be expressed as

$$\delta j^\mu = X^\nu\partial_\nu j^\mu - j^\nu\partial_\nu X^\mu, \tag{2.7}$$

where the first term is a translation parametrized by the coefficients X^ν and the second term is a rotation with the parameter $\Lambda_{\mu\nu} = \frac{1}{2}(\partial_\mu X_\nu - \partial_\nu X_\mu)$ due to Killing equation in (2.6). Condition (2.7) can be rewritten as follows:

$$\Delta_X j^\mu \equiv X^\nu\partial_\nu j^\mu = \Lambda^\mu_\rho j^\rho, \tag{2.8}$$

which implies that the translation of the current j^μ is compensated by a rotation. In the same way, the variation of a_μ can be cast in the form

$$\delta a_\mu = \Delta_X a_\mu + R_\mu{}^\nu a_\nu \equiv X^\nu\partial_\nu a_\mu + \Lambda_\mu{}^\rho a_\rho. \tag{2.9}$$

Then, computing the variation of the action under a translation, we have

$$\begin{aligned}\Delta_X S &= \int(\Delta_X j^\mu a_\mu + j^\mu\Delta_X a_\mu + \Delta_X f(j^2))\\ &= \int(\Lambda_\mu{}^\nu j^\mu a_\nu + j^\mu X^\nu\partial_\nu a_\mu) = \int(j^\mu\delta a_\mu)\\ &= \int(j^\mu(-F_{\mu\nu}X^\nu + \partial_\mu(a_\rho X^\rho))).\end{aligned} \tag{2.10}$$

In the first line, we have used Eq. (2.8) and the Lorentz invariance of $f(j^2)$. From the second line to the third line, we have used the definition of the variation of the gauge potential a_μ under isometry (2.6) combined with a gauge variation. Thus, the second term vanishes because j^μ is divergenceless and from the first term, comparing with the definition of the energy-momentum tensor obtained by the Nöther theorem $\Delta_X S = \int X^\mu\partial^\nu T_{\mu\nu}$, it yields

$$j^\mu F_{\mu\nu} = \partial^\mu T_{\mu\nu} = 0. \tag{2.11}$$

As a consistency condition, we must have $j^\nu\partial^\mu T_{\mu\nu} = 0$, which can be easily verified using its explicit form (2.3).

B. Clebsch parametrization of a_μ

One may wonder why we adopt the above derivation of NS equations instead of computing directly the equations of motion by functional derivatives. Actually, it is possible to obtain them by means of variational principles, considering the auxiliary field a_μ as parametrized by a set of potentials. Moreover, since a_μ is an auxiliary field we have to avoid any nontrivial solution for it, then we impose the constraint

$$F \wedge F = 0, \tag{2.12}$$

where $F = d\mathcal{A}$ ($\mathcal{A} \equiv a_\mu dx^\mu$) which, in components, becomes $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 0$. This constraint is equivalent to $\mathcal{A} \wedge F = d\Omega$ where Ω is a generic 2-form. It can be easily shown [23] that the most general solution in four dimensions to (2.12) is

$$\mathcal{A} = d\lambda + \alpha d\beta, \tag{2.13}$$

where λ, α, and β are zero forms. This implies that $F = d\alpha \wedge d\beta$ and the constraint (2.12) follows immediately. This means that out of the four components of a_μ only 3 of them survive the constraint and inserting them in the Lagrangian (2.1) we get

$$\mathcal{L} = (j^\mu(\partial_\mu\lambda + \alpha\partial_\mu\beta) + f(j^2)). \tag{2.14}$$

The equations of motion are

$$\partial_\mu j^\mu = 0, \qquad j^\mu\partial_\mu\beta = 0, \qquad j^\mu\partial_\mu\alpha = 0,$$
$$\partial_\mu\lambda + \alpha\partial_\mu\beta + 2j_\mu f'(j^2) = 0. \tag{2.15}$$

With simple algebraic manipulations, one derives NS Eq. (2.4).

There is another way to parametrize the solution of (2.12). Introducing one complex field ϕ and a real function $K(\phi, \bar\phi)$, consequently a_μ becomes

$$a_\mu = \partial_\mu\lambda + i(\partial K\partial_\mu\phi - \bar\partial K\partial_\mu\bar\phi). \tag{2.16}$$

If K is identified with a Kähler potential for the complex manifold spanned by ϕ, the second term in a_μ is the Kähler connection. Computing the field strength F, we get

$$F = -2i\partial\bar\partial K d\phi \wedge d\bar\phi. \tag{2.17}$$

Namely, the manifold is a Hodge manifold where the $U(1)$ connection is related to the canonical 2-form of the complex manifold. By the Bianchi identity, it follows that the canonical 2-form $2i\partial\bar\partial K d\phi \wedge d\bar\phi$ must be closed and therefore the space is Kähler. Notice that for a one-dimensional complex manifold, no constraint on K is due to its closure.

The two parametrizations (2.13) and (2.16) are equivalent. This can be verified by assuming that α and β are real functions of ϕ and $\bar\phi$. It yields

$$\alpha\partial\beta = i\partial K, \qquad \alpha\bar\partial\beta = -i\bar\partial K. \tag{2.18}$$

By dividing both equations by α and by computing the derivative, we get

$$2\partial\bar\partial K = (\partial K\bar\partial + \bar\partial K\partial)\ln\alpha. \tag{2.19}$$

This equation can be brought to quadrature. For example, assuming that α and K are functions of the modulus $|\phi|^2$, one can easily bring the above equation to an integral form. If $K(\phi, \bar\phi) = |\phi|^2$, then we get $\alpha = |\phi|^2$ and $\beta = i\ln(\phi/\bar\phi)$. On the other hand, if $K(\phi, \bar\phi) = \ln(1 + |\phi|^2)$, then we get $\alpha = |\phi|^2/(1 + |\phi|^2)$ and $\beta = i\ln(\phi/\bar{\phi})$. See also [18] for a discussion on this point.

III. SUPERSYMMETRIC LAGRANGIAN

A. Superfields, action, and superfield expansion

We are now ready for the supersymmetrized version of the Lagrangian. We first construct the action reproducing the usual bosonic action (2.1) in the limit in which the fermions and the additional bosonic field are set to zero. A conserved current is a component of a linear multiplet in four dimensions and therefore we introduce a superfield J for it. The auxiliary field a_μ is a component of the vector multiplet and we introduce a real superfield A. Again, we face with the problem of deriving the equations of motion since the superfield A is constrained and, for that, we adopt a Clebsch parametrization. In the present case, it becomes natural to identify the Abelian real superfield A with a Kähler potential [12] which is a real function of a chiral superfield ϕ.

J and A are defined as follows:[1]

$$\bar{D}DJ = 0, \qquad \bar{A} = A, \tag{3.1}$$

where $D = -\frac{\partial}{\partial\bar{\theta}} - (\gamma^\mu\theta)\partial_\mu$ and $\bar{D} = \frac{\partial}{\partial\theta} + (\gamma^\mu\bar{\theta})\partial_\mu$ are the superderivatives. Using a linear superfield J, we automatically implement the conservation of the current j^μ which is its θ^2 component. The component expansion is given by

$$J = C - i\bar{\theta}\gamma_5\omega + \frac{i}{2}\bar{\theta}\gamma_5\gamma_\mu\theta j^\mu + \frac{i}{2}\bar{\theta}\gamma_5\theta\bar{\theta}\gamma^\mu\partial_\mu\omega + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box C, \tag{3.2}$$

and for the real superfield in the Wess-Zumino gauge

$$A = \frac{i}{2}\bar{\theta}\gamma_5\gamma^\mu\theta a_\mu - i\bar{\theta}\gamma_5\theta\bar{\theta}\lambda - \frac{1}{4}(\bar{\theta}\gamma_5\theta)^2 D. \tag{3.3}$$

The linear superfield contains one constrained vector j^μ, one scalar field C, and one Majorana spinor ω. The vector can be dualized as $j^\mu = \epsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$ where $H_{\mu\nu\rho}$ is the field strength of a 2-form potential $B_{\mu\nu}$. The latter can be further dualized into a scalar and therefore the linear multiplet has the same degree of freedom of an on-shell Wess-Zumino multiplet.

Supersymmetry transformations are given by $\delta\Phi = \bar{\alpha}Q\Phi$ or, in component,

$$\delta j^\mu = -\bar{\alpha}\gamma^{\mu\nu}\partial_\nu\omega, \qquad \delta a_\mu = \bar{\alpha}\gamma_\mu\lambda,$$
$$\delta\omega = (-i\gamma_5\gamma^\mu\partial_\mu C + \gamma_\mu j^\mu)\alpha,$$
$$\delta\lambda = -(iD\gamma_5 + F_{\mu\nu}\gamma^{\mu\nu})\alpha, \qquad \delta C = i\bar{\alpha}\gamma_5\omega,$$
$$\delta D = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda.$$

Using the properties listed in Appendix A, it is possible to show that

$$\int d^4x \int d^4\theta[-JA] = \int d^4x[j^\mu a_\mu + \bar{\omega}\lambda - CD], \tag{3.4}$$

which is the supersymmetric generalization of (2.1). In order to reproduce also the second term in (2.1), we need to introduce a new superfield defined as

$$\mathcal{J}_\mu = \frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J, \tag{3.5}$$

which contains j^μ as the first component and its expansion is

$$\begin{aligned}\mathcal{J}_\mu &= \frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J \\ &= j_\mu + \bar{\theta}\gamma_{\mu\nu}\partial^\nu\omega - \frac{i}{2}\bar{\theta}\gamma_5\gamma^\nu\theta(\partial_\mu\partial_\nu C - g_{\mu\nu}\Box C) \\ &\quad - \frac{1}{2}\bar{\theta}\gamma_5\theta\bar{\theta}\gamma_5\gamma^\nu(g_{\mu\nu}\Box\omega - \partial_\mu\partial_\nu\omega) \\ &\quad + \frac{1}{8}(\bar{\theta}\gamma_5\theta)^2\Box j_\mu.\end{aligned} \tag{3.6}$$

It should be noted that all the terms in the above expansion are divergenceless. This can also be proven directly by the D-algebra and because of the linearity of the superfield J. Moreover, the new superfield $\mathcal{J}_\mu$ is itself a linear superfield. This can be seen by observing that each component of the superfield $\mathcal{J}_\mu$ is in the same relation with higher terms of the expansion as the components of the superfield J, or it can be checked by direct use of superderivatives.

Therefore, the complete supersymmetric action is given by

$$S = \int d^4x \int d^4\theta(-JA + F(\mathcal{J}_\mu\mathcal{J}^\mu)J^2). \tag{3.7}$$

The minus sign in front of the first term is chosen to reproduce the normalization of the bosonic Lagrangian. The coefficients are chosen in order that Eq. (3.7) coincides with the normalization of the bosonic Lagrangian where $f(j^2) = F(j^2)j^2$. The argument of F, namely, $\mathcal{J}_\mu\mathcal{J}^\mu$, is a dimensionful superfield and therefore it would be convenient to rescale it by a dimensionful parameter. In the following, we will discard that parameter and we set it to 1.

As discussed above, we would like to deal with superconformal fluid. For that, we require the theory to be conformal and supersymmetric, thus superconformal invariance follows. In particular, we first impose the dilatation properties of F and it turns out that $F(x) = Cx^{-1/3}$.

[1] In the following, we use Weinberg notation [22]. Nevertheless, we recall that in the language of [27] a linear superfield is defined as $D^2J = 0$ and $\bar{D}^2\bar{J} = 0$. If J is a real linear superfield, $\bar{J} = J$, then the second condition follows from the first one.

That guarantees the conformal invariance of the action. The superconformal transformation rules for J are deduced by its geometrical properties.

To compute the component action, we need the expansion of $\mathcal{J}_\mu \mathcal{J}^\mu$ and, using (3.6), we get

$$
\begin{aligned}
\mathcal{J}_\mu \mathcal{J}^\mu = {} & j^2 + 2\bar{\theta} j_\mu \gamma^{\mu\nu}\partial_\nu \omega + \bar{\theta}\theta(-\tfrac{1}{2}\partial_\mu \bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega - \tfrac{3}{4}\partial_\mu\bar{\omega}\partial^\mu\omega) + (\bar{\theta}\gamma_5\theta)(-\tfrac{1}{2}\partial_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega - \tfrac{3}{4}\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega) \\
& + \bar{\theta}\gamma_5\gamma^\mu\theta(ij_\mu \Box C - ij\cdot\partial\partial_\mu C + \partial_\mu\bar{\omega}\gamma_5 \partial\!\!\!/ \omega - \tfrac{1}{4}\partial^\nu\bar{\omega}\gamma_5\gamma_\mu\partial_\nu\omega) + (\bar{\theta}\gamma_5\theta)\bar{\theta}\gamma_5(j\cdot\partial \partial\!\!\!/ \omega - j\cdot\gamma\Box\omega) \\
& + \bar{\theta}\gamma_5\theta\bar{\theta}(2i\partial\!\!\!/ \omega\Box C + i\gamma^\mu\partial^\nu\omega\partial_\mu\partial_\nu C) + \tfrac{1}{4}(\bar{\theta}\gamma_5\theta)^2(j_\mu\Box j^\mu + \partial_\mu\partial_\nu C\partial^\mu\partial^\nu C + 2\Box C\Box C + \partial_\mu\bar{\omega}\partial^\mu\omega \\
& + \partial^\mu\bar{\omega}\gamma_{\mu\nu}\partial^\nu\omega - 2\Box\bar{\omega}\partial\!\!\!/ \omega),
\end{aligned}
\tag{3.8}
$$

and similarly, for J^2, we have

$$
\begin{aligned}
J^2 = {} & C^2 - 2iC\bar{\theta}\gamma_5\omega + \tfrac{1}{4}(\bar{\theta}\theta)\bar{\omega}\omega + \tfrac{1}{4}(\bar{\theta}\gamma_5\theta)\bar{\omega}\gamma_5\omega \\
& + \bar{\theta}\gamma_5\gamma_\mu\theta(iCj^\mu + \tfrac{1}{4}\bar{\omega}\gamma_5\gamma^\mu\omega) + i\bar{\theta}\gamma_5\theta\bar{\theta}\partial\!\!\!/ \omega C \\
& + \bar{\theta}\gamma_5\gamma_\mu\theta\bar{\theta}\gamma_5\omega j^\mu + \tfrac{1}{4}(\bar{\theta}\gamma_5\theta)^2(C\Box C + j^2 - \bar{\omega}\partial\!\!\!/ \omega).
\end{aligned}
\tag{3.9}
$$

Notice that the choice $f(j^2) = F(j^2)j^2$ does not spoil the generality of (3.7) since it coincides with bosonic Lagrangian if $f(j^2)$ is defined up to an unessential constant. Action (3.7) is chosen such that, by setting C and ω to zero, it exactly reproduces the bosonic Lagrangian (2.1) and the corresponding NS equations. The presence of two different superfields, namely, J and $\mathcal{J}_\mu$, in the Lagrangian is needed because of dimensional reasons or, equivalently, because J's lowest component is not j^μ.

In components, the supersymmetric Lagrangian turns out to be

$$
\int d^4x\Big[j^\mu a_\mu + \bar{\omega}\lambda - CD + \int d^4\theta J^2 \sum_{i=0}^{4}\frac{1}{i!}F^{(i)}(j^2) \times (\mathcal{J}_\mu\mathcal{J}^\mu - j^2)^i\Big],
\tag{3.10}
$$

where we expanded the function F around the first bosonic component of $\mathcal{J}_\mu\mathcal{J}^\mu$. The first term in the expansion reproduces the bosonic Lagrangian, while the other terms are classified according to their dimensions. Notice that the computation of the component action is made unhandy by the fact that there is a product of two or more superfields $(\mathcal{J}_\mu\mathcal{J}^\mu - j^2)^i J^2$. After the θ expansion is taken, one needs to compute all Fierz identities to simplify the expressions and, finally, the integration over the θ variables can be taken.

The first two terms in the expansion of FJ^2 are

$$
\begin{aligned}
& \int d^4\theta \int d^4x[F^{(0)}(j^2)J^2 + F^{(1)}(j^2)(\mathcal{J}_\mu\mathcal{J}^\mu - j^2)J^2] \\
& = \int d^4x\Big\{[F^{(0)}(j^2)(C\Box C + j_\mu j^\mu - \bar{\omega}\gamma_\mu\partial^\mu\omega)] + \Big[F^{(1)}(j^2)\Big(-C^2[j_\mu\Box j^\mu + (\partial_\mu\partial_\nu C\partial^\mu\partial^\nu C + 2\Box C\Box C)] \\
& \quad + 4Cj^\mu j^\nu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C - C^2\partial_\mu\bar{\omega}\partial^\mu\partial\!\!\!/ \omega + 2C^2\Box\bar{\omega}\partial\!\!\!/ \omega - 2iCj^\mu\partial_\mu\bar{\omega}\gamma_5\partial\!\!\!/ \omega - iCj^\mu\partial_\nu\bar{\omega}\gamma_5\gamma_\mu\partial^\nu\omega \\
& \quad + 4C\Box C\bar{\omega}\partial\!\!\!/ \omega + 2C\partial_\mu\partial_\nu C\bar{\omega}\gamma^\mu\partial^\nu\omega + 2Cj_\mu\partial_\nu\bar{\omega}\gamma_\rho\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} - 2iCj^\mu\bar{\omega}\gamma_5\partial_\mu\partial\!\!\!/ \omega + 2iCj^\mu\bar{\omega}\gamma_5\gamma_\mu\Box\omega \\
& \quad + 2j^2\bar{\omega}\partial\!\!\!/ \omega - 2j^\mu j^\nu\bar{\omega}\gamma_\mu\partial_\nu\omega - ij^\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C\bar{\omega}\gamma_5\gamma^\nu\omega - \frac{3}{4}\bar{\omega}\omega\partial_\mu\bar{\omega}\partial^\mu\omega - \frac{1}{2}\bar{\omega}\omega\partial_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega \\
& \quad + \frac{3}{4}\bar{\omega}\gamma_5\omega\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega + \frac{1}{2}\bar{\omega}\gamma_5\omega\partial_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega - \bar{\omega}\gamma_5\gamma^\mu\omega\partial_\mu\bar{\omega}\gamma_5\partial\!\!\!/ \omega + \frac{1}{4}\bar{\omega}\gamma_5\gamma_\mu\omega\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial^\nu\omega\Big)\Big]\Big\}.
\end{aligned}
\tag{3.11}
$$

As can be seen from this expression, they contain the interaction between the current j^μ and the fields C and ω. The part proportional to $F^{(1)}$ contains terms with four fields ω and therefore their self-interactions. In the forthcoming section, we will discuss the implications of those terms. Even though the action might seem bulky, it is a good starting point for the perturbation theory since the expansion is done in terms of higher-derivative terms.

Since the resulting action is rather cumbersome, we find it convenient also to provide its bosonic truncation

$$
\begin{aligned}
& \int d^4x[j^\mu a_\mu - CD + F^{(0)}(j^2)(C\Box C + j_\mu j^\mu)] + \int d^4x[F^{(1)}(j^2)(-C^2 j_\mu\Box j^\mu - C^2(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)C \\
& + 4Cj^\mu j^\nu(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C)] + \int d^4x[F^{(2)}(j^2)(-4C^2 j^\mu j^\nu(\partial_\mu\partial_\rho - g_{\mu\rho}\Box)C(\partial_\nu\partial^\rho - \delta_\nu^\rho\Box)C)].
\end{aligned}
\tag{3.12}
$$

The bosonic action truncates at the second order in F, since all other terms are purely fermionic. This is due to the fact that in the expansion of the third power and of the fourth power, only those terms with a single θ contribute to the expansion since we have decided to expand around j^μ. This simplifies the derivation of the energy-momentum tensor for the bosonic sector as we are going to discuss in the forthcoming section. In Appendix B, all other terms are given.

B. Clebsch parametrization for the supersymmetric case

We discuss here the Clebsch parameterization for the supersymmetric case. Here, the gauge field a_μ is replaced by the real superfield A and therefore we have to parametrize it using a Clebsch parametrization as above. As suggested in [26] and in [28], we identify

$$A = \chi + \bar{\chi} + K(\phi, \bar{\phi}), \tag{3.13}$$

where χ, ϕ and $\bar{\chi}$, $\bar{\phi}$ are chiral and antichiral fields, respectively. $K(\phi, \bar{\phi})$ is a Kähler potential represented by a real function of the superfields ϕ and $\bar{\phi}$. The condition for the complex manifold spanned by ϕ and $\bar{\phi}$ to be Kähler is $d\mathcal{K} = 0$, where $\mathcal{K}$ is the canonical 2-form. Since the complex manifold is one dimensional, no interesting condition emerges from this constraint.

The identification in (3.13) implies that the Fayet-Ilioupoulos term induced by the Abelian gauge field A is given by

$$S_{F-I} = \int d^4x d^4\theta A = \int d^4x d^4\theta K(\phi, \bar{\phi}), \tag{3.14}$$

and it generates the dynamical equations of motion for the chiral fields (see, for example, [26]). In our case, however, this term is replaced by

$$S = \int d^4x d^4\theta(-JA + \ldots) = \int d^4x d^4\theta(-J(K(\phi, \bar{\phi}) + \chi + \bar{\chi}) + \ldots), \tag{3.15}$$

so that a naive kinetic term for ϕ and $\bar{\phi}$ is absent, being replaced by the superfield expansion of JK. The chiral field χ and the antichiral field $\bar{\chi}$ implement the linearity condition on J.

Let us now consider the first term of action (3.15) which, after Berezin integration, reads

$$\begin{aligned} S = \int d^4x \frac{1}{2}K(\varphi, \bar{\varphi})\Box C - \partial K\Big(ij^\mu\partial_\mu\varphi + \frac{1}{2}C\Box\varphi - i\frac{\sqrt{2}}{2}\bar{\psi}_L\not{\partial}\lambda + i\frac{\sqrt{2}}{2}\bar{\lambda}\not{\partial}\psi_L\Big) + c.c. \\ - \frac{1}{2}\partial^2K(C\partial_\mu\varphi\partial^\mu\varphi - \sqrt{2}i\partial_\mu\varphi\bar{\psi}_L\gamma^\mu\lambda) + c.c. - \partial\bar{\partial}K\Big(2|P|^2C - C\partial_\mu\varphi\partial^\mu\bar{\varphi} + \sqrt{2}iP\bar{\psi}_R\lambda - \sqrt{2}i\bar{P}\bar{\psi}_L\lambda \\ - C\bar{\psi}_L\not{\partial}\psi_R - C\bar{\psi}_R\not{\partial}\psi_L + ij^\mu\bar{\psi}_L\gamma_\mu\psi_R + \frac{\sqrt{2}}{2}i\partial_\mu\bar{\varphi}\bar{\psi}_L\gamma^\mu\lambda - \frac{\sqrt{2}}{2}i\partial_\mu\varphi\bar{\psi}_R\gamma^\mu\lambda\Big) \\ - \frac{1}{3}\partial^2\bar{\partial}K(-2C\bar{P}\bar{\psi}_L\psi_L + 2C\partial_\mu\varphi\bar{\psi}_L\gamma^\mu\psi_R - \sqrt{2}i\bar{\psi}_L\psi_L\bar{\psi}_R\lambda) + c.c. - \frac{1}{2}\partial^2\bar{\partial}^2KC\bar{\psi}_R\psi_R\bar{\psi}_L\psi_L, \end{aligned} \tag{3.16}$$

where the chiral and antichiral superfields ϕ (respectively $\bar{\phi}$) are defined by conditions

$$\frac{1-\gamma_5}{2}D\phi = 0, \qquad \frac{1+\gamma_5}{2}D\bar{\phi} = 0, \tag{3.17}$$

and their components include a left-chiral spinor field $\psi_L = (\frac{1+\gamma_5}{2})\psi$ (respectively right-chiral ψ_R) and two scalar complex fields φ and P (respectively $\bar{\varphi}$ and $\bar{P}$). The expression $Q_\mu \equiv i(\partial K\partial_\mu\varphi - \bar{\partial}K\partial_\mu\bar{\varphi}) - i\partial\bar{\partial}K\bar{\psi}_L\gamma_\mu\psi_R$ is known as the Kähler connection. Action (3.7) contains a piece which depends upon the superfield A. Inserting the above expressions into (3.10), we get an action which depends upon the components φ, ψ_L, and F of the superfield ϕ (and their conjugated). Differentiation w.r.t. those fields leads to the equations of motion. Truncating the action to its bosonic part, the first term in (3.7) reads

$$\begin{aligned} S = \int d^4x\Big[\frac{1}{2}K\Box C - ij^\mu(\partial K\partial_\mu\varphi - \bar{\partial}K\partial_\mu\bar{\varphi}) \\ - \frac{1}{2}C(\partial K\Box\varphi + \bar{\partial}K\Box\bar{\varphi}) - \frac{1}{2}C(\partial^2K\partial_\mu\varphi\partial^\mu\varphi \\ + \bar{\partial}^2K\partial_\mu\bar{\varphi}\partial^\mu\bar{\varphi}) - C\partial\bar{\partial}K(2|P|^2 - \partial_\mu\varphi\partial^\mu\bar{\varphi})\Big], \end{aligned} \tag{3.18}$$

where the Kähler potential K is evaluated on φ and its conjugate. Notice that, if we integrate by parts $K\Box C$, the above expression considerably simplifies and becomes

$$S = \int d^4x[ij^\mu(\bar{\partial}K\partial_\mu\varphi - \partial K\partial_\mu\bar{\varphi}) - 2C\partial\bar{\partial}K(|P|^2 - \partial_\mu\varphi\partial^\mu\bar{\varphi})]. \tag{3.19}$$

The Lagrangian is diagonal in the auxiliary fields P, $\bar{P}$ and their equations of motion (at the lowest level) imply either

$C = 0$ (fluid dynamics approximation) or $P = \bar{P} = 0$ (which is the supersymmetric dynamics).

To compute the equations of motion, we recall the expansion of F, given in (3.12). Varying w.r.t. φ, we get

$$2\partial\bar{\partial}K\partial_\mu\bar{\varphi}(ij^\mu - \partial^\mu C) - 2C\partial\bar{\partial}K\Box\bar{\varphi} - 2C\partial\bar{\partial}K\partial_\mu\bar{\varphi}\partial^\mu\bar{\varphi} = 0. \tag{3.20}$$

Analogously, we can get the equation of motion for $\bar{\varphi}$. The one for j^μ reads

$$\begin{aligned} &i(\bar{\partial}K\partial_\mu\bar{\varphi} - \partial K\partial_\mu\varphi) + 2F^{(1)}j^\mu(C\Box C + j^2) \\ &\quad + 2Fj_\mu - 2F^{(2)}C^2 j_\mu j_\nu\Box j^\nu - \Box(F^{(1)}C^2 j_\mu) \\ &\quad - F^{(1)}C^2\Box j_\mu + 8F^{(1)}j^\nu(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C \\ &\quad - 8F^{(3)}C^2 j_\mu - 4F^{(2)}C^2 j^\nu(\partial_\mu\partial_\rho - g_{\mu\rho}\Box) \\ &\quad \times C(\partial_\nu\partial^\rho - \delta^\rho_\nu\Box)C = 0, \end{aligned} \tag{3.21}$$

and finally, the one for C is

$$\begin{aligned} &2\partial\bar{\partial}K\partial_\mu\varphi\partial^\mu\bar{\varphi} + F\Box C + \Box F C - 2F^{(1)}Cj_\mu\Box j^\mu \\ &\quad + 2F^{(1)}\partial_\mu\partial_\nu C\partial^\mu\partial^\nu C + 2\partial_\mu\partial_\nu(F^{(1)}C^2\partial^\mu\partial^\nu C) \\ &\quad + 4F^{(1)}C(\Box C)^2 + 4\Box(F^{(1)}C^2\Box C) \\ &\quad + 4F^{(1)}j^\mu j^\nu(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C + 4(\partial_\mu\partial_\nu - g_{\mu\nu}\Box) \\ &\quad \times (F^{(1)}j^\mu j^\nu C) - 4F^{(2)}Cj^\mu j^\nu(\partial_\mu\partial_\rho - g_{\mu\rho}\Box) \\ &\quad \times C(\partial_\nu\partial^\rho - \delta^\rho_\nu\Box)C - 4(\partial_\mu\partial_\rho - g_{\mu\rho}\Box) \\ &\quad \times (F^{(2)}C^2 j^\mu j^\nu(\partial_\nu\partial^\rho - \delta^\rho_\nu\Box)C) = 0. \end{aligned} \tag{3.22}$$

C. Superfield equations

Action (3.7) is written in terms of a linear superfield J and a real superfield A. For those superfields, the usual functional derivative cannot be used and therefore we cannot obtain the equations of motion by usual means (see [27] for a complete discussion). To overcome such a problem, we add two auxiliary generic superfields Z, S^μ, one chiral superfield χ, and one antichiral superfield $\bar{\chi}$.

The following action,

$$\begin{aligned} S &= \int d^4x d^4\theta\Big(-J(A + \chi + \bar{\chi})) \\ &\quad + F\Big[\Big(\frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J\Big)^2\Big]J^2\Big) \\ &= \int d^4x d^4\theta\Big(-J(A + \chi + \bar{\chi})) + F[\mathcal{J}^2]J^2 \\ &\quad + S^\mu\Big[\frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J - \mathcal{J}_\mu\Big]\Big), \end{aligned} \tag{3.23}$$

turns out to be equivalent to (3.7). The chiral and antichiral superfields χ, $\bar{\chi}$ impose the linearity condition on the superfield J.

As already discussed above, in order to get the correct equations of motion, we replace the superfield A with the Kähler potential. Then, we have

$$\begin{aligned} S_K &= \int d^4x d^4\theta\Big(-J(K(\phi, \bar{\phi}) + \chi + \bar{\chi}) + F[\mathcal{J}^2]J^2 \\ &\quad + S^\mu\Big[\frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J - \mathcal{J}_\mu\Big]\Big), \end{aligned} \tag{3.24}$$

from which we can get the equations of motion by taking the functional (unconstrained) derivatives with respect to superfields J, ϕ, $\bar{\phi}$, S^μ, χ, $\bar{\chi}$ to get

$$\begin{aligned} &\bar{D}DJ = 0, \qquad \mathcal{J}_\mu - \frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)J = 0, \\ &S^\mu + 2\mathcal{J}^\mu F'[\mathcal{J}^2]J^2 = 0, \qquad \bar{D}D\Big(J\frac{\partial K}{\partial\phi}\Big) = 0, \\ &\bar{D}D\Big(J\frac{\partial K}{\partial\bar{\phi}}\Big) = 0, \\ &K(\phi, \bar{\phi}) + \chi + \bar{\chi} - 2JF[\mathcal{J}^2] - \frac{1}{4i}(\bar{D}\gamma_5\gamma_\mu D)S^\mu = 0. \end{aligned} \tag{3.25}$$

To study the above equations, we proceed as follows. The first equation in (3.25) implies the linearity of J [and therefore its θ expansion is given by (3.2)]. Then, we plug J into the second equation f to compute the vector superfield $\mathcal{J}_\mu$. Subsequently, we plug $\mathcal{J}_\mu$ into the third equation to evaluate S^μ and finally, using all those results, we can express K in terms of the superfields ϕ and $\bar{\phi}$. Given that, equations (3.25) become the new NS equations, written in terms of the linear superfield J which contains the physical degrees of freedom of the superfluid.

D. Bosonic sector

In the present section, we study the model by setting to zero the fermions. We first write the Lagrangian as a function of the fields j^μ and C and then we provide a new Lagrangian with new auxiliary fields which simplifies the derivation of the energy-momentum tensor.

The bosonic part of the Lagrangian is (up to a factor $\sqrt{-g}$)

$$\begin{aligned} \mathcal{L}_{\rm bos} &= j^\mu a_\mu - CD + F^{(0)}(j^2)(C\Box C + j_\mu j^\mu) \\ &\quad + F^{(1)}(j^2)[-C^2 j_\mu\Box j^\mu + C^2\partial_\mu\partial_\nu C\partial^\mu\partial^\nu C \\ &\quad + 2C^2(\Box C)^2 + 4j^\mu j^\nu C(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C] \\ &\quad + \tfrac{1}{2}F^{(2)}(j^2)[-4C^2 j^\mu j^\nu(\partial_\mu\partial_\rho - g_{\mu\rho}\Box)C \\ &\quad \times(\partial^\rho\partial_\nu - \delta^\rho_\nu\Box)C]. \end{aligned} \tag{3.26}$$

We define the quadratic differential operator

$$\begin{aligned} &M_{\mu\nu} = \partial_\mu\partial_\nu - g_{\mu\nu}\Box, \qquad \partial^\mu M_{\mu\nu} = 0, \\ &\Box = -\tfrac{1}{3}g^{\mu\nu}M_{\mu\nu}. \end{aligned} \tag{3.27}$$

and we rewrite (3.26) with the Lagrangian multiplier $S^{\mu\nu}$,

$$\mathcal{L}_{\rm bos} = j^\mu a_\mu - CD + F^{(0)}(j^2)\Big(-\tfrac{1}{3}g^{\mu\nu}B_{\mu\nu}C + j_\mu j^\mu\Big) + F^{(1)}(j^2)[-C^2 j_\mu \Box j^\mu + C^2 B_{\mu\nu}B_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} + 4Cj^\mu j^\nu B_{\mu\nu}] + \tfrac{1}{2}F^{(2)}(j^2) \times [-4C^2 j^\mu j^\nu B_{\mu\rho}B_{\nu\sigma}g^{\rho\sigma}] + S^{\mu\nu}(B_{\mu\nu} - M_{\mu\nu}C). \tag{3.28}$$

In this way, we restrict the covariantization of the differential operator $M_{\mu\nu}$ in a single term and the derivation of the energy-momentum tensor is greatly simplified. We now compute the equations of motion for C, $B_{\mu\nu}$, and j^μ, respectively,

$$D = -2F^{(1)}C + 2F^{(1)}B_{\mu\nu}B^{\mu\nu} + 4F^{(1)}j^\mu j^\nu B_{\mu\nu} - 4F^{(2)}Cj^\mu j^\nu B_{\mu\rho}B_{\nu\sigma}g^{\rho\sigma} - \tfrac{1}{3}F^{(0)}g^{\mu\nu}B_{\mu\nu} - M_{\mu\nu}S^{\mu\nu}, \tag{3.29}$$

$$S^{\mu\nu} = -2F^{(1)}B^{\mu\nu}C^2 - 4Cj^\mu j^\nu + 4F^{(2)}C^2 j^\mu j^\rho B_{\rho\sigma}g^{\nu\sigma} + \tfrac{1}{3}F^{(0)}Cg^{\mu\nu}, \tag{3.30}$$

$$a_\mu = -F^{(2)}j_\mu N_{[0]} + F^{(1)}C^2\Box j^\mu + \Box(F^{(1)}C^2 j^\mu) - 8F^{(1)}CB_{\mu\nu}j^\nu - F^{(3)}N_{[1]} + 4F^{(2)}C^2 B_{\mu\rho}B_{\nu\sigma}g^{\rho\sigma}j^\nu - 2F^{(1)}j_\mu N_{[2]} - 2F^{(0)}j_\mu, \tag{3.31}$$

where $N_{[0]}$, $N_{[1]}$, and $N_{[2]}$ are the terms in (3.28) proportional to $F^{(0)}$, $F^{(1)}$, and $F^{(2)}$, respectively.

In the case $j^\mu = 0$, the Lagrangian (3.28) coupled to worldline metric is (we set $F^{(0)} = F^{(1)} = 1$)

$$\mathcal{L}_{\rm bos}|_{j=0} = \sqrt{-g}[C^2 g^{\mu\rho}g^{\nu\sigma}B_{\mu\nu}B^{\rho\sigma} - CD - \tfrac{1}{3}Cg^{\mu\nu}B_{\mu\nu} + S^{\mu\nu}(B_{\mu\nu} - M_{\mu\nu}C)]. \tag{3.32}$$

The equations of motion for C and $B_{\mu\nu}$ are

$$D = 2Cg^{\mu\rho}g^{\nu\sigma}B_{\mu\nu}B^{\rho\sigma} - \tfrac{1}{3}g^{\mu\nu}B_{\mu\nu} - M_{\mu\nu}S^{\mu\nu}, \tag{3.33}$$

and

$$S^{\mu\nu} = -2C^2B^{\mu\nu} + \tfrac{1}{3}Cg^{\mu\nu}. \tag{3.34}$$

Finally, for this simplified Lagrangian we derive the energy-momentum tensor. We obtain

$$T^{\mu\nu} = -g^{\mu\nu}C\Box C - \tfrac{1}{2}g^{\mu\nu}\partial^\rho C\partial_\rho C + \partial^\mu C\partial^\nu C - \tfrac{5}{2}g^{\mu\nu}C^2\nabla^\rho\partial^\sigma C\nabla_\rho\partial_\sigma C - 7g^{\mu\nu}C^2\Box C\Box C - 2g^{\mu\nu}C\partial_\rho C\partial_\sigma C\nabla^\rho\partial^\sigma C - 14g^{\mu\nu}C^2\partial_\rho C\partial^\rho\Box C - 3g^{\mu\nu}C^3\Box\Box C - 8g^{\mu\nu}C\partial_\rho C\partial^\rho C\Box C - C^2\Box C\nabla^\mu\partial^\nu C + 4C\partial_\rho C\nabla^\rho\partial^{(\mu}C\partial^{\nu)}C - 2C\partial^\rho C\partial_\rho C\nabla^\mu\partial^\nu C + 6C^2\partial^{(\mu}C\nabla^{\nu)}\Box C - C^2\nabla^\rho\nabla^\mu\partial^\nu C\partial_\rho C + 8C\partial^\mu C\partial^\nu C\Box C. \tag{3.35}$$

We prefer to analyze only the equations of motion with the Clebsch parametrization and in the case $\omega = 0$. This gives novel dynamical equations.

E. Comparison with fluid/gravity correspondence

In order to compare the action we have obtained so far with the result obtained in a previous publication [11], we consider a further limit: we set the scalar field $C = 0$. Indeed, in [11] no scalar deformation of the black hole in AdS_5 has been considered, but there are taken into account the bosonic deformations which are interpreted as the fluid degrees of freedom that are collectively described by the current j^μ and the fermionic deformations which are collected into bilinears. Therefore, to compare with those results, we need to reproduce the same settings. Here, we cannot present a complete discussion since the result provided in [11] are still valid at the linear level and they are easily captured by the second term of the action below by setting $j = 0$. Nevertheless, it is instructive to see how the structures analyzed in [11] emerge in the present effective analysis. Furthermore, we can have a guideline for further corrections.

The first two terms in the expansion of FJ^2 are

$$\begin{aligned}\int d^4\theta \int d^4x[F^{(0)}(j^2)J^2 + F^{(1)}(j^2)(\mathcal{J}_\mu\mathcal{J}^\mu - j^2)J^2] &= \int d^4x\Big\{[F^{(0)}(j^2)(j_\mu j^\mu - \bar\omega\gamma_\mu\partial^\mu\omega)] + \Big[F^{(1)}(j^2)\Big(2j^2\bar\omega\not\partial\omega - 2j^\mu j^\nu\bar\omega\gamma_\mu\partial_\nu\omega - \frac{3}{4}\bar\omega\omega\partial_\mu\bar\omega\partial^\mu\omega \\ &\quad - \frac{1}{2}\bar\omega\omega\partial_\mu\bar\omega\gamma^{\mu\nu}\partial_\nu\omega + \frac{3}{4}\bar\omega\gamma_5\omega\partial_\mu\bar\omega\gamma_5\partial^\mu\omega + \frac{1}{2}\bar\omega\gamma_5\omega\partial_\mu\bar\omega\gamma_5\gamma^{\mu\nu}\partial_\nu\omega - \bar\omega\gamma_5\gamma^\mu\omega\partial_\mu\bar\omega\gamma_5\not\partial\omega \\ &\quad + \frac{1}{4}\bar\omega\gamma_5\gamma_\mu\omega\partial_\nu\bar\omega\gamma_5\gamma^\mu\partial^\nu\omega\Big)\Big]\Big\}.\end{aligned} \tag{3.36}$$

Assuming that the fermions ω are factorized in a given black hole background (see in [11] for the derivation) as $\omega = \epsilon \otimes \eta$. Inserting them into the action, we can observe that from the first line we obtain the structures of the type

$$\lambda \partial_i N^i, \qquad \partial \lambda_i N^i, \tag{3.37}$$

where λ_i and N_i are the bilinears written in terms of the fermion fields ϵ and η. The index labels the different types of bilinears that can be formed. From the second and the third line, we infer that higher power in bilinears can indeed emerge from the present action. Those terms are not yet comparable with the fermionic corrections computed in [11] since it requires the construction of the full backreacted metric [29].

Note that already in the second line there are new interaction terms between j^μ (the bosonic degrees of freedom of the fluid) and the fermionic bilinears

$$j^2 \lambda \partial_i N^i, \qquad j^2 \partial \lambda_i N^i. \tag{3.38}$$

In order to reproduce them in the computation of fluid/gravity correspondence, one has to compute the correlation functions of the sources coupled to those operators and this is at the moment not yet done. Nevertheless, we provide here a systematical tool to describe the expected form of the effective action.

F. Dependence on the Kähler potential

We have to discuss the dependence of the equations of motion upon the Kähler potential. For that, we discuss only the bosonic sector and we observe the following identity,

$$- j^\mu F_{\mu\nu} + C\partial_\nu D = -4\partial_\mu[\partial\bar{\partial}KC(\partial^\mu\bar{\varphi}\partial_\nu\varphi + \partial^\mu\varphi\partial_\nu\bar{\varphi})], \tag{3.39}$$

where the right-hand side can be also be written as $\partial_\mu(CG^{\mu\nu})$ where $G^{\mu\nu}$ is the inverse of the Kähler metric. It appears as a total derivative. However, we cannot discard such term. The reason is that it does not follow directly from the action, namely, it is not a total derivative term derived from the action. Nevertheless, we can show that it is harmless and, at least in the rigid case, can be discarded.

The left-hand side of (3.39) can be obtained by the same method as in Sec. II A. Indeed, by requiring the invariance under an isometry and using the same equations as above we get a new equation of the form

$$\begin{aligned}&\int d^4xX^\nu(-j^\mu F_{\mu\nu} + C\partial_\mu D)\\ &\quad = -4\int d^4xX^\nu\partial_\mu[\partial\bar{\partial}KC(\partial^\mu\bar{\varphi}\partial_\nu\varphi + \partial^\mu\varphi\partial_\nu\bar{\varphi})].\end{aligned} \tag{3.40}$$

Now, we can use the integration by parts in the right-hand side and by using the fact that X^μ must be a Killing vector for the flat metric we can easily conclude that the left-hand side of (3.39) is effectively a total derivative and it can be discarded. A complete proof of this statement would be very interesting since it would show that the dynamical equations of motion are independent of the parametrization of the gauge field A.

IV. CONCLUSIONS

We propose a new supersymmetric action for the supersymmetric fluid dynamics, discussing some of its aspects, such as the new NS equations and their derivation. A discussion on the Clebsch parametrization is proposed and the derivation of the superfield equations is done in that framework. There are several open issues: (1) What is the dynamics described by the present action? (2) What is the role of the boson C? (3) A fluid described only in terms of fermionic field can be discussed by setting to zero both j^μ and C. We believe that the study of the present system in the context of supergravity might shed some light on the coupling with the world-volume metric and finally the supersymmetry partner of $T^{\mu\nu}$ can be computed. We leave the discussion on supergravity to a forthcoming publication.

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APPENDIX A: FIERZ IDENTITIES

We list here some of the properties of Majorana spinors and some useful Fierz identities:

$$\begin{aligned}&\bar{s}_1 M s_2 = \bar{s}_2 M s_1 \quad \text{if } M = 1, \gamma_5, \gamma_5\gamma^\mu,\\ &\bar{s}_1 M s_2 = -\bar{s}_2 M s_1 \quad \text{if } M = \gamma^\mu, \gamma^{\mu\nu}.\end{aligned} \tag{A1}$$

The Fierz identities for 2 identical spinors read

$$\theta\bar{\theta} = -\tfrac{1}{4}(\bar{\theta}\theta + \bar{\theta}\gamma_5\theta\gamma_5 - \bar{\theta}\gamma_5\gamma_\mu\theta\gamma_5\gamma^\mu), \tag{A2}$$

while those for 3 spinors are

$$\theta(\bar{\theta}\theta) = -\gamma_5\theta\bar{\theta}\gamma_5\theta, \qquad \theta(\bar{\theta}\gamma_5\gamma_\mu\theta) = -\gamma_\mu\theta\bar{\theta}\gamma_5\theta. \tag{A3}$$

Using (A3), it is easy to show that the following identities also hold:

$$\begin{aligned}(\bar{\theta}\theta)^2 &= -(\bar{\theta}\gamma_5\theta)^2,\\ (\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta) &= -\eta_{\mu\nu}(\bar{\theta}\gamma_5\theta)^2,\\ (\bar{\theta}\theta)(\bar{\theta}\gamma_5\theta) = (\bar{\theta}\theta)(\bar{\theta}\gamma_5\gamma_\mu\theta) &= (\bar{\theta}\gamma_5\theta)(\bar{\theta}\gamma_5\gamma_\mu\theta) = 0.\end{aligned} \tag{A4}$$

Finally, the integration measure for Grassmann variables is

$$\int d^4\theta(\bar{\theta}\gamma_5\theta)^2 = -4. \tag{A5}$$

APPENDIX B: COMPLETE LAGRANGIAN

Here, we present the complete expansion of the supersymmetric Lagrangian (3.7). This can be rewritten as

$$\mathcal{L} = \int d^4x \int d^4\theta\Big(-JA + \sum_{i=0}^{4}\frac{1}{i!}F^{(i)}L_i\Big), \tag{B1}$$

where $F^{(i)}$ is the order i derivative of $F(\mathcal{J}_\mu\mathcal{J}^\mu)$ computed at $\mathcal{J}_\mu\mathcal{J}^\mu = j_\mu j^\mu$ and

$$L_i = (\mathcal{J}_\mu\mathcal{J}^\mu - j_\mu j^\mu)^i J^2. \tag{B2}$$

In the following, we show the explicit form of the four L_i. To perform the computation, we developed a program written in FORM language (see [30] and references therein) which, given a set of superfields expanded in components, returns as a result any desired combination of these fields, integrated over $d^4\theta$. The subroutine structure of the program allows us to check every intermediate passage, or to use each single procedure to perform different calculations such as Fierz identities or gamma manipulations.

Notice that only L_1 and L_2 have purely bosonic terms (B3a) and (B4a).

$$\begin{aligned} L_1 = {} & -C^2[j_\mu\Box j^\mu + (\partial_\mu\partial_\nu C\partial^\mu\partial^\nu C + 2\Box C\Box C)] + 4Cj^\mu j^\nu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C && \text{(B3a)}\\ & - C^2\partial_\mu\bar{\omega}\partial^\mu\not{\partial}\omega + 2C^2\Box\bar{\omega}\not{\partial}\omega - 2iCj^\mu\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega - iCj^\mu\partial_\nu\bar{\omega}\gamma_5\gamma_\mu\partial^\nu\omega + 4C\Box C\bar{\omega}\not{\partial}\omega + 2C\partial_\mu\partial_\nu C\bar{\omega}\gamma^\mu\partial^\nu\omega \\ & + 2Cj_\mu\partial_\nu\bar{\omega}\gamma_\rho\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} - 2iCj^\mu\bar{\omega}\gamma_5\partial_\mu\not{\partial}\omega + 2iCj^\mu\bar{\omega}\gamma_5\gamma_\mu\Box\omega + 2j^2\bar{\omega}\not{\partial}\omega - 2j^\mu j^\nu\bar{\omega}\gamma_\mu\partial_\nu\omega \\ & - ij^\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C\bar{\omega}\gamma_5\gamma^\nu\omega - \tfrac{3}{4}\bar{\omega}\omega\partial_\mu\bar{\omega}\partial^\mu\omega - \tfrac{1}{2}\bar{\omega}\omega\partial_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega + \tfrac{3}{4}\bar{\omega}\gamma_5\omega\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega \\ & + \tfrac{1}{2}\bar{\omega}\gamma_5\omega\partial_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega - \bar{\omega}\gamma_5\gamma^\mu\omega\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega + \tfrac{1}{4}\bar{\omega}\gamma_5\gamma_\mu\omega\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial^\nu\omega, && \text{(B3b)} \end{aligned}$$

$$\begin{aligned} L_2 = {} & -4C^2j^\mu j^\nu(\partial_\mu\partial_\rho - \eta_{\mu\rho}\Box)C(\partial_\nu\partial^\rho - \delta_\nu^\rho\Box)C && \text{(B4a)}\\ & - 2C^2j^\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C[\partial_\rho\bar{\omega}\gamma_\sigma\partial_\tau\omega\varepsilon_{\nu\rho\sigma\tau} - 6i\partial^\nu\bar{\omega}\gamma_5\not{\partial}\omega + i\partial_\rho\bar{\omega}\gamma_5\gamma^\nu\partial^\rho\omega] \\ & + 6C^2\Box Cj_\mu[\partial_\nu\bar{\omega}\gamma_\rho\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} + 2i\partial^\mu\bar{\omega}\gamma_5\not{\partial}\omega - 2i\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial^\nu\omega] + 2C^2j_\mu(\partial_\alpha\partial_\nu - \eta_{\alpha\nu}\Box)C\partial_\rho\bar{\omega}\gamma^\alpha\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} \\ & - 4iC^2(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)Cj^\rho\partial^\mu\bar{\omega}\gamma_5\gamma_\rho\partial^\nu\omega + \tfrac{9}{4}C^2\partial_\mu\bar{\omega}\partial^\mu\omega\partial_\nu\bar{\omega}\partial^\nu\omega + 3C^2\partial_\mu\bar{\omega}\partial^\mu\omega\partial_\nu\bar{\omega}\gamma^{\nu\rho}\partial_\rho\omega \\ & - \tfrac{9}{4}C^2\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega\partial_\nu\bar{\omega}\gamma_5\partial^\nu\omega - 3C^2\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega\partial_\nu\bar{\omega}\gamma_5\gamma^{\nu\rho}\partial_\rho\omega - 2C^2\partial_\mu\bar{\omega}\gamma_5\gamma^\nu\partial^\mu\omega\partial_\nu\bar{\omega}\gamma_5\not{\partial}\omega \\ & + \tfrac{1}{4}C^2\partial_\mu\bar{\omega}\gamma_5\gamma^\nu\partial^\mu\omega\partial_\rho\bar{\omega}\gamma_5\gamma_\nu\partial^\rho\omega + 4C^2\partial^\mu\bar{\omega}\gamma_5\not{\partial}\omega\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega + C^2\partial_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\gamma^{\rho\sigma}\partial_\sigma\omega \\ & - C^2\partial_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\gamma_5\gamma^{\rho\sigma}\partial_\sigma\omega + 4C^2j^\mu j^\nu\partial_\mu\bar{\omega}\gamma_\nu\Box\omega - 4C^2j^\mu j^\nu\partial_\mu\partial_\nu\bar{\omega}\not{\partial}\omega - 4C^2j^\mu j^\nu\partial_\rho\bar{\omega}\gamma_\mu\partial_\nu\partial_\rho\omega \\ & + 4C^2j^2\Box\bar{\omega}\not{\partial}\omega + 4iC^2j^\tau j_\mu\partial_\nu\bar{\omega}\gamma_5\gamma_\rho\partial_\tau\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} - 4Cj^2[j_\mu\partial_\nu\bar{\omega}\gamma_\sigma\partial_\rho\omega\varepsilon^{\mu\nu\rho\sigma} - 2ij^\mu\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega \\ & + 2ij^\nu\partial_\mu\bar{\omega}\gamma_5\gamma_\nu\partial^\mu\omega] + 4iCj^\mu j^\nu j^\rho\partial_\mu\bar{\omega}\gamma_5\gamma_\nu\partial_\rho\omega - 8iCj^2j^\mu\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega + 4iCj^2j^\mu\partial_\nu\bar{\omega}\gamma_5\gamma_\mu\partial^\nu\omega \\ & - 8Cj^\mu j^\nu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)C\bar{\omega}\not{\partial}\omega + 8Cj^\mu j^\nu(\partial_\nu\partial_\rho - \eta_{\nu\rho}\Box)C\bar{\omega}\gamma_\nu\partial^\rho\omega + 8iCj^\tau j_\mu(\partial_\tau\partial_\nu - g_{\tau\nu}\Box)C\bar{\omega}\gamma_5\gamma_\rho\partial_\sigma\omega\varepsilon^{\mu\nu\rho\sigma} \\ & - 8iCj^\nu\bar{\omega}\gamma_\nu\partial_\mu\omega\partial^\mu\bar{\omega}\gamma_5\not{\partial}\omega + 2iCj^\rho\bar{\omega}\gamma_\rho\partial_\mu\omega\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial^\nu\omega + 8iCj^\mu\bar{\omega}\not{\partial}\omega\partial_\mu\bar{\omega}\gamma_5\not{\partial}\omega - 2iCj^\mu\bar{\omega}\not{\partial}\omega\partial_\nu\bar{\omega}\gamma_5\gamma_\mu\partial^\nu\omega \\ & - 2Cj_\mu\bar{\omega}\gamma_5\gamma_\nu\partial_\rho\omega\partial_\tau\bar{\omega}\gamma_5\gamma_\sigma\partial^\tau\omega\varepsilon^{\mu\nu\rho\sigma} + 8Cj_\mu\bar{\omega}\gamma_5\gamma_\nu\partial_\rho\omega\partial_\sigma\bar{\omega}\gamma_5\not{\partial}\omega\varepsilon^{\mu\nu\rho\sigma} + 6iCj_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\gamma_5\partial^\rho\omega \\ & + 4iCj_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\gamma_5\gamma^{\rho\sigma}\partial_\sigma\omega - 6iCj_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\partial^\rho\omega - 4iCj_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega\partial_\rho\bar{\omega}\gamma^{\rho\sigma}\partial_\sigma\omega \\ & + \bar{\omega}\omega[j^\mu j^\nu\partial_\mu\bar{\omega}\partial_\nu\omega - j^2\partial_\mu\bar{\omega}\partial^\mu\omega + 2j^\mu j_\nu\partial_\mu\bar{\omega}\gamma^{\nu\rho}\partial_\rho\omega - j^2\partial_\mu\bar{\omega}\gamma^{\mu\nu}\partial_\nu\omega] + \bar{\omega}\gamma_5\omega[-j^\mu j^\nu\partial_\mu\bar{\omega}\gamma_5\partial_\nu\omega \\ & + j^2\partial_\mu\bar{\omega}\gamma_5\partial^\mu\omega - 2j^\mu j_\nu\partial_\mu\bar{\omega}\gamma_5\gamma^{\nu\rho}\partial_\rho\omega + j^2\partial_\mu\bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu\omega] + j^\mu\bar{\omega}\gamma_5\gamma_\mu\omega[-i\varepsilon^{\nu\rho\sigma\tau}j_\nu\partial_\rho\bar{\omega}\gamma_\sigma\partial_\tau\omega + 2j^\nu\partial_\nu\bar{\omega}\gamma_5\not{\partial}\omega \\ & - 2j^\nu\partial_\rho\bar{\omega}\gamma_5\gamma_\nu\partial^\rho\omega] + \bar{\omega}\gamma_5\gamma_\mu\omega[-i\varepsilon^{\mu\nu\rho\sigma}j^\tau j_\nu\partial_\rho\bar{\omega}\gamma_\tau\partial_\sigma\omega - j^\nu j^\rho\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial_\rho\omega + 2j^\nu j^\rho\partial_\nu\bar{\omega}\gamma_5\gamma_\rho\partial^\mu\omega \\ & - 2j^2\partial^\mu\bar{\omega}\gamma_5\not{\partial}\omega + j^2\partial_\nu\bar{\omega}\gamma_5\gamma^\mu\partial^\nu\omega], && \text{(B4b)} \end{aligned}$$

$$
\begin{aligned}
L_3 = {}& +12C^2(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)C[j^\mu j^\nu j_\tau \partial_\rho \bar{\omega}\gamma_\lambda \partial_\sigma \omega \varepsilon^{\tau\rho\sigma\lambda} - 2ij^\mu j^\nu j^\rho \partial_\rho \bar{\omega}\gamma_5 \not{\partial}\omega + 2ij^\mu j^\nu j^\rho \partial^\sigma \bar{\omega}\gamma_5\gamma_\rho \partial_\sigma \omega \\
& + j^\mu j^\rho j^\sigma \partial_\alpha \bar{\omega}\gamma_\rho \partial_\beta \omega \varepsilon^{\sigma\nu\alpha\beta} + ij^\mu j^\rho j^\sigma \partial_\rho \bar{\omega}\gamma_5\gamma^\nu \partial_\sigma \omega - 2ij^\mu j^\rho j^\sigma \partial_\rho \bar{\omega}\gamma_5\gamma_\sigma \partial^\nu \omega + 2i(j\cdot j)j^\mu \partial^\nu \bar{\omega}\gamma_5 \not{\partial}\omega \\
& - i(j\cdot j)j^\mu \partial^\rho \bar{\omega}\gamma_5\gamma^\nu \partial_\rho \omega] - 6C^2 j^\nu j^\mu \partial_\mu \bar{\omega}\partial_\nu \omega \partial_\rho \bar{\omega}\gamma^{\rho\sigma}\partial_\sigma \omega - 9C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\partial_\rho \omega \partial_\mu \bar{\omega}\partial_\nu \omega \\
& + 9C^2(j\cdot j)\partial^\mu \bar{\omega}\partial_\mu \omega \partial^\nu \bar{\omega}\partial_\nu \omega - 18C^2 j^\nu j_\rho \partial^\mu \bar{\omega}\partial_\mu \omega \partial_\nu \bar{\omega}\gamma^{\rho\sigma}\partial_\sigma \omega - 15C^2(j\cdot j)\partial^\mu \bar{\omega}\partial_\mu \omega \partial_\nu \bar{\omega}\gamma^{\rho\nu}\partial_\rho \omega \\
& - 6C^2 j^\mu j^\nu \partial_\mu \bar{\omega}\gamma_5 \partial_\nu \omega \partial_\rho \bar{\omega}\gamma_5\gamma^{\sigma\rho}\partial_\sigma \omega + 9C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\gamma_5 \partial_\rho \omega \partial_\nu \bar{\omega}\gamma_5 \partial_\rho \omega - 9C^2(j\cdot j)\partial^\mu \bar{\omega}\gamma_5 \partial_\mu \omega \partial_\nu \bar{\omega}\gamma_5 \partial_\nu \omega \\
& + 18C^2 j^\nu j_\rho \partial^\mu \bar{\omega}\gamma_5 \partial_\mu \omega \partial_\nu \bar{\omega}\gamma_5\gamma^{\rho\sigma}\partial_\sigma \omega - 15C^2(j\cdot j)\partial^\mu \bar{\omega}\gamma_5 \partial_\mu \omega \partial_\nu \bar{\omega}\gamma_5\gamma^{\nu\rho}\partial_\rho \omega \\
& + 3iC^2 j^\mu j_\nu \partial_\alpha \bar{\omega}\gamma_\mu \partial_\rho \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\lambda \partial_\sigma \omega \varepsilon^{\nu\alpha\rho\lambda} - 12iC^2 j^\mu j_\alpha \partial_\nu \bar{\omega}\gamma_\mu \partial_\rho \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\sigma \partial_\lambda \omega \varepsilon^{\alpha\nu\rho\lambda} \\
& - 12iC^2 j_\mu j^\lambda \partial_\nu \bar{\omega}\gamma_\rho \partial_\sigma \omega \partial_\lambda \bar{\omega}\gamma_5 \not{\partial}\omega \varepsilon^{\mu\nu\sigma\rho} + 3iCj^\mu j_\nu \partial_\rho \bar{\omega}\gamma_\sigma \partial_\lambda \omega \partial^\alpha \bar{\omega}\gamma_5\gamma_\mu \partial_\alpha \omega \varepsilon^{\nu\rho\lambda\sigma} \\
& - 6C^2 j^\mu j^\nu \partial_\mu \bar{\omega}\gamma_5\gamma_\rho \partial_\rho \omega \partial_\nu \bar{\omega}\gamma_5 \not{\partial}\omega - 6C^2 j^\mu j^\nu \partial_\rho \bar{\omega}\gamma_5\gamma_\mu \partial_\nu \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\sigma \partial_\rho \omega \\
& + 3C^2 j^\nu j^\rho \partial_\mu \bar{\omega}\gamma_5\gamma_\nu \partial_\rho \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\mu \partial_\sigma \omega - 18C^2 j^\mu j^\nu \partial_\mu \bar{\omega}\gamma_5 \not{\partial}\omega \partial_\nu \bar{\omega}\gamma_5 \not{\partial}\omega + 30C^2 j^\mu j^\nu \partial_\mu \bar{\omega}\gamma_5 \not{\partial}\omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\nu \partial_\sigma \omega \\
& - 6C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\gamma_5\gamma_\mu \partial_\rho \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\nu \partial_\sigma \omega - 3C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\gamma_5\gamma^\sigma \partial_\rho \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\sigma \partial_\mu \omega \\
& + 3C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\gamma_5\gamma^\sigma \partial_\rho \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\mu \partial_\sigma \omega - 18C^2(j\cdot j)\partial^\mu \bar{\omega}\gamma_5\gamma^\nu \partial_\mu \omega \partial_\nu \bar{\omega}\gamma_5 \not{\partial}\omega \\
& + 3C^2(j\cdot j)\partial^\mu \bar{\omega}\gamma_5\gamma^\nu \partial_\mu \omega \partial^\rho \bar{\omega}\gamma_5\gamma_\nu \partial_\rho \omega + 12C^2 j^\mu j^\nu \partial^\sigma \bar{\omega}\gamma_5 \not{\partial}\omega \partial_\mu \bar{\omega}\gamma_5\gamma_\sigma \partial_\nu \omega - 6C^2 j^\mu j^\nu \partial^\sigma \bar{\omega}\gamma_5 \partial\omega \partial_\nu \bar{\omega}\gamma_5\gamma_\mu \partial_\sigma \omega \\
& + 24(j\cdot j)C^2 \partial^\mu \bar{\omega}\gamma_5 \not{\partial}\omega \partial_\mu \bar{\omega}\gamma_5 \not{\partial}\omega - 12C^2 j^\mu j^\nu \partial^\rho \bar{\omega}\gamma_5 \not{\partial}\omega \partial_\nu \bar{\omega}\gamma_5\gamma_\mu \partial_\rho \omega - 12C^2 j_\mu j^\nu \partial_\nu \bar{\omega}\gamma^{\mu\rho}\partial_\rho \omega \partial_\sigma \bar{\omega}\gamma^{\sigma\lambda}\partial_\lambda \omega \\
& + 6C^2(j\cdot j)\partial_\mu \bar{\omega}\gamma^{\mu\nu}\partial_\nu \omega \partial_\rho \bar{\omega}\gamma^{\rho\sigma}\partial_\sigma \omega + 12C^2 j_\mu j^\nu \partial_\nu \bar{\omega}\gamma_5\gamma^{\mu\rho}\partial_\rho \omega \partial_\sigma \bar{\omega}\gamma_5\gamma^{\sigma\lambda}\partial_\lambda \omega \\
& - 6C^2(j\cdot j)\partial_\mu \bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu \omega \partial_\rho \bar{\omega}\gamma_5\gamma^{\rho\sigma}\partial_\sigma \omega - 4Cj^\mu j^\nu j_\rho \bar{\omega}\gamma_\mu \partial_\nu \omega \partial_\sigma \bar{\omega}\gamma_\beta \partial_\alpha \omega \varepsilon^{\rho\sigma\alpha\beta} \\
& + 8iCj^\mu j^\nu j^\rho \bar{\omega}\gamma_\mu \partial_\nu \omega \partial_\rho \bar{\omega}\gamma_5 \not{\partial}\omega - 8iCj^\mu j^\nu j^\rho \bar{\omega}\gamma_\mu \partial_\nu \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\rho \partial_\sigma \omega - 4Cj^\mu j^\nu j^\alpha \bar{\omega}\gamma_\mu \partial_\rho \omega \partial_\sigma \bar{\omega}\gamma_\alpha \partial_\beta \omega \varepsilon^{\nu\rho\sigma\beta} \\
& - 4iCj^\mu j^\rho j^\nu \bar{\omega}\gamma_\mu \partial^\sigma \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\sigma \partial_\rho \omega + 8iCj^\mu j^\rho j^\nu \bar{\omega}\gamma_\mu \partial^\sigma \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\rho \partial_\sigma \omega - 8iC(j\cdot j)j^\mu \bar{\omega}\gamma_\mu \partial^\nu \omega \partial_\nu \bar{\omega}\gamma_5 \not{\partial}\omega \\
& + 4iC(j\cdot j)j^\mu \bar{\omega}\gamma_\mu \partial^\nu \omega \partial^\rho \bar{\omega}\gamma_5\gamma_\nu \partial_\rho \omega + 4C(j\cdot j)j_\mu \bar{\omega}\not{\partial}\omega \partial_\nu \bar{\omega}\gamma_\rho \partial_\lambda \omega \varepsilon^{\mu\nu\lambda\rho} - 4iCj^\mu j^\nu j^\rho \bar{\omega}\not{\partial}\omega \partial_\rho \bar{\omega}\gamma_5\gamma_\mu \partial_\nu \omega \\
& + 4iC(j\cdot j)j^\mu \bar{\omega}\not{\partial}\omega \partial^\rho \bar{\omega}\gamma_5\gamma_\mu \partial_\rho \omega + 8iCj^\mu j^\nu j^\rho \bar{\omega}\gamma_5\gamma^\sigma \partial_\mu \omega \partial_\rho \bar{\omega}\gamma_\nu \partial_\sigma \omega - 8iCj^\mu j^\nu j^\rho \bar{\omega}\gamma_5\gamma_\mu \partial^\sigma \omega \partial_\rho \bar{\omega}\gamma_\nu \partial_\sigma \omega \\
& - 8iC(j\cdot j)j^\mu \bar{\omega}\gamma_5\gamma_\nu \partial^\rho \omega \partial_\rho \bar{\omega}\gamma_\mu \partial_\nu \omega - 4Cj^\mu j^\nu j_\rho \bar{\omega}\gamma_5\gamma_\sigma \partial_\lambda \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\alpha \partial_\mu \omega \varepsilon^{\rho\lambda\sigma\alpha} \\
& + 8Cj^\mu j^\nu j_\rho \bar{\omega}\gamma_5\gamma_\sigma \partial_\lambda \omega \partial_\nu \bar{\omega}\gamma_5\gamma_\mu \partial_\alpha \omega \varepsilon^{\rho\lambda\sigma\alpha} + 4C(j\cdot j)j_\mu \bar{\omega}\gamma_5\gamma_\nu \partial_\rho \omega \partial^\sigma \bar{\omega}\gamma_5\gamma_\alpha \partial_\sigma \omega \varepsilon^{\mu\rho\nu\alpha} \\
& - 8C(j\cdot j)j_\mu \bar{\omega}\gamma_5\gamma_\nu \partial_\rho \omega \partial_\alpha \bar{\omega}\gamma_5 \not{\partial}\omega \varepsilon^{\mu\rho\nu\alpha} - 4iCj_\mu j^\nu j^\rho \bar{\omega}\gamma^{\mu\sigma}\partial_\sigma \omega \partial_\rho \bar{\omega}\gamma_5 \partial_\nu \omega + 4iC(j\cdot j)j_\mu \bar{\omega}\gamma^{\mu\nu}\partial_\nu \omega \partial^\rho \bar{\omega}\gamma_5 \partial_\rho \omega \\
& - 8iCj_\mu j_\nu j^\rho \bar{\omega}\gamma^{\mu\sigma}\partial_\sigma \omega \partial_\rho \bar{\omega}\gamma_5\gamma^{\nu\alpha}\partial_\alpha \omega + 4iC(j\cdot j)j_\mu \bar{\omega}\gamma^{\mu\nu}\partial_\nu \omega \partial_\rho \bar{\omega}\gamma_5\gamma^{\rho\sigma}\partial_\sigma \omega \\
& + 4iCj_\mu j^\nu j^\rho \bar{\omega}\gamma_5\gamma^{\mu\sigma}\partial_\sigma \omega \partial_\nu \bar{\omega}\partial_\rho \omega - 4iC(j\cdot j)j^\mu \bar{\omega}\gamma_5\gamma_{\mu\nu}\partial_\nu \omega \partial^\rho \bar{\omega}\partial_\rho \omega \\
& + 8iCj_\mu j_\nu j^\rho \bar{\omega}\gamma_5\gamma^{\mu\sigma}\partial_\sigma \omega \partial_\rho \bar{\omega}\gamma^{\nu\alpha}\partial_\alpha \omega - 4iC(j\cdot j)j_\mu \bar{\omega}\gamma_5\gamma^{\mu\nu}\partial_\nu \omega \partial_\rho \bar{\omega}\gamma^{\rho\sigma}\partial_\sigma \omega, \qquad \text{(B5)}
\end{aligned}
$$

$$
\begin{aligned}
L_4 = {} & +4C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\mu \bar\omega \partial_\nu \omega \partial_\rho \bar\omega \partial_\sigma \omega + 16C^2 j^\mu j^\nu j_\rho j^\sigma \partial_\mu \bar\omega \partial_\nu \omega \partial_\sigma \bar\omega \gamma^{\rho\lambda} \partial_\lambda \omega - 8C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \partial_\nu \omega \partial_\rho \bar\omega \gamma^{\rho\sigma} \partial_\sigma \omega \\
& - 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \partial_\rho \omega \partial_\mu \bar\omega \partial_\nu \omega + 4C^2 (j\cdot j)^2 \partial^\mu \bar\omega \partial_\mu \omega \partial^\nu \bar\omega \partial_\nu \omega - 16C^2 (j\cdot j) j_\mu j^\nu \partial^\rho \bar\omega \partial_\rho \omega \partial_\nu \bar\omega \gamma^{\mu\sigma} \partial_\sigma \omega \\
& + 8C^2 (j\cdot j)^2 \partial^\mu \bar\omega \partial_\mu \omega \partial_\rho \bar\omega \gamma^{\rho\sigma} \partial_\sigma \omega - 4C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\mu \bar\omega \gamma_5 \partial_\nu \omega \partial_\rho \bar\omega \gamma_5 \partial_\sigma \omega \\
& - 16C^2 j^\mu j^\nu j_\rho j^\sigma \partial_\mu \bar\omega \gamma_5 \partial_\nu \omega \partial_\sigma \bar\omega \gamma_5 \gamma^{\rho\lambda} \partial_\lambda \omega + 8C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma_5 \partial_\nu \omega \partial_\rho \bar\omega \gamma_5 \gamma^{\rho\sigma} \partial_\sigma \omega \\
& + 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_5 \partial_\rho \omega \partial_\mu \bar\omega \gamma_5 \partial_\nu \omega - 4C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma_5 \partial_\mu \omega \partial^\nu \bar\omega \gamma_5 \partial_\nu \omega \\
& + 16C^2 (j\cdot j) j_\mu j^\nu \partial^\rho \bar\omega \gamma_5 \partial_\rho \omega \partial_\nu \bar\omega \gamma_5 \gamma^{\mu\sigma} \partial_\sigma \omega - 8C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma_5 \partial_\mu \omega \partial_\nu \bar\omega \gamma_5 \gamma^{\nu\rho} \partial_\rho \omega \\
& - 4C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma^\rho \partial^\sigma \omega \partial_\nu \bar\omega \gamma_\rho \partial_\sigma \omega - 4C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\mu \bar\omega \gamma_\nu \partial^\lambda \omega \partial_\rho \bar\omega \gamma_\sigma \partial_\lambda \omega \\
& + 4C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma^\rho \partial^\sigma \omega \partial_\nu \bar\omega \gamma_\sigma \partial_\rho \omega + 12C^2 j^\mu j^\nu j^\rho j^\sigma \partial^\lambda \bar\omega \gamma_\mu \partial_\nu \omega \partial_\rho \bar\omega \gamma_\sigma \partial_\lambda \omega \\
& + 12C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma^\sigma \partial_\mu \omega \partial_\nu \bar\omega \gamma_\sigma \partial_\rho \omega - 12C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma^\sigma \partial_\mu \omega \partial_\nu \bar\omega \gamma_\rho \partial_\sigma \omega \\
& - 24C^2 (j\cdot j) j^\mu j^\nu \partial^\sigma \bar\omega \gamma^\rho \partial_\mu \omega \partial_\rho \bar\omega \gamma_\nu \partial_\sigma \omega + 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_\mu \partial_\sigma \omega \partial_\rho \bar\omega \gamma^\sigma \partial_\nu \omega \\
& - 8iC^2 j^\mu j^\nu j^\rho j_\sigma \partial_\alpha \bar\omega \gamma_\mu \partial_\beta \omega \partial_\rho \bar\omega \gamma_5 \gamma_\lambda \partial_\nu \omega \varepsilon^{\sigma\alpha\beta\lambda} + 16iC^2 j^\mu j^\nu j^\rho j_\sigma \partial_\lambda \bar\omega \gamma_\mu \partial_\alpha \omega \partial_\rho \bar\omega \gamma_5 \gamma_\nu \partial_\beta \omega \varepsilon^{\sigma\lambda\alpha\beta} \\
& + 8iC^2 (j\cdot j) j^\mu j^\nu \partial_\rho \bar\omega \gamma_\mu \partial_\sigma \omega \partial^\lambda \bar\omega \gamma_5 \gamma_\alpha \partial_\lambda \omega \varepsilon^{\nu\rho\sigma\alpha} - 16iC^2 (j\cdot j) j^\mu j^\nu \partial_\rho \bar\omega \gamma_\mu \partial_\sigma \omega \partial_\alpha \bar\omega \gamma_5 \not{\partial} \omega \varepsilon^{\nu\rho\sigma\alpha} \\
& + 8C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma^\nu \partial^\rho \omega \partial_\mu \bar\omega \gamma_\nu \partial_\rho \omega - 16C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma^\nu \partial^\rho \omega \partial_\mu \bar\omega \gamma_\rho \partial_\nu \omega \\
& + 8iC^2 j^\mu j^\nu j^\rho j^\sigma \partial_\lambda \bar\omega \gamma_\alpha \partial_\beta \omega \partial_\rho \bar\omega \gamma_5 \gamma_\mu \partial_\nu \omega \varepsilon^{\sigma\lambda\beta\alpha} - 8iC^2 (j\cdot j) j^\mu j^\nu \partial_\rho \bar\omega \gamma_\sigma \partial_\lambda \omega \partial^\alpha \bar\omega \gamma_5 \gamma_\mu \partial_\alpha \omega \varepsilon^{\nu\rho\lambda\sigma} \\
& + 4C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\nu \bar\omega \gamma_5 \gamma^\lambda \partial_\mu \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\lambda \partial_\rho \omega - 8C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\nu \bar\omega \gamma_5 \gamma_\mu \partial^\lambda \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\lambda \partial_\rho \omega \\
& + 4C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\nu \bar\omega \gamma_5 \gamma_\mu \partial^\lambda \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\rho \partial_\lambda \omega + 8C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\mu \bar\omega \gamma_5 \not{\partial} \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\nu \partial_\rho \omega \\
& - 4C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma_5 \not{\partial} \omega \partial_\nu \bar\omega \gamma_5 \not{\partial} \omega - 8C^2 j^\mu j^\nu j^\rho j^\sigma \partial^\lambda \bar\omega \gamma_5 \gamma_\mu \partial_\nu \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\lambda \partial_\rho \omega \\
& + 12C^2 j^\mu j^\nu j^\rho j^\sigma \partial^\lambda \bar\omega \gamma_5 \gamma_\mu \partial_\nu \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\rho \partial_\lambda \omega - 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_5 \gamma_\mu \partial_\nu \omega \partial_\rho \bar\omega \gamma_5 \not{\partial} \omega \\
& + 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_5 \gamma_\mu \partial_\nu \omega \partial^\sigma \bar\omega \gamma_5 \gamma_\rho \partial_\sigma \omega + 8C^2 j^\mu j^\nu j^\rho j^\sigma \partial_\mu \bar\omega \gamma_5 \not{\partial} \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\nu \partial_\rho \omega \\
& - 12C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma_5 \not{\partial} \omega \partial_\nu \bar\omega \gamma_5 \not{\partial} \omega + 8C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma_5 \not{\partial} \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\nu \partial_\sigma \omega \\
& - 16C^2 j^\mu j^\nu j^\rho j^\sigma \partial^\lambda \bar\omega \gamma_5 \gamma_\mu \partial_\lambda \omega \partial_\sigma \bar\omega \gamma_5 \gamma_\nu \partial_\rho \omega + 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_5 \gamma_\mu \partial_\rho \omega \partial_\nu \bar\omega \gamma_5 \not{\partial} \omega \\
& - 8C^2 (j\cdot j) j^\mu j^\nu \partial_\mu \bar\omega \gamma_5 \gamma^\rho \partial_\nu \omega \partial^\sigma \bar\omega \gamma_5 \gamma^\rho \partial_\sigma \omega + 8C^2 (j\cdot j) j^\mu j^\nu \partial^\rho \bar\omega \gamma_5 \gamma^\sigma \partial_\rho \omega \partial_\nu \bar\omega \gamma_5 \gamma_\mu \partial_\sigma \omega \\
& - 16C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma_5 \gamma^\nu \partial_\mu \omega \partial_\nu \bar\omega \gamma_5 \not{\partial} \omega + 4C^2 (j\cdot j)^2 \partial^\mu \bar\omega \gamma_5 \gamma^\nu \partial_\mu \omega \partial^\rho \bar\omega \gamma_5 \gamma_\nu \partial_\rho \omega \\
& + 16C^2 (j\cdot j) j^\mu j^\nu \partial^\sigma \bar\omega \gamma_5 \not{\partial} \omega \partial_\nu \bar\omega \gamma_5 \gamma_\sigma \partial_\mu \omega - 24C^2 (j\cdot j) j^\mu j^\nu \partial^\sigma \bar\omega \gamma_5 \not{\partial} \omega \partial_\nu \bar\omega \gamma_5 \gamma_\mu \partial_\sigma \omega \\
& + 16C^2 (j\cdot j)^2 \partial^\nu \bar\omega \gamma_5 \not{\partial} \omega \partial_\nu \bar\omega \gamma_5 \not{\partial} \omega + 16C^2 j_\mu j^\nu j_\rho j^\sigma \partial_\nu \bar\omega \gamma^{\mu\lambda} \partial_\lambda \omega \partial_\sigma \bar\omega \gamma^{\rho\alpha} \partial_\alpha \omega \\
& - 16C^2 (j\cdot j) j_\mu j^\nu \partial_\nu \bar\omega \gamma^{\mu\rho} \partial_\rho \omega \partial_\sigma \bar\omega \gamma^{\sigma\lambda} \partial_\lambda \omega + 4C^2 (j\cdot j)^2 \partial_\mu \bar\omega \gamma^{\mu\nu} \partial_\nu \omega \partial_\rho \bar\omega \gamma^{\rho\sigma} \partial_\sigma \omega \\
& - 16C^2 j_\mu j^\nu j_\rho j^\sigma \partial_\nu \bar\omega \gamma_5 \gamma^{\mu\lambda} \partial_\lambda \omega \partial_\sigma \bar\omega \gamma_5 \gamma^{\rho\alpha} \partial_\alpha \omega + 16C^2 (j\cdot j) j^\mu j^\nu \partial_\nu \bar\omega \gamma_5 \gamma^{\mu\rho} \partial_\rho \omega \partial_\sigma \bar\omega \gamma_5 \gamma^{\sigma\lambda} \partial_\lambda \omega \\
& - 4C^2 (j\cdot j)^2 \partial_\mu \bar\omega \gamma_5 \gamma^{\mu\nu} \partial_\nu \omega \partial_\rho \bar\omega \gamma_5 \gamma^{\rho\sigma} \partial_\sigma \omega.
\end{aligned} \tag{B6}
$$

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