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Renormalization: The observable-state model. II

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The purpose of this work is to rewrite the generating functional of ϕ^4 theory for the $n = 0$ and $n = 4$ correlation functions as the inner product of a state with an observable, as we did in a previous work, for the two-points correlation function. The observables are defined through the external sources and the states are defined through the correlation function itself. In this sense, the divergences of Quantum Field Theory (QFT) appear in the reduced state by taking the partial trace of the state with respect to the internal vertices that appear in the perturbation expansion. From this viewpoint, the renormalization can be substituted by applying a projector on the internal quantum state. The advantage of this new insight is that we can obtain finite contributions to the correlation functions without introducing counterterms in the Lagrangian or by manipulating complex divergent quantities.

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I. INTRODUCTION

This paper, as its predecessor, develops the perturbation expansion of any correlation function in terms of the mean values of some observables, in particular, states as we did in [1].[1] In fact, our formalism produces unphysical infinities in the form of $[\delta(0)]^k$ that will be represented in a dimensional regularization scheme by the poles $\frac{1}{\epsilon^k}$, where $\epsilon = d - 4$ and d is the space-time dimension.[2] These infinities arise because the quantum state associated to the internal vertices of the perturbation expansion has a diagonal part in the coordinate basis. In [1] we have shown that we can simply disregard these unphysical infinities by applying a projection operator on the quantum states. The finite results found coincide with those of the usual renormalized QFT in several models (and we will present more coincidences in this and forthcoming papers). In this sense, it seems that throwing away the unphysical infinities due to the short-distance behavior through the projector is, after all, a good method. These ideas agree with those introduced in [3] (vol. 1, page 499): QFT yields divergent integrals "but these infinities cancel when we express all the parameters of the theory in renormalized quantities, such as the masses and the charges that we actually measure". Moreover, it also coincides with [4], since we believe that the process of subtracting infinities is really a matter of subtracting the irrelevant effect of the "perhaps poorly understood physics at high energy or short scale to obtain the meaningful physics at the scales actually studied in the laboratory" ([4], page 254). In this sense, the constraining is done by neglecting the physics of high energy or short scale.

[1]This idea has been called "the observable-state model".
[2]The equivalence between $\delta(0)$ and $\frac{1}{\epsilon}$ can be found in Quantum Field Theory textbooks, like [2], page 352, below eq. (11.55). In Appendix A we show how to obtain this equivalence in a formal way.

A. List of sections

The paper is organized as follows: In Sec. II we will explicitly show how to define the observables and states in a general way and the projection procedure. In Sec. III we will show how to describe the $n = 0$ correlation function in ϕ^4 theory using the observables and states. In Sec. IV we show in a similar way how to handle the observable-state model for the $n = 4$ correlation function. In particular, we show how the renormalization group of the coupling constant arises. In Sec. V we show how to obtain the renormalization group equations for the mass and the coupling constant using the finite contribution of the correlation function obtained by application of the projector on the quantum state. In Sec. VI we briefly discuss the conceptual meaning of the reduced state and partial traces and its relation with the nonphysical virtual particles and we introduce some general ideas of the observable-state model. In Sec. VII we present the conclusions. The Appendix A shows the relation between the Dirac delta and the pole parameter representation of the dimensional regularization. The appendix B shows how to obtain the relation between the vacuum energy and the space volume. Finally, in Appendix C we analyze the properties of the projector that gives the finite contribution in each correlation function.

II. OBSERVABLES AND STATES IN QUANTUM FIELD THEORY: THE MAIN IDEA

Let us recall the main idea of the observable-state model of paper [1], that can be considered as the first part of this paper, and that will be used in this section. The starting point is some (symmetric) n-point functions $\tau^{(n)}(x_1, \ldots, x_n)$ (like Feynman or Euclidean functions), and its corresponding generating functional [[5], Eq. (II.2.21), [6], Eq. (3.2.11)]. Then, the main equation reads

$$iZ[J] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{i^n}{n!}\frac{i^p}{p!} \times \int \langle\Omega_0|T\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_I^0(y_1)\ldots\mathcal{L}_I^0(y_p)|\Omega_0\rangle \times J(x_1)\ldots J(x_n)\prod_{i=1}^{n}d^4x_i\prod_{i=1}^{p}d^4y_i, \tag{1}$$

where y_i are the internal vertices of the perturbation expansion and $\mathcal{L}_I^0(y_p)$ is the Lagrangian interaction density [see Eq. (II.2.33) of [5]].

This last equation will be our starting point, we will write $Z[J]$ as an mean value of an observable defined through the $J(x_n)$ sources in a quantum state defined by the correlation function $\langle\Omega_0|T\phi(x_1)\ldots\phi(x_n)\mathcal{L}_I^0(y_1)\ldots\mathcal{L}_I^0(y_p)|\Omega_0\rangle$.[3] This procedure will be done for each correlation function of n external points.

Using dimensional regularization (see [8]) we can write the one-particle irreducible contribution to the correlation function such that (see [9] for ϕ^4 theory):

$$\int\langle\Omega_0|T\phi(x_1)\ldots\phi(x_n)\mathcal{L}_I^0(y_1)\ldots\mathcal{L}_I^0(y_p)|\Omega_0\rangle\prod_{i=1}^{p}d^4y_i = f_0^{(n)}(x_1,\ldots,x_n)\sum_{l=-L(n,p)}^{+\infty}\beta_l^{(n,p)}(m_0^2,\mu)\epsilon^l, \tag{2}$$

where $f_0^{(n)}$ is some function of the external points, $\beta_l^{(n,p)}(m_0^2,\mu)$ are some coefficients of the dimensional regularization that depends on the external momentum, the mass factor μ used to keep the coupling constant dimensionless and the mass of the field m_0. The parameter ϵ is $\epsilon = d-4$, where d is the dimension of space-time. The sum in l starts at $-L(n,p)$, where $L(n,p)$ is the number of loops at order p in the correlation functions of n external points (see Appendix A, Eq. A6 of [1]). The functions $f_0^{(n)}$ and $L(n,p)$ are very simple in the case of ϕ^4 theory, for example,

(i) $n=0$

$$f_0^{(0)} = 1, \qquad L(0,p) = p+1 \tag{3}$$

(ii) $n=2$

$$f_0^{(2)} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m_0^2)^2}, \quad L(2,p) = p \tag{4}$$

(iii) $n=4$

$$f_0^{(4)} = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_4)}}{p^2-m_0^2}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_2-x_4)}}{q^2-m_0^2} \times\int\frac{d^4l}{(2\pi)^4}\frac{e^{-il(x_3-x_4)}}{(l^2-m_0^2)((p+q+l)^2-m_0^2)},$$
$$L(4,p) = p-1. \tag{5}$$

In general

$$f_0^{(n)} = \prod_{i=1}^{n}\int\frac{d^4p_i}{(2\pi)^4}\frac{e^{-ip_ix_i}}{p_i^2-m_0^2}\delta\Big(\sum_{j=1}^{n}p_j^2\Big), \quad L(n,p) = p-\frac{n}{2}+1 \tag{6}$$

Inserting Eq. (2) in Eq. (1) we obtain[4]

$$iZ[J] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{i^n}{n!}\frac{i^p}{p!}\sum_{l=-L(n,p)}^{0}\beta_l^{(n,p)}\epsilon^l \times\int f_0^{(n)}(x_1,\ldots,x_n)J(x_1)\ldots J(x_n)\prod_{i=1}^{n}d^4x_i. \tag{7}$$

The observable-state model consist in the assumption that the generating functional of the last equation can be rewritten as a mean value of the following observable:

$$O^{(n,p)} = O_{\rm ext}^{(n)}\otimes I_{\rm int}^{(p)}, \tag{8}$$

in the following quantum state:

$$\rho^{(n,p)} = \rho_{\rm ext}^{(n)}\otimes\rho_{\rm int}^{(n,p)}, \tag{9}$$

where $O_{\rm ext}^{(n)}$ is some observable that acts on the external coordinates x_i and $I_{\rm int}^{(p)}$ is the identity operator that acts on the internal vertices due to the perturbation expansion. In a similar way, $\rho_{\rm ext}^{(n)}$ is the quantum state of the external part and $\rho_{\rm int}^{(n,p)}$ is the quantum state of the internal part.

Then, the mean value of $O^{(n,p)}$ in $\rho^{(n,p)}$ reads

$$\mathrm{Tr}(\rho^{(n,p)}O^{(n,p)}) = \mathrm{Tr}(\rho_{\rm ext}^{(n)}O_{\rm ext}^{(n)})\mathrm{Tr}(\rho_{\rm int}^{(n,p)}). \tag{10}$$

[3] In some sense, these observables will be the particle detector [see [7], page 6, below Eq. (2.6)].

[4] The infinite sum in the l index in Eq. (2) can be truncated in $l=0$, because the remaining terms are proportional to ϵ^l and the final result must be computed by taking the $\epsilon\to 0$ limit. In this sense, what concern us is the principal part plus the constant term of the Laurent series with poles $d-4$.

Using the last equation, the generating functional of Eq. (7) can be written as

$$iZ[J] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{i^n}{n!}\frac{i^p}{p!}\mathrm{Tr}(\rho^{(n,p)}O^{(n,p)}) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{i^n}{n!}\frac{i^p}{p!}\mathrm{Tr}(\rho_{\mathrm{int}}^{(n,p)})\mathrm{Tr}(\rho_{\mathrm{ext}}^{(n)}O_{\mathrm{ext}}^{(n)}), \quad (11)$$

where

$$\rho_{\mathrm{ext}}^{(n)} = \int f_0^{(n)}(x_1,\ldots,x_n)|x_1,\ldots,x_{(n/2)}\rangle\langle x_{(n/2)+1},\ldots,x_n|\prod_{i=1}^{n}d^4x_i \quad (12)$$

and

$$O_{\mathrm{ext}}^{(n)} = \int J(x_1)\ldots J(x_n)|x_1,\ldots,x_{(n/2)}\rangle\langle x_{(n/2)+1},\ldots,x_n|\prod_{i=1}^{n}d^4x_i. \quad (13)$$

In turn

$$\mathrm{Tr}\,(\rho_{\mathrm{int}}^{(n,p)}) = \sum_{l=-L(n,p)}^{+\infty}\beta_l^{(n,p)}\epsilon^l \quad (14)$$

which implies that the divergences of the quantum field theory are the consequence of taking the trace of the internal quantum state $\rho_{\mathrm{int}}^{(n,p)}$. This point is relevant; because the trace of an operator is an invariant quantity, this means that it is the same in different bases. This implies that if we want to obtain a finite contribution $\beta_0^{(n,p)}$, we must apply a nonunitary transformation on $\rho_{\mathrm{int}}^{(n,p)}$ that changes its trace, i.e., we must project to another ρ_{int}.

A. Internal quantum state

To define the internal quantum state we will just recall some considerations (see Sec. 6 in [1]): the algebra of observables $\mathcal{O}$ is represented by *-algebra $\mathcal{A}$ of self-adjoint elements and states are represented by functionals on $\mathcal{O}$, that is, by elements of the dual space $\mathcal{O}'$, $\rho \in \mathcal{O}'$. We will construct a C^*-algebra of operators defined in terms of elements with the property $\mathrm{Tr}(A^*A) < \infty$. As is well known, a C^*-algebra can be represented in a Hilbert space $\mathcal{H}$ (GNS theorem)[5] and, in this particular case $\mathcal{O} = \mathcal{O}'$; therefore $\mathcal{O}$ and $\mathcal{O}'$ are represented by $\mathcal{H}\otimes\mathcal{H}$ that will be called $\mathcal{N}$, the Liouville space.

As we are interested in the diagonal and nondiagonal elements of a matrix state we can define a subalgebra of $\mathcal{N}$, that can be called a van Hove algebra [11] since such a structure appears in his work as

$$\mathcal{N}_{vh} = \mathcal{N}_S \oplus \mathcal{N}_R \subset \mathcal{N}, \quad (15)$$

where the vector space $\mathcal{N}_R$ is the space of operators with $O(x) = 0$ and $O(x,x')$ is a regular function. Moreover $\mathcal{O} = \mathcal{N}_{vhS}$ is the space of self-adjoint operators of $\mathcal{N}_{vh}$, which can be constructed in such a way it could be dense in $\mathcal{N}_S$ (because any distribution can be approximated by regular functions) (for the details see [1], Sec. II.B and Sec. VI). Therefore essentially the introduced restriction is the minimal possible coarse-graining. Now the $\oplus$ is a direct sum because $\mathcal{N}_S$ contains the factor $\delta(x-x')$ and $\mathcal{N}_R$ contains just regular functions and a kernel cannot be both a δ and a regular function. Moreover, as our observables must be self-adjoint, the space of observables must be

$$\mathcal{O} = \mathcal{N}_{vhS} = \mathcal{N}_S \oplus \mathcal{N}_R \subset \mathcal{N}. \quad (16)$$

The states must be considered as linear functionals over the space $\mathcal{O}$ ($\mathcal{O}'$ the dual of space $\mathcal{O}$),

$$\mathcal{O}' = \mathcal{N}'_{vhS} = \mathcal{N}'_S \oplus \mathcal{N}'_R \subset \mathcal{N}' \quad (17)$$

The set of these generalized states is the convex set $\mathcal{S} \subset \mathcal{O}'$.

Having this in mind, we can define the internal quantum state in the following way:

$$\rho_{\mathrm{int}}^{(n,p)} = \int \prod_{i=1}^{L(n,p)}(\rho_D^{(n,p,i)}(y_i)\delta(y_i - w_i) + \rho_{ND}^{(n,p,i)}(y_i, w_i)) \times |y_1,\ldots,y_{L(n,p)}\rangle\langle w_1,\ldots,w_{L(n,p)}|\prod_{i=1}^{L(n,p)}d^4y_i d^4w_i. \quad (18)$$

The trace reads [see Appendix B, Eq. (A7)]

$$\mathrm{Tr}\,(\rho_{\mathrm{int}}^{(n,p)}) = \prod_{i=1}^{L(n,p)}\left(\frac{\rho_D^{(n,p,i)}}{\pi\epsilon} + \rho_{ND}^{(n,p,i)}\right), \quad (19)$$

where

$$\rho_D^{(n,p,i)} = \int \rho_D^{(n,p,i)}(y_i)d^4y_i \qquad \rho_{ND}^{(n,p,i)} = \int \rho_{ND}^{(n,p,i)}(y_i,y_i)d^4y_i. \quad (20)$$

We can see from the last equation that $\rho_D^{(n,p,i)}$ and $\rho_{ND}^{(n,p,i)}$ are merely normalization factors. Equation (19) can be written as

$$\mathrm{Tr}\,(\rho_{\mathrm{int}}^{(n,p)}) = \sum_{l=-L(n,p)}^{0}\gamma_l^{(n,p)}\epsilon^l, \quad (21)$$

where

$$\gamma_0^{(n,p)} = \prod_{i=1}^{L(n,p)}\rho_{ND}^{(n,p,i)},\ldots, \qquad \gamma_{L(n,p)}^{(n,p)} = \frac{1}{\pi^{L(n,p)}}\prod_{i=1}^{L(n,p)}\rho_D^{(n,p,i)}. \quad (22)$$

[5] Gelfand, Naimark, and Segal [10].

All the terms $\gamma_l^{(n,p)}$ with $l > 0$ that are multiplied by ϵ^l contain at least one $\rho_D^{(n,p,i)}$, that is, the diagonal part of the state of the i-internal quantum system. In particular, we can make the following equality:

$$\beta_l^{(n,p)} = \gamma_l^{(n,p)}. \tag{23}$$

In this sense, the coefficients obtained by the dimensional regularization can be associated with the products of the diagonal and nondiagonal parts of the internal quantum state. In particular, the coefficient that is not multiplied by a ϵ is $\gamma_0^{(n,p)}$ which depends exclusively on the nondiagonal quantum state.

B. Projection over the finite contribution

As we saw in Eqs. (21) and (22), the finite result exclusively depends on the nondiagonal quantum state, so we can construct a projector that projects over the nondiagonal quantum state. This projector reads[6]

$$\begin{aligned}\Pi_p(\rho_{\rm int}^{(n,p)}) = \rho_{\rm int}^{(n,p)} &- \int \rho_D^{(n,p,1)}(y_1)\rho_D^{(n,p,2)}(y_2)\ldots\rho_D^{(n,p,L(n,p))}(y_{L(n,p)})|y_1,\ldots,y_{L(n,p)}\rangle\langle y_1,\ldots,y_{L(n,p)}|\prod_{i=1}^{L(n,p)} d^4y_i \\ &+ \int \rho_D^{(n,p,1)}(y_1)\rho_D^{(n,p,2)}(y_2)\ldots\rho_D^{(n,p,L(n,p)-1)}(y_{L(n,p)-1})\rho_{ND}^{(n,p,L(n,p))}(y_{L(n,p)}, w_{L(n,p)}) \\ &\times |y_1,\ldots,y_{L(n,p)}\rangle\langle y_1,\ldots,w_{L(n,p)}|d^4w_{L(n,p)}\prod_{i=1}^{L(n,p)-1} d^4y_i + \ldots + \int \rho_D^{(n,p,1)}(y_1)\rho_{ND}^{(n,p,2)}(y_2,w_2)\ldots\rho_{ND}^{(n,p,L(n,p))} \\ &\times (y_{L(n,p)}, w_{L(n,p)})|y_1,\ldots,y_{L(n,p)}\rangle\langle y_1,\ldots,w_{L(n,p)}|d^4y_1\prod_{i=2}^{L(n,p)} d^4y_i d^4w_i).\end{aligned} \tag{24}$$

The projection procedure consists in the subtraction of the part of the state that contains at least one internal diagonal quantum state. This projector acting on the state $\rho^{(n,p)}$ yields

$$\Pi_p(\rho_{\rm int}^{(n,p)}) = \int \prod_{i=1}^{L(n,p)} \rho_{ND}^{(n,p,i)}(y_i,w_i)|y_1,\ldots,y_{L(n,p)}\rangle \times \langle w_1,\ldots,w_{L(n,p)}|\prod_{i=1}^{L(n,p)} d^4y_i d^4w_i. \tag{25}$$

Then, using the equivalence of Eq. (23), the mean value of $O^{(n,p)}$ in the state $\Pi_p(\rho^{(n,p)})$ reads

$$\mathrm{Tr}(\Pi_p(\rho^{(n,p)}O^{(n,p)}) = \beta_0^{(n,p)}\int f_0^{(n)}(x_1,\ldots,x_n)O_{ext}^{(n)}(x_1,\ldots,x_n)\prod_{i=1}^{n} d^4x_i, \tag{26}$$

where $O_{\rm ext}^{(n)}(x_1,\ldots,x_n) = J(x_1)\ldots J(x_n)$ [see Eq. (13)]. Multiplying by $\frac{i^p}{p!}$ and summing in p we obtain[7]

$$\begin{aligned}\mathrm{Tr}(\rho^{(n)}O_{\rm ext}^{(n)}) &= \sum_{p=0}^{+\infty}\frac{i^p}{p!}\mathrm{Tr}(\Pi_p(\rho^{(n,p)})O^{(n,p)}) \\ &= \sum_{p=0}^{+\infty}\frac{i^p}{p!}\beta_0^{(n,p)}\int f_0^{(n)}(x_1,\ldots,x_n)O_{\rm ext}^{(n)}(x_1,\ldots,x_n) \\ &\quad\times\prod_{i=1}^{n} d^4x_i,\end{aligned} \tag{27}$$

where

$$\rho^{(n)} = \left(\sum_{p=0}^{+\infty}\frac{i^p}{p!}\beta_0^{(n,p)}\right)\rho_{\rm ext}^{(n)}, \tag{28}$$

where $\frac{i^p}{p!}\beta_0^{(n,p)}$ is the coefficient of the quantum state $\rho_{\rm ext}^{(n)}$.

In this way, we can eliminate all the divergences of the observable-state model by the application of the projector over a well-defined Hilbert subspace. This formalism has been applied to the two-point correlation function for ϕ^4 theory (see [1]) and the idea of this work is to apply it to $n = 0$ and $n = 4$ correlation function of external points. In Appendix C we briefly show the relation between the projector and the R-operation of the *BPHZ* subtraction method in QFT.

III. EXAMPLES: ϕ^4 THEORY, $n=0$

In this section we will briefly study the vacuum amplitude for the ϕ^4 theory. When there are interactions, the vacuum amplitude reads (see [12], page 87):

[6] Is not difficult to show that it is a projector: linearity implies that $\Pi(a+b) = \Pi(a) + \Pi(b)$, then, if $\Pi(a) = a - G$, then, $\Pi^2(a) = \Pi(a-G) = \Pi(a) - \Pi(G)$, but $\Pi(G) = G - G = 0$, then $\Pi^2(a) = \Pi(a)$.

[7] The factor $\frac{i^p}{p!}$ is introduced for later convenience, but its meaning could be that in the observable-state model, the quantum state is invariant under an exchange of internal vertices.

$$\langle\Omega|\Omega\rangle = (|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T})^{-1} \times \left\langle \Omega_0 \left| \exp\left(-i\int_{-T}^{T} dt H_I(t)\right) \right| \Omega_0 \right\rangle, \quad (29)$$

where $|\Omega\rangle$ is the vacuum vector for the interacting theory, $|\Omega_0\rangle$ is the vacuum vector for the free theory, $E_0 = \langle\Omega|H|\Omega\rangle$ is the energy of the vacuum state of the interacting theory, H is the full Hamiltonian $H = H_0 + H_I$, where H_I is the interacting Hamiltonian, and $2T$ is the time interval where the process occurs. The brackets in Eq. (29) can be written in terms of the perturbation expansion in the coupling constant λ_0:

$$\left\langle \Omega_0 \left| \exp\left(-i\int_{-T}^{T} dt H_I(t)\right) \right| \Omega_0 \right\rangle = 1 + (-i\lambda_0)\int d^4y_1 \langle\Omega_0|\phi^4(y_1)|\Omega_0\rangle + (-i\lambda_0)^2 \times \int d^4y_1 d^4y_2 \langle\Omega_0|\phi^4(y_1)\phi^4(y_2)|\Omega_0\rangle + \ldots + (-i\lambda_0)^p \times \int d^4y_1 \ldots d^4y_p \langle\Omega_0|\phi^4(y_1)\ldots\phi^4(y_p)|\Omega_0\rangle + \ldots . \quad (30)$$

The structure of the vacuum amplitude in terms of the perturbation expansion can be obtained, to do so we will consider the first order in the perturbation expansion. We just recall that we will compute the connected diagrams and not the products of them.

The first order $p = 1$ reads

$$(-i\lambda_0)\int d^4y_1 \langle\Omega_0|\phi^4(y_1)|\Omega_0\rangle = i\lambda_0[\Delta(0)]^2 \int d^4y_1 = i\lambda_0[\Delta(0)]^2 2TV, \quad (31)$$

where V is the volume of space and $\Delta(0)$ is the Feynman propagator of a scalar field. Using dimensional regularization, Eq. (31) reads

$$(-i\lambda_0)\int d^4y_1 \langle\Omega_0|\phi^4(y_1)|\Omega_0\rangle = i\lambda_0 2TV\left(\frac{\beta_2^{(0,1)}}{\epsilon^2} + \frac{\beta_1^{(0,1)}}{\epsilon} + \beta_0^{(0,1)}\right), \quad (32)$$

where the coefficients $\beta_i^{(0,1)}$ are some constants that can be obtained from the regularized propagator $\Delta(0)$ and depend on a mass factor μ that is introduced to keep the coupling constant dimensionless, this is, we must replace λ_0 by $\lambda_0(\mu^{-\epsilon})$.[8] The first superscript 0 in β refers to the number of external points and the second superscript 1 refers to the order in the perturbation expansion. The subscript refers to the power of the $\epsilon = d - 4$ factor, where d is the dimension of space-time. Using Eq. (A.44) of Appendix A.4 of [12], page 807, the coefficients $\beta_k^{(0,1)}$ read

$$\beta_2^{(0,1)} = \frac{m_0^4}{64\pi^4}$$
$$\beta_1^{(0,1)} = \frac{m_0^4}{64\pi^4}\left(\gamma - 1 + \ln\left(\frac{m_0^2}{4\pi\mu}\right)\right)$$
$$\beta_0^{(0,1)} = \frac{m_0^4}{24\cdot 64\pi^4}\left(18 - 24\gamma + 12\gamma^2 + \pi^2 + 12(\ln^2(m_0^2) - \ln^2(4\pi) + \ln^2(\mu)) + 24(1-\gamma+\ln(4\pi))\ln\left(\frac{4\pi\mu}{m_0^2}\right)\right). \quad (33)$$

The second order $p = 2$ in the perturbation expansion has three terms, where two of them are connected,

$$(-i\lambda_0)^2 \int d^4y_1 d^4y_2 \langle\Omega_0|\phi^4(y_1)\phi^4(y_2)|\Omega_0\rangle = (-i\lambda_0)^2[\Delta(0)]^2 \int d^4y_1 d^4y_2 [\Delta(y_1 - y_2)]^2 + (-i\lambda_0)^2 \int d^4y_1 d^4y_2 [\Delta(y_1 - y_2]^4. \quad (34)$$

It can be shown that the following orders for the connected Feynman diagrams in the perturbation expansion can be accommodated following Eq. (32)[9]:

$$(-i\lambda_0)^p \int d^4y_1 \ldots d^4y_p \langle\Omega_0|\phi^4(y_1)\ldots\phi^4(y_p)|\Omega_0\rangle = \sum_{j=0}^{p+1} (-i\lambda_0)^p i^{2p}(2TV)\frac{\beta_j^{(0,p)}}{\epsilon^j}, \quad (35)$$

where i^{2p} comes from $2p$ propagators that can be obtained from the vacuum expectation values of the $4p$ quantum fields. If we want to compute Eq. (29) we must consider the nonconnected Feynman diagrams that can be constructed by multiplying the connected ones. For example, for the second order $p = 2$ we can obtain the nonconnected Feynman diagram by multiplying by itself the first order

[8] Is not difficult to show that the coupling constant has dimension $[\lambda_0] = [\text{mass}]^{4-d}$ where d is the dimension of space-time (see [12], page 322). Then, the mass factor $\mu^{-(4-d)}$ multiplied to λ_0 maintains the new coupling constant dimensionless. A dimensionless coupling constant is necessary because it is the parameter we use to apply the perturbation expansion.

[9] The general solution showed in Eq. (35) can be traced to general results which appear in the dimensional regularization scheme (see [9], pages 103–130 and [13], page 686).

$p = 1$. This procedure can be done for all the orders, in particular, to obtain the nonconnected Feynman diagrams at order p we must multiply all the lowest orders where the sum of them gives p. If we call the result of Eq. (35) as $f(p)$, then, the sum of the connected diagrams and the nonconnected diagrams reads

$$\sum_{k=0}^{+\infty}\frac{1}{k!}\left(\sum_{p=1}^{+\infty}f(p)\right)^k = f(1)+f(2)+\ldots+f(p)+\ldots + \frac{1}{2!}(f(1)+f(2)+\ldots)(f(1)+f(2)+\ldots)+\ldots, \quad (36)$$

the factor $\frac{1}{k!}$ is introduced to avoid double counting, for example, $f(i)f(j)$ and $f(j)f(i)$. With this result, we can proceed to evaluate Eq. (29),

$$|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T} = \sum_{k=0}^{+\infty}\frac{1}{k!}\left(\sum_{p=1}^{+\infty}\sum_{j=0}^{p+1}(-i\lambda_0)^p i^{2p}(2TV)\frac{\beta_j^{(0,p)}}{\epsilon^j}\right)^k, \quad (37)$$

where we have put $\langle\Omega|\Omega\rangle = 1$ and we have introduced the result of Eq. (35) in $f(p)$. The projection procedure will be given by only keeping the $j = 0$ term in Eq. (37) as we will show in the following section. We then have

$$|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T} = \sum_{k=0}^{+\infty}\frac{(2TV)^k}{k!}\left(\sum_{p=1}^{+\infty}(-i\lambda_0)^p i^{2p}\beta_0^{(0,p)}\right)^k. \quad (38)$$

In Appendix A we show how to obtain the relation between the vacuum energy E_0 and the volume of space V in a formal way. This result has no direct relation with the aim of this work, but is a contribution to the observable-state model.

A. The observable-state model for $n=0$ in ϕ^4

Now we can apply this mathematical structure to the case of vacuum bubbles in ϕ^4 theory, where we can use Eq. (18) in the case $n = 0$, then,

$$\rho^{(0,p)} = \rho_{\mathrm{int}}^{(0,p)} = \int\prod_{i=1}^{p+1}(\rho_D^{(0,p,i)}(y_i)\delta(y_i-w_i)+\rho_{ND}^{(0,p,i)}(y_i,w_i)) \times|y_1,\ldots,y_{p+1}\rangle\langle w_1,\ldots,w_{p+1}|\prod_{i=1}^{p+1}d^4y_i d^4w_i, \quad (39)$$

where $\rho_D^{(0,p,i)}$ and $\rho_{ND}^{(0,p,i)}$ are some regular functions. The trace $\mathrm{Tr}(\rho_{\mathrm{int}}^{(0,p)})$ reads

$$\mathrm{Tr}\,(\rho_{\mathrm{int}}^{(0,p)}) = \sum_{l=-(p+1)}^{0}\gamma_l^{(0,p)}\epsilon^l, \quad (40)$$

where in particular,

$$\gamma_0^{(0,p)} = \prod_{i=1}^{p+1}\rho_{ND}^{(0,p,i)},\ldots, \qquad \gamma_{p+1}^{(0,p)} = \frac{1}{\pi^{p+1}}\prod_{i=1}^{p+1}\rho_D^{(0,p,i)}, \quad (41)$$

and the remaining coefficients $\gamma_l^{(0,p)}$ with $p+1 > l > 1$ contains at least one ρ_D.

Comparing Eq. (41) with Eq. (35) we can see that the coefficients $\gamma_l^{(0,p)}$ read

$$\gamma_l^{(0,p)} = (-i\lambda_0)^p i^{2p}(2TV)\beta_l^{(0,p)}. \quad (42)$$

In the first order in the perturbation expansion, using Eq. (40) and (41) we have

$$\sum_{l=0}^{2}\gamma_l^{(0,2)}\epsilon^{-l} = \gamma_0^{(0,2)} + \gamma_1^{(0,2)}\epsilon^{-1} + \gamma_2^{(0,2)}\epsilon^{-2} = \rho_{ND}^{(0,2,1)}\rho_{ND}^{(0,2,2)} + (\rho_D^{(0,2,1)}\rho_{ND}^{(0,2,2)} + \rho_D^{(0,2,2)}\rho_{ND}^{(0,2,1)})\epsilon^{-1} + \rho_D^{(0,2,1)}\rho_D^{(0,2,2)}\epsilon^{-2}. \quad (43)$$

Using Eq. (33) and (42) we have that

$$\begin{aligned}
\rho_D^{(0,2,1)}\rho_D^{(0,2,2)} &= \frac{i\lambda_0 2TVm_0^4}{64\pi^4}\\
\rho_D^{(0,2,1)}\rho_{ND}^{(0,2,2)} + \rho_D^{(0,2,2)}\rho_{ND}^{(0,2,1)} &= \frac{i\lambda_0 2TVm_0^4}{64\pi^4}\left(-1+\gamma+\ln\left(\frac{m_0^2}{4\pi\mu}\right)\right)\\
\rho_{ND}^{(0,2,1)}\rho_{ND}^{(0,2,2)} &= \frac{i\lambda_0 2TVm_0^4}{24\cdot 64\pi^4}\bigg(18-24\gamma+12\gamma^2+\pi^2+12(\ln^2(m_0^2)-\ln^2(4\pi)+\ln^2(\mu))\\
&\quad + 24(1-\gamma+\ln(4\pi))\ln\left(\frac{4\pi\mu}{m_0^2}\right)\bigg).
\end{aligned} \quad (44)$$

This implies that the diagonal and nondiagonal quantum states are not well determined. In this case, we have four unknown quantities and three equations. As we saw in Sec. II, the finite contribution for the correlation function comes from the nondiagonal quantum states, so the indetermination can be translate to an arbitrary election of one of the nondiagonal quantum states. The indetermination will grow up with the order of the perturbation expansion;

in fact, at order p we will have p diagonal states and p nondiagonal states, so we have $2p$ unknown quantities, but we have $p+1$ equations, so the indetermination grows like $2p-p-1=p-1$. In general, for the correlation function of n external points we will have $2L(n,p)$ unknown quantities and $L+1$ equations, so the indetermination will grow as $2L-L-1=L-1$.

The finite contribution of Eq. (40) can be obtained by the application of the projector on the quantum state of Eq. (39)

$$\Pi_p(\rho_{\rm int}^{(0,p)}) = \rho_{\rm int}^{(0,p)} - \int \rho_D^{(0,p,1)}(y_1)\rho_D^{(0,p,2)}(y_2)\ldots\rho_D^{(0,p,p+1)}(y_{p+1})|y_1,\ldots,y_{p+1}\rangle\langle y_1,\ldots,y_{p+1}|\prod_{i=1}^{p+1}d^4y_i$$
$$+\int \rho_D^{(0,p,1)}(y_1)\rho_D^{(0,p,2)}(y_2)\ldots\rho_{ND}^{(0,p,p+1)}(y_{p+1},w_{p+1})|y_1,\ldots,y_{p+1}\rangle\langle y_1,\ldots,w_{p+1}|d^4w_{p+1}\prod_{i=1}^{p}d^4y_i+\ldots$$
$$+\int \rho_D^{(0,p,1)}(y_1)\rho_{ND}^{(0,p,2)}(y_2,w_2)\ldots\rho_{ND}^{(0,p,p+1)}(y_{p+1},w_{p+1})|y_1,\ldots,y_{p+1}\rangle\langle y_1,\ldots,w_{p+1}|d^4y_1\prod_{i=2}^{p+1}d^4y_id^4w_i. \quad (45)$$

This projector eliminates all the diagonal parts of the quantum state. Then, the trace with the projected state reads

$$\mathrm{Tr}\,(\Pi_p(\rho_{\rm int}^{(0,p)})) = \beta_0^{(0,p)}. \quad (46)$$

Adding all the orders in the perturbation expansion we finally obtain

$$\mathrm{Tr}\,(\Pi(\rho_{\rm int}^{(0)})) = 1 + \sum_{p=1}^{+\infty}(-i\lambda_0)^p i^{2p}(2TV)\beta_0^{(0,p)}. \quad (47)$$

Then, multiplying the nonconnected Feynman diagrams, we obtain Eq. (38).

In the case of no external points, the renormalization is a normalization of the quantum state itself. In the observable-state model, this normalization is explicit, because the projection changes the trace of the quantum state [see Eqs. (40) and (46)]. From this point of view, the renormalization is a change of the norm of the quantum state by a projection, in a similar manner in which the projection postulate occurs in nonrelativistic quantum mechanics.

IV. EXAMPLE: ϕ^4 THEORY, $n = 4$

The four-point correlation function, when there are interactions, reads

$$\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle = \langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega_0\rangle$$
$$+(-i\lambda_0)\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)|\Omega_0\rangle d^4y_1$$
$$+(-i\lambda_0)^2\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)\phi^4(y_2)|\Omega_0\rangle d^4y_1d^4y_2+\ldots$$
$$+(-i\lambda_0)^p\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)\ldots\phi^4(y_p)|\Omega_0\rangle\prod_{i=1}^{p}d^4y_i. \quad (48)$$

The first term of the last equation reads

$$\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega_0\rangle = \Delta(x_1-x_2)\Delta(x_3-x_4)+\Delta(x_1-x_3)\Delta(x_2-x_4)+\Delta(x_1-x_4)\Delta(x_2-x_3) \quad (49)$$

where $\Delta(x-y)$ is the scalar propagator. This term does not contribute to the scattering amplitude because it describes a trivial process where the initial and final states are identical.

The first order in the perturbation expansion reads

$$(-i\lambda_0)\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)|\Omega_0\rangle d^4y_1$$
$$=f_0^{(4)}(x_1,x_2,x_3,x_4)=(-i\lambda_0)\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x_1-x_4)}}{p^2-m_0^2}\int\frac{d^4q}{(2\pi)^4}\frac{ie^{-iq(x_2-x_4)}}{q^2-m_0^2}\int\frac{d^4l}{(2\pi)^4}\frac{ie^{-il(x_3-x_4)}}{(l^2-m_0^2)}\frac{i}{((p+q-l)^2-m_0^2)}. \quad (50)$$

In this case, the first order does not have any loops.

The second order in the perturbation expansion reads

$$(-i\lambda_0)^2\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)\phi^4(y_2)|\Omega_0\rangle d^4y_1d^4y_2$$
$$=f_0^{(4)}(x_1,x_2,x_3,x_4)\lambda_0^2\int\frac{d^4r}{(2\pi)^4}\frac{1}{(r^2-m_0^2)((p+q-r)^2-m_0^2)}$$
$$=f_0^{(4)}(x_1,x_2,x_3,x_4)\lambda_0^2\left(\frac{\beta_1^{(4,2)}}{\epsilon}+\beta_0^{(4,2)}\right), \tag{51}$$

where $\beta_1^{(4,2)}$ and $\beta_0^{(4,2)}$ read [see [14], pages 120–122 or Eq. (4.4.16)]

$$\beta_1^{(4,2)}=\frac{1}{32\pi^2}$$
$$\beta_0^{(4,2)}=\frac{1}{2}\frac{3}{32\pi^2}\left(\ln(\mu^2)-\gamma+2+\ln\left(\frac{4\pi\mu^2}{m_0^2}\right)-\frac{1}{3}\sum_{z=s,t,u}\sqrt{1+\frac{4m_0^2}{z}}\ln\left(\frac{\sqrt{1+\frac{4m_0^2}{z}}+1}{\sqrt{1+\frac{4m_0^2}{z}}-1}\right)\right), \tag{52}$$

where s, t, and u are Mandelstam variables $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, and $u=(p_1+p_4)^2$ and $\frac{1}{2}$ is the symmetry factor and the μ factor appears by changing the coupling constant λ_0 to $\lambda_0\mu^{-\epsilon}$. Is not difficult to show that the higher orders in the perturbation expansion obey the following rule:

$$(-i\lambda_0)^p\int\langle\Omega_0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(y_1)\ldots\phi^4(y_p)|\Omega_0\rangle\times\prod_{i=1}^{p}d^4y_i=f_0^{(4)}(x_1,x_2,x_3,x_4)\sum_{l=0}^{p-1}\frac{(-i\lambda_0)^p i^{2+2p}\beta_l^{(4,p)}}{\epsilon^l}, \tag{53}$$

where $p-1$ is the number of loops in the case of ϕ^4 theory with four external points.

Following the idea of our work, we will apply the observable-state model to the four-point correlation function.

A. The observable-state model for $n=4$ in ϕ^4 theory

The state and the observable reads

$$\rho^{(4,p)}=\int f_0^{(4)}(x_1,x_2,x_3,x_4)\prod_{i=1}^{p-1}(\rho_D^{(4,p,i)}(y_i)\delta(y_i-w_i)+\rho_{ND}^{(4,p,i)}(y_i,w_i))|x_1,x_2,y_1\ldots,y_{p-1}\rangle\times\langle x_3,x_4,w_1,\ldots,w_{p-1}|\prod_{i=1}^{4}d^4x_i\prod_{i=1}^{p-1}d^4y_id^4w_i, \tag{54}$$

$$O^{(4,p)}=\int J(x_1)J(x_2)J(x_3)J(x_4)\prod_{i=1}^{p-1}\delta(y_i-w_i)\times|x_1,x_2,y_1,\ldots,y_{p-1}\rangle\langle x_3,x_4,w_1,\ldots,w_{p-1}|\times\prod_{i=1}^{4}d^4x_i\prod_{i=1}^{p-1}d^4y_id^4w_i. \tag{55}$$

Then, the trace reads

$$\mathrm{Tr}(\rho^{(4,p)}O^{(4,p)})=\sum_{l=0}^{p-1}\frac{\gamma_l^{(4,p)}}{\epsilon^l}\int f_0^{(4)}(x_1,x_2,x_3,x_4)J(x_1)J(x_2)J(x_3)J(x_4)\times\prod_{i=1}^{4}d^4x_i, \tag{56}$$

where

$$\gamma_l^{(4,p)}=(-i\lambda_0)^p i^{2+2p}\beta_l^{(4,p)}. \tag{57}$$

In particular

$$\gamma_0^{(4,p)}=\prod_{i=1}^{p-1}\rho_{ND}^{(4,p,i)},\ldots,\quad \gamma_{p-1}^{(4,p)}=\frac{1}{\pi^{p-1}}\prod_{i=1}^{p-1}\rho_D^{(4,p,i)}. \tag{58}$$

For the order $p=2$, using Eqs. (52) and (57), the $\gamma_l^{(4,2,1)}$ coefficients read

$$\gamma_1^{(4,2)}=\rho_D^{(4,2,1)}=\frac{\lambda_0^2}{32\pi^2}$$
$$\gamma_0^{(4,2)}=\rho_{ND}^{(4,2,1)}=\frac{\lambda_0^2}{32\pi^2}\left(-\frac{1}{2}\ln(\mu)-\gamma+2+\ln\left(\frac{4\pi\mu}{m_0^2}\right)-\sqrt{1+\frac{4m_0^2}{(p+q)^2}}\ln\left(\frac{\sqrt{1+\frac{4m_0^2}{(p+q)^2}}+1}{\sqrt{1+\frac{4m_0^2}{(p+q)^2}}-1}\right)\right). \tag{59}$$

The projector over the finite contribution reads

$$\Pi_p(\rho_{\rm int}^{(4,p)}) = \rho_{\rm int}^{(4,p)} - \int \rho_D^{(4,p,1)}(y_1)\rho_D^{(4,p,2)}(y_2)\ldots\rho_D^{(4,p,p-1)}(y_{p-1})|y_1,\ldots,y_{p-1}\rangle\langle y_1,\ldots,y_{p-1}|\prod_{i=1}^{p-1}d^4y_i$$
$$+\int \rho_D^{(4,p,1)}(y_1)\rho_D^{(4,p,2)}(y_2)\ldots\rho_{ND}^{(4,p,p-1)}(y_{p-1},w_{p-1})|y_1,\ldots,y_{p-1}\rangle\langle y_1,\ldots,w_{p-1}|d^4w_{p-1}\prod_{i=1}^{p-2}d^4y_i+\ldots$$
$$+\int \rho_D^{(4,p,1)}(y_1)\rho_{ND}^{(4,p,2)}(y_2,w_2)\ldots\rho_{ND}^{(4,p,p-1)}(y_{p-1},w_{p-1})|y_1,\ldots,y_{p-1}\rangle\langle y_1,\ldots,w_{p-1}|d^4y_1\prod_{i=2}^{p-1}d^4y_id^4w_i. \quad (60)$$

Then, the trace of the observable in the projected state reads

$$\mathrm{Tr}(\Pi_p\rho^{(4,p)}O^{(4,p)}) = \gamma_0^{(4,p)}\int \rho_{\rm ext}^{(4,1)}(x_1,x_2,x_3,x_4)J(x_1)J(x_2)J(x_3)J(x_4)\prod_{i=1}^{4}d^4x_i \quad (61)$$

Summing all the perturbation expansion terms we obtain

$$\mathrm{Tr}(\Pi\rho^{(4)}O^{(4)}) = \int f_0^{(4)}(x_1,x_2,x_3,x_4)J(x_1)J(x_2)J(x_3)J(x_4)\prod_{i=1}^{4}d^4x_i = \sum_{p=0}^{+\infty}(-i\lambda_0)^p i^{2+2p}\beta_0^{(4,p)}, \quad (62)$$

where we have replaced $\gamma_0^{(4,p)}$ by $(-i\lambda_0)^p i^{2+2p}\beta_0^{(4,p)}$ [see Eq. (57)].

B. Renormalization of λ

We can proceed by summing the perturbation expansion, but without taking account the $p = 0$ order, because it describes a trivial process in which the initial and final states are identical. Only fully connected diagrams contribute to the scattering amplitude. Then

$$\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle = f_0^{(4)}(x_1,x_2,x_3,x_4)\sum_{p=1}^{+\infty}\sum_{l=0}^{p-1}\frac{(-i\lambda_0)^p i^{2+2p}\beta_l^{(4,p)}}{\epsilon^l}. \quad (63)$$

We can then put $x_4 = 0$ and take the Fourier transform on both sides of last equation,

$$\int d^4x_1d^4x_2d^4x_3e^{-ipx_1}e^{-iqx_2}e^{-ilx_3} \times\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(0)|\Omega\rangle = \frac{1}{(p^2-m_0^2)}\frac{1}{(q^2-m_0^2)}\frac{1}{(l^2-m_0^2)}\frac{1}{((p+q-l)^2-m_0^2)} \times\sum_{p=1}^{+\infty}\sum_{l=0}^{p-1}\frac{(-i\lambda_0)^p i^{2+2p}\beta_l^{(4,p)}}{\epsilon^l}. \quad (64)$$

If we remove the propagators of the external lines we obtain the four-point proper vertex $\Gamma^{(4)}$. We can write $i\Gamma^{(4)}(0) = \lambda$, this is, the renormalized coupling constant is equal to the magnitude of the scattering amplitude at zero momentum (see [12], page 325). But from dimensional regularization we know that the coupling constant depends on the mass factor μ, so in the most general case, $i\Gamma^{(4)} = \lambda(\mu)$, then

$$i\lambda(\mu) = \sum_{p=1}^{+\infty}(-i\lambda_0)^p i^{2+2p}\sum_{l=0}^{p-1}\frac{\beta_l^{(4,p)}}{\epsilon^l}, \quad (65)$$

where $\beta_l^{(4,p)}$ depends on μ and the external momentum. The last equation is identical to Eq. (2.3.b) of [15]. Once renormalized, we must only keep the $l = 0$ term, then

$$i\lambda(\mu) = \sum_{p=1}^{+\infty}(-i\lambda_0)^p i^{2+2p}\beta_0^{(4,p)}. \quad (66)$$

In terms of the observable-state model, this reads (see Eq. (57)):

$$i\lambda(\mu) = \sum_{p=1}^{+\infty}\gamma_0^{(4,p)} \quad (67)$$

In this sense, the nondiagonal functions of the quantum state of Eq. (54), that is, the renormalized coupling constant.

V. THE RENORMALIZATION GROUP

In this last section we will see how the renormalization group arises in the context of the observable-state model. As we see in [1] and this paper, the $n = 2$ and $n = 4$ correlation functions give the mass and coupling constant renormalization. Those equations read [see Eq. (B18) of [1] and Eq. (66) of this paper][10]

$$m^2 = m_0^2 + \sum_{p=1}^{+\infty}(-i\lambda_0)^p i^{1+2p}\hbar^p\beta_0^{(2,p)}(m_0^2,\mu) = m_0^2 - \lambda_0\hbar\beta_0^{(2,1)}(m_0^2,\mu) + \ldots, \quad (68)$$

[10] In the following equations we will restore the Planck constant $\hbar$ for later convenience.

$$\lambda = \lambda_0 + \sum_{p=2}^{+\infty}(-i\lambda_0)^p i^{2+2p}\hbar^{p-1}\beta_0^{(4,p)}(m_0^2,\mu)$$
$$= \lambda_0 + \lambda_0^2\hbar\beta_0^{(4,2)}(m_0^2,\mu) + \ldots . \qquad (69)$$

In the other side, since m_0^2 and λ_0 do not depend on μ in the absence of loop correction, we have

$$\frac{dm_0^2}{d\mu} = O(\hbar), \qquad \frac{d\lambda_0}{d\mu} = O(\hbar). \qquad (70)$$

The renormalization group can be obtained by imposing the fact that the dressed masses m^2 and λ do not depend on μ, this is, $\frac{dm^2}{d\mu} = 0$ and $\frac{d\lambda}{d\mu} = 0$. Using the chain rule in Eq. (68), we have for m^2:

$$\frac{dm^2}{d\mu} = \frac{\partial m^2}{\partial m_0^2}\frac{dm_0^2}{d\mu} + \frac{\partial m^2}{\partial\lambda_0}\frac{d\lambda_0}{d\mu} + \frac{\partial m^2}{\partial\mu} = 0 \qquad (71)$$

using Eqs. (68) and (70), Eq. (71) reads at order $\hbar$:

$$\frac{dm_0^2}{d\mu} - \lambda_0\frac{\partial\beta_0^{(2,1)}}{\partial\mu} = 0. \qquad (72)$$

From Eq. (82) of [1]

$$\beta_0^{(2,1)} = \frac{m_0^2}{16\pi^2}\left[1 - \gamma + 2\ln\left(\frac{4\pi\mu^2}{m_0^2}\right)\right] \qquad (73)$$

then

$$\frac{\partial\beta_0^{(2,1)}}{\partial\mu} = \frac{m_0^2}{8\pi^2}\frac{1}{\mu}. \qquad (74)$$

Replacing Eq. (74) in Eq. (72) we obtain a differential equation for m_0^2 at order $\hbar$:

$$\frac{dm_0^2}{d\mu} = \lambda_0\frac{m_0^2}{8\pi^2}\frac{1}{\mu}. \qquad (75)$$

We can solve it and obtain

$$m_0^2 = m_S^2\left(\frac{\mu}{\mu_S}\right)^{(\lambda_0/8\pi^2)}, \qquad (76)$$

where m_S^2 is the value of the mass when $\mu = \mu_S$. This result is in concordance with Eq. (4.6.20) and Eq. (4.6.22), page 142 of [14] at order $\hbar$. In a similar way, we can obtain the change of λ_0 in terms of μ at order $\hbar$. To do so, we must impose that the dressed coupling constant does not depend on μ:

$$\frac{d\lambda}{d\mu} = \frac{\partial\lambda}{\partial m_0^2}\frac{dm_0^2}{d\mu} + \frac{\partial\lambda}{\partial\lambda_0}\frac{d\lambda_0}{d\mu} + \frac{\partial\lambda}{\partial\mu} = 0. \qquad (77)$$

Using Eqs. (69) and (70), the last equation reads at order $\hbar$:

$$\lambda_0^2\frac{\partial\beta_0^{(4,2)}}{\partial\mu} - \frac{d\lambda_0}{d\mu} = 0. \qquad (78)$$

Using the result of Eq. (52), the last equation reads

$$\frac{d\lambda_0}{d\mu} + \frac{3}{16\pi^2}\frac{\lambda_0^2}{\mu} = 0. \qquad (79)$$

The last equation can be solved with the following result:

$$\lambda_0 = \frac{\lambda_S}{1 - \frac{3\lambda_S}{16\pi^2}\ln\left(\frac{\mu}{\mu_S}\right)}, \qquad (80)$$

which is identical to Eq. (4.6.15), page 139 of [14]. This last equation is the one-loop correction to the coupling constant that arises from Eq. (78).[11]

Thus, we can see that the projection method not only allows finite perturbation expansions, but also, finite values that are consistent with the results shown in textbooks and the renormalization group.

VI. DISCUSSION

The formalism introduced in Sec. I has a physical content which can be traced to the decoherence formalism (see [16–21]) and to systems with continuous spectrum (see [17,18,22–25]). The trace of the internal quantum state of Eq. (14) can be interpreted as a reduced state, since the observable is an identity operator in the Hilbert space of the internal vertices. This has a physical meaning. It is well known that the reduction of a state decreases the information available to the observer about the composite system. In this case, the reduction is done over the internal vertices where the interaction occurs due to the perturbation expansion. In QFT, the particles that are created in these vertices are virtual particles because they are off-shell, that is, they do obey the conservation laws, but the propagators must be integrated out, which implies that the momentum of the particle associated with each internal propagator may not obey the mass-energy relation $p_\mu p^\mu = m_0^2$. In this sense, the conceptual meaning of the partial trace of the internal degrees of freedom is to neglect these virtual nonphysical particles. This is consistent with the experiments of scattering because basically what is seen are the in and out states. However, perturbation theory introduces off-shell intermediate states whose existence depends on the uncertainty principle $\Delta E\Delta t \geq \frac{\hbar}{2}$. In turn, these give us an interpretation of this integration as a reduction of the degrees of freedom of the theory. In the conventional interpretation of this integration The integral d^4z instructs us to sum over all points where this process can occur. This is just the superposition principle of quantum mechanics: when a process can happen in alternative ways, we add the amplitudes for each possible way ([12], page 94). In our case, the integration over the internal vertices reflects the fact that we are neglecting the degrees

[11] The power of the Planck constant counts the number of loops, so at order $O(\hbar)$, we obtain the one-loop correction (see [13], page 623).

of freedom of this virtual particles and what we finally obtain is a reduced state which is divergent.

Summarizing, the main idea of this work is that in the p order in the perturbation expansion of any quantum field theory, we can define a quantum state as

$$\rho^{(n,p)} = \rho_{\rm ext}^{(n)} \otimes \rho_{\rm int}^{(n,p)} \tag{81}$$

and an observable

$$O^{(n,p)} = O_{\rm ext}^{(n)} \otimes I_{\rm int}^{(p)}, \tag{82}$$

then the trace reads

$$\mathrm{Tr}\,(\rho^{(n,p)} O^{(n,p)}) = \mathrm{Tr}(\rho_{\rm int}^{(n,p)})\mathrm{Tr}(\rho_{\rm ext}^{(n)} O_{\rm ext}^{(n)}). \tag{83}$$

The divergences of the quantum field theory occur in the trace of the internal quantum state $\mathrm{Tr}(\rho_{\rm int}^{(n,p)})$. These divergences appear because the internal quantum state contains diagonal functions multiplied by Dirac deltas that cannot be avoided unless we remove the diagonal functions by a projection. This is the only available transformation that can cure the divergences, because the trace is an invariant quantity that does not depend on the basis in which the state is written. The projector reads

$$\Pi^{(n,p)} = I_{\rm ext}^{(n)} \otimes \Pi_{\rm int}^{(n,p)} = I_{\rm ext}^{(n)} \otimes (\rho_{\rm int}^{(n,p)} - \rho_D^{(n,p)}), \tag{84}$$

where $\rho_D^{(n,p)}$ is the sum of all the states that has a diagonal part of the quantum state $\rho_{\rm int}^{(n,p)}$. Then, the trace of $\Pi \rho^{(n,p)}$ reads

$$\begin{aligned}&\mathrm{Tr}(\Pi^{(n,p)} \rho^{(n,p)} O^{(n,p)})\\ &\quad = (\mathrm{Tr}(\rho_{\rm int}^{(n,p)}) - \mathrm{Tr}(\rho_D^{(n,p)}))\mathrm{Tr}(\rho_{\rm ext}^{(n)} O_{\rm ext}^{(n)}),\end{aligned} \tag{85}$$

which is our finite desired physical contribution. Basically, the projection is a translation of the quantum state by an amount given by the diagonal state. In this work, the renormalization procedure is done by the projection method, but without introducing counterterms, which in principle is much more advantageous, because it can be applied to nonrenormalizable theories, like ϕ^6 in four space-time dimensions, or the quantum field theory of a massless particle with spin 2, such as gravitation. These two theories will be worked out in future works.

A. A general procedure

Suppose we define the following projector that acts on the external quantum state $\rho^{(n)}$ of Eq. (28):

$$\Pi_0^{(n)} = I_1 \otimes I_2 \otimes \ldots I_{n-1} \otimes |0\rangle\langle 0|, \tag{86}$$

where $|0\rangle$ corresponds to $x_n = 0$. When we apply it to $\rho^{(n)}$ we obtain

$$\begin{aligned}\rho^{(n)}\Pi_0^{(n)} = \sum_{p=0}^{+\infty} \frac{i^p}{p!} \beta_0^{(n,p)} \int f_0^{(n)}(x_1, x_2, \ldots, x_{n-1}, 0)\\ \times |x_1, \ldots, x_{(n/2)}\rangle\langle x_{(n/2)+1}, \ldots, x_{n-1}, 0| \prod_{i=1}^{n-1} d^4 x_{i-1}.\end{aligned} \tag{87}$$

The trace with $O_{\rm ext}^{(n)}$ reads

$$\begin{aligned}\mathrm{Tr}(\rho^{(n)}\Pi_0^{(n)} O_{\rm ext}^{(n)}) = \sum_{p=0}^{+\infty} \frac{i^p}{p!} \beta_0^{(n,p)} \int f_0^{(n)}(x_1, x_2, \ldots, x_{n-1}, 0)\\ \times J(x_1)\ldots J(x_{n-1}) J(0) \prod_{i=1}^{n-1} d^4 x_{i-1}.\end{aligned} \tag{88}$$

If we allow the currents to be plane waves[12]

$$J(x_k) = e^{-ip_k x_k}, \tag{89}$$

then, the trace reads

$$\mathrm{Tr}(\rho^{(n)}\Pi_0^{(n)} \tilde{O}_{\rm ext}^{(n)}) = \sum_{p=0}^{+\infty} \frac{i^p}{p!} \beta_0^{(n,p)} \mathcal{F}[f_0^{(n)}(x_1, x_2, \ldots, x_{n-1}, 0)](k_1, \ldots, k_{n-1}), \tag{90}$$

where $\tilde{O}_{\rm ext}^{(n)}$ is the plane wave observable and $\mathcal{F}[f]$ is the Fourier transform of the function f.

(i) *The mass shift*

In the case $n = 2$,

$$f_0^{(2)}(x_1, 0) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x_1 - x_2)}}{(p^2 - m_0^2)^2}, \tag{91}$$

then

$$\begin{aligned}\mathrm{Tr}(\rho^{(2)}\Pi_0^{(2)} \tilde{O}_{\rm ext}^{(2)}) &= \frac{1}{(p^2 - m_0^2)^2} \sum_{p=1}^{+\infty} (-i\lambda_0)^p \beta_0^{(2,p)}\\ &= \frac{M}{(p^2 - m_0^2)^2},\end{aligned} \tag{92}$$

where $M = \sum_{p=1}^{+\infty} (-i\lambda_0)^p \beta_0^{(2,p)}$. Then, this equation implies that

$$(p^2 - m_0^2)^2\, \mathrm{Tr}(\rho^{(2)}\Pi_0^{(2)} \tilde{O}_{\rm ext}^{(2)}) = M. \tag{93}$$

The mass renormalization is obtained by having in mind that the last equation is the result of the one-particle irreducible diagrams.[13] The full contribution

[12]This idea is in concordance with [7], page 19, "For an ingoing particle, we use a source function $J(x)$ whose Fourier components emit a positive amount of energy k_0. For an outgoing particle the source emits a negative k_0."

[13]A one-particle irreducible diagram is any diagram that cannot be split in two by removing a single line.

of the $n=2$ correlation function is equal to the following geometric series [see [12], Eq. (10.27), page 328]:

$$\int \langle\Omega|\phi(x_1)\phi(x_0)|\Omega\rangle e^{-ipx_1}d^4x_1 = \frac{1}{p^2-m_0^2}+\frac{M}{(p^2-m_0^2)^2}+\frac{M^2}{(p^2-m_0^2)^3}+\ldots = \frac{1}{p^2-(m_0^2+M)}. \quad (94)$$

On the other side, [using Eq. (92)] we have

$$\int \langle\Omega|\phi(x_1)\phi(x_0)|\Omega\rangle e^{-ipx_1}d^4x_1 = \frac{1}{p^2-m_0^2-(p^2-m_0^2)^2Tr(\rho^{(2)}\Pi_0^{(2)}\tilde{O}_{\rm ext}^{(2)})}, \quad (95)$$

which implies the mass shift reads

$$\Delta m = m^2 - m_0^2 = M = (p^2-m_0^2)^2\,{\rm Tr}(\rho^{(2)}\Pi_0^{(2)}\tilde{O}_{\rm ext}^{(2)}). \quad (96)$$

(ii) *The coupling constant*
In the $n=4$ case

$$f_0^{(4)}(x_1,x_2,x_3,0) = \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ipx_1}}{p^2-m_0^2}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iqx_2}}{q^2-m_0^2} \times\int\frac{d^4l}{(2\pi)^4}\frac{e^{-ilx_3}}{(l^2-m_0^2)}\frac{1}{((p+q-l)^2-m_0^2)} \quad (97)$$

then

$$[(l^2-m_0^2)(p^2-m_0^2)(q^2-m_0^2)((p+q-l)^2-m_0^2)] \times{\rm Tr}(\rho^{(4)}\Pi_0^{(4)}\tilde{O}_{\rm ext}^{(4)}) = \sum_{p=1}^{+\infty}(-i\lambda_0)^p\beta_0^{(4,p)} = \lambda, \quad (98)$$

which has the same structure of Eq. (93).
In a general way we can write

$${\rm Tr}(\rho^{(n)}\Pi_0^{(n)}\tilde{O}_{\rm ext}^{(n)})\int\prod_{i=1}^{n}(p_i^2-m_0^2)\delta\Big(p_n-\sum_{i=1}^{n-1}p_i\Big)d^4p_n = C_n, \quad (99)$$

where C_n is the renormalized quantity.

This last equation is important, because it can be applied to nonrenormalizable theories. In [26], the renormalization group has been generalized to Lagrangians of arbitrary form, in particular, to nonrenormalizables theories. The idea of this work and [1] follows the same line of thought because the observable-state model treats on equal footing the nonrenormalizable theories and the renormalizable ones.

VII. CONCLUSIONS

The aim of this work was to extend the observable-state model in ϕ^4 theory to the $n=0$ and $n=4$ external points in the correlation function, showing how to build a projector that eliminates all the divergences that appear in the perturbation expansion. This procedure allows us to renormalize the quantum field theory of ϕ^4 without introducing counterterms in the Lagrangians. Besides this, we have shown how the renormalization group arise in this context obtaining the same results as the conventional renormalized QFT.

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APPENDIX A: THE DIRAC DELTA AND THE DIMENSIONAL REGULARIZATION POLES

To understand the relation between the Dirac delta and the poles of the dimensional regularization we can use the following representation of the Dirac delta (see [27], page 35)[14]:

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}, \quad (A1)$$

where ϵ is some parameter that tends to zero. In particular, we can assume that this parameter is the pole parameter of the dimensional regularization, that is, $\epsilon = d-4$.

Consider now for simplicity, the following quantum state:

$$\rho = \int[\rho_D(x)\delta(x-x')+\rho_{ND}(x,x')]|x\rangle\langle x'|dxdx'. \quad (A2)$$

Replacing the representation of the Dirac delta of Eq. (A1) in last equation we obtain

$$\rho = \frac{1}{\pi}\lim_{\epsilon\to 0}\int\rho_D(x)\frac{\epsilon}{(x-x')^2+\epsilon^2}|x\rangle\langle x'|dxdx' + \int\rho_{ND}(x,x')|x\rangle\langle x'|dxdx'. \quad (A3)$$

Taking the trace of ρ we obtain

[14]The relation between the Dirac delta and the dimensional regularization pole in this appendix is introduced by formal mathematical operations, but we must warm the reader that this development is not mathematically rigorous.

$$\mathrm{Tr}(\rho) = \int \langle x''|\rho|x''\rangle dx'' = \frac{1}{\pi}\lim_{\epsilon\to 0}\int \rho_D(x)\frac{\epsilon}{(x-x')^2+\epsilon^2}\delta(x'-x)dxdx' + \int \rho_{ND}(x,x')\delta(x'-x)dxdx'. \quad (A4)$$

We can proceed with the integral of the Dirac delta in both terms, so finally we obtain

$$\mathrm{Tr}\,(\rho) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{1}{\epsilon}\rho_D + \rho_{ND}, \quad (A5)$$

where

$$\rho_D = \int \rho_D(x)dx \quad \rho_{ND} = \int \rho_{ND}(x,x)dx. \quad (A6)$$

In the case of the quantum state of Eq. (19) we will have (we do not put the $\lim_{\epsilon\to 0}$ for simplicity)

$$\mathrm{Tr}\,(\rho^{(n,p)}_{\mathrm{int}}) = \prod_{i=1}^{L(n,p)}\left(\frac{\rho_D^{(n,p,i)}}{\pi\epsilon} + \rho_{ND}^{(n,p,i)}\right) = \sum_{j=-L(n,p)}^{0}\gamma_j^{(n,p)}\epsilon^j, \quad (A7)$$

where in particular

$$\gamma_0^{(n,p)} = \prod_{i=1}^{L(n,p)}\rho_{ND}^{(n,p,i)},\ldots,\quad \gamma_{L(n,p)}^{(n,p)} = \frac{1}{\pi^{L(n,p)}}\prod_{i=1}^{L(n,p)}\rho_D^{(n,p,i)}. \quad (A8)$$

In [1] we suggest the relation between the Dirac delta valuated at zero and the pole of the dimensional regularization but we do not prove it.[15]

APPENDIX B: RELATION BETWEEN THE VACUUM ENERGY AND THE SPACE VOLUME

To obtain the relation between the energy of the vacuum and the space volume V we can recall the renormalized result of Eq. (38),

$$|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T} = \sum_{k=0}^{+\infty}\frac{(2TV)^k}{k!}\left(\sum_{p=1}^{+\infty}(-i\lambda_0)^p\beta_0^{(0,p)}\right)^k, \quad (B1)$$

then we can call

$$(-i)^k R(k) = \left(\sum_{p=1}^{+\infty}(-i\lambda_0)^p\beta_0^{(0,p)}\right)^k, \quad (B2)$$

which implies that

$$R(k) = [R(1)]^k, \quad (B3)$$

where

$$R(1) = \sum_{p=1}^{+\infty}(-i\lambda_0)^p\beta_0^{(0,p)}, \quad (B4)$$

then, Eq. (B1) reads

$$|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T} = \sum_{k=0}^{+\infty}\frac{1}{k!}(-i2TVR(1))^k = e^{-i2TVR(1)}, \quad (B5)$$

then the vacuum energy reads

$$E_0 = VR(1) - \frac{i}{2T}\ln(|\langle\Omega|\Omega_0\rangle|^2), \quad (B6)$$

in particular, for $T\to\infty$

$$E_0 \sim V, \quad (B7)$$

which is the desired result (see [12], page 98). This result is valid if the $R(1)$ as a sum converges. In fact, the ratio test applied to argument of the sum in Eq. (B4) implies that

$$\lim_{p\to\infty}\frac{|\beta_0^{(0,p+1)}|}{|\beta_0^{(0,p)}|} < \frac{1}{\lambda_0}. \quad (B8)$$

This inequality can be tested on the l.h.s. step by step using dimensional regularization. Is not the purpose of this work to prove the convergence of the $n = 0$ correlation function of ϕ^4 theory, besides that it would be a long task.

APPENDIX C: THE PROJECTION IN ALGEBRAIC TERMS

Let us remember the transformation of Eq. (24). For simplicity we will describe it when there are only one diagonal state and one nondiagonal state, in this case, the transformation act in the following way:

$$\Pi(\rho) = \rho - \rho_D. \quad (C1)$$

This transformation is linear

$$\Pi(\rho^{(1)}+\rho^{(2)}) = \rho^{(1)}+\rho^{(2)} - (\rho_D^{(1)}+\rho_D^{(2)}) = \rho^{(1)} - \rho_D^{(1)} + \rho^{(2)} - \rho_D^{(2)} = \Pi(\rho^{(1)}) + \Pi(\rho^{(2)}). \quad (C2)$$

Then it is a projector because

$$\Pi^2(\rho) = \Pi(\Pi(\rho)) = \Pi(\rho-\rho_D) = \Pi(\rho) - \Pi(\rho_D) = \rho - \rho_D - (\rho_D - \rho_D) = \Pi(\rho) \quad (C3)$$

or by using that the diagonal part of the transformed state $\Pi(\rho)_D$ is zero

$$\Pi(\Pi(\rho)) = \Pi(\rho) - \Pi(\rho)_D = \rho - \rho_D - \Pi(\rho)_D = \Pi(\rho). \quad (C4)$$

[15] From a different point of view, if we expand in Taylor series the representation of the Dirac delta of Eq. (A1) we obtain $\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}(\frac{1}{\epsilon} - \frac{x^2}{\epsilon^3} + \frac{x^4}{\epsilon^5} + \ldots)$. Taking the trace of the quantum state implies to replace $x = 0$ in the representation of the Dirac delta.

In this sense, the projector can be written as

$$\Pi = I - Q, \tag{C5}$$

where

$$Q(\rho) = \rho - \rho_{ND} \tag{C6}$$

then, Eq. (C5) is the relation of orthogonal projections. In fact

$$\begin{aligned} Q\Pi(\rho) &= Q(\rho - \rho_D) = Q(\rho) - Q(\rho_D) \\ &= \rho_D + \rho_{ND} - \rho_{ND} - \rho_D = 0 \end{aligned} \tag{C7}$$

which implies that $\Pi(\rho)$ is the null space of Q.

What the projector does is to subtract from ρ its diagonal part, which gives a divergent structure when we compute the trace with the observable. In this sense, to subtract the ϵ^{-l} terms via a projection is similar to the minimal subtraction, where an operator K is defined to pick out the pure poles terms of the dimensional regularization [see [9], Eq. 9.76)]:

$$K\left[\sum_{n=-k}^{+\infty} A_n \epsilon^n\right] = \sum_{n=-k}^{-1} A_n \epsilon^n \tag{C8}$$

then

$$(I - K)\left[\sum_{n=-k}^{+\infty} A_n \epsilon^n\right] = \sum_{n=0}^{+\infty} A_n \epsilon^n = A_0 + A_1 \epsilon + \dots . \tag{C9}$$

In fact, $K^2 = K$, then K is a projector. The main difference is that our projector acts on a quantum state and not over a Laurent series. It will be source of future works to study the relationship between the projection procedure and the *BPHZ* subtraction method [28].

Finally, we can rewrite the projector that acts on the whole Liouville space in algebraic language. For this, in the order p of the perturbation expansion we have the following Hilbert spaces:

$$\mathcal{H}^{(n,p)} = \mathcal{H}_{\text{ext}} \oplus_{i=0}^{L(n,p)} \mathcal{H}^{(i)}. \tag{C10}$$

The total Hilbert space to all orders in the perturbation theory reads

$$\mathcal{H} = \mathcal{H}^{(n,0)} \oplus \mathcal{H}^{(n,1)} \oplus \ldots \oplus \mathcal{H}^{(n,p)} = \oplus_{i=0}^{p} \mathcal{H}^{(n,i)}. \tag{C11}$$

The observables are defined in the Liouville space $\mathcal{N}$:

$$\mathcal{N} = \mathcal{H} \otimes \mathcal{H} = (\oplus_{i=0}^{p} \mathcal{H}^{(n,i)}) \otimes (\oplus_{i=0}^{p} \mathcal{H}^{(n,i)}) = \oplus_{i=0}^{p} \mathcal{N}^{(i)}. \tag{C12}$$

We can decompose as [see Eq. (C13)],

$$\mathcal{N}_{vh} = \mathcal{N}_S \oplus \mathcal{N}_R \subset \mathcal{N}. \tag{C13}$$

Then, the relevant Liouville space will read

$$\mathcal{N}_{vh} = \oplus_{i=0}^{p} (\mathcal{N}_S^{(i)} \oplus \mathcal{N}_R^{(i)}). \tag{C14}$$

Because the states must be considered as linear functionals over the space $\mathcal{N}_{vh}$ ($\mathcal{N}'_{vh}$ the dual of space $\mathcal{N}_{vh}$),

$$\mathcal{N}'_{vh} = \oplus_{i=0}^{p} (\mathcal{N}_S^{\prime(i)} \oplus \mathcal{N}_R^{\prime(i)}), \tag{C15}$$

then, the projector will be a map from $\mathcal{N}'_{vh}$ to $\mathcal{N}'_R$,

$$\Pi = \Pi^{(0)} \oplus \ldots \oplus \Pi^{(p)} : \mathcal{N}'_{vhS} \to \mathcal{N}'_R. \tag{C16}$$

This is the simple trick that allows us to neglect the singularities [i.e. the $\delta(x - x')$*] in a rigorous mathematical way and to obtain correct physical results*. Essentially we have defined a new dual space $\mathcal{N}'_{vh}$ (that contains the states ρ without divergences) that are adapted to solve our problem.

So, essentially we have substituted an *ad hoc* counterterm procedure (or an *ad hoc* subtraction procedure [28]) with a clear physical motivated theory. These are the essential features of the proposed formalism, where the deltas are absent.

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[1] J. S. Ardenghi and M. Castagnino, Phys. Rev. D **85**, 025002 (2012).
[2] W. Greiner and J. Reinhardt, *Field Quantization* (Springer-Verlag, Berlin Heidelberg, 1996).
[3] S. Weinberg, *The Quantum Theory of Fields* (Cambridge Uinv. Press, Cambridge, 1995), Vol. I.
[4] G. Folland, *Quantum Field Theory: A Tourist Guide for Mathematicians* (American Mathematical Society, Rhode Island, 2008).
[5] R. Haag, *Local Quantum Physics* (Springer Verlag, Berlin, 1993).
[6] L. Brown, *Quantum Field Theory* (Cambridge Univ. Press, Cambridge, 1992).
[7] G. 't Hooft, in *Handbook of the Philosophy of Science, Philosophy of Physics* (Elsevier, Cambridge, to be published), Part A, p. 661.
[8] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).
[9] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of* ϕ^4 *Theories* (World Scientific, Freie Universität Berlin, 2000).
[10] I. M. Gelfand and M. A. Naimark, Mathematics of the USSR-Sbornik **12**, 197 (1943).
[11] L. Van Hove, Physica (Amsterdam) **21**, 901 (1955); **22**, 343 (1956); **23**, 441 (1957); **23**, 441 (1957); **25**, 268 (1959).

[12] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, Reading, 1995).
[13] A. Das, *Lectures on Quantum Field Theory* (World Scientific, University of Rochester, 2008).
[14] P. Ramond, *Fields Theory, a Modern Primer* (Benjamin, London, 1981).
[15] G. T'hooft, Nucl. Phys. **B61**, 455 (1973).
[16] J. P. Paz and W. Zurek, arXiv:quant-ph/0010011.
[17] M. Castagnino and R. Laura, Phys. Rev. A **62**, 022107 (2000).
[18] M. Castagnino, M. Gadella, R. Laura, and R. Id Betan, Phys. Lett. A **282**, 245 (2001); J. Phys. A **34**, 10067 (2001); M. Castagnino, R. Laura, R. Id Betan, and R. Liotta, J. Phys. A **35**, 6055 (2002).
[19] M. Castagnino and A. Ordoñez, Int. J. Theor. Phys. **43**, 695 (2004); G. Murgida and M. Castagnino, Physica A (Amsterdam) **381**, 170 (2007).
[20] M. Castagnino, S. Fortin, R. Laura, and O. Lombardi, "Decoherence in closed and open systems, an unifying approach," Classical Quantum Gravity (to be published).
[21] M. Castagnino and S. Fortin, arXiv:1009.0535v2.
[22] R. Laura and M. Castagnino, Phys. Rev. A **57**, 4140 (1998).
[23] R. Laura and M. Castagnino, Phys. Rev. E **57**, 3948 (1998).
[24] M. Castagnino and M. Gadella, Found. Phys. **36**, 920 (2006).
[25] M. Castagnino and O. Lombardi, Chaos, Solitons Fractals **28**, 879 (2006).
[26] D. I. Kazakov and G. S. Vartanov, J. Phys. A **39**, 8051 (2006).
[27] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964) Vol. I.
[28] N. N. Bogoliubow and O. S. Parasiuk, Acta Math. **97**, 227 (1957); K. Hepp, Commun. Math. Phys. **2**, 301 (1966); **6**, 161 (1967); W. Zimmerman, Commun. Math. Phys. **15**, 208 (1969).