Classical duals of derivatively self-coupled theories

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Solutions to scalar theories with derivative self-couplings often have regions where nonlinearities are important. Given a classical source, there is usually a region, demarcated by the Vainshtein radius, inside of which the classical nonlinearities are dominant, while quantum effects are still negligible. If perturbation theory is used to find such solutions, the expansion generally breaks down as the Vainshtein radius is approached from the outside. Here we show that it is possible, by integrating in certain auxiliary fields, to reformulate these theories in such a way that nonlinearities become small inside the Vainshtein radius, and large outside it. This provides a complementary, or classically dual, description of the same theory—one in which nonperturbative regions become accessible perturbatively. We consider a few examples of classical solutions with various symmetries, and find that in all the cases the dual formulation makes it rather simple to study regimes in which the original perturbation theory fails to work. As an illustration, we reproduce by perturbative calculations some of the already known nonperturbative results, for a pointlike source, cosmic string, and domain wall, and derive a new one. The dual formulation may be useful for developing the parametrized post Newtonian formalism in the theories of modified gravity that give rise to such scalar theories.

I. INTRODUCTION AND SUMMARY

Perturbation theory is often the only analytic tool available to extract detailed information from interacting theories. The regime in which perturbation theory is valid is usually limited. In certain cases, however, it is possible to reformulate a theory in terms of new, dual, variables that allow perturbative calculations in the regime where the original formulation was nonperturbative.

In this note, we discuss certain special nonlinear theories, and show that at the classical level they admit a dual description. These are field theories of a scalar, ϕ, with purely derivative nonlinear terms, that nevertheless give equations of motion with no more than two time derivatives.

Our main motivation for considering such models, and their classical duals, stems from the theories that modify general relativity (GR) in the infrared—the five-dimensional Dvali-Gabadadze-Porrati (DGP) model [1], and four-dimensional ghost-free massive gravity [2,3]. The four-dimensional scalar Lagrangians discussed here capture parts of the full gravitational theory, as shown in [4] for DGP and [2] for massive gravity (for reviews and experimental limits, see Ref. [5], and for a recent theoretical review of massive gravity see Ref. [6]).

Our analysis may have broader applications though: derivatively self-coupled theories, in particular, the Galileons [4,7,8], can also be obtained in the probe-brane limit of higher dimensional constructions [9], and their extensions [10–14], while their three-dimensional

counterparts are obtained in the context of three-dimensional "new massive gravity" [15], as shown in [16].

In the present work we shall focus on the so-called cubic Galileon, $\sim \Box \phi (\partial \phi)^2 / \Lambda^3$, [4,7] by which the free Lagrangian, $-(\partial \phi)^2 / 2$, is supplemented in four dimensions. The state described by ϕ can be thought of as a Nambu-Goldstone boson, nonlinearly realizing (a limit of) broken higher dimensional Poincaré or diffeomorphism invariance

$$\phi \rightarrow \phi + c + c_\mu x^\mu,$$

with c_μ denoting a constant vector. In parallel, we will also consider—mainly as toy examples—theories of an "ordinary" Nambu-Goldstone (NG) boson with the self-interaction terms, such as $-(\partial \phi)^{2n} / \Lambda^{4n-4}$, $n \geq 2$.

The only physical scale in these models is Λ. Such theories are usually regarded as effective field theories valid at energies/momenta below Λ. One reason for this is that scatterings of the ϕ-quanta—when treated in conventional perturbation theory—exhibit nonperturbative behavior at/above the scale Λ. On the other hand, the Galileons do not seem to represent garden variety effective field theories. They are special—for instance, they do not get renormalized by quantum corrections [4,7,8,10] (although, other higher-derivative terms may be generated). One may wonder then, if there may be some hidden structure in the Galileon theories that would enable one to deal in a controllable way with scales above Λ, by a resummation of perturbative diagrams, or by a dual

description. Although, in the present work, we will not explore the above important question, we will make a step in that direction.

What we shall show, instead, is that a dual description is possible for these theories in the classical regime. To enable a window in which the classical description is meaningful, we introduce a minimal coupling of the scalar ϕ to the trace of an external classical stress-tensor T (planets, stars, etc.)

$$\sim \frac{\phi}{M_{\mathrm{Pl}}} T, \qquad (1)$$

where M_{Pl} is the Planck mass. Coupling to the stress-tensor can be nonminimal in a more general context [2], with certain interesting observational consequences [17]; our analysis should straightforwardly apply to those cases too.

The presence of the Planck mass, in addition to Λ, gives rise to a new *derived* scale, referred to as the Vainshtein scale [18]. For a static spherically symmetric classical source of mass M, $r_* \sim \Lambda^{-1} (M/M_{\mathrm{Pl}})^{1/3}$ [19]. This scale is much greater than Λ^{-1}. The conventional perturbative expansion can be used to compute the field configuration outside the Vainshtein radius, $r \gg r_*$. Inside the radius, $r \ll r_*$, classical nonlinearities in ϕ are dominant, and the perturbative expansion breaks down. More formally speaking, external classical sources introduce a new expansion parameter, α_{cl}, that captures the strength of classical nonlinearities; for the Galileon, $\alpha_{\mathrm{cl}} = \partial^2 \phi / \Lambda^3$, while for the NG-type theories, $\alpha_{\mathrm{cl}} = (\partial \phi)^2 / \Lambda^4$. The parameter α_{cl} is source dependent, and for theories considered here, there is generically a broad region in space where $\alpha_{\mathrm{cl}} \gtrsim 1$, while energies and momenta are still well below Λ. Therefore, the classical field enters a highly nonlinear regime, while the quantum corrections are still negligible, as long as we stay at the distance scales greater than Λ^{-1}.

We show how these theories can be dualized by integrating in certain auxiliary variables. The dual theory is classically equivalent to the original one, however, it no longer has any higher-dimensional derivatively coupled terms. Instead, the dual theory is nonlocal, in the sense that it contains lower dimensional nonderivative terms with fractional classical dimensions.

Perturbation theory in the dual version has a regime of validity opposite to the original one: there is still a Vainshtein radius, but now nonlinearities are small *inside* the Vainshtein radius, and large outside of it. Hence, the nonperturbative regime in the original variables is perturbative in the dual picture, and *vice versa*.

We point out that, whether in the dual description or the original description, in both the perturbative and nonperturbative regimes the classical fields are *weak* in Planckian units, $\phi_{\mathrm{cl}} \ll M_{\mathrm{Pl}}$. This should be compared to the case of a black hole in GR, where nonlinearities near the Schwarzschild radius, r_g, are due to classical fields that

are not small in Planckian units. These nonlinearities can be resummed into the Schwarzschild solution.[1]

We would like to make a few important comments on the literature. First, our dual description is what captures the properties of the small-mass expansion used by Vainshtein [18] in massive gravity. The latter expansion is replaced here by a series governed by positive powers of Λ. This parameter, in the context of massive gravity, is derived from the graviton mass and Planck scale. Second, the derivatively coupled theory of a ϕ field containing a non-linear ghost (for instance the Lagrangian $\sim (\Box \phi)^3 / \Lambda_5^5$ describing the longitudinal mode of earlier, ghostly massive gravity theories), was decomposed by Deffayet and Rombouts [20] by means of the Ostrogradskii method to manifestly exhibit the ghost in the linear theory. Our construction is similar, but not identical, since we are dealing with the theories without ghosts; this essential distinction gives rise to significant differences in the two cases, as will be seen below. In spite of the differences, we have been inspired by both [18,20].

The work is organized as follows: In Sec. II we consider the simplest NG-type model. We discuss its classical dual and perform calculations for sources with spherical and cylindrical symmetry. This serves demonstrational purposes, as the main focus of the present work is on Galileons which are relevant to the theories of IR modified gravity. In Sec. III we study the cubic Galileon. Again, we present the dual theory, and use it for perturbative calculations with spherical, cylindrical, and planar symmetries. Section IV contains conclusions and outlook. In the Appendix we discuss more general NG-type theories.

II. NAMBU-GOLDSTONE TYPE THEORIES

As mentioned before, the Galileons [4,8], are perhaps the most remarkable derivatively self-coupled scalar field theories: their special structure guarantees a good Cauchy formulation, as well as nonrenormalization of these terms [4,8] in the quantum theory with no additional fields. Quantum effects may generate other derivative terms, such as $(\Box \phi)^k$, $k \geq 2$, however, the effects of the latter are suppressed in the classical regime considered here. Therefore, our restriction to a single cubic Galileon term in the next section can be justified even in the full quantum theory.

On the other hand, there is no similar argument for the NG-like theories. Also, there is no known principle which would lead to the resummation of the $(\partial \phi)^{2n}$-type NG interactions, except in the case leading to the Dirac-Born-Infeld action. Dirac-Born-Infeld, however, has no well-behaved static solutions [21], and we will not consider it here. Therefore, restricting only to the NG-type term with $n = 2$, as it will be done in this section, is not justified

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[1]It may be interesting to attempt to dualize GR along the lines discussed here.

in the full quantum theory. Nevertheless, we consider this example as a starting point in this section and regard it as a toy model where calculations are easier, keeping in mind that generically one should be retaining terms with all possible integer values for n (other values of n are considered in the Appendix).

Thus, we consider a theory of a scalar field ϕ with a NG-type derivative quartic self-interaction

$$\mathcal{L}_1 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{4\Lambda^4}(\partial\phi)^4 + \frac{1}{M_P}\phi T. \quad (2)$$

Around the trivial background, $T = 0$, $\phi = 0$, perturbation theory for the amplitudes of the ϕ-quanta starts to fail as energies reach the scale Λ. We now turn to nontrivial classical backgrounds with nonzero T.[2]

The equation of motion reads

$$\Box\phi + \frac{1}{\Lambda^4}\partial_\mu[(\partial\phi)^2\partial^\mu\phi] = -\frac{T}{M_P}. \quad (4)$$

For a static pointlike source, $T = -M\delta^3(\mathbf{x})$, the solution for ϕ is spherically symmetric and static, and the equation of motion reduces to

$$\vec{\nabla}\cdot\left(\vec{\nabla}\phi + \frac{1}{\Lambda^4}(\vec{\nabla}\phi)^2\vec{\nabla}\phi\right) = \frac{M}{M_P}\delta^3(\mathbf{x}), \quad (5)$$

which can readily be integrated once to obtain a cubic algebraic equation for the radial derivative ϕ',

$$\phi' + \frac{1}{\Lambda^4}(\phi')^3 = \frac{M}{4\pi M_P}\frac{1}{r^2}. \quad (6)$$

This can be solved exactly. The exact solution has two regimes, depending on which of the two terms on the left hand side of (6) dominates. The scale that separates the two regimes is denoted r_*, this being the distance at which the two terms become comparable,

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[2]Note that we consider this model with a "wrong sign" in front of the nonlinear term—the sign that does not admit a conventional UV completion [22], while it exhibits the Vainshtein mechanism [23]. We thank Lasha Berezhiani for pointing out that the "right-sign" nonlinear term for the NG, $+(\partial\phi)^4/\Lambda^4$, does not admit the Vainshtein mechanism. The theory (3) with $T = 0$ can be regarded as the decoupling limit of a massive Abelian vector field with a quartic interaction [24],

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}m_A^2A^\mu A_\mu - \frac{g^4}{4}(A^\mu A_\mu)^2. \quad (3)$$

At energies parametrically above m_A, ϕ describes the helicity-0 component of A_μ, extracted through the Stükelberg replacement, $A_\mu \to \tilde{A}_\mu - \frac{1}{m_A}\partial_\mu\phi$, and the decoupling limit is defined as follows: $m_A \to 0$, $g \to 0$, $\Lambda \equiv \frac{m_A}{g}$ fixed The effective theory (2) is valid at distance scales $\Lambda^{-1} \ll r \ll m_A^{-1}$.

$$r_* \sim \left(\frac{M}{M_P}\right)^{1/2}\frac{1}{\Lambda}. \quad (7)$$

At scales larger than r_*, the linear term on the left-hand side dominates, leading to the usual Newtonian potential for the scalar

$$\phi \simeq -\frac{M}{4\pi M_P}\frac{1}{r}, \qquad r \gg r_*. \quad (8)$$

Note that the value of the classical field is small in Planckian units: $\frac{\phi}{M_{Pl}} \sim \frac{r_g}{r} \ll 1$, for $r > r_*$. At distances shorter than r_* on the other hand, the nonlinear term is more important, and the solution reads

$$\phi \simeq 3\left(\frac{M}{4\pi M_P}\right)^{1/3}\Lambda^{4/3}r^{1/3} + \text{const}, \qquad r \ll r_*. \quad (9)$$

It is straightforward to check that even in this regime the value of the classical field (9) is sub-Planckian, $\frac{\Phi}{M_{Pl}} \sim \frac{r_g}{r_*}(\frac{r}{r_*})^{1/3} \ll 1$, and decreases with decreasing r.

If the exact solution were not known, and we wanted to set up a perturbation theory to find it, we could perform an expansion in powers of the interaction. We could do this by expanding the field into powers of the nonlinear interaction,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots. \quad (10)$$

Plugging this expansion into the equation of motion (4) and equating powers of $\frac{1}{\Lambda}$, one generates a series of equations

$$\Box\phi_0 = -\frac{T}{M_P}, \quad (11)$$

$$\Box\phi_1 + \frac{1}{\Lambda^4}\partial_\mu[(\partial\phi_0)^2\partial^\mu\phi_0] = 0, \quad (12)$$

$$\vdots \quad (13)$$

For the static pointlike source $T = -M\delta^3(\mathbf{x})$, the leading order solution is $\phi_0 = -\frac{M}{4\pi M_P}\frac{1}{r}$. By simple power counting one can see that the series is nothing but an expansion in powers of the parameter

$$\left(\frac{r_*}{r}\right)^4, \quad (14)$$

where r_* is the Vainshtein radius (7). The expansion is good for large radii, and starts to fail as we approach the Vainshtein radius from the outside.

A. Dual formulation

The theory in the form given above naturally yields a perturbative expansion which is valid in the IR but breaks down in the UV. As discussed above, the region

of transition between the classically perturbative and nonperturbative regimes lies around the Vainshtein radius.[3]

We would like to rewrite the theory in a form that makes it possible to perform a perturbative expansion which is valid in the UV, rather than in the IR. It is straightforward to check that the following Lagrangian written in terms of the original scalar ϕ, and an auxiliary vector field ψ_μ:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{3}{4}\Lambda^{4/3}(\psi_\mu^2)^{2/3} - \psi^\mu\partial_\mu\phi + \frac{1}{M_P}\phi T,$$
(15)

recovers back the original theory (2) upon integrating out ψ_μ.

By simply introducing a new variable, ψ_μ, we seem to have arrived at an equivalent action in which none of the coupling constants (never mind Planck's mass) are of negative mass dimension, naively pointing towards the possibility of a perturbative expansion which is good in the UV. Note however that there is now a potential term for ψ_μ with a fractional power, and that this fraction is less than two (in the Appendix, we show that more general interactions also lead to such terms and that the fractional power is always less than two).

In the limit $\Lambda \to 0$, $M_{Pl} \to \infty$, the theory (15) reduces to the one governed by the nondynamical equations, $\partial_\mu\phi = 0$, $\partial^\mu\psi_\mu = 0$.

Note that the second term in (15), if regarded as a potential, has a "tachyonic" sign. This is a consequence of the minus sign in front of the nonlinear term in (2), and is a cause of the existence of superluminal modes [8,21,22,25] in the theory (2). Thus, in the dual version the superluminality is related to the tachyonic instability of the nonanalytic potential. The latter could be stabilized, e.g., by supplementing (15) with carefully chosen higher powers of ψ_μ^2. However, we will not pursue this completion here, since we just use the theory (2) as a toy example to demonstrate the trick of dualization in an easier setup.

The equations of motion that follow from (15) are

$$\Box\phi + \partial_\mu\psi^\mu = -\frac{1}{M_P}T,$$

$$\Lambda^{4/3}((\psi_\nu)^2)^{-(1/3)}\psi_\mu - \partial_\mu\phi = 0.$$
(16)

At this point we choose to make the field decomposition

$$\psi_0 = \chi, \qquad \psi_i = \psi_i^T + \partial_i\psi, \qquad \sigma = \phi + \psi, \quad (17)$$

under which the equations become

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[3]We mention again that quantum mechanically, we should really be considering all operators of the form $(\partial\phi)^{2n}$, which become of the same order as $(\partial\phi)^2$ once the Vainshtein radius is approached from the outside. Since we are concerned here with a mere illustration of how dual theories work (before a much more stable analysis of Galileons in the next section), and for simplicity, we choose to ignore these operators.

$$\Box\sigma + \partial_0^2\psi - \partial_0\chi = -\frac{1}{M_P}T,$$

$$\Lambda^{4/3}(-\chi^2 + (\psi_i^T + \partial_i\psi)^2)^{-(1/3)}\chi + \partial_0\psi = \partial_0\sigma,$$

$$\Lambda^{4/3}(-\chi^2 + (\psi_i^T + \partial_i\psi)^2)^{-(1/3)}(\psi_i^T + \partial_i\psi) + \partial_i\psi = \partial_i\sigma.$$
(18)

One can see that the vanishing transverse component of ψ, $\psi_i^T = 0$, is a consistent ansatz for the solution. Moreover, for static field configurations, $\chi = 0$ and σ obeys a linear equation sourced by T, while σ in turn determines ψ. Instead of the irrelevant operator of the original theory (2), we have only the self-interaction of the field ψ which looks like a *relevant* operator, and we could expect it to be subdominant in the UV. In the IR, on the other hand, things are ill-defined because the interactions above the trivial ground state are nonanalytic.[4]

For the static point source $T = -M\delta^3(\mathbf{x})$, the only excited degrees of freedom are σ and the longitudinal component of ψ_i. The first equation in (18) tells us that the exact value of σ is its linear Newtonian value, $\sigma = -\frac{M}{4\pi M_P}\frac{1}{r}$, while the equation for ψ reduces to the following one:

$$\psi' + \Lambda^{4/3}(\psi')^{1/3} = \frac{M}{4\pi M_P}\frac{1}{r^2}.$$
(19)

If we ignore the nonlinear interaction by setting $\Lambda = 0$, ψ will have the zeroth order linear solution, $\psi_0 = -\frac{M}{4\pi M_P}\frac{1}{r}$. The full solution for the original field ϕ_0 is then trivial, since there is a cancellation $\phi_0 = \sigma - \psi_0 = 0$. It is easy now to estimate the distance for which the nonlinear term in (19) becomes important. We find that this scale is again the Vainshtein radius r_* (7). However, in contrast to the original formulation of the theory, the self-interactions are now important only at distances larger than the Vainshtein radius, $r \gtrsim r_*$.

We can see this reversal in the region of strong coupling more explicitly by solving the ψ equation perturbatively using the dual formulation. We set up the expansion by expanding in powers of the interaction coupling $\Lambda^{4/3}$,

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots,$$
(20)

plugging into (19) and equating powers of Λ. The solution to lowest nontrivial order is

$$\psi_0 + \psi_1 = -\frac{M}{4\pi M_P}\frac{1}{r} - 3\Lambda^{4/3}\left(\frac{M}{4\pi M_P}\right)^{1/3}r^{1/3} + \text{const},$$
(21)

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[4]All the above statements refer to the classical theory. We point out, however, that noncovariant decomposition (17), which does not introduce extra time derivatives, is likely to be a good starting point for quantization of this theory.

while σ has the linear Newtonian $1/r$ profile to all orders. Recalling the definition of the physical field ϕ, we have,

$$\phi = \sigma - \psi = 3\Lambda^{4/3}\left(\frac{M}{4\pi M_P}\right)^{1/3} r^{1/3} + \text{const} + \cdots,$$

(22)

this shows that the expansion is in powers

$$\left(\frac{r}{r_*}\right)^{4/3}.$$

(23)

This expansion is inverse to the expansions of the original theory (14). As a result, the dual expansion breaks down as we approach the Vainshtein radius from the inside.

B. Profile of an infinite string

The use of the dual formulation is not restricted to spherically symmetric static solutions. It should provide a complementary description for any classical solution. For illustrative purposes, we now consider a cylindrically symmetric solution sourced by a uniform string, with the mass-per-unit-length denoted by κ

$$T = -\kappa \frac{\delta(r)}{2\pi r}.$$

(24)

The exact solution again has two regimes, separated by a Vainshtein radius

$$r_* \sim \frac{\kappa}{M_P \Lambda^2}.$$

(25)

The leading behavior of ϕ in the two regimes is,

$$\phi = \begin{cases} \frac{3}{2}\left(\frac{\kappa}{2\pi M_P}\right)^{1/3} \Lambda^{4/3} r^{2/3} + \text{const}, & r \ll r_*, \\ \frac{\lambda}{2\pi M_P} \ln\left(\frac{r}{r_s}\right), & r \gg r_*, \end{cases}$$

(26)

where r_s is a UV regulator scale—the transverse width of the string in this case. Using perturbation theory in the original formulation, we recover the logarithmic profile for $r \gg r_*$ as the leading term, and perturbation theory breaks down as we approach the Vainshtein radius from the outside.

On the other hand, the dual form of the equations of motion, (18), yields the following expressions for the fields σ and ψ:

$$\chi = 0, \qquad \sigma = \frac{\lambda}{2\pi M_P} \ln\left(\frac{r}{r_s}\right),$$

$$\psi \simeq \frac{\lambda}{2\pi M_P} \ln\left(\frac{r}{r_s}\right) - \frac{3}{2}\left(\frac{\lambda}{2\pi M_P}\right)^{1/3} \Lambda^{4/3} r^{2/3} + \text{const}. \quad (27)$$

The expression for σ here is exact as above, while the series for ψ breaks down in the IR, as the Vainshtein scale r_* is approached from the inside. Recalling the definition $\phi = \sigma - \psi$, one again finds an agreement with the result obtained in the original formulation of (26).

III. THE CUBIC GALILEON

We have illustrated how the simplest model of a single Nambu-Goldstone scalar can be rewritten in a form for which classical perturbation theory has a region of validity opposite to that of the original formulation. The method however is quite general. The essence of the method is to introduce auxiliary fields in such a way as to replace the nonrenormalizable derivative interactions with (generally nonanalytic) nonderivative terms. We now consider the cubic Galileon —an example of a scalar field theory that is relevant for modifications of gravity,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{\Lambda^3}(\partial\phi)^2 \Box\phi + \frac{1}{M_P}\phi T. \quad (28)$$

As mentioned before, in the $M_{\text{Pl}} \to \infty$ limit, the Galileon term in the above action does not get renormalized in the full quantum theory; also, no higher Galileons [8] will be induced if they are not introduced to begin with. Moreover, no NG-type terms, $(\partial\phi)^n$, of the previous section will be generated since the latter do not respect the "Galilean symmetry" $\phi \to \phi + c + c_\mu x^\mu$.

In spite of the presence of higher derivatives, the equations of motion that follow from the above action are second-order, leading to a well-defined Cauchy problem, and the absence of additional ghostly degrees of freedom,

$$\Box\phi - \frac{2}{\Lambda^3}[(\partial_\mu \partial_\nu \phi)^2 - (\Box\phi)^2] = -\frac{T}{M_P}. \quad (29)$$

Concentrating again on radial profiles for ϕ sourced by a pointlike source, $T = -M\delta^3(\mathbf{x})$, the equation of motion (29) reduces to the following:

$$\vec{\nabla}\cdot\left[\vec{\nabla}\phi + \frac{1}{\Lambda^3}(2\vec{\nabla}\phi\vec{\nabla}^2\phi - \vec{\nabla}(\vec{\nabla}\phi)^2)\right] = \frac{M}{M_P}\delta^3(\mathbf{x}). \quad (30)$$

Integrating once, we obtain a quadratic equation for the radial derivative of the Galileon field,

$$\frac{\phi'}{r} + \frac{4}{\Lambda^3}\frac{\phi'^2}{r^2} = \frac{M}{M_P}\frac{1}{4\pi r^3}. \quad (31)$$

Like in the model of the previous section, the exact solution has two regimes, separated by the Vainshtein radius r_*,

$$\phi(r) = \begin{cases} \frac{1}{2\sqrt{\pi}}\Lambda^3 r_*^2 \left(\frac{r}{r_*}\right)^{1/2}\left[1 + \mathcal{O}\left(\frac{r^{3/2}}{r_*^{3/2}}\right)\right] + \text{const} & r \ll r_*, \\ -\frac{M}{M_P}\frac{1}{4\pi r}\left[1 + \mathcal{O}\left(\frac{r^3}{r^3}\right)\right] & r \gg r_*, \end{cases}$$

(32)

where

$$r_* \equiv \left(\frac{M}{M_P}\right)^{1/3}\frac{1}{\Lambda}. \quad (33)$$

The Vainshtein mechanism is therefore at work in the cubic Galileon theory as well, screening the scalar potential significantly within r_*. As in the previous section, the classical field is weak (sub-Planckian) in both the linear and nonlinear regimes, $(\phi/M_{\rm Pl}) \ll 1$.

Perturbation theory in this formulation allows us to compute the corrections to the $1/r$ solution for $r \gg r_*$ via a $1/\Lambda^3$ expansion of Eq. (29). We thus write,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots, \qquad (34)$$

so that after plugging into the equation of motion and equating powers of $1/\Lambda$, we find the series of equations

$$\Box \phi_0 = -\frac{T}{M_P}. \qquad (35)$$

$$\Box \phi_1 - \frac{2}{\Lambda^3}[(\partial_\mu \partial_\nu \phi_0)^2 - (\Box \phi_0)^2] = 0. \qquad (36)$$

$$\vdots \qquad (37)$$

This gives an expansion in powers of $(r_*/r)^3$, which is valid outside the Vainshtein radius and starts to fail as the Vainshtein radius is approached from the outside.

A. The dual Galileon theory

We would now like to find a dual formulation of the Galileon, one whose classical perturbative expansion is valid *inside* the Vainshtein radius. Starting with the original Lagrangian (28), we introduce two auxiliary scalar fields b_μ and λ and write an equivalent version of the theory as follows:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \Lambda^{3/2}\sqrt{\lambda b_\mu^2} - b^\mu \partial_\mu \phi - \lambda \Box \phi + \frac{1}{M_P}\phi T. \qquad (38)$$

Again, we have succeeded in representing the cubic Galileon in a form in which all terms look *relevant* at the expense of introducing fractional dimensions on fields. This has a more complicated structure than the dual Lagrangian of the previous section (15). However, there are some similarities, such as that the nonanalytic potential term in (38) also has a tachyonic sign, and the fact the ϕ field becomes trivial in both (15) and (38) in the limit $\Lambda \to 0$, $M_{\rm Pl} \to \infty$. This, and the spectrum of the theory, is best seen by using a noncovariant (i.e., a $3 + 1$) decomposition of the vector field b_μ, as given below.[5]

The equations for ϕ, λ and b^μ that follow from the latter Lagrangian are given as follows:

$$\Box\phi + \partial_\mu b^\mu - \Box\lambda = -\frac{1}{M_{\rm P}}T,$$

$$-\Box\phi + \frac{1}{2}\Lambda^{3/2}\sqrt{\frac{b_\mu^2}{\lambda}} = 0, \qquad (39)$$

$$-\partial_\mu\phi + \Lambda^{3/2}\sqrt{\frac{\lambda}{b_\nu^2}}b_\mu = 0.$$

Resorting again to the $3 + 1$ decomposition of the vector field, taking the divergence of the last of the equations of motion (39),[6] forming suitable linear combinations and (re) defining fields as follows:

$$b_0 = \beta, \quad b_i = b_i^T + \partial_i b, \quad \lambda = \bar{\lambda} + b, \quad \phi = \bar{\phi} + \bar{\lambda}, \quad (40)$$

one can reduce (39) to the following system:

$$\Box\bar{\phi} - \partial_0\beta + \partial_0 b = -\frac{1}{M_{\rm P}}T,$$

$$\partial_0\left(\partial_0\bar{\lambda} - \Lambda^{3/2}\sqrt{\frac{\bar{\lambda}+b}{b_\nu^2}}\beta\right) - \partial_i\left(\partial_i\bar{\lambda} - \Lambda^{3/2}\sqrt{\frac{\bar{\lambda}+b}{b_\nu^2}}(b_i^T + \partial_i b)\right) + \partial_0^2 b - \partial_0\beta = -\frac{1}{M_{\rm P}}T, \qquad (41)$$

$$\frac{1}{2}\sqrt{\frac{b_\nu^2}{\bar{\lambda}+b}} - \partial_0\left(\sqrt{\frac{\bar{\lambda}+b}{b_\nu^2}}\beta\right) + \partial_i\left(\sqrt{\frac{\bar{\lambda}+b}{b_\nu^2}}(b_i^T + \partial_i b)\right) = 0.$$

As in the example of the previous section, in the presence of a point source, $T = -M\delta^3(\mathbf{x})$, the transverse component b_i^T as well as β vanish; moreover, one combination of the fields, denoted by $\bar{\phi}$, is free and has a linear equation everywhere in space, receiving an exact Newtonian profile

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[5]As noted in the previous section, this decomposition, unlike the covariant one, does not introduce artificially extra time derivatives which would have confused the counting of propagating modes already in the classical theory.

[6]For the static profiles at hand, retaining only the divergence of this equation is sufficient for obtaining the complete solution.

$$\bar{\phi} = -\frac{M}{4\pi M_{\mathrm{P}}}\frac{1}{r}. \tag{42}$$

The last equation from the system (41) on the other hand gives

$$b = -\bar{\lambda} - \frac{1}{4}r\bar{\lambda}', \tag{43}$$

which, when plugged into the second of (41), gives an equation for $\bar{\lambda}$ after integrating once,[7]

$$-\bar{\lambda}' + \Lambda^{3/2}\sqrt{-\frac{1}{4}r\bar{\lambda}'} = \frac{M}{4\pi M_{\mathrm{P}}}\frac{1}{r^2}. \tag{44}$$

We have now arrived at an equation which achieves the goal of the dual formulation. The interaction term, proportional to $\Lambda^{3/2}$, becomes important only at distances *larger* than the Vainshtein radius (33). We again set up perturbation theory by expanding the $\bar{\lambda}$ profile in powers of $\Lambda^{3/2}$,

$$\bar{\lambda} = \bar{\lambda}_0 + \bar{\lambda}_1 + \bar{\lambda}_2 + \cdots. \tag{45}$$

Plugging the expansion into (44) and equating powers of Λ, we obtain

$$\bar{\lambda}_0 = \frac{M}{4\pi M_{\mathrm{P}}}\frac{1}{r}, \quad \bar{\lambda}_1 = \Lambda^{3/2}\left(\frac{M}{4\pi M_{\mathrm{P}}}\right)^{1/2}r^{1/2} + \mathrm{const}, \quad \cdots \tag{46}$$

Finally, recalling the definition of the physical field ϕ, we have

$$\phi = \bar{\phi} + \bar{\lambda} = \frac{1}{2\sqrt{\pi}}\Lambda^3 r_*^2\left(\frac{r}{r_*}\right)^{1/2} + \mathrm{const} + \cdots, \tag{47}$$

in complete agreement with the result (32) of the original theory well within the Vainshtein radius. The perturbative expansion in the dual formulation is an expansion in the ratio

$$\left(\frac{r}{r_*}\right)^{3/2}, \tag{48}$$

and so the expansion is valid inside the Vainshtein radius, complementary to the expansion in the original formulation which is valid outside r_*.

B. Domain wall and infinite string

Similarly to the case of the Nambu-Goldstone theory, the dual formulation should be useful in reorganizing the perturbation expansion for any classical solution. It is interesting to see this on examples other than that of a point source—such as a domain wall or a string.

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[7]One has to be careful with the square root at this point. Jumping a bit ahead, we note that (in complete analogy to the Goldstone-Stückelberg case considered above) the leading term in the expression for $\bar{\lambda}$ well within the Vainshtein radius is of the form A/r, with A denoting some positive constant. This uniquely fixes all signs in front of square roots, as well as makes the $\sqrt{\bar{\lambda} + b}$ expression in these equations well defined.

It has been shown that domain walls do not possess a Vainshtein scale in DGP and massive gravity [26] (this scale is of the order of the wall width), so they give rise to a fifth force at all distances. This fact should be captured by the cubic Galileon theory (28) and therefore by its dualized version, presented above. The absence of an r_* distance can be easily seen in the equations of motion of the original theory (29). Indeed, in the presence of an infinite domain wall in the x-y plane at $z = 0$, the problem becomes one dimensional, with ϕ depending on a single coordinate z. One can then easily see that all nonlinearities in the original equations of motion vanish and one is left with the Newtonian profile of a one-dimensional source for the scalar. On the other hand, the last of the equations of motion (41) that follow from the dual theory implies $\lambda = 0$ for only z-dependent profiles. Recalling the definition of the original Galileon ϕ in terms of the free scalar $\bar{\phi}$ of the dual theory and λ, $\phi = \bar{\phi} + \lambda$, one obtains agreement between the two representations of the theory (as must be the case since the representations are classically equivalent). Both the original and dual formulations are free of nonlinearities, and so there is no perturbative expansion to be done in either case.

Next consider an infinite string source, $T = -\kappa\frac{\delta(r)}{2\pi r}$. In the original Galileon theory, the equation of motion can be integrated to yield the following algebraic equation for the radial derivative of the axially symmetric ϕ profile:

$$\phi' + \frac{2}{\Lambda^3}\frac{(\phi')^2}{r} = \frac{\kappa}{2\pi M_{\mathrm{P}}r}. \tag{49}$$

One can immediately read off the Vainshtein radius from this equation,[8]

$$r_* \sim \left(\frac{\kappa}{M_{\mathrm{P}}\Lambda^3}\right)^{1/2}. \tag{50}$$

Using perturbation theory, one can readily solve for ϕ well outside the Vainshtein radius to obtain the leading behavior of the scalar profile,

$$\phi \simeq \frac{\kappa}{2\pi M_{\mathrm{P}}}\ln\left(\frac{r}{r_s}\right), \qquad r \gg r_*, \tag{51}$$

where again r_s is a cutoff scale the finite thickness of the string. Well within the Vainshtein radius, the nonlinear term in (49) dominates, leading to

$$\phi \simeq \left(\frac{\kappa\Lambda^3}{4\pi M_{\mathrm{P}}}\right)^{1/2}r + \mathrm{const}, \qquad r \ll r_*. \tag{52}$$

In the dual theory on the other hand, we can proceed in complete analogy to the above analysis. The last of the system (41) yields the following identity for such profiles:

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[8]Note that this decoupling-limit expression for the Vainshtein radius coincides with the one derived in the full DGP model in [27].

$$b = -\bar{\lambda} - \tfrac{1}{2}r\bar{\lambda}'. \tag{53}$$

Plugging this into the second of these equations and integrating, one finds that the equation determining $\bar{\lambda}$ is given as follows:

$$-\bar{\lambda}' + (\Lambda)^{3/2}\sqrt{\bar{\lambda} + b} = \frac{\kappa}{2\pi M_{\mathrm{P}}r}. \tag{54}$$

One can now solve this equation perturbatively, but perturbation theory now works well *within* the Vainshtein radius,

$$\bar{\lambda} = -\frac{\kappa}{2\pi M_{\mathrm{P}}} \ln\left(\frac{r}{r_s}\right) + \Lambda^{3/2}\left(\frac{\kappa}{4\pi M_{\mathrm{P}}}\right)^{1/2} r + \mathrm{const}, \tag{55}$$

arriving at an expression for ϕ which is valid in the UV,

$$\phi = \bar{\phi} + \bar{\lambda} = \Lambda^{3/2}\left(\frac{\kappa}{4\pi M_{\mathrm{P}}}\right)^{1/2} r + \mathrm{const}. \tag{56}$$

This is in complete agreement with the expression (52), obtained from the original formulation.

IV. CONCLUSIONS

We have studied a few examples of classical dualization for Nambu-Goldstone and Galileon theories, which allows for a perturbative formulation of the regimes in which the original theory becomes classically nonperturbative. Quantum mechanically, such a formulation is only valid in special cases, where certain symmetries make it possible for classical nonlinearities to be strong, while keeping quantum corrections under complete control.

Among scalar field theories, Galileons perhaps represent the most remarkable examples of this, due to a powerful nonrenormalization theorem [4,8,10] protecting the leading part of the Lagrangian from quantum corrections. Hence, our results are justified for these theories. Moreover, it should be possible to generalize our approach to the higher Galileon terms.

In addition to capturing many features of the DGP model, Galileons have emerged as an essential ingredient in the recently formulated ghost-free massive gravity models [2,3]. The decoupling limit of these theories represents a certain (scalar-tensor) extension of the Galileon with a more general structure, which however retains all the nice properties of the Galileons, such as the presence of the Vainshtein mechanism and the nonrenormalization theorems (for studies of cosmological and spherically symmetric solutions in the decoupling limit and beyond in ghost-free massive gravity, see [28–40] and references therein). Our method should have a straightforward algorithmic generalization to those more general Lagrangians obtained in [2], and should be useful for the development of the analog of the parametrized post Newtonian formalism in massive gravity [3] (or for generic modified gravity models with extra scalar fields, such as the recently proposed fab four theory, [41]) and for systematically determining the

observational consequences of the Vainshtein mechanism [42,43].

The method presented above might as well be useful in studies of the proposal of Ref. [23] (see [44] for treatment of the scattering problem in this context for the theories considered above).

Finally, it remains to be seen if quantization of the classical duals considered in the present work can lead to duals of the full quantum theory. Given the nonanalytic nature of the dual theories that we obtained, quantization seems to be a nontrivial task. We expect the noncovariant decomposition of the auxiliary fields used in Secs. II and III to be a good starting point for bookkeeping of the degrees of freedom. The quantization procedure, may or may not force us to introduce additional dynamical degrees of freedom at the scale Λ.

APPENDIX: MORE GENERAL INTERACTIONS

In this appendix we study a more general derivative self-interaction which is an arbitrary power of the field ϕ, in order to show that the dual formulation always has non-analytic self-interactions with fractional powers of the fields, and that this fractional power is always less than 2, i.e., less than that of a mass term.

Consider a (ghost-free) Goldstone–Stückelberg-type theory with a shift symmetry and a Z_2 symmetry. A general interaction term containing only one derivative per field is given by the following:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{((\partial\phi)^2)^n}{2n\Lambda^{4n-4}} + \frac{1}{M_{\mathrm{P}}}\phi T. \tag{A1}$$

The theory can be equivalently rewritten by integrating in the vector field ψ_μ,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \psi^\mu \partial_\mu \phi$$
$$+ \left(1 - \frac{1}{2n}\right)\Lambda^{4n-4/2n-1}((\psi_\mu)^2)^{n/2n-1} + \frac{1}{M_{\mathrm{P}}}\phi T. \tag{A2}$$

Integrating out ψ_μ recovers (A1). Note that the power of ψ_μ in the interaction term is always <2.

The equations of motion that follow from (A2) can be reduced to a form similar to the system (19). Using the same decomposition of the vector field as before,

$\psi_\mu \to (\chi, \psi_i^T + \partial_i \psi)$, one finds that $\chi = \psi_i^T = 0$. One combination of the fields, $\sigma = \phi + \psi$, in the presence of a pointlike external source has the usual $1/r$ profile at all distances, whereas ψ has a crossover Vainshtein scale due to nonlinearities. Moreover, the form of the dual action suggests that at small distances, nonlinearities in ψ should be subdominant, providing a small perturbation over the $1/r$ potential. This can be checked explicitly by perturbatively solving the static equation of motion for ψ,

$$-\vec{\nabla} \cdot \left(\vec{\nabla}\psi + \Lambda^{4n-4/2n-1} \vec{\nabla}\psi((\vec{\nabla}\psi)^2)^{-(n-1/2n-1)} \right) = \frac{1}{M_\mathrm{P}} T.$$
(A3)

For a static pointlike source $T = -M\delta^3(\mathbf{x})$, this reduces to

$$\psi' + \Lambda^{4n-4/2n-1}(\psi')^{1/2n-1} = \frac{M}{4\pi M_\mathrm{P} r^2}.$$
(A4)

Expanding ψ into contributions of different order in powers of the nonlinear interaction,

$$\psi = \psi_0 + \psi_1 + \ldots,$$
(A5)

with $\psi_0 = -\frac{M}{4\pi M_\mathrm{P} r}$, one finds that the first perturbation should satisfy the following equation:[9]

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[9]Note that the fractional powers of $-M/M_\mathrm{P}r^2$ in this equation are well defined for any n.

$$\psi_1' = -\Lambda^{4n-4/2n-1}\left(\frac{M}{4\pi M_\mathrm{P} r^2} \right)^{1/2n-1},$$
(A6)

which is solved by the following expression:

$$\psi_1 = -\frac{(2n-1)}{(2n-3)}\Lambda^{4n-4/2n-1}\left(\frac{M}{4\pi M_\mathrm{P}} \right)^{1/2n-1} r^{2n-3/2n-1}.$$
(A7)

One can estimate the crossover distance r_* for the ψ profile as the one for which the perturbation theory breaks down,

$$r_* \sim \left(\frac{M}{M_P} \right)^{1/2} \frac{1}{\Lambda}.$$
(A8)

This scale is of the same order for any n, up to an weakly n-dependent multiplicative factor of order one. The expression for the physical field ϕ inside the Vainshtein radius is

$$\phi = \sigma - \psi = -\psi_1$$
$$= \frac{(2n-1)}{(2n-3)}\Lambda^{4n-4/2n-1}\left(\frac{M}{4\pi M_\mathrm{P}} \right)^{1/2n-1} r^{2n-3/2n-1}.$$
(A9)

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