

**Schrödinger models for solutions of the Bethe–Salpeter equation in Minkowski space**Richard L. Hall<sup>1,\*</sup> and Wolfgang Lucha<sup>2,†</sup><sup>1</sup>*Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Boulevard West, Montréal, Québec, Canada H3G 1M8*<sup>2</sup>*Institute for High Energy Physics, Austrian Academy of Sciences, Nikolsdorfergasse 18, A-1050 Vienna, Austria*  
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By application of the “geometric spectral inversion” technique, which we have recently generalized to accommodate also *singular* interaction potentials, we construct from spectral data emerging from the solution of the Minkowski-space formulation of the homogeneous Bethe–Salpeter equation describing bound states of two spinless particles a Schrödinger approach to such states in terms of nonrelativistic potential models. This spectrally equivalent modeling of bound states yields their *qualitative* features (masses, form factors, etc.) without having to deal with the more involved Bethe–Salpeter formalism.

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**I. INTRODUCTION: MOTIVATION AND INCENTIVE**

Within the framework of relativistic quantum field theory, the appropriate tool for the description of bound states is, in principle, the Bethe–Salpeter (BS) formalism [1–3]. In this approach, a bound state  $B(P)$ , of momentum  $P$  and mass  $M$ , is described by its BS amplitude, which in configuration-space representation is defined by the matrix element of the time-ordered product of the field operators of all bound-state constituents between vacuum  $|0\rangle$  and bound state  $|B(P)\rangle$ . In momentum-space representation, the BS amplitude, upon splitting off the center-of-momentum motion of the bound state and suppression of all indices generically denoted by  $\Phi(p, P)$ , encodes the distribution of the relative momenta  $p$  of the bound-state constituents. It satisfies a formally exact BS equation involving two kinds of dynamical ingredients, namely, for bound states composed of  $n$  constituents, (a) the propagators  $S_i(p_i)$  ( $i = 1, 2, \dots, n$ ) of the  $n$  constituents of respective individual momenta  $p_i$  and (b) its BS interaction kernel  $K$ , a fully truncated  $(2n)$ -point Green function of the  $n$  bound-state constituents, perturbatively defined as the sum (of the countable infinity) of all “BS-irreducible” Feynman graphs for  $n$  particle into  $n$ -particle scattering. For two bound-state constituents, the BS equation is of the generic form

$$\Phi(p, P) = \frac{i}{(2\pi)^4} S_1(p_1) \int d^4q K(p, q, P) \Phi(q, P) S_2(-p_2). \quad (1)$$

Physical considerations provide a profound motivation to formulate one’s BS framework in *Minkowski space*, with the pseudo-Euclidean space-time metric tensor  $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ . In the Minkowski-space formulation, however, finding solutions to the BS equation may be

heavily impeded by the presence of singularities induced by the propagators of the bound-state constituents or its BS interaction kernel. As a remedy, by assuming that analytic continuation of the Minkowski-space formalism is possible and the Cauchy integral theorem is applicable, it has been proposed to study the BS equation in Euclidean space, with metric  $g_{\mu\nu} = \delta_{\mu\nu}$ , reached by a procedure misleadingly labeled Wick “rotation” [4]. The Euclidean-space formulation facilitates making contact with lattice field theory, usually defined in Euclidean space. Solutions of the BS equation provide the set of mass eigenvalues  $M$  and associated BS amplitudes  $\Phi$  of the bound states. The mass eigenvalues arising in Minkowski-space and Euclidean-space formulations of a given BS equation are identical. For BS amplitudes, however, complicated analyticity structures of the BS equation in the complex plane cause troubles: The BS solutions derived in one formulation may differ from those obtained in the other one. That is to say, the analytic continuation to Minkowski space of some solution to the BS equation in Euclidean space may bear no resemblance to its counterpart found as solution to the same BS equation in Minkowski space. Since the BS amplitudes determine physical observables such as decay constants and form factors, knowledge of the Minkowski-space amplitudes is highly desirable.

This dilemma between, on the one hand, the comparative ease of deriving solutions to a Euclidean-space BS equation and, on the other hand, the need of physical applications for BS amplitudes constituting solutions to a Minkowski-space BS equation can be tentatively resolved by developing—of course, only approximately equivalent—Schrödinger models. This may be effected by fixing the interaction potential entering in the Schrödinger Hamiltonian by spectral inversion of the bound-state mass eigenvalues  $M$  arising from the easier-to-accomplish solution of the Euclidean-space BS equation. The wave functions obtained as solutions of the resulting Schrödinger equation allow us to compute decay constants and form factors of bound states, provided we succeed in

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acquiring control of the uncertainties introduced by such modeling. Obviously, we may estimate the accuracy of the envisaged Schrödinger models by applying our proposal for resolution of the dilemma to some known solutions of the Minkowski-space BS equation in order to extract an associated Schrödinger potential and comparing the outcome of the corresponding Schrödinger equation with the findings of the BS framework.

Recently, renewed attempts of solving the BS equation for two-particle bound states in Minkowski-space formulation have been undertaken [5–13]. The basic idea advocated for in Ref. [5] is to remove the singularities of the BS amplitudes by considering an equivalent integral transform of the BS equation (1) obtained by projection onto the light-front plane, and to take advantage of a particular integral representation of the BS amplitude  $\Phi(p, P)$  proposed by Nakanishi [14], in order to obtain a nonsingular integral equation that is straightforward to solve numerically. This route has been applied to bound states consisting either of two identical spin-0 bosons [5–10] or of a spin- $\frac{1}{2}$  fermion and its antiparticle [11–13]. Any such bound state arises, in the case of two scalar constituents, from their couplings to a further scalar boson or, in the case of fermionic constituents, from their couplings to a scalar, a pseudoscalar, or a massless vector boson; for bound states of scalars, the interactions are taken into account in the BS equation by considering the BS kernel either in ladder approximation, which amounts to the iteration of single-boson exchanges, or in ladder-plus-cross-ladder approximation. Moreover, for the bound-state constituents the studies [5–13] employ, for simplicity, the free-propagator approximation.

In order to get an idea of the behavior of such Schrödinger interaction potentials to be expected to arise in the course of spectral inversion, we recall that there is a well-paved path of simplifications leading from the relativistic BS equation to its nonrelativistic or “static” Schrödinger reduction. The sequence of necessary steps involves several well-defined and thoroughly studied approximations to the BS formalism (for brief reviews of this reduction, consult, e.g., Refs. [15–17]):

- (1) In some instantaneous limit, realizable if in the bound state’s center-of-momentum frame fixed by  $P = (M, \mathbf{0})$  the BS kernel takes the form  $K(p, q, P) = K(\mathbf{p}, \mathbf{q})$ , the BS equation may be reduced to the *instantaneous BS equation* (for attempts in these directions, consult, for instance, Ref. [18] and references therein) for the Salpeter amplitude

$$\phi(\mathbf{p}) \equiv \frac{1}{2\pi} \int dp_0 \Phi(p).$$

- (2) The additional assumption of free propagation of all bound-state constituents with effective masses encompassing the dynamical self-energy effects leads

to the *Salpeter equation* [19]. (Note, however, that in quantum field theory the Dyson–Schwinger equations relate every  $n$ -point Green function to at least one ( $m > n$ )-point Green function. This means, in particular, that the propagators, i.e., the 2-point Green functions, and the  $n$ -point Green functions entering in the BS kernel cannot be chosen independently: the use of free propagators might be incompatible with the feature of confinement exhibited by quantum chromodynamics, the theory describing the strong interactions.)

- (3) Dropping all negative-energy contributions simplifies Salpeter’s equation to the *reduced Salpeter equation* [20–24].
- (4) Furthermore, ignoring all spin degrees of freedom of all bound-state constituents and assuming the BS interaction kernel  $K$  to be of convolution type, i.e., to depend only on the difference of the involved relative momenta  $\mathbf{p}$  and  $\mathbf{q}$ ,  $K(\mathbf{p}, \mathbf{q}) = K(\mathbf{p} - \mathbf{q})$ , yields the *spinless Salpeter equation*. Therein, the interactions manifest in form of a potential arising, in configuration space, as the Fourier transform of this kernel  $K(\mathbf{p} - \mathbf{q})$ . This bound-state equation may be viewed as a generalization of the Schrödinger equation towards relativistic kinematics. (Concise reviews of various aspects and facets of semirelativistic approaches to the bound-state problem may be found in, e.g., Refs. [25–27].)<sup>1</sup>
- (5) In an ultimate static limit, replacing in the latter equation of motion the relativistic form of all one-particle kinetic energies by the corresponding nonrelativistic approximation, we eventually end up with the *Schrödinger equation*.

Let us begin our analysis by inspecting the simplest case: bound states of two scalar constituents [5–10]. In the ladder approximation, the only contribution to the BS interaction kernel derives from single-particle exchange. Apart from the couplings of the exchanged particle to the bound-state constituents, the BS kernel is then nothing but the propagator of the exchanged particle. For a scalar boson with mass  $\mu$ , its free propagator is given by  $S(k) = i(k^2 - \mu^2)^{-1}$ . The Fourier transform in three dimensions of the instantaneous approximation to this propagator, i.e., of its remnant  $i(\mathbf{k}^2 + \mu^2)^{-1}$ , is proportional to the configuration-space Yukawa potential  $V(r) = -\exp(-\mu r)/r$ , thus singular at the origin  $r \equiv |\mathbf{x}| = 0$ . As a consequence, depending, clearly, on the proximity of the system described by the BS equation to the nonrelativistic Schrödinger limit, the outcome of any spectral inversion may be potentials resembling, to some extent, the Yukawa type but modified, of course, by the various effects ignored on the way down to the static limit, such as

<sup>1</sup>A related approach is the quasipotential formalism devised by Todorov [28].

relativistic kinematics or higher-order contributions to the BS interaction kernel; cross-ladder terms are but the simplest example of the latter. Accordingly, we have to devise and utilize an inversion technique that is capable of dealing also with singular potentials.

To this end, we recently generalized the earlier geometric spectral inversion [29–33] to treat singular potentials [34]. We suppose that  $f(r)$  is the shape of the potential and  $v > 0$  is the coupling parameter in the Schrödinger Hamiltonian  $H = -\Delta/(2m) + vf(r)$ . In this inversion technique, a functional sequence is built which starts from a seed  $f^{[0]}(r)$  and reconstructs the potential shape  $f(r)$  from a given spectral function  $E = F(v)$  that defines how a discrete eigenvalue  $E$  of  $H$  depends on the parameter  $v$ . The most relevant earlier paper is Ref. [34], which also includes a proof of uniqueness for the inverse for a large class of singular potentials. Here, the inversion sequence is defined in Sec. II C; a statement of the uniqueness theorem may be found in Sec. II D. The principal goal of the present paper is to take for  $F(v)$  the solutions to the Minkowski-space BS equation [5–10,13] for bound states of two scalar constituents, and to reconstruct directly from this set of data an effective potential shape  $f(r)$  in the Schrödinger model defined by the Hamiltonian  $H$ .

The outline of this paper is as follows. In Sec. II, we summarize enough of the geometric spectral inversion theory to make this paper essentially self-contained. In Sec. III, we present the spectral data  $F(v)$  from the BS solutions [5–10]. In Sec. IV, we apply the functional inversion sequence of Sec. II C to these data to construct the effective potential shape. In the Appendix, we sketch the nonrelativistic reduction of the Bethe–Salpeter formalism for scalar bound-state constituents along a route which mimics to the utmost possible extent the case of fermionic bound-state constituents.

## II. GEOMETRIC SPECTRAL INVERSION

We consider the discrete spectrum of a Schrödinger Hamiltonian operator

$$H = -\Delta + vf(r), \quad r \equiv \| \mathbf{x} \|, \quad (2)$$

where  $f(r)$  is the shape of an attractive central potential, and  $v > 0$  is a coupling parameter. We shall assume that the potentials are monotone nondecreasing and no more singular than the Coulomb potential  $f(r) = -1/r$ . The arguments we use apply generally to the problem in  $d > 1$  spatial dimensions, but, for definiteness, we shall usually assume that  $d = 3$ . The operator inequality [35,36]

$$-\Delta \geq \left( \frac{d/2 - 1}{r} \right)^2, \quad d \geq 3, \quad (3)$$

implies that a discrete spectrum exists for sufficiently large coupling  $v > 0$ . For  $d = 3$ , the Hamiltonian  $H$  is bounded below by

$$E \geq \min_{r>0} \left[ \frac{1}{4r^2} + vf(r) \right], \quad (4)$$

and a simple trial function can be used to establish an upper bound to  $E$ . Thus, we may assume, in particular, that the ground-state energy may be written as a function  $E = F(v)$ . An explicit example of the class of problems we consider is provided by the Hulthén potential, whose shape is given by  $f(r) = -1/(e^r - 1)$  and whose  $s$ -state ( $\ell = 0$ ) eigenvalues  $E_n$  are given [37] exactly for  $d = 3$  by the formula

$$E_n = F_n(v) = -\left( \frac{v - n^2}{2n} \right)^2, \quad v > n^2, \quad n = 1, 2, 3, \dots \quad (5)$$

The problem discussed in the present paper may be stated as follows: given, for example, the curve  $F_1(v)$ , can we use this spectral data to reconstruct the potential shape  $f(r)$ ? We call this reconstruction a “geometric spectral inversion”.

### A. Exact representation of spectral functions by kinetic potentials

The discrete spectra of operators bounded from below, such as  $H = -\Delta + vf(r)$ , may be characterized variationally [38]. Thus, the ground-state energy may be written

$$F(v) = \inf_{\substack{\psi \in \mathcal{D}(H) \\ \|\psi\|=1}} (\psi, H\psi). \quad (6)$$

Since  $H$  depends on the coupling  $v$ , so therefore does the domain  $\mathcal{D}(H)$ . However, for the problems considered, either  $H$  has discrete eigenvalues, perhaps for  $v$  greater than some critical coupling  $v_1$ , or the entire spectrum of  $H$  is discrete for  $v > 0$ . The kinetic potential  $\tilde{f}(s)$  associated with a given potential shape  $f(r)$  is defined (for the ground state  $\psi$ ) by a constrained minimization in which the mean kinetic energy  $s \equiv \langle -\Delta \rangle$  is kept constant:

$$\tilde{f}(s) = \inf_{\substack{\psi \in \mathcal{D}(H) \\ \|\psi\|=1 \\ (\psi, -\Delta\psi)=s}} (\psi, f\psi). \quad (7)$$

The eigenvalue  $F(v)$  of  $H$  is then recovered from  $\tilde{f}(s)$  by a final minimization over  $s$ :

$$F(v) = \min_{s>0} [s + v\tilde{f}(s)]. \quad (8)$$

The spectral function  $F(v)$  is concave ( $F''(v) < 0$ ); moreover, it has been shown [29] that

$$F''(v)\tilde{f}''(s) = -\frac{1}{v^3} < 0. \quad (9)$$

Hence,  $F(v)$  and  $\bar{f}(s)$  have opposite convexities and are related by the following Legendre transformations  $\bar{f} \leftrightarrow F$  [39]:

$$\bar{f}(s) = F'(v), \quad s = F(v) - vF'(v), \quad (10)$$

$$\frac{1}{v} = -\bar{f}'(s), \quad \frac{F(v)}{v} = \bar{f}(s) - s\bar{f}'(s). \quad (11)$$

$F(v)$  is not necessarily monotone, but the kinetic potential  $\bar{f}(s)$  is monotone decreasing. Equation (10) enables us also to use the coupling as a minimization parameter. For this purpose, we write the coupling as  $u$  and we have from Eq. (8)

$$F(v) = \min_{u>0} [F(u) - uF'(u) + vF'(u)]. \quad (12)$$

This is particularly useful in cases where  $\bar{f}(s)$  is difficult to find explicitly.

Another form of expression, useful for our present task, is obtained if we change the kinetic-energy parameter from  $s$  to  $r$  itself, by inverting the (monotone) function  $\bar{f}(s)$  to define the associated  $K$  function by

$$K^{[f]}(r) = s = (\bar{f}^{-1} \circ f)(r). \quad (13)$$

Now the energy formula Eq. (8) becomes

$$F(v) = \min_{r>0} [K^{[f]}(r) + vf(r)]. \quad (14)$$

A sleight of hand may be perceived here since  $K$  depends on  $f$ . However, we do now have a relation that has  $F$  on one side and  $f$  on the other: our goal is to invert this expression, to effect  $F \rightarrow f$ . We shall do this below by constructing a sequence of approximate  $K$  functions which do not depend on  $f$ .

### B. Smooth transformations and envelope approximations

In this section, we consider potential shapes  $f(r)$  that may be written as smooth transformations  $f(r) = g(h(r))$  of a ‘‘basis potential’’  $h(r)$ . The idea is that we know the spectrum of  $-\Delta + \nu h(r)$  and we try to exploit this to study the spectrum of  $-\Delta + \nu f(r)$ . When the transformation function  $g$  has definite convexity ( $g''$  does not change sign), the kinetic-potential formalism immediately allows us to derive energy bounds. This is a consequence of Jensen’s inequality [40], which may be expressed in our context by the following:

$$\begin{aligned} g \text{ is convex } (g'' \geq 0) &\Rightarrow (\psi, g(h)\psi) \geq g((\psi, h\psi)), \\ g \text{ is concave } (g'' \leq 0) &\Rightarrow (\psi, g(h)\psi) \leq g((\psi, h\psi)). \end{aligned} \quad (15)$$

More specifically, we have for the kinetic potentials

$$g'' \geq 0 \Rightarrow \bar{f}(s) \geq g(\bar{h}(s)); \quad g'' \leq 0 \Rightarrow \bar{f}(s) \leq g(\bar{h}(s)). \quad (16)$$

We can summarize these results by writing  $\bar{f}(s) \approx g(\bar{h}(s))$  and remembering that the relation  $\approx$  indicates an inequality whenever  $g$  has definite convexity. The expression of these results in terms of  $K$  functions is even simpler, for we have

$$K^{[f]} = \bar{f}^{-1} \circ f \approx (g \circ \bar{h})^{-1} \circ (g \circ h) = \bar{h}^{-1} \circ h = K^{[h]}. \quad (17)$$

Thus,  $K^{[f]} \approx K^{[h]}$  is the approximation we sought, that no longer depends on  $f$ . The corresponding energy bounds are provided by

$$E = F(v) \approx \min_{s>0} [s + \nu g(\bar{h}(s))] = \min_{r>0} [K^{[h]}(r) + \nu f(r)]. \quad (18)$$

### C. The envelope inversion sequence

We suppose that an eigenvalue  $E$  of  $H = -\Delta + \nu f(r)$  is known as function  $E = F(v)$  of the coupling parameter  $\nu > 0$ . In some cases, such as the square well, the discrete eigenvalue may exist only for sufficiently large coupling,  $\nu > \nu_1$ . The kinetic potential  $\bar{f}(s)$  may be obtained by inverting the Legendre transformation in Eq. (10). Thus

$$F(v) = \min_{s>0} [s + \nu \bar{f}(s)] \rightsquigarrow \bar{f}(s) = \max_{\nu>\nu_1} \left[ \frac{F(v)}{\nu} - \frac{s}{\nu} \right]. \quad (19)$$

We shall also need to invert the relation (14) between  $F^{[n]}$  and  $K^{[n]}$  by means of

$$K(r) = \max_{\nu>\nu_1} [F(v) - \nu f(r)]. \quad (20)$$

We begin with a seed potential shape  $f^{[0]}(r)$  from which we generate a sequence  $\{f^{[n]}(r)\}_{n=0}^{\infty}$  of improving potential approximations. The idea behind this sequence is that we search for a transformation  $g$  so that  $g(f^{[n]}(r))$  is close to  $f(r)$  in the sense that the eigenvalue generated is close to  $F(v)$ . The envelope approximation is used at each stage. The best transformation  $g^{[n]}$  at stage  $n$  is given by using the current potential approximation  $f^{[n]}(r)$  as an envelope basis. We have:

$$\bar{f} = g^{[n]} \circ \bar{f}^{[n]} \Rightarrow g^{[n]} = \bar{f} \circ \bar{f}^{[n]-1}.$$

Thus

$$f^{[n+1]} = g^{[n]} \circ f^{[n]} = \bar{f} \circ K^{[n]}.$$

The resulting inversion algorithm may be summarized by the following:

TABLE I. Couplings  $\nu$  and binding energies  $E$  arising from Bethe–Salpeter equations in either ladder or ladder-plus-cross-ladder approximation for common mass  $m = 1$  of the bound scalar bosons and mass  $\mu = 0.15$  or  $\mu = 0.5$  of the exchanged scalar boson, or from the ladder-approximation nonrelativistic limit, the Schrödinger equation with Yukawa potential  $V(r) = -\nu \exp(-\mu r)/r$ .

$\nu$			$E$	
Minkowski-space Bethe–Salpeter equation			Schrödinger equation	
Ladder		Ladder + cross-ladder	Yukawa potential	
$\mu = 0.15$ [5,7,13]	$\mu = 0.50$ [5–7,13]	$\mu = 0.50$ [6,7]	$\mu = 0.50$	
0.5716	1.440	1.21	1.034	−0.01
-	2.01	1.62	1.285	−0.05
1.437	2.498	1.93	1.532	−0.10
2.100	3.251	2.42	1.848	−0.20
3.611	4.901	3.47	2.204	−0.50
5.315	6.712	4.56	2.918	−1.00

*inversion algorithm*

$$f^{[n]}(r) \rightarrow F^{[n]}(\nu) \rightarrow K^{[n]}(r) = \max_{u > \nu_1} [F^{[n]}(u) - u f^{[n]}(r)], \quad (21)$$

$$f^{[n+1]}(r) = \max_{\nu > \nu_1} \left[ \frac{F(\nu)}{\nu} - \frac{K^{[n]}(r)}{\nu} \right]. \quad (22)$$

The step  $f^{[n]}(r) \rightarrow F^{[n]}(\nu)$  is effected by solving  $(-\Delta + \nu f^{[n]})\psi = E\psi$  numerically for  $E = F^{[n]}(\nu)$ .

#### D. Uniqueness

We consider now a singular potential  $f(r)$  of the form

$$f(r) = \frac{g(r)}{r}, \quad \text{where } g(0) < 0, \quad g'(r) \geq 0, \quad (23)$$

and  $g(r)$  is not constant. Examples of this class of singular potential shapes  $f(r)$  are Yukawa  $g(r) = -e^{-ar}$ , Hulthén  $g(r) = -r/(e^{ar} - 1)$ , and linear-plus-Coulomb  $g(r) = -a + br^2$ , with  $a, b > 0$ . With these assumptions, we have proved in Ref. [34] the following

*Theorem* The potential shape  $f(r)$  in  $H = -\Delta + \nu f(r)$  is uniquely determined by the ground-state energy function  $E = F(\nu)$ .

### III. SPECTRAL DATA FROM MINKOWSKI-SPACE BETHE–SALPETER EQUATION

#### A. The raw data

In Table I, we exhibit the binding energy  $E$  versus coupling  $\nu$  results from numerical solutions of the Bethe–Salpeter equation for a system of two scalar particles each of mass  $m$ , bound by single or multiple exchange of a scalar particle of mass  $\mu$ , computed by Carbonell

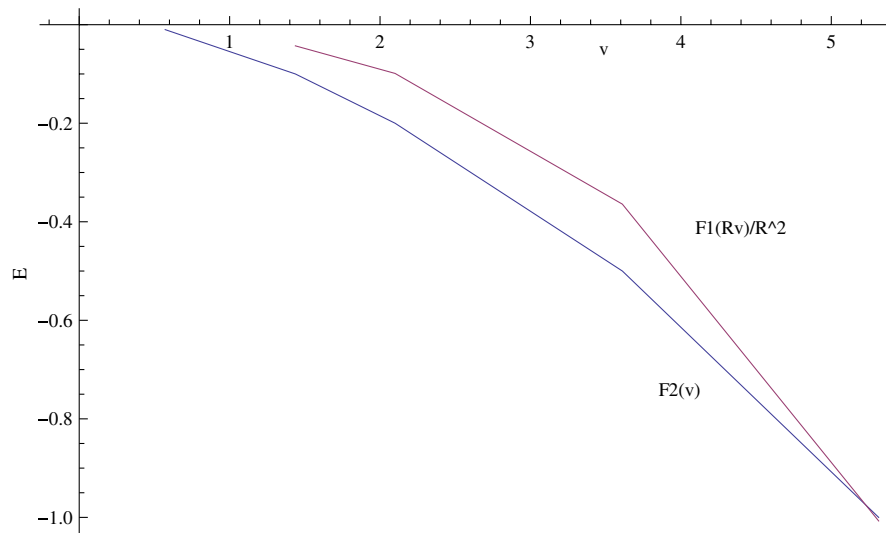


FIG. 1 (color online). Approximate exchange-mass dependence from scaling arguments: Table I data  $F_2(\nu)$  vs scaled Table I data  $F_1(R\nu)/R^2$ .

and Karmanov in Refs. [5], Table 1, [6], Table 1, [7], Tables 1 and 2, and [13], Table 1. For comparison, we add corresponding results of the Schrödinger equation with an interaction potential of Yukawa form  $V(r) = v \exp(-\mu r)/r$ , which, as shown in the Appendix, constitutes the nonrelativistic limit of the ladder BS equation. Interestingly, the nonrelativistic binding energies emerging from the Schrödinger equation with Yukawa potential seem to reproduce better the ladder-plus-cross-ladder approximation findings than the ones from mere ladder approximation.

### B. Exchange-mass dependence

In this subsection, we demonstrate that the spectral data of Table I for  $\mu = 0.5$  can be obtained approximately from the corresponding data shown in Table I for  $\mu = 0.15$  by a scale change in the potential of a Schrödinger model. This is interesting if one expects to find that a potential such as the Yukawa  $V(r) = -ve^{-\mu r}/r$  would account approximately for the spectral dependence of the problem on the exchange mass  $\mu$ . Let us consider a Schrödinger operator given by

$$-\Delta + v \frac{f(r)}{r} \rightarrow E = F(v), \quad (24)$$

where  $E$  is a discrete eigenvalue. A simple scaling argument applied to the operator

$$H = -\frac{1}{2m} \Delta + v \frac{f(\mu r)}{r} \quad (25)$$

shows that a corresponding discrete eigenvalue of  $H$  is given in terms of  $F(v)$  by the formula

$$E = \frac{\mu^2}{2m} F\left(\frac{2mv}{\mu}\right). \quad (26)$$

Thus, if we compare two different  $\mu$  values,  $\mu_1$  and  $\mu_2$  with  $R \equiv \mu_1/\mu_2$ , and we write for the  $\mu_1$  case  $E_1 = F_1(v)$ , then, under the scaling rule (26), for the  $\mu_2$  case we would have  $E_2 = F_1(Rv)/R^2$ . For our present problem, we have  $\mu_1 = 0.5$ ,  $\mu_2 = 0.15$ , and therefore  $R = \mu_1/\mu_2 = 10/3$ . Hence, given the second column of Table I expressed as  $E = E_1 = F_1(v)$ , we would expect to generate data consistent with the first column of Table I by the formula  $E_2 = (9/100)F_1(10v/3)$ . Graphs of  $F_2(v)$  and its approximation in terms of the scaled  $F_1(v)$  are shown in Fig. 1. The scaling law seems to yield a rough approximation.

## IV. THE CONSTRUCTION OF EFFECTIVE POTENTIALS

We now consider the BS spectral data collected in Table I and we use our inversion theory [34] to answer the question what potential shape  $f(r)$  in the Schrödinger

Hamiltonian  $H = -1/(2m)\Delta + vf(r)$  would generate the corresponding binding energies  $E$  for the given values of the coupling parameter  $v$ ? For this purpose, we take  $m = m_1 m_2 / (m_1 + m_2)$  with  $m_1 = m_2 = 1$ . We adopt the

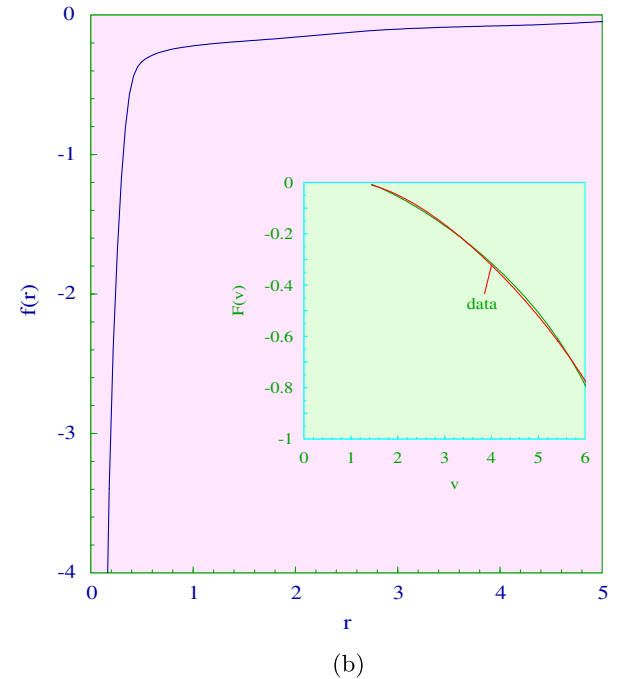
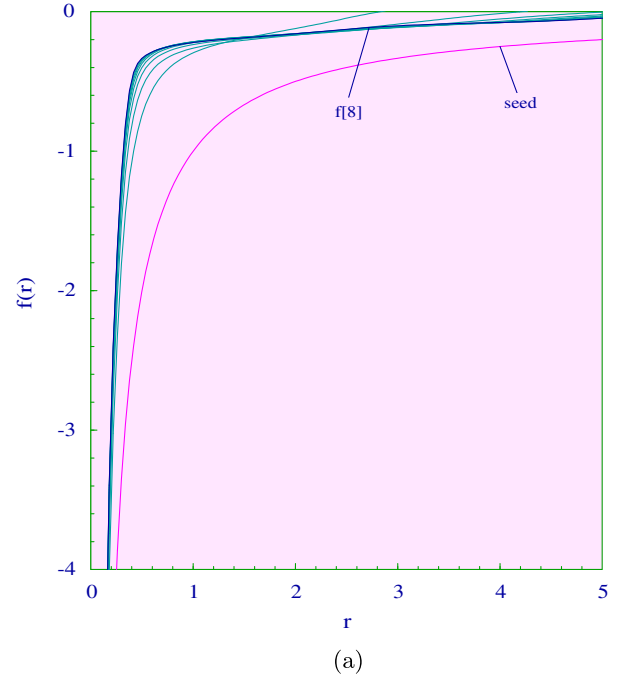


FIG. 2 (color online). Geometric inversion of spectral data of bound states of two scalar bosons described by a Minkowski-space Bethe–Salpeter equation in ladder approximation for mass  $\mu = 0.5$  of the exchanged particle [5–7,13]. Graph (a) shows the first eight iterations  $f[n]$ ,  $n = 1, 2, \dots, 8$ , starting from the seed, while graph (b) shows the resulting potential shape  $f(r)$  and, in the inner graph, the corresponding spectral curve  $F(v)$  along with the input data.

*inversion algorithm* of Eq. (22) with the pure Coulomb seed potential  $f^{[0]}(r) = -1/r$ . For the three sets of BS data in Table I, our results are exhibited, respectively, in Figs. 2–4. In each case, we first show a sequence of

eight iterations and then depict the last potential iteration,  $f^{[8]}(r)$ , along with the eigenvalue curve  $F(v)$  of the Bethe–Salpeter data and the corresponding eigenvalue curve of the Hamiltonian  $H = -\Delta + v f^{[8]}(r)$ . For comparison,

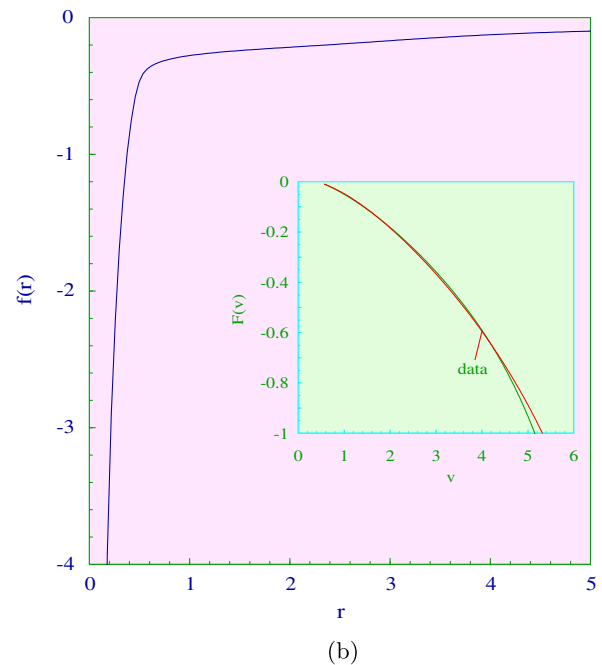
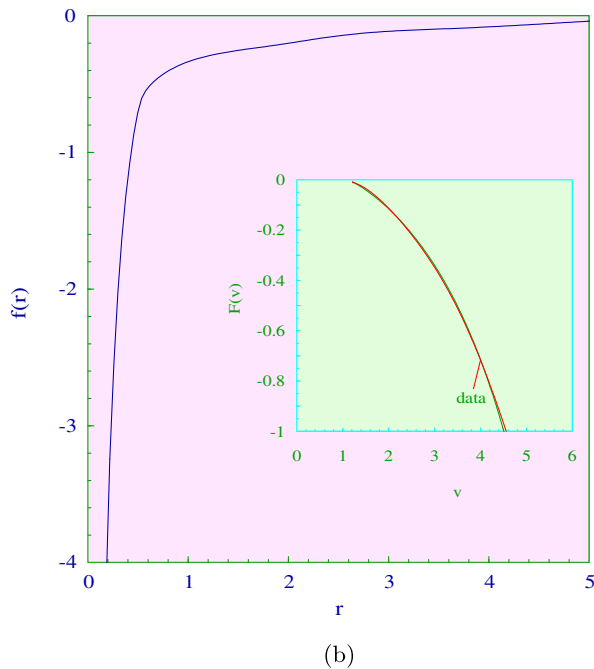
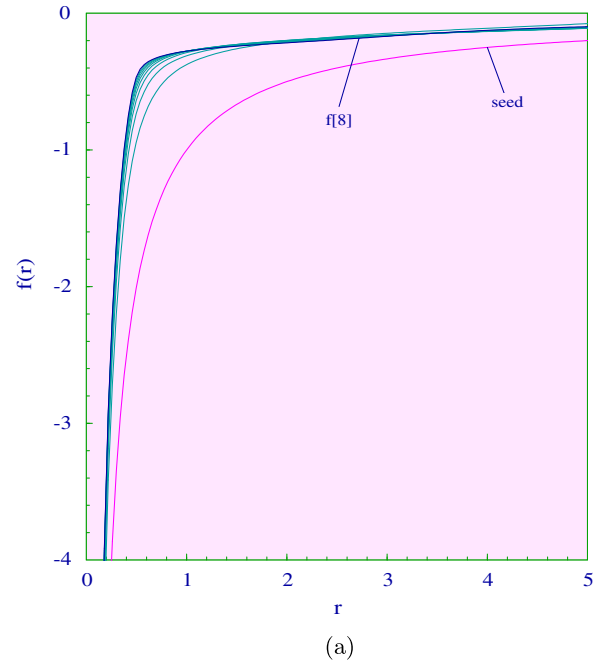
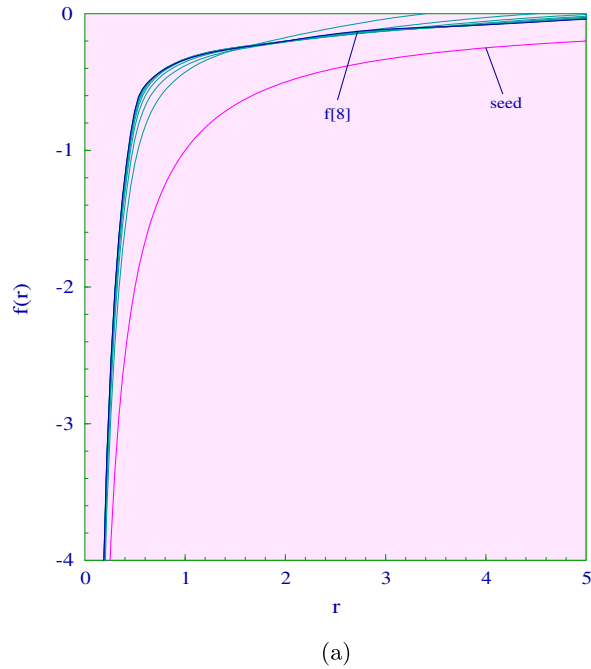


FIG. 3 (color online). Geometric inversion of spectral data of bound states of two scalar bosons described by a Minkowski-space Bethe–Salpeter equation in ladder-plus-cross-ladder approximation for mass  $\mu = 0.5$  of the exchanged particle [6,7]. Graph (a) shows the first eight iterations  $f[n]$ ,  $n = 1, 2, \dots, 8$ , starting from the seed, while graph (b) shows the resulting potential shape  $f(r)$  and, in the inner graph, the corresponding spectral curve  $F(v)$  along with the input data.

FIG. 4 (color online). Geometric inversion of spectral data of bound states of two scalar bosons described by a Minkowski-space Bethe–Salpeter equation in ladder approximation for mass  $\mu = 0.15$  of the exchanged particle [5,7,13]. Graph (a) shows the first eight iterations  $f[n]$ ,  $n = 1, 2, \dots, 8$ , starting from the seed, while graph (b) shows the resulting potential shape  $f(r)$  and, in the inner graph, the corresponding spectral curve  $F(v)$  along with the input data.

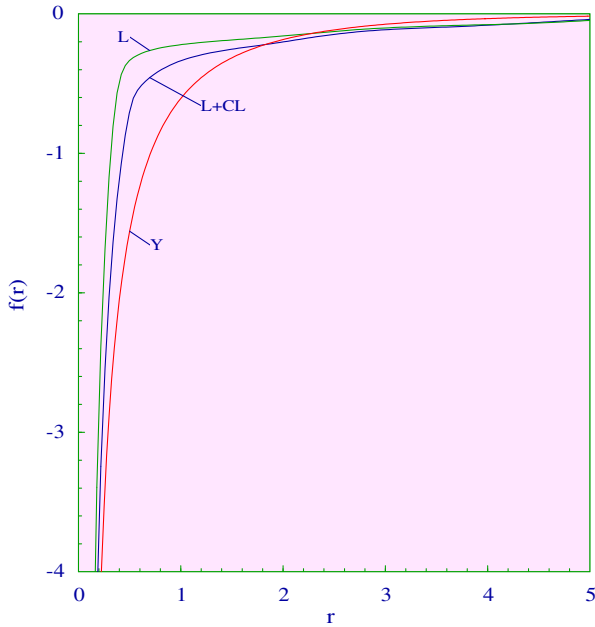


FIG. 5 (color online). Potential shapes  $f(r)$  from Figs. 2 and 3 resulting from geometric inversion of the ladder (L) and ladder-plus-cross-ladder (L + CL) Bethe–Salpeter findings [5–7,13] for the masses of two-scalar-boson bound states, with exchanged-boson mass  $\mu = 0.5$ , compared with the corresponding Yukawa potential  $Y(r) = -\exp(-\mu r)/r$ , arising in the nonrelativistic limit of the ladder case.

Fig. 5 shows the reconstructed potentials for both ladder (L) and ladder-plus-cross-ladder (L + CL) kernels with an exchanged-particle mass  $\mu = 0.5$ , along with the corresponding Yukawa-potential shape  $Y(r) \equiv -\exp(-\mu r)/r$ . Judged by the eye, the Yukawa potential seems to be closer to the inversion output for the ladder-plus-cross-ladder case than for the mere ladder case. In view of the ordering of couplings  $\nu$  in Table I, this observation is no genuine surprise.

## V. FORM FACTORS

In quantum mechanics, the three-dimensional form factor  $F(\mathbf{k})$  of a bound state described by its configuration-space wave function  $\psi(\mathbf{x})$  is nothing else but the Fourier transform of the corresponding charge density  $\rho(\mathbf{x}) \equiv |\psi(\mathbf{x})|^2$ . For a given coupling strength  $\nu$  and common mass  $m$  of the two bound-state constituents, the reduced radial wave functions  $u(r)$  of  $s$  states satisfy an ordinary differential equation that determines the associated binding-energy eigenvalues  $E$ :

$$-\frac{1}{m}u''(r) + \nu f(r)u(r) = Eu(r), \quad u(0) = 0. \quad (27)$$

It proves convenient to normalize the radial wave functions  $u(r)$  such that  $\int_0^\infty dr |u(r)|^2 = 1$ . In momentum space, any corresponding form factor  $F(k)$  is given by the Fourier–Bessel transform of the radial density  $u^2(r)$ , that is to say, by

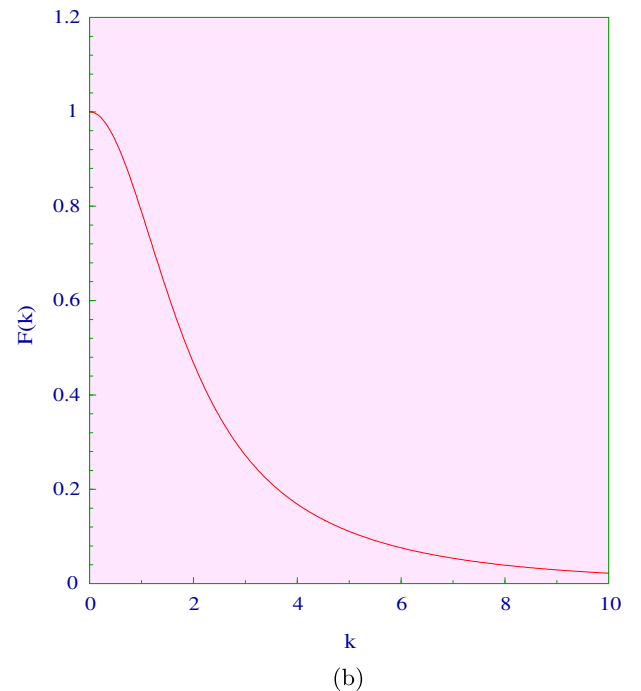
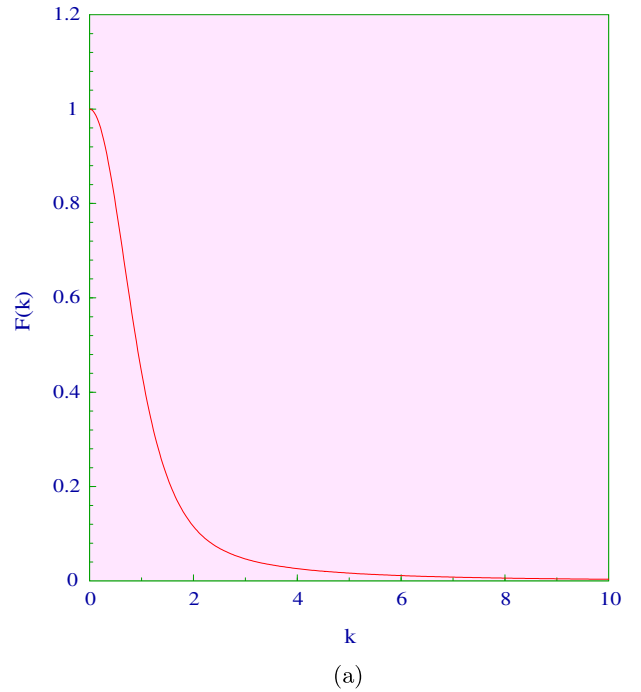


FIG. 6 (color online). Momentum-space form factor  $F(k)$  of the ground-state Schrödinger solution mimicking the results of the Bethe–Salpeter equation in ladder (a) and ladder-plus-cross-ladder (b) approximation with coupling  $\nu = 5$  and exchanged-particle mass  $\mu = 0.5$ .

$$F(k) = \frac{1}{k} \int_0^\infty dr \frac{\sin(kr)}{r} u^2(r), \quad F(0) = 1. \quad (28)$$

Figures 6 and 7 depict the ground-state form factors  $F(k)$  for  $\nu = 5$  and  $m = 1$  for each of the three potential shapes  $f(r)$



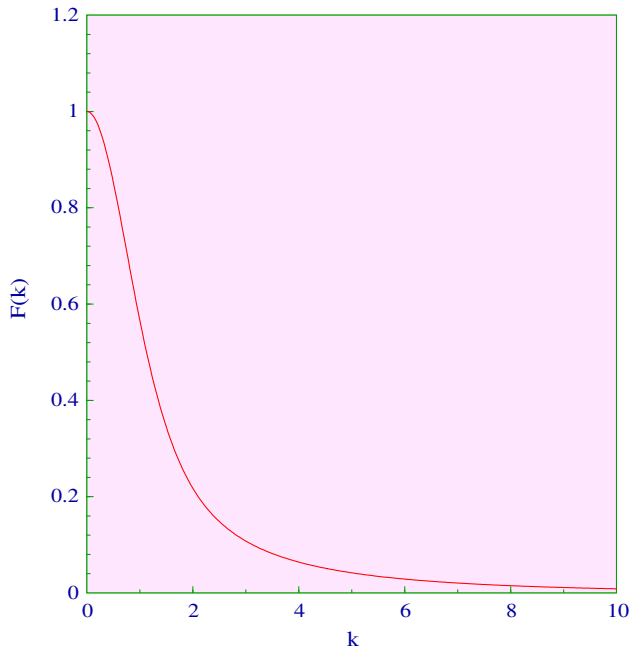


FIG. 7 (color online). Momentum-space form factor  $F(k)$  of the ground-state Schrödinger solution mimicking the results of the Bethe–Salpeter equation in ladder approximation with coupling  $\nu = 5$  and exchanged-particle mass  $\mu = 0.15$ .

shown in Figs. 2–4, obtained by application of our geometrical inversion technique to the spectral data of Table I. By comparing Figs. 6(a) and 7, we see that the quantum-mechanical form factors broaden with increasing mass  $\mu$  of the exchange particle. Likewise, a comparison of the two plots in Fig. 6 reveals that the quantum-mechanical form factors broaden when taking into account higher-order corrections in the BS interaction kernel, that is to say, when working in the somewhat more sophisticated ladder-plus-cross-ladder approximation instead of in the naïve ladder approximation. These form factors  $F(k)$  constitute the three-dimensional counterparts of the four-dimensional form factor  $F(Q^2)$  found from the ladder-plus-cross-ladder Minkowski-space BS equation with exchanged-boson mass  $\mu = 0.5$  in Refs. [8–10,13].

## VI. CONCLUSION

For a *quantum-mechanical* description of relativistic systems in which a single particle is bound to a fixed center, one uses the Klein–Gordon equation or the Dirac equation. For the analysis of systems composed of more than one particle within *quantum field theory*, the Bethe–Salpeter formalism is required. In a series of papers, Carbonell *et al.* discussed the bound states of a system of two charged scalar bosons by means of the Bethe–Salpeter equation in Minkowski-space representation, in contrast to most studies of this kind which rely on the Euclidean-space formulation of this equation. These investigations report some numerical results for the binding

energies, in a variety of cases, as functions  $E(\nu)$  of a coupling parameter  $\nu$ . What we have done in the present analysis is to employ a geometric spectral inversion theory to reconstruct, in each case, the potential shape  $f(r)$  in a Schrödinger model  $H = -\Delta/(2m) + \nu f(r)$  which would have the same energy curve  $E(\nu)$ . As more complete quantum-field-theoretic spectral data becomes available, we shall be able to reveal more details of such spectrally equivalent potential models.

## ACKNOWLEDGMENTS

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## APPENDIX: NONRELATIVISTIC REDUCTION OF THE BETHE–SALPETER EQUATION

For the sake of completeness we briefly sketch how, by successive application of a sequence of simplifying assumptions and approximations, the Bethe–Salpeter equation for two bound-state constituents both of spin zero may be reduced to an equation of motion of Schrödinger form with all interactions represented by a static potential. Regarding *kinematics*, for a bound state of two particles discriminated by a label  $i = 1, 2$  the relation between total momentum  $P$  and relative momentum  $p$  of the constituents, on the one hand, and the individual-particle momenta  $p_1, p_2$ , on the other hand, reads

$$\begin{aligned} P &\equiv p_1 + p_2, & p &\equiv \eta_2 p_1 - \eta_1 p_2 \\ \Leftrightarrow p_1 &= \eta_1 P + p, & p_2 &= \eta_2 P - p, \end{aligned}$$

where  $\eta_1, \eta_2$  denote two real parameters satisfying  $\eta_1 + \eta_2 = 1$ . In the center-of-momentum frame of some bound state of mass  $M = \sqrt{P^2}$ , defined by  $\mathbf{P} \equiv \mathbf{p}_1 + \mathbf{p}_2 = 0$  and therefore  $P = (M, 0)$ , the individual-particle momenta  $p_1, p_2$ , become

$$\begin{aligned} p_1^0 &= \eta_1 M + p^0, & p_2^0 &= \eta_2 M - p^0, \\ \mathbf{p}_1 &= \mathbf{p}, & \mathbf{p}_2 &= -\mathbf{p}. \end{aligned}$$

Our starting point of the nonrelativistic reduction is the Bethe–Salpeter equation in momentum-space representation

$$\Phi(p, P) = \frac{i}{(2\pi)^4} S_1(p_1) \int d^4 q K(p, q, P) \Phi(q, P) S_2(-p_2). \quad (\text{A1})$$

Its *instantaneous approximation* assumes that in the center-of-momentum frame of the bound state the Bethe–Salpeter

interaction kernel  $K(p, q, P)$  depends only on the (initial and final) spatial relative momenta  $\mathbf{p}, \mathbf{q}$ :  $K(p, q, P) = K(\mathbf{p}, \mathbf{q})$ . Then, integrating over  $p_0$  reduces the Bethe–Salpeter equation (A1) to a kind of instantaneous Bethe–Salpeter equation

$$\phi(\mathbf{p}) = \frac{i}{2\pi} \int dp_0 S_1(p_1) S_2(-p_2) \int \frac{d^3q}{(2\pi)^3} K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q})$$

for the “Salpeter amplitude”  $\phi(\mathbf{p})$ , defined as integral of  $\Phi(p, P)$  over the time component  $p_0$  of the relative momentum  $p$ :

$$\phi(\mathbf{p}) \equiv \frac{1}{2\pi} \int dp_0 \Phi(p, P).$$

Replacing the propagator  $S_i(p)$  of bound-state constituent  $i$  by its free counterpart  $S_i^{(0)}(p)$  entails the Salpeter equation

$$\phi(\mathbf{p}) = \frac{i}{2\pi} \int dp_0 S_1^{(0)}(p_1) S_2^{(0)}(-p_2) \int \frac{d^3q}{(2\pi)^3} K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q}). \quad (\text{A2})$$

The free Feynman propagator  $S_i^{(0)}(p_i)$  in momentum space of a *scalar boson* of mass  $m_i$  and momentum  $p_i$  is given by

$$S_i^{(0)}(p_i) = S_i^{(0)}(-p_i) = \frac{i}{p_i^2 - m_i^2 + i\varepsilon}, \quad \varepsilon \downarrow 0, \quad i = 1, 2.$$

In terms of relativistic free-particle energies  $E_i(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + m_i^2}$  the unique<sup>2</sup> *partial fraction decomposition* of  $S_i^{(0)}(p_i)$  is

$$S_i^{(0)}(p_i) = \frac{i}{2E_i(\mathbf{p})} \left[ \frac{1}{p_i^0 - E_i(\mathbf{p}) + i\varepsilon} - \frac{1}{p_i^0 + E_i(\mathbf{p}) - i\varepsilon} \right],$$

$\varepsilon \downarrow 0, \quad i = 1, 2.$

Evaluating the integral over the two propagators in Eq. (A2) by contour integration and Cauchy’s residue theorem gives

$$\begin{aligned} & \int dp_0 S_1^{(0)}(p_1) S_2^{(0)}(-p_2) \\ &= \frac{2\pi i}{4E_1(\mathbf{p})E_2(\mathbf{p})} \\ & \quad \times \left[ \frac{1}{M - E_1(\mathbf{p}) - E_2(\mathbf{p})} - \frac{1}{M + E_1(\mathbf{p}) + E_2(\mathbf{p})} \right]. \end{aligned}$$

<sup>2</sup>A different decomposition of the free scalar-boson propagator  $S_i^{(0)}(p_i)$  that is, however, not compatible with the Cauchy residue theorem is

$$S_i^{(0)}(p_i) = \frac{i}{2p_i^0} \left[ \frac{1}{p_i^0 - E_i(\mathbf{p}) + i\varepsilon} + \frac{1}{p_i^0 + E_i(\mathbf{p}) - i\varepsilon} \right].$$

The three-dimensional reduction of the relativistically covariant Bethe–Salpeter equation (A1) to a Schrödinger-type equation is applicable to nearly nonrelativistic and *weakly bound* states composed of sufficiently heavy constituents. For such systems, the first contribution to the above integral over propagators may be assumed to dominate the second one,

$$\frac{1}{M - E_1(\mathbf{p}) - E_2(\mathbf{p})} \gg \frac{1}{M + E_1(\mathbf{p}) + E_2(\mathbf{p})},$$

since in this situation  $M \approx E_1(\mathbf{p}) + E_2(\mathbf{p})$ , such that the second term in the propagator integral may be safely neglected:

$$\int dp_0 S_1^{(0)}(p_1) S_2^{(0)}(-p_2) \approx \frac{2\pi i}{4E_1(\mathbf{p})E_2(\mathbf{p})} \frac{1}{M - E_1(\mathbf{p}) - E_2(\mathbf{p})}.$$

Adopting this standard approximation, one arrives at the reduced Salpeter equation for spin-0 bound-state constituents

$$\begin{aligned} & [E_1(\mathbf{p}) + E_2(\mathbf{p})] \phi(\mathbf{p}) - \frac{1}{4E_1(\mathbf{p})E_2(\mathbf{p})} \int \frac{d^3q}{(2\pi)^3} K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q}) \\ &= M \phi(\mathbf{p}). \end{aligned} \quad (\text{A3})$$

Moreover, assuming that, in the first term of the propagator integral, the factor  $[M - E_1(\mathbf{p}) - E_2(\mathbf{p})]^{-1}$  varies faster than the factor  $[E_1(\mathbf{p})E_2(\mathbf{p})]^{-1}$  justifies the substitution  $E_i(\mathbf{p}) \approx m_i$  in the denominator of the *interaction term* in Eq. (A3). Finally, the nonrelativistic expansion  $E_i(\mathbf{p}) \approx m_i + \mathbf{p}^2/(2m_i)$  of the free energies leads to the Schrödinger-type equation

$$\begin{aligned} & \left( m_1 + m_2 + \frac{\mathbf{p}^2}{2m_1} + \frac{\mathbf{p}^2}{2m_2} \right) \phi(\mathbf{p}) - \frac{1}{4m_1 m_2} \int \frac{d^3q}{(2\pi)^3} K(\mathbf{p}, \mathbf{q}) \phi(\mathbf{q}) \\ &= M \phi(\mathbf{p}). \end{aligned} \quad (\text{A4})$$

The Schrödinger equation governing the dynamics of two particles, of masses  $m_1, m_2$ , interacting via a potential  $V(\mathbf{x})$  involving their relative coordinate  $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$ , in a bound state of mass  $M$  reads, in configuration-space representation,

$$\left[ m_1 + m_2 - \frac{\Delta_x}{2m_1} - \frac{\Delta_x}{2m_2} + V(\mathbf{x}) \right] \psi(\mathbf{x}) = M \psi(\mathbf{x}).$$

Upon introduction, for configuration-space wave function  $\psi(\mathbf{x})$  and interaction potential  $V(\mathbf{x})$ , their Fourier transforms

$$\tilde{\psi}(\mathbf{p}) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{x}), \quad \tilde{V}(\mathbf{p}) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} V(\mathbf{x}),$$

the latter proving to be a very convenient choice, the Schrödinger equation becomes in momentum-space representation

$$\begin{aligned} & \left( m_1 + m_2 + \frac{p^2}{2m_1} + \frac{p^2}{2m_2} \right) \tilde{\psi}(p) + \int \frac{d^3q}{(2\pi)^3} \tilde{V}(p-q) \tilde{\psi}(q) \\ & = M \tilde{\psi}(p). \end{aligned} \quad (\text{A5})$$

Assuming the kernel  $K(p, q)$  to be of convolution type, i.e.,  $K(p, q) = K(p - q)$ , the comparison of the reduced Salpeter equation in nonrelativistic limit (A4) with the momentum-space Schrödinger equation (A5) allows for the identification

$$\tilde{V}(p - q) = -\frac{1}{4m_1 m_2} K(p - q) \iff \tilde{V}(p) = -\frac{1}{4m_1 m_2} K(p).$$

As an illustration of this relationship, let us demonstrate how the *Yukawa potential* arises from *single-boson exchange* between our two bound-state constituents  $i = 1, 2$ . Let  $g_i$  denote the interaction strength (having the mass dimension 1) of the three-boson coupling of the spin-0 bound-state constituent  $i$  (of mass  $m_i$ ) to some spin-0 force mediator of mass  $\mu$ . Introducing the momentum transfer  $k \equiv p - q$ , the resulting one-boson exchange contribution to the interaction kernel is

$$iK(p, q, P) = iK(k) = \frac{i(g_1)(g_2)}{k^2 - \mu^2};$$

contenting oneself with this form entails the *ladder approximation* to Eq. (A1). In instantaneous limit, this kernel reads

$$K(k) = \frac{g_1 g_2}{k^2 + \mu^2}.$$

As consequence of the spherical symmetry of  $K(k)$ , the Fourier transformation of  $\tilde{V}(k)$  yields the spherically symmetric configuration-space *Yukawa potential*  $V(x) = V(r)$ ,  $r \equiv |x|$ . For convenience, we represent it in the form  $V(r) \equiv v f(r)$ :

$$\begin{aligned} V(x) &= \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \tilde{V}(k) = - \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x} K(k)}{4m_1 m_2} \\ &= -\frac{g_1 g_2}{16\pi m_1 m_2} \frac{e^{-\mu r}}{r} = V(r) \equiv v f(r), \\ f(r) &= -\frac{e^{-\mu r}}{r}. \end{aligned}$$

Hence our coupling constant  $v$  is related to the mass and coupling parameters of the underlying quantum field theory by

$$v = \frac{g_1 g_2}{16\pi m_1 m_2}.$$

Especially, for *identical* bound-state constituents, clearly satisfying  $m_1 = m_2 = m$  and  $g_1 = g_2 = g$ , this result becomes

$$v = \frac{g^2}{16\pi m^2}.$$

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