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Nonsingular Brans-Dicke-Λ cosmology

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We discuss a Brans-Dicke model with a cosmological constant, a negative value of the w parameter, and an arbitrary (in general nonvanishing) scale factor at the big bang. The Friedmann equations for a flat universe are considered. The current observational values for Hubble constant H_0 and deceleration parameter q_0 play the role of initial conditions. We follow the approach of Uehara and Kim in order to solve field equations analytically. In K. Uehara and C. W. Kim [Phys. Rev. D **26**, 2575 (1982)] only positive values of w were considered; we extend the study to a complete set of possible w values. Our main result is that the scale factor (during its evolution back in time direction) may not vanish, unlike in the standard ΛCDM case. In other words, the considered model demonstrates a cosmological bounce instead of the initial singularity. The famous formula (24), that leads to the bounce, is valid only for the dust-filled universe with $p = 0$ and, therefore, is not adequate for the early universe hot stage when the bounce happens. So, our results are qualitative in nature and must be used to obtain initial values for the hot stage of the Universe.

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I. INTRODUCTION

The cosmic acceleration is now a well observationally established fact [1–6]; however its physical reasons remain open. So hence, the most studied model for the moment is the cold dark matter one with cosmological constant (ΛCDM). Providing good quantitative agreement with observational data, this model however does not explain the nature of the dark matter and dark energy. Another weakness of ΛCDM is the absence of explanation of the smallness of Λ value if it is assumed to be the so-called "vacuum energy." All these arguments lead to the idea of a dynamical theory of dark energy creation (see, for example, [7]). The most widely discussed candidates are quintessence (a slowly rolling scalar field [8]) and higher order curvature gravity [including so-called $f(R)$ gravity models [9]].

The Brans-Dicke model (BD) is one of the first gravity models with a scalar field [10]. It was suggested in 1961 and contains an additional parameter ω whose value has to be determined by observational data. Large values of w mean an important contribution from the tensor part (Ricci scalar), and smaller values of w mean an increasing role for the scalar field contribution. In the limit $|w| \to \infty$ BD theory leads to general relativity (GR). In the BD model the value of Newton's constant is proportional to the inverse scalar field ($G \approx 1/\phi$), proving additional coupling between the model parameters. The most accurate limit on w comes from Cassini-Huygens mission data on the post-Newtonian parameter γ and is $|W| > 50\,000$ [11].

BD theory is the most natural GR extension. It is interesting because, first, this model could be the low energy effective limit of grand unification (and super unification) approaches [from the latest LHC data (see [12]) this possibility is not completely closed yet]. Second, because the scalar field in BD theory can be reinterpreted as a dilation field in string theory. Finally, because the BD model is the simplest GR extension and is useful to investigate any supertheory, so as to gauge the difference with GR [13]. In addition, BD gravity is widely used in cosmology as one needs a scalar field for inflation and such a field is necessary in the BD model. A large set of inflationary models

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[14–16] is based on BD gravity and more generic scalar-tensor approaches. Brans-Dicke theory is also closely related to the widely discussed $f(R)$ gravity (see, for example, [17]).

It is necessary to stress that there is no accelerated expansion in the standard version of the BD model, so one has to study its extended versions. One of the most common extensions is a scalar field potential addition. As the accurate shape of this potential is not yet known [18], one can consider a Λ term as the effective contribution instead of a potential (so we obtain the BDΛ model). The explanation of the Λ term smallness in the framework of BDΛ is possible and was suggested in [19]. In Ref. [20], with the help of the scalar field in BDΛ context, a dark matter hallo around galaxies is modeled.

An analytical accurate solution for Friedmann equations in the BDΛ model was also obtained in Ref. [21] where positive values of ω and initial conditions for the scale factor in the form $a(t_{\min}) = 0$, where $t_{\min}$ is big bang time, were considered. Partial solutions in this model with scalar field power dependence versus the scale factor were presented in [22–25]. Vacuum solutions were obtained in [26–28]. Some papers discussed a Λ-term dependence as a function of the scalar field (for example, [29]). Numerical integration and stability analysis of BDΛ + matter solutions were carried out in [30]. The big rip solution in BDΛ is discussed in [31]. The analytical solution in the pure BD model with negative w, avoiding the initial singularity, was obtained in 1973 by Gurevich *et al.* [32]. There is no cosmic acceleration in the Gurevich *et al.* solution, so nowadays one has to extend BD theory to include cosmic acceleration. In this paper we study the Einstein-Friedmann equation solutions in BDΛ theory for $w < 0$ with a scale factor with initial value $a(t_{\min}) = a_{\min}$. Generally $a_{\min} \neq 0$. Friedmann equations are studied for a flat universe. We take the current values for the Hubble parameter and its derivative (deceleration parameter) as their initial conditions. In our study, we use the approach suggested in [21]. We would like to point out that only positive values of w are considered in [21], so our solution with $w < 0$ represents a new branch. As opposed to the standard ΛCDM model, in the considered case, the scale factor never vanishes during backward time evolution. A so-called "bounce" (a snap back from the minimal value of $a_{\min}$) corresponds to this situation. The expression (24) leading to the bounce is obtained for a cold universe with $p = 0$ and is not valid for a hot universe. Therefore all the values in the bounce region are only qualitative estimates for the initial values for the transition to the hot stage.

This paper is organized as follows: in Sec. II we discuss the choice of the space-time metric and the corresponding field equations; in Sec. III the initial values for cosmological parameters are obtained; in Sec. IV we obtain an analytical solution with a bounce for a dust-filled universe ($p = 0$); Sec. V contains a preliminary discussion of the results of Sec. IV; in Sec. VI we explore the case of a ultrarelativistic state of matter (hot phase); and Sec. VII is devoted to the conclusions.

II. FIELD EQUATIONS

The Friedmann-Robertson-Walker (FRW) metrics reads[1]

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{1}$$

where $k = 0, \pm 1$.

The action of the BDΛ theory can be written as

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\Phi(R + 2\Lambda) - \frac{w}{\Phi}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + 16\pi L_{\text{matter}}\right]. \tag{2}$$

Here w is the BD theory parameter, $\Phi(t)$ is the scalar field, and Λ is the cosmological constant.[2]

Variation of the action with respect to the metric $g_{\mu\nu}$ and the scalar field Φ gives the following field equations:

$$G_{\mu\nu} = \frac{8\pi}{\Phi}T_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{w}{\Phi^2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\sigma\lambda}\partial_\sigma\Phi\partial_\lambda\Phi\right) + \frac{\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla_\lambda\nabla^\lambda\Phi}{\Phi}, \tag{3}$$

$$\frac{8\pi}{\Phi}T^\mu_\mu + 2\Lambda = \frac{3+2w}{\Phi}\nabla_\lambda\nabla^\lambda\Phi, \tag{4}$$

where ∇_μ is a covariant derivative,

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}, \\ T_{\mu\nu} &= (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \\ \partial_\mu\Phi &= \delta^t_\mu\partial_t\Phi. \end{aligned} \tag{5}$$

Here $\rho(t)$ and $p(t)$ are the matter density and pressure, respectively, the stress-energy tensor corresponds to a barotropic fluid, and $G_{\mu\nu}$ is the Einstein tensor.

Here it is convenient to introduce new dimensionless variables[3]:

[1]Here and below we set light speed $c = 1$.

[2]Λ here can differ from the one in ΛCDM theory.

[3]Here and below the present time is denoted by the subscript 0, so G_0 is the current value of the gravitational constant. From now on we consider the current time moment as the initial one, so that $t_0 = 0$. Current values of cosmological parameters are taken as initial conditions. New variables lead to $\phi_0 = 1$, which is convenient for further calculations.

$$\begin{aligned}&\Phi(t) \equiv \phi(t)/G_0,\\ &\epsilon(t) \equiv \partial_t\phi/(\sqrt{\Lambda}\phi),\\ &\tilde{H}(t) \equiv H(t)/\sqrt{\Lambda} = \partial_t a/(\sqrt{\Lambda}a),\\ &\tilde{\rho}(t) = 4\pi G_0\rho/\Lambda,\\ &\tilde{p}(t) = 4\pi G_0 p/\Lambda.\end{aligned} \tag{6}$$

Here H is the Hubble parameter, and $\tilde{H}$ is its dimensionless value. In these notations, Friedmann equations for a flat universe ($k = 0$) in a comoving frame ($u_\mu = [1, 0, 0, 0]$) are

$$\frac{G_t^t}{\Lambda} = 3\tilde{H}^2 = \frac{2\tilde{\rho}}{\phi} + 1 + \frac{w}{2}\epsilon^2 - 3\tilde{H}\epsilon, \tag{8}$$

$$\frac{G_r^r}{\Lambda} = 2\dot{\tilde{H}} + 3\tilde{H}^2 = -\frac{2\tilde{p}}{\phi} + 1 - \frac{w}{2}\epsilon^2 - \frac{\ddot{\phi}}{\phi} - 2\tilde{H}\epsilon. \tag{9}$$

The Klein-Gordon equation (4) can be rewritten as

$$\frac{2\tilde{\rho} - 6\tilde{p}}{\phi} + 2 = (3 + 2w)\left[\frac{\ddot{\phi}}{\phi} + 3\tilde{H}\epsilon\right]. \tag{10}$$

Here and below the dot denotes the derivative with respect to the dimensionless time $\tilde{t} \equiv \sqrt{\Lambda}t$.

Equations (8)–(10) lead to the continuity one in the form:

$$\frac{\dot{\tilde{\rho}}}{\tilde{\rho} + \tilde{p}} + 3\tilde{H} = 0, \tag{11}$$

which is consistent with the equivalence principle.

III. INITIAL VALUES OF THE MODEL PARAMETERS

We introduce the deceleration parameter q and the dimensionless matter density β for the initial time in the following form:

$$\begin{aligned}&\dot{\tilde{H}} \equiv -(1 + q)\tilde{H}^2,\\ &\beta \equiv \frac{4\pi G_0(\rho_0 - p_0)}{H_0^2} = \frac{\tilde{\rho}_0 - \tilde{p}_0}{\tilde{H}_0^2}.\end{aligned} \tag{12}$$

Combining Eqs. (8)–(10) to exclude ϵ and $\ddot{\phi}/\phi$, we obtain for $p = 0$ at the current moment t_0 the following equation:

$$\begin{aligned}&w[\tilde{H}_0^2(2 - q_0 - \beta z) - z]^2 - 2\tilde{H}_0^2(3z - 1)\\ &\quad + \tilde{H}_0^4(6 - 6q_0 - 6\beta z + 4\beta) = 0,\\ &z \equiv \frac{2 + 2w}{3 + 2w}.\end{aligned} \tag{13}$$

This equation defines H_0 as a function of β, q_0, and ω. In the $|w| \gg 1$ approximation[4] Eq. (13) yields

$$\frac{1}{\tilde{H}_0^2} \to (2 - q_0 - \beta) \pm \sqrt{\frac{2(1 + q_0 - \beta)}{w}}. \tag{14}$$

In the GR limit ($|w| \to \infty$) the second term can be neglected, so current cosmological parameter values can be established as [33,34] $H_0 \approx 2.3 \times 10^{-18}\ \mathrm{s}^{-1}$, $\rho_0 \approx 0.27 \times 10^{-29}\ \mathrm{g/cm^3}$ (accounting for baryonic and dark matter), $q_0 \approx -0.6$. We consider a dust-filled universe, thus neglecting the pressure. In the zeroth approximation we rewrite the above expression for the cosmological constant:

$$\Lambda \to (2 - q_0)H_0^2 - 4\pi G_0(\rho_0 - p_0) \approx 11.3 \times 10^{-36}\ \mathrm{s}^{-2}. \tag{15}$$

From the lunar ranging experiment (LLR) data [35] one can extract the following limitations: $|\partial_t G/G|_{(0)} \le 4 \times 10^{-20}\ \mathrm{s}^{-1}$; hence $|\epsilon_0|$ is a small value $|\epsilon_0| < 0.01$. When $|w| \gg 1$ we have

$$\tilde{H}_0 \approx 0.68, \qquad \tilde{\rho}_0 \approx 0.2, \qquad \beta \approx 0.4. \tag{16}$$

Combining Eqs. (8) and (9), multiplying the result by $1/\tilde{H}_0^2$ and substituting $\ddot{\phi}/\phi$ from (10), we obtain for the initial moment t_0 the following expression:

$$\frac{\epsilon_0}{\tilde{H}_0} = \frac{1}{\tilde{H}_0^2} - (2 - q_0 - \beta) + \frac{\beta + 1/\tilde{H}_0^2}{3 + 2w}. \tag{17}$$

Substituting the H value from (14), we obtain the order of[5] $1/\sqrt{|w|}$:

$$\epsilon_0 \to \pm\sqrt{\frac{2(1 + q_0 - \beta)}{w(2 - q_0 - \beta)}}. \tag{18}$$

IV. DUST-FILLED UNIVERSE SOLUTION

First of all we consider a dust-filled universe, i.e., $p = 0$. As in Ref. [21], we rewrite field equations using $f \equiv \phi a^3$ and take into account that the expression (11) leads to $\tilde{\rho}/\phi = \tilde{\rho}_0 f_0/f$.

Considering $\ddot{f}/f = \ddot{\phi}/\phi + 6\tilde{H}\epsilon + 3\dot{\tilde{H}} + 9\tilde{H}^2$, we combine field equations in the following way: $\frac{3}{2}[(8) + (9)] + (10)/[6 + 4w]$. This yields

$$\ddot{f} - \eta^2(f + \tilde{\rho}_0 f_0) = 0, \qquad \eta^2 \equiv \frac{8 + 6w}{3 + 2w}. \tag{19}$$

The obtained equation can be straightforwardly integrated:

$$\frac{f(\tilde{t})}{f_0} = c^+E + c^-/E - \tilde{\rho}_0, \qquad E(\tilde{t}) \equiv \exp(\eta\tilde{t}), \tag{20}$$

[4]Here and below (unless otherwise noted) the arrow denotes the $|w| \gg 1$ approximation.

[5]When calculating the right hand of the expression (17) we considered only the terms of order $1/\sqrt{|w|}$ from (14); the last term from (17) was neglected due to the taken accuracy.

where c^+ and c^- can be easily obtained from the initial data.

One can rewrite (10) as

$$2f + 2\tilde{\rho}_0 f_0 = (3 + 2w)(\dot{\phi}\dot{a}^3). \tag{21}$$

With the help of Eq. (20) one gets the expression for the Hubble parameter from Eq. (20)[6]:

$$\begin{aligned} 3\tilde{H} &= \frac{\dot{f}}{f} - \frac{\dot{\phi}}{\phi} = \frac{\dot{f}}{f} - \frac{2f_0}{f(3+2w)}\int_{\text{const}}^{\tilde{t}}\left(\frac{f}{f_0} + \tilde{\rho}_0\right)d\tilde{t} \\ &= \frac{\dot{f}}{f} - \frac{2(c^+E - c^-/E + c_H)}{\eta(3+2w)(c^+E + c^-/E - \tilde{\rho}_0)} \\ &= \frac{6(1+w)(c^+E - c^-/E) - 2c_H}{\eta(3+2w)(c^+E + c^-/E - \tilde{\rho}_0)}. \end{aligned} \tag{22}$$

Here c_H can also be determined from initial data.

Solving (20) and (22) for the present time, one obtains the coefficient values as

$$\begin{aligned} c^+ &= \frac{1+\tilde{\rho}_0}{2} + \frac{\epsilon_0 + 3\tilde{H}_0}{2\eta}, \\ c^- &= \frac{1+\tilde{\rho}_0}{2} - \frac{\epsilon_0 + 3\tilde{H}_0}{2\eta}, \\ c_H &= \frac{\eta\epsilon_0(3+2w)}{2} - \frac{\epsilon_0 + 3\tilde{H}_0}{\eta}, \end{aligned} \tag{23}$$

resulting in the following expression for the scale factor:

$$\begin{aligned} \frac{a}{a_0} &= (c^+E + c^-/E - \tilde{\rho}_0)^{1/3} \\ &\quad\times \exp\left[\frac{-1}{3(4+3w)}\int_1^E\left(\frac{c^+E^2 + c_HE - c^-}{c^+E^2 - \tilde{\rho}_0E + c^-}\right)\frac{dE}{E}\right] \\ &= \left(c^+E + \frac{c^-}{E} - \tilde{\rho}_0\right)^{(1+w)/(4+3w)}\exp\left[\frac{-2c_H(A - A_0)}{3(4+3w)\sqrt{\Delta}}\right], \end{aligned} \tag{24}$$

where

$$\begin{aligned} \Delta &\equiv 4c^+c^- - \tilde{\rho}_0^2 = 1 + 2\tilde{\rho}_0 - (3\tilde{H}_0 + \epsilon_0)^2/\eta^2 \\ &= \frac{-3}{8+6w}[\tilde{H}_0 - \epsilon_0(1+w)]^2, \end{aligned} \tag{25}$$

$$A(E) \equiv \arctan[(2c^+E - \tilde{\rho}_0)/\sqrt{\Delta}].$$

$$\phi = (c^+E + c^-/E - \tilde{\rho}_0)^{1/(4+3w)}\exp\left[\frac{2c_H(A - A_0)}{(4+3w)\sqrt{\Delta}}\right]. \tag{26}$$

In order to keep Δ positive, it is necessary to set w to be rather large ($|w| \gg 1$) and negative.[7]

[6]Note that $d\tilde{t} = dE/(\eta E)$.

[7]It is also important to mention that the $w < 0$ case in the BD model opens the possibility for wormholes existence without energy conditions violation; see [36,37] for details.

In the GR case $|w| \to \infty$ and Eq. (24) tends to the well-known Friedmann solution with a cosmological constant:

$$\begin{aligned} H_{\text{Fr}} &= \frac{1}{\sqrt{3}}\cdot\frac{E + E_{cr}}{E - E_{cr}}, \qquad E_{\text{cr}} \equiv \frac{\sqrt{3}\tilde{H}_0 - 1}{\sqrt{3}\tilde{H}_0 + 1}, \\ \eta_{\text{Fr}} &= \sqrt{3}, \end{aligned} \tag{27}$$

$$\frac{a_{\text{Fr}}}{a_0} = \frac{(\sqrt{3}\tilde{H}_0 + 1)^{2/3}(E - E_{cr})^{2/3}}{(4E)^{1/3}}. \tag{28}$$

It is necessary to note that in this case $E = E_{\text{cr}}$, $\Delta = 0$, $a = 0$, and the scale factor $a(t)$ experiences a kink (which is absent when $\Delta > 0$). The big bang corresponds to the moment $t_1 \approx -1.46\Lambda^{-1/2}$, $\Lambda^{-1/2} \approx 10^{10}$ yr (see Fig. 1).

V. NONSINGULAR COSMOLOGY

In BDΛ models, the scale factor may not vanish during its evolution back in time unlike in the standard ΛCDM one. This is a bounce of the scale factor from its minimal value $a_m \neq 0$. The bounce appears in case there is a local minimum of the scale factor greater than zero. The parameter phase space for the bounce case starts from $a_m(E) = 0$, so it is possible to obtain the condition for a bounce from Eq. (24). It has the following form:

$$\Delta > 0. \tag{29}$$

Further, the time estimation for E at the bounce is

$$E_{\text{min}} = \sqrt{c^-/c^+}. \tag{30}$$

The exact equality $E_m = \sqrt{c^-/c^+}$ is satisfied when $\Delta = 0$, e.g., when the scale factor vanishes at the local minimum. Such a scenario excludes the initial singularity and leaves the scale factor regular and continuous everywhere, including during the bounce [see Eq. (22)].

It is possible to estimate the numerical values of the scale factor $\tilde{a}(t)$ and Δ at the bounce from the following arguments. The cosmological microwave background radiation indicates that the Universe was hot and radiation dominated at early stages of its evolution [38]. Using the expression for an adiabatic expansion $a_{\text{hot}}/a_0 = 4 \times 10^{-5}$ [39] we can obtain the value of Δ at the bounce for $|w| \gg 1$ from Eq. (18), so that

$$\Delta \approx 2\tilde{\rho}_0 a_m^3/a_0^3 < 2\tilde{\rho}_0 a_{\text{hot}}^3/a_0^3 \approx 2.6 \times 10^{-14}. \tag{31}$$

This tiny value of Δ can be achieved only in a nearly flat universe, i.e., when $1 + q_0 - \beta \approx 0$. This result states that the LLR bound on w remains in agreement with the cosmological one in the BDΛ model, as well as with the flatness of the Universe.

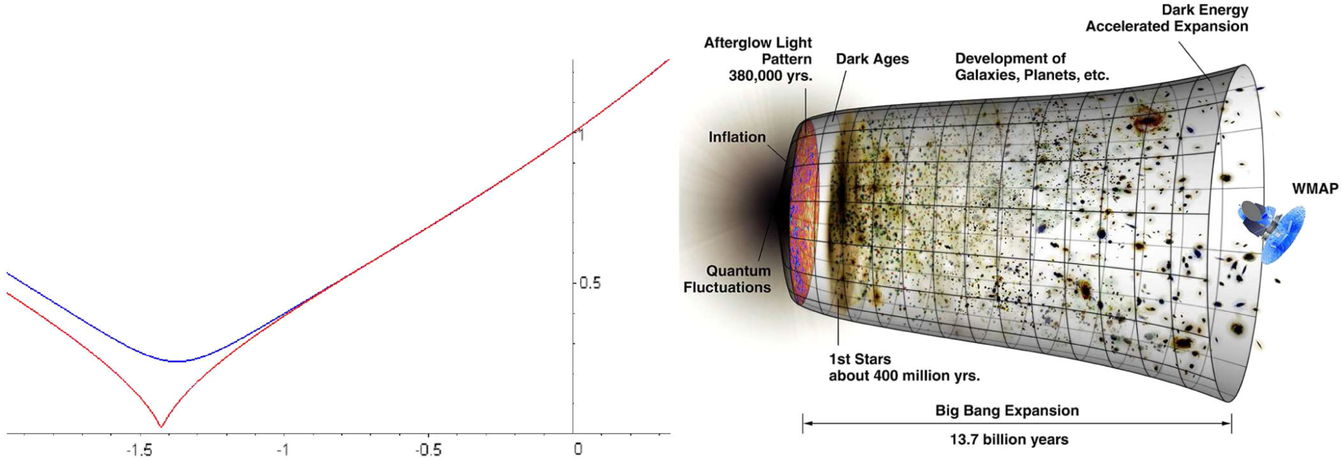

FIG. 1 (color online). The left-hand side represents $a(\tilde t)/a_0$ according to Eq. (24) for a dust-filled universe with a bounce for the following parameter values: $w = -1000$, $q = -0.6$, $\beta = 0.45$ for the upper line and $\beta = 0.436\,53$ for the lower one. The time unit is 10^{10} yr. Right-hand side: Illustration (taken from NASA: http://map.gsfc.nasa.gov) for $a(\tilde t)/a_0$ in ΛCDM theory, corresponding to the Friedmann solution.

The model under consideration as well as the usual singular cosmology[8] of [21] is not applicable for the hot stage of the Universe. Thus the discussed results should be used only to obtain initial values for the hot universe during the evolution back in time study.

VI. SOLUTION FOR A HOT UNIVERSE

The analytical study of the functions behavior near the bounce can be done only for an ultrarelativistic pressure. During the hot phase when $p = \frac{1}{3}\rho$ Eqs. (8)–(10) lead to the following expression:

$$\dot{\tilde H} + 2\tilde H^2 = \frac{1}{6}\left(-w\epsilon^2 + \frac{6+8w}{3+2w}\right) \equiv Q(\tilde t). \tag{32}$$

When $w < -1.5$, we obtain a positive value of Q.

In the FRW case (when $\epsilon = 0$ and $|w| \to \infty$), from Eq. (32) it is possible to obtain expressions for the hot stage that are similar to Eqs. (27) and (28):

$$H_{\rm Fr} = \frac{1}{\sqrt{3}}\cdot\frac{U+1}{U-1}, \qquad U(\tilde\tau) \equiv \exp\left(\frac{4\tilde\tau}{\sqrt{3}}\right),$$
$$\frac{a_{\rm Fr}}{a_{\rm hot}} = \left[\frac{(U-1)^2 U_{\rm hot}}{(U_{\rm hot}-1)^2 U}\right]^{1/4}. \tag{33}$$

Here a new variable $\tilde\tau$ is introduced. It represents a dimensionless time; that is a measure from the scale factor minimum $(\tilde\tau - \tilde t)/\sqrt{\Lambda}$ is equal to the age of the Universe and the subscript "hot" corresponds to the transition from the hot stage to the cold one.

The derivatives of the scale factor (33) are singular [as well as for the FRW case in the matter-dominated universe—see (27) and (28)]. Remarkably, when $\Delta \to 0$ is in the cold phase, the second derivative $\ddot a$ goes to $+\infty$ at the bounce while it goes to $-\infty$ in the vicinity of the bounce. Therefore the Hubble function appears to be rapidly growing when $\Delta \to 0$. The situation has to be similar to the case of the radiation-dominated universe. Since the hot phase matches large values of $\tilde H$, it ends for a short time interval $\tilde\tau$. Therefore, when $|w| \gg 1$ and $\epsilon \ll 1$ the solution of BDΛ is nearly indistinguishable from the FRW one (except in the bounce region).

Further, we consider the series expansion of the scale factor $a(\tilde\tau)$ against $\tilde\tau$ near the local minimum (bounce). Keeping the terms up to the fourth order, it is possible to obtain

$$a = a_m + \tfrac{1}{2}a_m\dot{\tilde H}_m\tilde\tau^2 - \tfrac{1}{12}a_m b^2\dot{\tilde H}_m^2\tilde\tau^4 + \cdots. \tag{34}$$

Here $\dot{\tilde H}_m$ and b are constants; $\dot{\tilde H}_m$ corresponds to the second derivative of the scale factor at the bounce, hence it should be positive when $a_m > 0$: $\dot{\tilde H}_m > 0$. Therefore, the equations for the Hubble function and its first derivative up to second order on $\tilde\tau$ are

$$\tilde H^2 = \dot{\tilde H}_m^2\tilde\tau^2, \qquad \dot{\tilde H} = \frac{\dot{\tilde H}_m(1 - b^2\dot{\tilde H}_m\tilde\tau^2)}{1 + \dot{\tilde H}_m\tilde\tau^2/2} - \dot{\tilde H}_m^2\tilde\tau^2. \tag{35}$$

After substituting this into (32), one gets

$$\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} = \frac{\dot{\tilde H}_m[1 + \tilde\tau^2\dot{\tilde H}_m(\frac{3}{2} - b^2)]}{(1 + \dot{\tilde H}_m\tilde\tau^2/2)^2} = Q > 0. \tag{36}$$

The last inequality is satisfied automatically [see Eq. (32)] and is valid only when $0 < b^2 < 3/2$.

From (34), one notices that at $\tilde\tau_1 = 1/\sqrt{\dot{\tilde H}_m b^2}$ the second derivative of the scale factor changes its sign ($\tilde\tau_1$ is an

[8] The article [21] presents an analytical solution for $\Delta < 0$, $w > 0$.

inflection point). So we consider an additional scale factor inflection point compared to the FRW case. At the time $\tilde{\tau}_2 = \sqrt{3}/\sqrt{\dot{\tilde{H}}_m b^2} = \sqrt{3}\tilde{\tau}_1$, the first derivative of the scale factor changes its sign. Hence, the solutions for the hot phase and the cold one should be matched along the $\tilde{\tau}_1$ to $\tilde{\tau}_2$ interval.

When $\dot{\tilde{H}}_m$ is large[9] and b is small, starting from the time $\tilde{\tau}_1$, the second derivative of the scale factor rapidly goes to a large negative value (during the time interval of the order of $\tilde{\tau}_1$). Meanwhile the Hubble function remains positive (up to the moment $\tilde{\tau}_2$). Therefore, along the $\tilde{\tau}_1$ to $\tilde{\tau}_2$ interval the solution for the hot phase can be matched to the cold phase solution. Varying the values of a_m, $\dot{\tilde{H}}_m$, and b, one could achieve a smooth connection.

VII. DISCUSSION AND CONCLUSIONS

In this paper, we have demonstrated that the Friedmann solution with a cosmological term is a degenerated case of a more generic cosmology (for example, the BDΛ one as a ground effective approximation).

In standard FRW cosmology, the graph of the solution of the scale factor $a(t)$ has a form of a vertical line before vanishing (the bounce is possible afterward); in the BDΛ case

(i) with $w > 0$ the graph of the scale factor $a(t)$ vanishes with a finite slope (first derivative remains finite at $a = 0$);

(ii) when $w < 0$ the graph of the scale factor $a(t)$ does not reach zero (and a bounce occurs), so all functions remain regular.

Further, an adequate model with a bounce can be obtained numerically because of the complicated structure of the theory. Here it is important to note that the parameter k (we considered the case $k = 0$ in this paper) describes the flatness type of the Universe and could provide a leading contribution near the bounce due to the scale factor smallness. The hot phase also implies the presence of a nonvanishing pressure (we considered $p = 0$) which leads to the inability of obtaining an analytical solution for the hot phase. The presence of the bounce in the BDΛ cosmological solution allows one to avoid one of the greatest problems of cosmology: the initial singularity.

Moreover, we would like to point out that the appearance of a bounce instead of a singularity is rather a common effect in gravity models when an additional scalar or tensor contribution is taken into account. For example, when studying the interplay between curvature and Maxwell terms in Gauss-Bonnet gravity one encounters an effect of the same nature when the singularity is changed by a local minimum [40]. The same effects occur in Gauss-Bonnet cosmology with additional fields [41]. A bounce appearance was also discovered in many new (sometimes exotic) models with additional terms (some examples of bounce appearance can be found at [42–45]) and in loop quantum gravity [46,47]. So, this effect is rather common and natural from the mathematical point of view (changing the balance between different term contributions) and can be used to obtain new (stronger) estimations of model parameters. So, based on the condition of existence of a bounce and, so, from Eq. (31) one can put a new limit on the BD parameter w in the form

$$|w| > 10^{40}, \qquad w < 0. \tag{37}$$

This limitation is much stronger than the existing experimental one ($|w| > 50\,000$). Note that these huge values (of order of 10^{40}) are rather common in theoretical physics and cosmology. Such values provide an addition argument on the Mach principle nonobservability (but, on the other hand, do not prohibit its existence). Future developments will show the connection of these values with reality.

Finally, we have to emphasis that one can treate BD theory as the GR one with scalar field including a potential in a specific form. For small values of w (when $w < -1/2$) the conventional inflation is prevented because of potential properties [48]. Further, when $w < -3/2$ it seems that the field has a negative kinetic term and, therefore, represents a phantom field. Solutions of such type, really, could be unphysical ones (see, for example, [49]). Unlike this in BD theory the scalar field is usually not coupled with stress-energy components and is treated as an independent (geometrical, for example) part of field equations. Therefore, the situation becomes a physical one. In the same situation Coule [50] suggested applying the Zeldovich argument to "close" bounces but [51] contained some entropy problems. In the case when $w < -4/3$ [Eq. (26)] the bounce appearance also becomes possible [52–54].

At last, it is possible to extend the consideration by taking the perturbations into account [55] and it could be a good test for the BDΛ model with additional ideas like, for example, [56].

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[9] Here and below we compare with the unit value.

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