Self-energy of a scalar charge near higher-dimensional black holes

Valeri P. Frolov[*] and Andrei Zelnikov[†]

Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2E1

We study the problem of self-energy of charges in higher-dimensional static spacetimes. Application of regularization methods of quantum field theory to calculation of the classical self-energy of charges leads to model-independent results. The correction to the self-energy of a scalar charge due to the gravitational field of black holes of the higher-dimensional Majumdar-Papapetrou spacetime is calculated exactly. It proves to be zero in even dimensions, but it acquires nonzero value in odd-dimensional spacetimes. The origin of the self-energy correction in odd dimensions is similar to the origin of the conformal anomalies in quantum field theory in even-dimensional spacetimes.

I. INTRODUCTION

There are several problems in theoretical physics that have quite a long history but still attract a lot of attention. The problem of the electromagnetic origin of the electron mass is one of them. It first was formulated in the classical theory when in 1881 Thompson [1] demonstrated that the self-energy of the electromagnetic field contributes to the inertial mass of a charged particle. This idea was then elaborated in the works by Lorentz [2,3], Abraham [4], Poincaré [5], Fermi [6], and others. For a simple model of a uniformly charged sphere of radius ε, the electrostatic energy is $E = e^2/(2\varepsilon)$. However, it was shown by Abraham [7,8] that the relation between energy and momentum for such a particle differs from the standard one by a factor 4/3. This factor disappears if one includes in the definition of the self-energy a contribution of additional (nonelectromagnetic) forces that are required to make the system stable. To solve the 4/3 problem, Poincaré [5,9] introduced special sort of nonelectromagnetic pressure. Max von Laue [10] formulated a general theorem, demonstrating that whenever a spatially extended system is stable, the total mass of such a system m_{tot} is always related to its rest mass as follows: $m_{tot} = E_{tot}/c^2$. The problem of self-energy and stability of a classical electron is discussed in many more recent papers (see e.g. [11–15]).

There are many different ways that a simple spherical-shell model of a classical electron can be modified. For example, instead of the shell, one can consider a charged ball, the shape of the shell or of the ball can be deformed, the distribution of the electric charge can be nonhomogeneous, and so on. Certainly, to be consistent, for each of these modifications one must also modify the nonelectromagnetic forces in order to satisfy the von Laue's theorem. Bopp [16] and Podolsky [17,18] proposed a covariant scheme for the calculation of the classical self-energy of the electron. Their idea was to start with a higher-derivative

modification of the corresponding field theory. For example, for a scalar massless field φ one starts with the equation

$$\left(1 - \frac{1}{\mu^2}\Box\right)\Box\varphi = -4\pi J.$$

This equation is equivalent to a set of relations

$$\varphi = \varphi' - \varphi'',$$

$$\varphi' = \left(1 - \frac{1}{\mu^2}\Box\right)\varphi, \qquad \varphi'' = -\frac{1}{\mu^2}\Box\varphi,$$

if the fields φ' and φ'' obey the equations

$$\Box\varphi' = -4\pi J, \qquad (\Box - \mu^2)\varphi'' = -4\pi J.$$

For a pointlike charge q, the infinite parts in the self-energy for both fields φ' and φ'' are identical, and as result of their subtraction the self-energy is finite $E_{self} = q^2\mu$. In this regularization μ plays the role of the cutoff parameter. In fact, in such an approach for a small size of a classical charged particle $\varepsilon \ll \mu^{-1}$ all the details of the charge distribution become unimportant. One can easily see that this approach has many common features with the Pauli-Villars regularization widely used in the modern quantum field theory [19].

In quantum electrodynamics the self-energy of the electron is divergent in the point-particle limit. In the second order of the perturbation theory this divergence is of the form

$$\Delta m \sim m_0 \frac{3e^2}{2\pi\hbar c} \ln(\hbar/(\varepsilon m_0 c)), \tag{1.1}$$

where m_0 is the "bare" mass of the electron, and ε is the cutoff radius. In the limit $\hbar \to 0$ the expression in Eq. (1.1) does not reproduce the classical result. The reason for this is that in order to derive this relation, one uses the expansion in $\alpha = e^2/(\hbar c)$. However, as demonstrated by Vilenkin and Fomin [20,21], there exists a correct

---

*vfrolov@ualberta.ca
†zelnikov@ualberta.ca

quantum-to-classical correspondence for the self-energy of the electron (see also [22]).

In the presence of the gravitational field the self-energy problem becomes more complicated. The reason is that the field of a pointlike charge is spread in space, and its contribution to the energy in a general case is nonlocal. The classical fields created by charges are not localized at the position of the charge. It means that the charges are to be treated as extended objects. Qualitatively, the origin of the classical electromagnetic self-force of charges can be explained by the deformation of the distribution of their classical fields in curved spacetime that leads to an extra force acting on the charge itself. Fermi [6] showed that for a special case of the homogeneous static gravitational field, the self-energy is the same as in the absence of the field. This result can be related to the equivalence principle. However, in a general case the electromagnetic (or scalar, or any other field) self-energy depends on the position of the particle. This may lead to an extra force acting on a charged particle.

In a generic case the self-force acting on a particle *moving* in the gravitational field contains both conservative and dissipative terms, with the dissipative terms being responsible for the radiation reaction. The fundamental problem of calculation of the radiation reaction of particles in the external gravitational field [23] has got much attention [24–26] in connection with the study of waveforms of gravitational radiation, especially because of possible applications in experiments for gravitational wave detection. One can find a nice review on the contemporary state of the problem in [27].

For a static particle in a static (stationary) gravitational field, the radiation force is absent, the problem is simplified, and in some special cases the self-energy can be calculated exactly. This becomes possible when the static Green functions of classical fields are known exactly. Fortunately, this is the case for some physically interesting systems like static charge near four-dimensional Schwarzschild or Reissner-Nordström black holes [28–30]. For the electric pointlike charge, a self-interaction energy is [31–33]

$$E = \left(m_{\text{bare}} + \frac{e^2}{2\epsilon}\right)|g_{00}|^{1/2} + \Delta E, \qquad \Delta E = \frac{e^2 M}{2r^2}, \quad (1.2)$$

which leads to an additional repulsive (directed from the black hole) self-force. Here ϵ is the classical radius of the electron, m_{bare} is its bare mass, and r is the radial distance to the black hole. It was also demonstrated that for a scalar charge near the Reissner-Nordström black hole, the self-energy correction ΔE vanishes [26,34]. In [35] the self-energy of scalar charges in the background geometry of wormholes was studied.

The aim of this paper is to analyze the self-energy problem for a static pointlike charge in a static higher-dimensional spacetime. Our motivations for this analysis are the following. In higher dimensions the fields near a pointlike source are stronger than in the four-dimensional (4D) case. Hence, one can expect much more dramatic dependence of the divergent part of the classical self-energy on details of a model of a classical source. We demonstrate that for the calculation of the self-energy, one can use methods similar to the ones adopted in quantum field theory. To be concrete we focus on the point-splitting method. It is well known for the calculations of the quantum vacuum polarization effects in a curved spacetime. We demonstrate that this regularization method works well for the calculation of the self-energy.

Another interesting question is why nonlocal self-energy correction ΔE for a scalar massless field vanishes in 4D. We shall demonstrate that this is a generic property of a wide class of even-dimensional spacetimes with a spatial metric conformal to the flat one, while in the odd-dimensional case there exists a nonvanishing extra force acting on a charged scalar source.

The paper is organized as follows. In Sec. II we discuss the self-energy problem for a scalar charge in a static gravitational field and obtain an expression for the corresponding shift of mass in terms of a static Green function. After this, we adapt the point-splitting formalism, well known in the quantum field theory, for the classical self-energy problem in a general higher-dimensional static gravitational field. This approach allows one (at least formally) to avoid problems connected with the details of the charge structure (Sec. III). We illustrate this method by calculations of the self-energy of a scalar massless charge. We apply the point-splitting approach for the calculation of the self-energy of a static source in 4D static black hole metrics and show that this formalism correctly reproduces known 4D results for the self-force (Sec. IV). In Sec. V we use the point-splitting method for the calculation of the self-energy of pointlike scalar charges at rest in the vicinity of a higher-dimensional extremely charged black hole, or a set of such black holes. For such gravitational backgrounds, the exact static Green functions are known [36], so that one is able to obtain an exact explicit expression for the self-energy. We demonstrate that in even-dimensional Majumdar-Papapetrou spacetimes the self-force vanishes, while in the odd-dimensional ones the self-force and self-energy can be related with conformal anomalies. Conclusions contain a discussion of the obtained results.

II. SELF-ENERGY OF A SCALAR CHARGE IN A STATIC SPACETIME

Let us consider a scalar massless field φ in D-dimensional spacetime with the metric

$$dS^2 = g_{\mu\nu}dy^\mu dy^\nu. \qquad (2.1)$$

It obeys the equation

$$\Box \varphi \equiv -4\pi J. \qquad (2.2)$$

We assume that the spacetime is static and $\boldsymbol{\xi}$ is its Killing vector, so that in the region where $\alpha^2 \equiv -\boldsymbol{\xi}^2 > 0$ one can write the metric in the form

$$dS^2 = -\alpha^2 dt^2 + g_{ab}dx^a dx^b, \qquad \partial_t \alpha = \partial_t g_{ab} = 0. \tag{2.3}$$

For a static source, the field equation takes the form

$$\triangle \varphi + g^{ab}\nabla_a(\ln\alpha)\nabla_b\varphi = -4\pi J. \tag{2.4}$$

Here ∇_a is a covariant derivative in $(D-1)$-dimensional metric g_{ab} and $\triangle = g^{ab}\nabla_a\nabla_b$.

For a pointlike scalar charge q located at $\boldsymbol{x}$, one has

$$J(\boldsymbol{x}) = q\int_{-\infty}^{\infty}d\tau\delta^{D-1}(\boldsymbol{x},\boldsymbol{x}')\frac{\delta(t-t'(\tau))}{\alpha(\boldsymbol{x}')} = q\delta^{D-1}(\boldsymbol{x},\boldsymbol{x}'),$$

$$\delta^{D-1}(\boldsymbol{x},\boldsymbol{x}') = \frac{\delta^{D-1}(\boldsymbol{x}-\boldsymbol{x}')}{\sqrt{g}}, \quad g = \det g_{ab}. \tag{2.5}$$

In what follows it is convenient to rewrite Eq. (2.4) in a self-adjoint form. For this purpose, we introduce the quantities

$$\varphi = \alpha^{-1/2}\tilde{\varphi}, \qquad J = \alpha^{-1/2}j, \tag{2.6}$$

and write the Eq. (2.4) in the form

$$\hat{F}\tilde{\varphi} \equiv (\triangle + V)\tilde{\varphi} = -4\pi j,$$

$$V = \frac{(\nabla\alpha)^2}{4\alpha^2} - \frac{\triangle\alpha}{2\alpha} \equiv -\frac{\triangle(\alpha^{1/2})}{\alpha^{1/2}}. \tag{2.7}$$

The energy E in a static spacetime is

$$E = \int_{\Sigma} T_{\mu\nu}\xi^{\mu}d\Sigma^{\nu}, \tag{2.8}$$

where Σ is a Cauchy surface and $d\Sigma^{\nu}$ is a future-directed volume element on it. The energy-momentum tensor for the minimally coupled massless scalar field is

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2}g_{\mu\nu}\varphi_{;\alpha}\varphi^{;\alpha}\right) + g_{\mu\nu}\varphi J. \tag{2.9}$$

For a static field, $\xi^{\mu}\varphi_{,\mu} = 0$ so that one has

$$E = \int_{\Sigma}\mathcal{T}\xi_{\mu}d\Sigma^{\mu}, \qquad \mathcal{T} = J\varphi - \frac{1}{8\pi}(\nabla\varphi)^2. \tag{2.10}$$

Since E does not depend on the choice of Σ, we choose this surface in the form $t = $ const. In the presence of a black hole one restricts the integration domain to the black hole exterior. For this choice of Σ, one has

$$d\Sigma^{\mu} = n^{\mu}\sqrt{g}d^{D-1}x, \tag{2.11}$$

where n^{μ} is a unit future-directed vector normal to Σ, and

$$\xi_{\mu}n^{\mu} = -\alpha. \tag{2.12}$$

Thus,

$$E = -\int_{t=\text{const}}d^{D-1}x\sqrt{g}\alpha\mathcal{T}. \tag{2.13}$$

Using the Stokes' theorem and the field equation Eq. (2.4), we get [37]

$$E = -\frac{1}{2}\int_{\Sigma}\alpha\varphi J\sqrt{g}d^{D-1}x = -\frac{1}{2}\int_{\Sigma}\tilde{\varphi}j\sqrt{g}d^{D-1}x. \tag{2.14}$$

Denote by $\mathcal{G}$ the Green function of the operator $\hat{F}$:

$$\hat{F}\mathcal{G}(\boldsymbol{x},\boldsymbol{x}') = -\delta^{D-1}(\boldsymbol{x},\boldsymbol{x}'). \tag{2.15}$$

Then Eq. (2.14) takes the form

$$E = -\frac{q^2}{2}\alpha(\boldsymbol{x})\mathcal{G}(\boldsymbol{x},\boldsymbol{x}). \tag{2.16}$$

As expected, the obtained expression for the self-energy of a pointlike charge is divergent. To deal with this problem, we shall use the point-splitting method, similar to the regularization schemes adopted in the quantum field theory. Namely, to regularize E, we use the regularized version of the Green function $\mathcal{G}_{\text{reg}}(\boldsymbol{x},\boldsymbol{x})$:

$$\mathcal{G}(\boldsymbol{x},\boldsymbol{x}) \to \mathcal{G}_{\text{reg}}(\boldsymbol{x},\boldsymbol{x}) = \lim_{\boldsymbol{x}\to\boldsymbol{x}'}[\mathcal{G}(\boldsymbol{x},\boldsymbol{x}') - \mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}')]. \tag{2.17}$$

We discuss this point-splitting procedure in the next section. Now let us make the following remark. The energy of the object of mass m at rest at a point $\boldsymbol{x}$ in a static gravitational is

$$E = -mu_{\mu}\xi^{\mu} = m\alpha(\boldsymbol{x}). \tag{2.18}$$

Using this relation, we obtain for the contribution Δm of the self-energy to the mass of a scalar charge the following expression:

$$\Delta m = -\frac{q^2}{2}\mathcal{G}_{\text{reg}}(\boldsymbol{x},\boldsymbol{x}). \tag{2.19}$$

III. POINT-SPLITTING REGULARIZATION OF OF THE SELF-MASS

A. Schwinger-DeWitt expansion

To obtain $\mathcal{G}_{\text{div}}$ it is convenient to start with the heat kernel expansion for the operator $\hat{F}$. We define the heat kernel $\mathcal{K}(s|\boldsymbol{x},\boldsymbol{x}')$ as a solution of the equation

$$\left[-\frac{\partial}{\partial s} + \hat{F}\right]\mathcal{K}(s|\boldsymbol{x},\boldsymbol{x}') = -\delta^{D-1}(\boldsymbol{x},\boldsymbol{x}')\delta(s). \tag{3.1}$$

The static Green function $\mathcal{G}(\boldsymbol{x},\boldsymbol{x}')$ defined by Eq. (2.15) is

$$\mathcal{G}(\boldsymbol{x},\boldsymbol{x}') = \int_0^{\infty}ds\mathcal{K}(s|\boldsymbol{x},\boldsymbol{x}'). \tag{3.2}$$

The divergent terms of $\mathcal{G}$ are determined by the behavior of the heat kernels at small s and can be found by using the standard Schwinger-DeWitt expansion

$$\mathcal{K}(s|\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{(4\pi s)^{(n+2)/2}} \exp\left(-\frac{\sigma(\boldsymbol{x},\boldsymbol{x}')}{2s}\right) \times \sum_{k=0}^{\infty} a_k(\boldsymbol{x},\boldsymbol{x}') s^k. \tag{3.3}$$

Here a_k are the Schwinger-DeWitt coefficients for the operator $\hat{F}$. The world function σ and Van Vleck-Morette determinant Δ are defined on the $(D-1)$-dimensional spatial metric g_{ab}.

The divergent part of the static Green function

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \int_0^{\infty} ds \, \mathcal{K}_{\text{div}}(s|\boldsymbol{x},\boldsymbol{x}') \tag{3.4}$$

comes from the first $[(D-1)/2]$ terms in this series Eq. (3.3). Denote $n = D-3$, then one has

$$\mathcal{K}_{\text{div}}(s|\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{(4\pi s)^{(n+2)/2}} \exp\left(-\frac{\sigma(\boldsymbol{x},\boldsymbol{x}')}{2s}\right)$$
$$\times \sum_{k=0}^{[n/2]} a_k(\boldsymbol{x},\boldsymbol{x}') s^k. \tag{3.5}$$

Therefore,

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}') \frac{1}{(2\pi)^{(n/2)+1}}$$
$$\times \sum_{k=0}^{[n/2]} \frac{\Gamma(\frac{n}{2}-k)}{2^{k+1}\sigma^{(n/2)-k}} a_k(\boldsymbol{x},\boldsymbol{x}'), \tag{3.6}$$

where brackets in the upper limit of the sum mean the integer part of $n/2$. When n is even the last term ($k = n/2$) in the sum should be replaced by

$$\frac{\Gamma(\frac{n}{2}-k)}{2^{k+1}\sigma^{(n/2)-k}} a_k(\boldsymbol{x},\boldsymbol{x}')\big|_{k=n/2} \to -\frac{\ln\sigma(\boldsymbol{x},\boldsymbol{x}') + \gamma - \ln 2}{2^{(n/2)+1}}$$
$$\times a_{n/2}(\boldsymbol{x},\boldsymbol{x}'). \tag{3.7}$$

B. Special cases

Let us illustrate the self-mass calculations by examples of special static black hole solutions in a spacetime with dimensions $D = 4$, 5, 6, and 7. For this purpose, let us present here the corresponding expressions for the divergent parts of the static Green functions for these cases

(i) *Four dimensions $D = 4$, $n = 1$*

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{4\pi} \frac{1}{(2\sigma)^{1/2}} a_0(\boldsymbol{x},\boldsymbol{x}'). \tag{3.8}$$

(ii) *Five dimensions $D = 5$, $n = 2$*

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{4\pi^2}\left[\frac{1}{2\sigma} a_0(\boldsymbol{x},\boldsymbol{x}')\right.$$
$$\left. -\frac{1}{4}(\ln\sigma + \gamma - \ln 2)a_1(\boldsymbol{x},\boldsymbol{x}')\right]. \tag{3.9}$$

(iii) *Six dimensions $D = 6$, $n = 3$*

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{8\pi^2}\left[\frac{1}{(2\sigma)^{3/2}} a_0(\boldsymbol{x},\boldsymbol{x}')\right.$$
$$\left. +\frac{1}{2(2\sigma)^{1/2}} a_1(\boldsymbol{x},\boldsymbol{x}')\right]. \tag{3.10}$$

(iv) *Seven dimensions $D = 7$, $n = 4$*

$$\mathcal{G}_{\text{div}}(\boldsymbol{x},\boldsymbol{x}') = \frac{\Delta^{1/2}(\boldsymbol{x},\boldsymbol{x}')}{8\pi^3}\left[\frac{1}{2\sigma^2} a_0(\boldsymbol{x},\boldsymbol{x}') + \frac{1}{4\sigma}\alpha_1(\boldsymbol{x},\boldsymbol{x}')\right.$$
$$\left. -\frac{1}{8}(\ln\sigma + \gamma - \ln 2)a_2(\boldsymbol{x},\boldsymbol{x}')\right]. \tag{3.11}$$

IV. SELF-ENERGY IN FOUR-DIMENSIONAL REISSNER-NORDSTRÖM SPACETIME

As a first example, let us apply the developed point-splitting method to calculation of the self-energy of a scalar charge near a four-dimensional Reissner-Nordström black hole. Namely, we shall demonstrate that these methods give he same answer as earlier calculations using a spherical shell model of a the classical charged particle.

Let M and Q be mass and electric charge Q of the lack hole. The Reissner-Nordström metric in the isotropic coordinates is

$$dS^2 = -\alpha^2 dt^2 + U^2 \delta_{ab} dx^a dx^b, \tag{4.1}$$

where

$$\alpha = \frac{4\rho^2 - (M^2 - Q^2)}{(2\rho + M + Q)(2\rho + M - Q)},$$
$$U = 1 + \frac{M}{\rho} + \frac{M^2 - Q^2}{4\rho^2}, \qquad \rho^2 = \delta_{ab}x^a x^b. \tag{4.2}$$

The standard radial coordinate r is related to the isotropic coordinate ρ as follows:

$$r = M + \rho + \frac{M^2 - Q^2}{4\rho}. \tag{4.3}$$

The spatial metric is conformally flat

$$g_{ab} = U^2 \delta_{ab}. \tag{4.4}$$

We denote a "coordinate distance" between two points as

$$|\boldsymbol{x} - \boldsymbol{x}'| \equiv \sqrt{\delta_{ab}(x^a - x'^a)(x^b - x'^b)}. \tag{4.5}$$

In spherical isotropic coordinates (t, ρ, θ, ϕ)

$$dS^2 = -\alpha^2 dt^2 + U^2[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.6}$$

it takes the form

$$|\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\lambda},$$
$$\lambda = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{4.7}$$

The static Green function for a scalar field in the Reissner-Nordström spacetime is known exactly [38–40]. In isotropic coordinates it takes the form

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{\sqrt{\alpha\alpha'}}{4\pi R(\boldsymbol{x}, \boldsymbol{x}')}, \quad \alpha = \alpha(\rho), \quad \alpha' = \alpha(\rho'). \tag{4.8}$$

Here

$$R(\boldsymbol{x}, \boldsymbol{x}')^2 = (\rho^2 + \rho'^2 - 2\rho\rho'\lambda)\left[1 - \frac{\lambda(M^2 - Q^2)}{2\rho\rho'}\right.$$
$$\left. + \frac{(M^2 - Q^2)^2}{16\rho^2\rho'^2}\right]. \tag{4.9}$$

In the limit $\boldsymbol{x}' \to \boldsymbol{x}$ one has the following expressions:

$$\sigma(\boldsymbol{x}, \boldsymbol{x}') = \frac{UU'}{2}|\boldsymbol{x} - \boldsymbol{x}'|^2 + O(|\boldsymbol{x} - \boldsymbol{x}'|^4),$$
$$a_0(\boldsymbol{x}, \boldsymbol{x}') = 1, \quad \Delta^{1/2}(\boldsymbol{x}, \boldsymbol{x}') = 1 + O(|\boldsymbol{x} - \boldsymbol{x}'|^2). \tag{4.10}$$

Thus, Eq. (3.8) leads to

$$\mathcal{G}_{\text{div}}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} + O(|\boldsymbol{x} - \boldsymbol{x}'|). \tag{4.11}$$

Similarly, when $\boldsymbol{x} \to \boldsymbol{x}'$, one can expand the exact expression Eq. (4.8) in series. It is easy to check that

$$\frac{\alpha\alpha'}{\left[1 - \frac{\lambda(M^2 - Q^2)}{2\rho\rho'} + \frac{(M^2 - Q^2)^2}{16\rho^2\rho'^2}\right]} = \frac{1}{UU'} + O(|\boldsymbol{x} - \boldsymbol{x}'|^2), \tag{4.12}$$

and hence,

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} + O(|\boldsymbol{x} - \boldsymbol{x}'|). \tag{4.13}$$

By comparing Eq. (4.13) with Eq. (4.11), one obtains

$$\mathcal{G}_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x}') = \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') - \mathcal{G}_{\text{div}}(\boldsymbol{x}, \boldsymbol{x}') = O(|\boldsymbol{x} - \boldsymbol{x}'|), \tag{4.14}$$

and hence,

$$\mathcal{G}_{\text{reg}}(x, x) = 0. \tag{4.15}$$

The corresponding self-energy E_{self} and the mass correction Δm vanish

$$\Delta m = \alpha^{-1} E_{\text{self}} = 0. \tag{4.16}$$

This result coincides with the corresponding result for a spherical shell model obtained earlier [25,26,32,34]. Let us emphasize that the point-splitting method is not only easier and simplifies calculations, but more importantly, it allows one to extract the finite part of the self-energy without discussing details of the classical charged particle model. It also can be used in arbitrary number of spacetime

dimensions. In order to illustrate the latter point we shall perform calculations of the self-energy in special higher-dimensional black hole metrics.

V. SELF-ENERGY OF A SCALAR CHARGE IN THE HIGHER-DIMENSIONAL MAJUMDAR-PAPAPETROU METRICS

A. Static Green function in the Majumdar-Papapetrou spacetime

There exist a wide class of higher-dimensional metrics where the static Green functions for scalar and electromagnetic field of a point charge are known in explicit form [36]. These are so-called higher-dimensional Majumdar-Papapetrou metrics [41]. They describe the field of a set of extremally charged black holes in equilibrium in a higher-dimensional asymptotically flat spacetime. The corresponding background metric and electric potential are of the form ($D = n + 3$)

$$dS^2 = -U^{-2}dt^2 + U^{2/n}\delta_{ab}dx^a dx^b,$$
$$A_\mu = \sqrt{\frac{n+1}{2n}}U^{-1}\delta^0_\mu. \tag{5.1}$$

Denote

$$\rho = |\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{\delta_{ab}(x^a - x'^a)(x^b - x'^b)}, \tag{5.2}$$

$$\triangle = \delta^{ab}\partial_a\partial_b. \tag{5.3}$$

Note that the flat Laplace operator $\triangle$ differs from the curved one $\triangle = g^{ab}\nabla_a\nabla_b$.

The function U in Eq. (5.1)

$$U = 1 + \sum_k \frac{M_k}{\rho_k^n}, \quad \rho_k = |\boldsymbol{x} - \boldsymbol{x}_k|. \tag{5.4}$$

The index $k = (1, \dots, N)$ enumerates the extremal black holes. x_k^a is the spatial position of the kth extremal black hole. The potential U obeys the equation

$$\triangle U = -\frac{4\pi^{1+(n/2)}}{\Gamma(\frac{n}{2})}\sum_k M_k\delta^{n+2}(\boldsymbol{x} - \boldsymbol{x}_k). \tag{5.5}$$

In the black holes exterior the function U is a harmonic function.

It is easy to heck that the static scalar field equation Eq. (2.4) in the metric Eq. (4.4) takes the form

$$\triangle\varphi = -4\pi U^{2/n}J. \tag{5.6}$$

For a pointlike charge, this equation can be easily solved. It is sufficient to use the following relations:

$$\triangle\left[\frac{1}{\rho^n}\right] = -\frac{4\pi^{1+(n/2)}}{\Gamma(\frac{n}{2})}\delta^{n+2}(\boldsymbol{x} - \boldsymbol{x}') = -n\frac{1}{\rho^{n+1}}\delta(\rho), \tag{5.7}$$

$$\delta^{n+2}(\boldsymbol{x} - \boldsymbol{x}') = \frac{\Gamma(1 + \frac{n}{2})}{2\pi^{1+(n/2)}} \frac{1}{\rho^{n+1}} \delta(\rho). \qquad (5.8)$$

The static Green function is [36]

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{\Gamma(\frac{n}{2})}{4\pi^{1+(n/2)}} \frac{1}{\sqrt{UU'}} \frac{1}{\rho^n}. \qquad (5.9)$$

B. Self-energy

Since the spatial part of the Majumdar-Papapetrou metric is conformally flat, the calculations of the divergent part of a static function in the higher-dimensional case are greatly simplified. In addition in this case the operator $\hat{F}$ happens to be conformally invariant. The details of the calculations can be found in Appendices A and B. Using these results and expressions for $\mathcal{G}_{\mathrm{div}}$ presented in Sec. III B, one obtains E_{self} and $\Delta m = E_{\mathrm{self}}/\alpha$. Here we collect the corresponding results for $D = 4$, 5, and 6 dimensional spacetimes:

(i) *Four dimensions $D = 4$, $n = 1$*

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi} \frac{1}{\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|},$$

$$\mathcal{G}_{\mathrm{div}}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi} \frac{1}{\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} + O(|\boldsymbol{x} - \boldsymbol{x}'|),$$

$$\mathcal{G}_{\mathrm{reg}}(\boldsymbol{x}, \boldsymbol{x}) = 0, \qquad (5.10)$$

$$\Delta m = 0. \qquad (5.11)$$

(ii) *Five dimensions $D = 5$, $n = 2$*

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi^2} \frac{1}{\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^2},$$

$$\mathcal{G}_{\mathrm{div}}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi^2} \frac{1}{\sqrt{UU'}} \left[\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^2} + \frac{1}{72} U\mathcal{R} \right]$$
$$+ O(|\boldsymbol{x} - \boldsymbol{x}'|^2),$$

$$\mathcal{G}_{\mathrm{reg}}(\boldsymbol{x}, \boldsymbol{x}) = -\frac{1}{288\pi^2} \mathcal{R}, \qquad (5.12)$$

$$\Delta m = \frac{q^2}{576\pi^2} \mathcal{R}. \qquad (5.13)$$

Here $\mathcal{R}$ is the Ricci scalar of the spatial metric g_{ab}

$$\mathcal{R} = \frac{3}{2} U^{-3}(U_{,a}U_{,b} - 2UU_{,ab})\delta^{ab}. \qquad (5.14)$$

(iii) *Six dimensions $D = 6$, $n = 3$*

$$\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{8\pi^2} \frac{1}{\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^3},$$

$$\mathcal{G}_{\mathrm{div}}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{8\pi^2} \frac{1}{\sqrt{UU'}} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^3} + O(|\boldsymbol{x} - \boldsymbol{x}'|),$$

$$\mathcal{G}_{\mathrm{reg}}(\boldsymbol{x}, \boldsymbol{x}) = 0, \qquad (5.15)$$

$$\Delta m = 0. \qquad (5.16)$$

Note that the obtained results are valid for geometries which are more general than the Majumdar-Papapetrou spacetimes, because in the derivation of these formulas we used the metric in the form Eq. (5.1) with an arbitrary function U. The Majumdar-Papapetrou spacetimes satisfy the Einstein equations which lead to the additional constraint Eq. (5.5) on the function U.

C. Self-force near five-dimensional Reissner-Nordström black hole

The self-force f^a acting on a static scalar charge can be read out from the variation of the self-energy over displacement of the charge (see, e.g. [25,31,34] for details)

$$\delta E_{\mathrm{self}} = -f_a \delta x^a. \qquad (5.17)$$

In even-dimensional asymptotically flat spacetimes of the type in Eq. (5.1) the self-energy vanishes and there is no corresponding self-force. In odd-dimensional spacetimes there appears a nontrivial self-force.

Consider a simple example of a self-force of a scalar charge near a single five-dimensional extremal Reissner-Nordström black hole. In this case

$$U = 1 + \frac{M}{\rho^2}, \qquad \mathcal{R} = 6\frac{M^2}{(\rho^2 + M)^3}. \qquad (5.18)$$

The self-energy

$$E_{\mathrm{self}} = \frac{q^2}{576\pi^2} U^{-1}\mathcal{R} = \frac{q^2}{96\pi^2} \frac{M^2\rho^2}{(\rho^2 + M)^4}. \qquad (5.19)$$

In terms of the Schwarzschild radial coordinates $r^2 = \rho^2 + M$ it reads

$$E_{\mathrm{self}} = \frac{q^2}{96\pi^2} \frac{M^2(r^2 - M)}{r^8}. \qquad (5.20)$$

Thus, the only nonvanishing component of the self-force is its radial component

$$f^\rho = \frac{q^2}{48\pi^2} \frac{M^2\rho^3(3\rho^2 - M)}{(\rho^2 + M)^6}. \qquad (5.21)$$

The force is repulsive at far distances, vanishes at $\rho = \sqrt{M/3}$ (or, equivalently, $r = 2\sqrt{M}/\sqrt{3}$) and becomes attractive at smaller radii. At the horizon $\rho = 0$, it vanishes again.

VI. CONCLUSIONS

In the paper we discussed the problem of self-energy of a classical charged particle in an external static gravitational field. Our main focus was on the case of higher-dimensional gravity. Classical self-energy of pointlike charges diverges and should be properly regularized and renormalized. Our approach is to use well established

regularization techniques of quantum field theory to deal with this problem. To single out divergences of the self-energy of a pointlike charge, we used the point-splitting method. This method is well known in quantum field theory and is convenient for our purposes. It has been intensively used to study the vacuum polarization effects in black hole physics and cosmology. We demonstrated that the application of a similar method for the classical problem allows one to reproduce the earlier published results of calculations where a special (uniformly charged shell) model of classical charged particle was used.

An important property of the point-splitting method is that it does not require a special model for the charged particle, and it is easily adapted for higher-dimensional calculations. We performed calculations of the self-energy for a static source of a minimally coupled scalar massless field. We showed that the contribution of the self-energy to the proper mass of the particle has the form

$$\Delta m = -\frac{1}{2} q^2 \mathcal{G}_{\text{reg}}(\boldsymbol{x}, \boldsymbol{x}). \qquad (6.1)$$

In other words, Δm is identical to

$$\Delta m = -\frac{1}{2} q^2 \langle \tilde{\varphi}^2 \rangle_{\text{ren}}, \qquad (6.2)$$

where $\tilde{\varphi}$ is the properly normalized Euclidean quantum field in $(D-1)$-dimensional space with metric g_{ab}. The classical point-splitting method practically coincides with calculations of $\langle \tilde{\varphi}^2 \rangle_{\text{ren}}$ in the corresponding $(D-1)$-quantum theory (see, e.g., [42,43]).

To illustrate the developed method of the calculations of the self-energy, we applied it to a special case of a charge in the higher-dimensional Majumdar-Papapetrou metrics, describing a set of static extremally charged black holes in equilibrium. For such spaces, the static Green functions for a static charge are known explicitly. We showed that for even-dimensional spacetimes $(D = 4, 6, \ldots)$ Δm vanishes identically. In five dimensions Δm does not vanish and is related to the local curvature invariants of the spatial metric. One can expect that this is a generic property of all odd-dimensional spacetimes. A natural explanation of these results might be the following. On the class of space-times of the form Eq. (5.1) characterized by an arbitrary function U, the corresponding operator $\hat{F}$ is invariant under the transformation of the function U. The spatial part of the Majumdar-Papapetrou metric is conformally flat. In a flat space Δm vanishes identically. Thus, the nontrivial value of Δm in a "physical" space arises as a result of the mechanism similar to the conformal anomalies. For odd-dimensional spaces with $(D - 1 = 3, 5, \ldots)$, such anomalies vanish, and Δm remains equal to zero. We are going to return to this interesting problem and will discuss this mechanism in detail in another publication.

APPENDIX A: CONFORMAL TRANSFORMATION OF THE DEWITT COEFFICIENTS

Consider $(n + 2)$-dimensional space with the conformally flat metric

$$g_{ab} = \Omega^2 \bar{g}_{ab}, \quad \sigma^c \sigma_c = 2\sigma, \quad \bar{\sigma}^c \bar{\sigma}_c = 2\bar{\sigma}. \qquad (A1)$$

Here $\sigma^a \equiv g^{ab}\sigma_{,b}$ and $\bar{\sigma}^a \equiv \bar{g}^{ab}\bar{\sigma}_{,b}$. The metric $\bar{g}_{ab}$ is a flat metric. Therefore,

$$\bar{\sigma}_{a;b} = \bar{g}_{ab}. \qquad (A2)$$

We can express σ in terms of $\bar{\sigma}$ and its derivatives. The result is

$$
\begin{aligned}
\sigma &= \bar{\sigma}\Bigg[\Omega^2 - \Omega\Omega_{;a}\bar{\sigma}^a + \frac{1}{12}(4\Omega\Omega_{;ab} \\
&\quad + 4\Omega_{;a}\Omega_{;b} - \Omega_{;c}\Omega^{;c}\bar{g}_{ab})\bar{\sigma}^a\bar{\sigma}^b \Bigg] + \ldots \\
&= \bar{\sigma}\Omega(\boldsymbol{x})\Omega(\boldsymbol{x}')\Bigg[1 + \frac{1}{12\Omega^2}(-2\Omega\Omega_{;ab} \\
&\quad + 4\Omega_{;a}\Omega_{;b} - \Omega_{;c}\Omega^{;c}\bar{g}_{ab})\bar{\sigma}^a\bar{\sigma}^b \Bigg] + \ldots \qquad (A3)
\end{aligned}
$$

For the determinant $\Delta^{1/2}(\boldsymbol{x}, \boldsymbol{x}')$, we have

$$
\begin{aligned}
\Delta^{1/2} &= 1 + \frac{1}{12}\mathcal{R}_{ab}\sigma^a\sigma^b + \ldots \\
&= 1 + \frac{1}{12\Omega^2}[-n\Omega\Omega_{;ab} + 2n\Omega_{;a}\Omega_{;b} - (\Omega\Omega^{;c}_{;c} \\
&\quad + (n-1)\Omega^{;c}\Omega_{;c})\bar{g}_{ab}]\bar{\sigma}^a\bar{\sigma}^b + \ldots \qquad (A4)
\end{aligned}
$$

Here we took into account

$$
\begin{aligned}
\mathcal{R}_{ab} &= \bar{\mathcal{R}}_{ab} + \frac{1}{\Omega^2}[-n\Omega\Omega_{;ab} + 2n\Omega_{;a}\Omega_{;b} \\
&\quad - (\Omega\Omega^{;c}_{;c} + (n-1)\Omega^{;c}\Omega_{;c})\bar{g}_{ab}], \qquad (A5)
\end{aligned}
$$

$$\mathcal{R} = \frac{1}{\Omega^2}\bar{\mathcal{R}} - \frac{n+1}{\Omega^4}[2\Omega\Omega^{;c}_{;c} + (n-2)\Omega^{;c}\Omega_{;c}], \quad (A6)$$

and that in our case $\bar{g}_{ab}$ is flat and, hence, $\bar{R}_{ab} = 0$. Here, on the right-hand side of these equations, all the covariant derivatives $\Omega_{;ab}$, $\Omega_{;a}$, etc. are defined in accordance with the flat metric $\bar{g}_{ab}$.

The first DeWitt coefficients corresponding to the operator

$$\hat{F} = \nabla^a \nabla_a + V \qquad (A7)$$

are

$$a_0(\boldsymbol{x}, \boldsymbol{x}') = 1, \tag{A8}$$

$$a_1(\boldsymbol{x}, \boldsymbol{x}') = V + \frac{1}{6}\mathcal{R} + \dots \tag{A9}$$

APPENDIX B: DEWITT COEFFICIENTS IN MAJUMDAR-PAPAPETROU SPACETIMES

The expressions Eq. (3.8), (3.9), and (3.10). for the UV divergent terms of the scalar Green function have been derived for generic curved spacetimes. Now, we apply these results to the class of Majumdar-Papapetrou metrics.

In the case of the metric Eq. (5.1) we have

$$\Omega = U^{1/n}, \tag{B1}$$

$$\mathcal{R} = \frac{1}{4\xi_{n+2}} U^{-2-(2/n)}(U_{,a}U_{,b} - 2UU_{,ab})\delta^{ab},$$
$$\xi_{n+2} = \frac{n}{4(n+1)}, \tag{B2}$$

$$V = -\xi_{n+2}\mathcal{R} = -\frac{1}{4}U^{-2-(2/n)}(U_{,a}U_{,b} - 2UU_{,ab})\delta^{ab}, \tag{B3}$$

$$a_1(\boldsymbol{x}, \boldsymbol{x}) = \frac{1}{6}\mathcal{R} + V = \left(\frac{1}{6} - \xi_{n+2}\right)\mathcal{R}. \tag{B4}$$

(i) In five dimensions $n = 2$, $\Omega = U^{1/2}$, $\xi_4 = 1/6$

$$a_1(\boldsymbol{x}, \boldsymbol{x}) \equiv 0, \tag{B5}$$

$$\frac{\Delta^{1/2}}{\sigma} = \frac{1}{\Omega\bar{\sigma}\Omega'} + \frac{1}{36}\mathcal{R} + \dots, \tag{B6}$$

and

$$\mathcal{R} = \frac{3}{2}U^{-3}(U_{,a}U_{,b} - 2UU_{,ab})\delta^{ab}. \tag{B7}$$

(ii) In six dimensions $n = 3$, $\Omega = U^{1/3}$, $\xi_5 = 3/16$

$$a_1 = -\frac{1}{48}\mathcal{R}, \tag{B8}$$

$$\mathcal{R} = -\frac{4}{\Omega^4}[2\Omega\Omega_{;c}^{:c} + \Omega^{:c}\Omega_{;c}], \tag{B9}$$

$$\Delta^{1/2} = 1 + \frac{1}{12\Omega^2}[-3\Omega\Omega_{;ab} + 6\Omega_{;a}\Omega_{;b} - (\Omega\Omega_{;c}^{:c} + 2\Omega^{:c}\Omega_{;c})\bar{g}_{ab}]\bar{\sigma}^a\bar{\sigma}^b + \dots, \tag{B10}$$

$$\Delta^{1/2}\left[\frac{1}{(2\sigma)^{3/2}}a_0 + \frac{1}{2(2\sigma)^{1/2}}a_1\right]$$
$$= \frac{1}{\Omega^{3/2}(2\bar{\sigma})^{3/2}\Omega'^{3/2}} + O(\bar{\sigma}^{1/2})$$
$$= \frac{1}{U^{1/2}(2\bar{\sigma})^{3/2}U'^{1/2}} + \dots \tag{B11}$$

In all odd-dimensional $(n = 2, 4, \dots)$ Majumdar-Papapetrou spacetimes the DeWitt coefficients

$$a_{n/2}(\boldsymbol{x}, \boldsymbol{x}') = 0. \tag{B12}$$

These coefficients appear as the factors before $\ln|\boldsymbol{x} - \boldsymbol{x}'|$ in the Hadamard representation. This is why there are no logarithmic divergences in the static Green functions in these spacetimes.

[1] J. J. Thomson, Philos. Mag. **5**, 229 (1881).
[2] H. A. Lorentz, in *Proceedings of the Royal Netherlands Academy of Arts and Sciences*, Vol. 1, 1899), p. 427 (unpublished).
[3] H. A. Lorentz, in *Proceedings of the Royal Netherlands Academy of Arts and Sciences*, Vol. 6, 1904, p. 809 (unpublished).
[4] M. Abraham, Ann. Phys. (Leipzig) **315**, 105 (1902).
[5] H. Poincaré, Comptes Rendus **140**, 1504 (1905).
[6] E. Fermi, Nuovo Cimento **22**, 176 (1921).
[7] M. Abraham, Phys. Z. **5**, 576 (1904).
[8] M. Abraham and A. Foppl, *Theorie der Elektrizität: Elektromagnetische Theorie der Strahlung* (B. G. Teubner, Leipzig, 1905), p. 201.
[9] H. Poincaré, Rend. Circ. Mat. Palermo **21**, 129 (1906).
[10] M. von Laue and , *Das Relativitätsbrinzip*, Wissenschaft (Braunschweig, Germany), Bd. 38 (F. Vieweg, Braunschweig, 1911).
[11] F. Rohrlich, Am. J. Phys. **28**, 639 (1960).
[12] F. R. Tangherlini, Nuovo Cimento **26**, 497 (1962).
[13] T. H. Boyer, Phys. Rev. D **25**, 3246 (1982).
[14] S. M. Blinder, Rep. Math. Phys. **47**, 269 (2001).
[15] S. M. Blinder, arXiv:physics/0208072.
[16] F. Bopp, Ann. Phys. (Leipzig) **38**, 345 (1940).
[17] B. Podolsky, Phys. Rev. **62**, 68 (1942).
[18] B. Podolsky and P. Schwed, Rev. Mod. Phys. **20**, 40 (1948).
[19] In the standard theory the bare mass of elementary particles is explained by the Higgs mechanism. However the self-interaction gives contribution to the renormalization of the mass.
[20] A. V. Vilenkin and P. I. Fomin, Sov. Phys. JETP **40**, 6 (1974).

[21] A. Vilenkin, Ann. Phys. (N.Y.) **120**, 471 (1979).
[22] G. V. Efimov, M. A. Ivanov, and O. A. Mogilevsky, Ann. Phys. (N.Y.) **103**, 169 (1977).
[23] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[24] Y. Mino, M. Sasaki, and Takahiro Tanaka, Phys. Rev. D **55**, 3457 (1997).
[25] T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997).
[26] T. C. Quinn, Phys. Rev. D **62**, 064029 (2000).
[27] E. Poisson, Living Rev. Relativity **7**, 6 (2004).
[28] E. T. Copson, Proc. R. Soc. A **118**, 184 (1928).
[29] B. Leaute and B. Linet, Phys. Lett. **58A**, 5 (1976).
[30] B. Linet, J. Phys. A **9**, 1081 (1976).
[31] A. G. Smith and C. M. Will, Phys. Rev. D **22**, 1276 (1980).
[32] A. I. Zelnikov and V. P. Frolov, Sov. Phys. JETP **55**, 191 (1982).
[33] D. Lohiya, J. Phys. A **15**, 1815 (1982).

[34] D. H. J. Cho, A. A. Tsokaros, and A. G. Wiseman, Classical Quantum Gravity **24**, 1035 (2007).
[35] V. B. Bezerra and N. R. Khusnutdinov, Phys. Rev. D **79**, 064012 (2009).
[36] V. P. Frolov and A. Zelnikov, Phys. Rev. D **85**, 064032 (2012).
[37] Notice that the boundary term at infinity in the Stock's formula vanished since the field $\bar{\varphi}$ decreases there rapidly enough. In the presence of a black hole there exists also a boundary term at its horizon. However for a static regular field it vanishes as well (for details see e.g. [32])
[38] B. Linet, C.R. Acad. Sci. A **284**, 215 (1977).
[39] B. Linet, Gen. Relativ. Gravit. **37**, 2145 (2005).
[40] V. P. Frolov and A. I. Zel'nikov, J. Phys. A **13**, L345 (1980).
[41] R. C. Myers, Phys. Rev. D **35**, 455 (1987).
[42] D. N. Page, Phys. Rev. D **25**, 1499 (1982).
[43] R. T. Thompson and J. P. S. Lemos, Phys. Rev. D **80**, 064017 (2009).