

Hidden quantum groups symmetry of super-renormalizable gravity

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In this paper, we consider the relation between the super-renormalizable theories of quantum gravity studied by Biswas, Gerwick, Koivisto, and Mazumdar [Phys. Rev. Lett. **108**, 031101 (2012)] and Modesto [arXiv:1107.2403; arXiv:1202.0008] and an underlying noncommutativity of space-time. For one particular super-renormalizable theory, we show that at the linear level (quadratic in the Lagrangian) the propagator of the theory is the same one we obtain starting from a theory of gravity endowed with θ -Poincaré quantum groups of symmetry. Such a theory is over the so-called θ -Minkowski noncommutative space-time. We shed new light on this link and show that, among the theories considered in these references, there exists only one nonlocal and Lorentz invariant super-renormalizable theory of quantum gravity that can be described in terms of a quantum-group symmetry structure. We also emphasize contact with preexistent works in the literature and discuss preservation of the equivalence principle in our framework.

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I. INTRODUCTION

In the recent papers [1,2], a modified theory of gravity has been introduced assuming a synthesis of minimal requirements: (i) there is a regularity of classical solutions; (ii) Einstein-Hilbert action should be the correct low energy limit; (iii) the space-time dimension has to decrease with the energy; (iv) the theory has to be perturbatively renormalizable at the quantum level; (v) the theory has to be unitary, with no other pole beyond the graviton in the propagator.

The theory we are going to summarize in the next section is power counting super-renormalizable at the quantum level at least perturbatively, and at the classical level the gravitational potential [1], black hole solutions [2–26], and the cosmological solutions are singularity-free [1,27,28]. The Lagrangian is a “nonlocal” extension of the renormalizable quadratic Stelle theory [29], but the nonlocality involves only positive powers of the d’Alembertian covariant operator. In other words, there are not operators like $1/\square^p$ ($p > 0$). The theory is not unique (we thus refer to super-renormalizable “theories”), but all the freedom present in the action can be read in an “entire function” of the d’Alembertian operator, $H(-\square/\Lambda^2)$ [30] (Λ is a physical mass-invariant scale introduced in the classical action).

The reason for this paper is not only to find an elegant reason for the nonlocal nature of the action but to find a way to fix uniquely the entire function which is mentioned above. In this paper, we show that the propagator of the theory, for a particular choice of the entire function $H(-\square/\Lambda^2)$, has exactly the same form of the propagator we obtain by starting from a theory of gravity endowed with θ -Poincaré quantum groups of symmetry. The right choice is much easier than we could think, i.e. $H(-\square/\Lambda^2) = -\square/\Lambda^2$. Any other entire function gives of course a well defined super-renormalizable theory of gravity (consistently with some particular properties [1,2]) but is not compatible with the requirement of having a nontrivial Hopf-algebra-like symmetry regulating the super-renormalizability of the theory. In particular, the Hopf algebra underlying the super-renormalizable model we discuss below is a quantum group associated to an associative noncommutative space-time. In particular, this is the only quantum group of (space-time) symmetry that can be accounted within the model presented in Ref. [2], if we do not relax the associativity of the space-time points’ coordinates. What emerges is therefore a new symmetric structure underlying the theory.

II. THE THEORY

A simplified version of the theory is a nonlocal generalization of the Stelle quadratic action for gravity [29] and can be written in the following compact form:

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$$\mathcal{L}_g = -\sqrt{-g} \left[\frac{\beta}{\kappa^2} R + R_{\mu\nu} F(-\square_\Lambda)^{\mu\nu\rho\sigma} R_{\rho\sigma} \right], \quad (1)$$

where the tensor $F(-\square_\Lambda)$ is a function of the covariant d'Alembertian operator $-\square_\Lambda := -\square/\Lambda^2$, Λ is a physical mass scale, and $\kappa^2 = 32\pi G$. To fix the notation we can write more explicitly the tensor $F(-\square_\Lambda)$ in terms of two entire functions h_2 and h_0 that we are going to define in this same section:

$$\begin{aligned} F(-\square_\Lambda)^{\mu\nu\rho\sigma} & := -(\beta_2 - h_2(-\square_\Lambda))g^{\mu\rho}g^{\nu\sigma} \\ & + \left(\frac{\beta_2}{3} + \beta_0 - h_0(-\square_\Lambda) - \frac{h_2(-\square_\Lambda)}{3} \right) g^{\mu\nu}g^{\rho\sigma}. \end{aligned} \quad (2)$$

The complete Lagrangian including also the gauge fixing and ghost terms is

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{\text{GF,GH}}, \quad (3)$$

where the gauge fixing and ghost Lagrangian terms are

$$\mathcal{L}_{\text{GF,GH}} = -\frac{1}{2\xi} F^\mu \omega(-\square_\Lambda^\eta) F_\mu + \bar{C}^\mu M_{\mu\nu} C^\nu. \quad (4)$$

The operator \square_Λ^η encapsulates the d'Alembertian of the flat fixed background, whereas F_μ is the gauge fixing function with the weight functional ω . The two functions h_2 and h_0 have not to be polynomial but “entire functions without poles or essential singularities” to avoid ghosts (states with negative norm) in the spectrum ω . \bar{C}^μ and C^μ are the ghost fields, and $M^\tau_\alpha = F^\tau_{\mu\nu} D_\alpha^{\mu\nu}$. The gauge fixing function $F^\tau_{\mu\nu}$ and the operator $D_\alpha^{\mu\nu}$ will be defined shortly, in (6).

We calculate now the graviton propagator. For this purpose we start by considering the quadratic expansion of the Lagrangian (1) in the graviton field fluctuation without specifying the explicit form of the functionals h_2 and h_0 (if not necessary). Following the Stelle paper [29], we expand around the Minkowski background $\eta^{\mu\nu}$ in the power of the graviton field $h^{\mu\nu}$ defined in the following way:

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \quad (5)$$

The form of the propagator depends not only on the gauge choice but also on the definition of the gravitational fluctuation [31]. The gauge choice is the familiar “harmonic gauge” $\partial_\nu h^{\mu\nu} = 0$ and, in (4), $F^\tau = F^\tau_{\mu\nu} h^{\mu\nu}$ with $F^\tau_{\mu\nu} = \delta^\tau_\mu \partial_\nu$. $D_\alpha^{\mu\nu}$ is the operator which generates the gauge transformations in the graviton fluctuation $h^{\mu\nu}$. Given the infinitesimal coordinates transformation $x^{\mu'} = x^\mu + \kappa \xi^\mu$, the graviton field transforms as follows:

$$\begin{aligned} \delta h^{\mu\nu} & = D_\alpha^{\mu\nu} \xi^\alpha \\ & = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu} \partial_\alpha \xi^\alpha + \kappa (\partial_\alpha \xi^\mu h^{\alpha\nu} \\ & + \partial_\alpha \xi^\nu h^{\alpha\mu} - \xi^\alpha \partial_\alpha h^{\mu\nu} - \partial_\alpha \xi^\alpha h^{\mu\nu}). \end{aligned} \quad (6)$$

We Taylor-expand now the gravitational part of the action (1) to the second order in the gravitational pertur-

bation $h^{\mu\nu}(x)$ to obtain the graviton propagator. In the momentum space, the action which is purely quadratic in the gravitational field reads

$$\mathcal{L}^{(2)} = \frac{1}{4} h^{\mu\nu}(-k) K^{\mu\nu\rho\sigma} h^{\rho\sigma}(k) + \mathcal{L}_{\text{GF}}, \quad (7)$$

where \mathcal{L}_{GF} is the gauge fixing Lagrangian at the second order in the graviton field

$$\begin{aligned} \mathcal{L}_{\text{GF}} & = \frac{1}{4\xi} h^{\mu\nu}(-k) (\omega(k^2/\Lambda^2) k^2 P_{\mu\nu\rho\sigma}^{(1)}(k) \\ & + 2\omega(k^2/\Lambda^2) k^2 P_{\mu\nu\rho\sigma}^{(0-\omega)}(k)) h^{\rho\sigma}(k). \end{aligned} \quad (8)$$

The kinetic operator $K_{\mu\nu\rho\sigma}$ is defined by

$$\begin{aligned} K_{\mu\nu\rho\sigma} & := -\bar{h}_2(z) k^2 P_{\mu\nu\rho\sigma}^{(2)}(k) + \frac{3}{2} k^2 \bar{h}_0(z) P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) \\ & + \frac{k^2}{2} \bar{h}_0(z) \{ P_{\mu\nu\rho\sigma}^{(0-s)}(k) + \sqrt{3} [P_{\mu\nu\rho\sigma}^{(0-\omega s)}(k) \\ & + P_{\mu\nu\rho\sigma}^{(0-s\omega)}(k)] \}, \end{aligned}$$

and we have introduced the following notation:

$$\begin{aligned} \bar{h}_2(z) & := \beta - \beta_2 \kappa^2 \Lambda^2 z + \kappa^2 \Lambda^2 z h_2(z), \\ \bar{h}_0(z) & := \beta - 6\beta_0 \kappa^2 \Lambda^2 z + 6\kappa^2 \Lambda^2 z h_0(z), \end{aligned} \quad (9)$$

where $z := -\square_\Lambda$. Notice that in (7) \square_Λ has to be identified with the d'Alembertian operator in flat space-time $-\square_\Lambda^\eta$. We have used the gauge $F^\tau = \partial_\mu h^{\mu\tau}$ and introduced the projectors $P^{(2)}$, $P^{(1)}$, $P^{(0-s)}$, $P^{(0-s\omega)}$, and $P^{(0-\omega s)}$ [32] (see also Appendix B). Using the orthogonality properties of the projectors we can now invert the kinetic matrix in (7) and obtain the graviton propagator. In the following expression the graviton propagator is expressed in the momentum space according to the quadratic Lagrangian (7):

$$D_{\mu\nu\rho\sigma}(k) = D_{\mu\nu\rho\sigma}^{\xi=0}(k) + D_{\mu\nu\rho\sigma}^\xi(k), \quad (10)$$

where the propagator in the gauge $\xi = 0$ is

$$D_{\mu\nu\rho\sigma}^{\xi=0}(k) = \frac{-i}{(2\pi)^4} \frac{2}{k^2 + i\epsilon} \left(\frac{P_{\mu\nu\rho\sigma}^{(2)}(k)}{\bar{h}_2(k^2/\Lambda^2)} - \frac{2P_{\mu\nu\rho\sigma}^{(0-s)}(k)}{\bar{h}_0(k^2/\Lambda^2)} \right)$$

and $D_{\mu\nu\rho\sigma}^\xi(k)$ is the gauge-dependent part of the propagator.

We are now in a position to find an upper bound to the divergences in quantum gravity. We consider a particular theory in which the two general entire functions $h_i(z)$ introduced in the action have the following asymptotic exponential behavior:

$$\begin{aligned} h_2(z) & = \frac{\alpha(e^z - 1) + \alpha_2 z}{\kappa^2 \Lambda^2 z}, \\ h_0(z) & = \frac{\alpha(e^z - 1) + \alpha_0 z}{6\kappa^2 \Lambda^2 z}, \end{aligned} \quad (11)$$

for three general parameters α , α_2 , and α_0 .

Given the ultraviolet exponential behavior of the two functions $h_i(z)$, let us study the high energy behavior of the quantum theory. The ultraviolet behavior of the propagator in momentum space (actually, we will see that this is the correct scaling of the propagator at any energy scale), omitting the tensorial structure, reads

$$D(k) \sim \frac{e^{-k^2/\Lambda^2}}{k^2}. \quad (12)$$

But also the n -graviton interaction has the same scaling in the momentum space, since it can be written in the following schematic way:

$$\begin{aligned} \mathcal{L}^{(n)} &\sim h^n \square_\eta h h_i(-\square_\Lambda) \square_\eta h \\ &\rightarrow h^n \square_\eta h \frac{e^{-\square_\eta}}{\square_\eta} \square_\eta h + \dots, \end{aligned} \quad (13)$$

in which “ \dots ” indicates other interaction terms coming from the covariant d’Alembertian and $\square_\eta = \eta^{\mu\nu} \partial_\mu \partial_\nu$.

Placing an upper bound to the amplitude with L loops, we find

$$\begin{aligned} A(L) &\leq \int (d^4 p)^L \left(\frac{e^{-p^2/\Lambda^2}}{p^2} \right)^L (e^{p^2/\Lambda^2} p^2)^V \\ &= \int (d^4 p)^{4L} \left(\frac{e^{-p^2/\Lambda^2}}{p^2} \right)^{L-V} = \int (d^4 p)^{4L} \left(\frac{e^{-p^2/\Lambda^2}}{p^2} \right)^{L-1}. \end{aligned} \quad (14)$$

In the last step we used again the topological identity $I = V + L - 1$. The L -loop amplitude is UV finite for $L > 1$, and it diverges as “ p^4 ” for $L = 1$.

Thus, only one-loop divergences exist and the theory is super-renormalizable.¹ In these super-renormalizable theories of quantum gravity (SRQG) theories, the quantities β , β_2 , β_0 , and eventually the cosmological constant are renormalized, namely,

$$\begin{aligned} \mathcal{L}_{\text{Ren}} &= \mathcal{L} - \sqrt{-g} \left\{ \frac{\beta(Z-1)}{\kappa^2} R + \lambda(Z_\lambda - 1) \right. \\ &\quad \left. - \beta_2(Z_2 - 1) \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + \beta_0(Z_0 - 1) R^2 \right\}, \end{aligned} \quad (15)$$

in which all the coupling must be understood as renormalized at an energy scale μ . On the other hand, the functions h_i are not renormalized because the upper limit $A(L) \leq 4$ in (14).

We assume that the theory is renormalized at an energy scale μ_0 . If we want the bare propagator to possess no other gauge-invariant pole than the transverse physical graviton pole, we have to set

¹A local super-renormalizable quantum gravity with a large number of metric derivatives was for the first time introduced in Ref. [33].

$$\alpha = \beta(\mu_0), \quad \frac{\alpha_2}{\kappa^2 \Lambda^2} = \beta_2(\mu_0), \quad \frac{\alpha_0}{6\kappa^2 \Lambda^2} = \beta_0(\mu_0). \quad (16)$$

If the energy scale μ_0 is taken as the renormalization point, then $\bar{h}_2 = \bar{h}_0 = \beta(\mu_0) \exp(z)$, and only the physical massless spin-2 graviton pole occurs in the bare propagator. In the gauge $\xi = 0$, the propagator in (10) reads

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4} \frac{e^{-k^2/\Lambda^2}}{\alpha(k^2 + i\epsilon)} (2P_{\mu\nu\rho\sigma}^{(2)}(k) - 4P_{\mu\nu\rho\sigma}^{(0-s)}(k)).$$

If we choose another renormalization scale μ , then the bare propagator acquires poles; however, these poles cancel in the dressed physical propagator, because the renormalization group invariance preserves unitarity in the dressed physical propagator at any energy scale and no other physical pole emerges at any other scale.

III. NONCOMMUTATIVE SPACE-TIME AND QUANTUM GROUPS

We unveil in this section the link between one of the SRQG theories analyzed above and the quantum-group structure of space-time symmetries proper to noncommutative space-times. The key point is that the two-point function of the super-renormalizable theory can be reexpressed in such a way to exhibit the hidden quantum-group-like structure in the momentum space through the Fourier transform of $\bar{h}_i(-\square_\Lambda)$ ($i = 0, 2$). We present, in particular, two procedures accounting for this result and leading to a particularly simple example of noncommutativity that is well known and has been studied mathematically in depth, namely, the θ -Minkowski space-time with its associated θ -Poincaré Hopf algebra of symmetries. We then move to scrutinize possible generalizations within the framework of space-times with noncommutativity of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu} (\hat{X}^\alpha)$$

and conclude with the theorem that, focusing on associative space-time algebras, there is only one possible choice of $\bar{h}_n(\square_\Lambda)$ compatible with a nontrivial Hopf-algebra structure of space-time symmetries.

A. Emergence of the quantum θ structure

Starting from the expression of the two-point function (10), we easily obtain, within an appropriate choice of the gauge, the scalar structure for the graviton propagator to be

$$D_{\mu\nu\rho\sigma}^{\xi=0}(k) \sim \frac{-i}{(2\pi)^4} \frac{e^{-H(k^2/\Lambda^2)}}{k^2 + i\epsilon} \times \text{TS}, \quad (17)$$

where we start considering $\bar{h}_2(z) = \bar{h}_0(z) := \exp H(z)$ to be as general as possible and where $H(k^2/\Lambda^2)$ is an entire function of the argument. TS means “tensorial structure.” In order to make explicit the mechanism underlying the

result we are going to show, we focus in this first part of the section on a Euclidean 2D space-time and then consider a phase-space noncommutativity involving \hat{X}_i space-coordinate operators and \hat{P}_j momentum operators characterized by the following Lie brackets:

$$[\hat{X}^i, \hat{X}^j] = i\theta^{ij}, \quad [\hat{X}^i, \hat{P}_j] = i\delta_j^i, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad (18)$$

namely, the Heisenberg noncommutativity between conjugated variables and the Moyal-plane noncommutativity between space coordinates. It has been shown in Refs. [34,35] that, for a particular choice of θ^{ij} involving noncommutativity in two of the space coordinates (e.g., say, $\theta^{3i} = -\theta^{i3} = 0$ and $\theta^{ab} = \theta\epsilon^{ab}$ with $a, b = 1, 2$), it is possible to cast the space noncommutativity on the 2D plane in terms of ladder operators and coherent states diagonalizing these latter. For instance, assuming $\theta^{ab} = \theta\epsilon^{ab}$, one can define $\sqrt{2}\hat{Z} = \hat{X}^1 + i\hat{X}^2$ and $\sqrt{2}\hat{Z}^\dagger = \hat{X}^1 - i\hat{X}^2$. These new operators fulfill the algebra $[\hat{Z}, \hat{Z}^\dagger] = \theta$, and their eigenstates are labeled as $|z\rangle$ and are such that $\hat{Z}|z\rangle = z|z\rangle$ and $\langle z|\hat{Z}^\dagger = \langle z|\bar{z}$, namely,

$$|z\rangle = \exp\left(-\frac{z\bar{z}}{\theta}\right)\exp\left(-\frac{z}{\theta}\hat{Z}^\dagger\right)|0\rangle. \quad (19)$$

These coherent states of the noncommutative plane satisfy the completeness relation $\int dz d\bar{z} |z\rangle\langle z| = \pi\theta$. In quantum field theory, the basic noncommutative variables are fields and their conjugated momenta. Coordinates are represented as labels and are commutative. Differently, for a quantum field theory grounded on (18), we must consider the expectation value of fields over coherent states (19) encoding space noncommutativity, in order to relate quantization results to standard commutative quantum field theory. Indeed, within the context of (18), noncommutativity emerges already at the level of classical fields not subjected to the canonical quantization of the symplectic phase space. In order to make contact with classical fields, we must therefore recur to the procedure of extracting the expectation value over semiclassical coherent states, which we called here $|z\rangle$, following the lines of Refs. [34,35]. Noncommuting coordinates, which may be treated as operators within the scheme of (18), are evaluated on the coherent states (19), as we show explicitly in the forthcoming relation, Eq. (20). We should now consider that noncommutative functions can be Fourier expanded by using complex exponential bases, such as

$$\exp\left(i\sum_j p_j \hat{X}^j\right) \quad \text{or} \quad \prod_j \exp(ip_j \hat{X}^j).$$

The two bases that we are mentioning among many others are equal on the standard commutative 2D space but differ among each other within the case specified by (18), because of the underlying noncommutativity of \hat{X}^j coordinates. This basic fact allows us to expand the expectation value over coherent states of noncommutative fields on the

expectation value $\langle z|\exp(ip_j \hat{X}^j)|z\rangle$ of the Fourier basis elements $\exp(i\sum_j p_j \hat{X}^j)$, yielding the crucial result [34,35]

$$\begin{aligned} \langle z|\exp(ip_1 \hat{X}^1 + ip_2 \hat{X}^2)|z\rangle \\ = \langle z|\exp(ip_+ \hat{Z}^\dagger)\exp(ip_- \hat{Z})\exp\left(\frac{p_- p_+}{2}[\hat{Z}^\dagger, \hat{Z}]\right)|z\rangle, \end{aligned} \quad (20)$$

in which the Baker-Campbell-Hausdorff formula has been used (see Appendix A) and the quantities $\sqrt{2}p_\pm = (p_1 \pm ip_2)$ have been defined. Notice also that shrinking to zero the deformation parameter θ accounts for considering the ‘‘classical limit’’ toward standard commutative quantum field theory.

We can now generalize this procedure to a noncommutative 4D space-time and find an energy-momentum exponential-dumping behavior as in (17), but only if $H(k^2/\Lambda^2) \sim k^2$. We start considering a phase space involving space-time coordinates and conjugated momenta of the type

$$\begin{aligned} [\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}, \quad [\hat{X}^\mu, \hat{P}_\nu] = i\delta_\nu^\mu, \\ [\hat{P}_\mu, \hat{P}_\nu] = 0. \end{aligned} \quad (21)$$

We recall that, for $\theta_{\mu 0} \neq 0$, any Lorentzian theory constructed on (21) is nonunitary [36]. For the moment we disregard this problem, perform a Wick rotation to the Euclidean space-time, and show that, assuming the only nonzero components are $\theta^{03} = -\theta^{30} \equiv \xi \neq 0$ and $\theta^{12} = -\theta^{21} \equiv \theta \neq 0$, it is possible to give sense to a graviton propagator whose scalar structure is expressed by (17). Let us see here below how it is possible to achieve this result. Together with the ladder operators \hat{Z} and \hat{Z}^\dagger , we consider the choice of $\theta^{\mu\nu}$ specified above and of another class of ladder operators involving \hat{X}_1 and \hat{X}_3 coordinates, namely, $\sqrt{2}\hat{T} = \hat{X}^0 + iX^3$ and $\sqrt{2}\hat{T}^\dagger = \hat{X}^0 - iX^3$. It follows that $[\hat{T}, \hat{T}^\dagger] = \xi$ and, from the type of space-time noncommutativity we assumed above, that $[\hat{T}, \hat{Z}] = [\hat{T}, \hat{Z}^\dagger] = 0$; i.e. the two sectors of ladder operators can be simultaneously diagonalized. The coherent states for the \hat{T} sector can be constructed in the same way as for the \hat{Z} sector, yielding eigenstates $|t\rangle$ such that $\hat{T}|t\rangle = t|t\rangle$ and $\langle t|\hat{T}^\dagger = \langle t|\bar{t}$, namely,

$$|t\rangle = \exp\left(-\frac{t\bar{t}}{\theta}\right)\exp\left(-\frac{t}{\theta}\hat{T}^\dagger\right)|0\rangle, \quad (22)$$

which is provided with the completeness relation $\int dt d\bar{t} |t\rangle\langle t| = \pi\theta$. We can therefore consider the coherent states $|z, t\rangle = |z\rangle|t\rangle$. The relevant formula for expanding quantum fields on a Fourier basis is given by the manipulation of $\langle z, t|\exp(ip_\mu \hat{X}^\mu)|z, t\rangle$. This is the expectation value over the coherent state $|z, t\rangle$ of wave exponentials entering the Fourier-mode expansion of quantum fields on noncommutative space-time and yields the crucial result

$$\begin{aligned}
& \langle z, t | \exp(ip_\mu \hat{X}^\mu) | z, t \rangle \\
&= \langle z | \exp(ip_+ \hat{Z}^\dagger) \exp(ip_- \hat{Z}) \exp\left(\frac{p_- p_+}{2} [\hat{Z}^\dagger, \hat{Z}]\right) | z \rangle \\
&\quad \times \langle t | \exp(i\tilde{p}_- \hat{T}^\dagger) \exp(i\tilde{p}_+ \hat{T}) \exp\left(\frac{\tilde{p}_- \tilde{p}_+}{2} [\hat{T}^\dagger, \hat{T}]\right) | t \rangle.
\end{aligned} \tag{23}$$

In (23) we have introduced the quantities $\sqrt{2}\tilde{p}_\pm = (p_0 \pm ip_3)$ and used the Baker-Campbell-Hausdorff formula. Once again we emphasize that, having performed a Wick rotation, our analysis focuses on the Euclidean signature $\text{sgn}(\eta_{\mu\nu}) = (+, +, +, +)$, for which problems of unitarity do not appear. Indeed, thanks to the analysis that has been carefully developed in Refs. [36,37] about unitarity being preserved in the Lorentzian case once $\theta^{0i} = 0$, based on the symmetry arguments we can convince ourselves that the Euclidean theory will not suffer for the lack of unitarity.² Among the super-renormalizable theories previously considered, there exists one that can be recast when $\xi = \theta$ as a theory over a noncommutative space-time, with non-commutativity of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}. \tag{24}$$

Indeed, when $\xi = \theta = 1/\Lambda$, the damping factor in (17), specialized to the case $H(z) = z$, is automatically recreated from the kinematical manipulations reviewed in (23). Moreover, the choice of $\xi = \theta$ preserves Lorentz covariance when we Wick rotate back to the noncommutative space-time with a Lorentzian signature. It is well known in the literature that the algebra of symmetries for the non-commutative space-time in (24) is a twisted Hopf algebra \mathcal{P}_θ called θ -Poincaré Hopf algebra and that a theory with a covariant action gives rise in this framework to conserved Noether charges (see e.g. [38], which extends the work done in [39] for the case of the κ -Poincaré Hopf algebra). Therefore, θ -Poincaré symmetry truly represents an external symmetry (Hopf) algebra, and the same holds for its Lorentz subalgebra. Moreover, we can construct a covariant theory under the action of the generators of \mathcal{P}_θ that has a new scale invariant, i.e. θ . We will discuss later these implications. For the moment, we want just to make clear the point that the θ -Poincaré symmetry does not imply a breakdown or a modification of the Poincaré symmetries in the algebraic sector, or better at the level of the Hilbert

²The link we are exploring between SRQG theories and theories of gravity on θ -Minkowski noncommutative space-time can be recovered on a solid basis focusing on the Euclidean case, in which a unitary and Lorentz covariant quantum field theory can be constructed on θ -Minkowski. Nevertheless, thanks to the work in Refs. [36,37], we could easily reach in the Lorentzian case with $\theta^{0i} = 0$ conclusions similar to the ones we have been discussing in this paper. Anyway, we prefer to leave this task to further and detailed investigations, in which also the delicate case $\theta^{0i} = 0$ will be treated.

space of the theory, but rather an enlargement of the Lie-algebra-type Poincaré symmetry, which can be recast as a trivial Hopf algebra, to a nontrivial Hopf-algebra, precisely a θ -Poincaré Hopf-algebra in our case. This latter reflects³ in the deformation of the trivial coalgebraic structure of the Poincaré algebra (Leibnitz rule) to a twisted coproduct structure (non-cocommutativity or θ -deformed Leibnitz rule), leaving unchanged the algebraic structure of the Hopf algebra.⁴ In other words, the enlargement of the symmetry structure we are dealing with will effect a modification of the Fock space of the theory (see, for instance, the last reference of [39]), without changing the Hilbert space of the theory. This is also reflected in the fact that the quantum groups structure emerges at the tree level, through the evaluation of the graviton propagator. We will come back later, in Sec. III D, to the issue of the relation between the θ -Poincaré and the Poincaré Hopf algebras.

We complete the discussion on the emergence of the θ -Poincaré symmetry by noticing that the tensorial structure in (10) does not affect the result of our analysis, as indeed this can be made fully consistent with the θ -Poincaré symmetry of this theory [41]. We emphasize that this property relies on a remarkable feature of the θ -Poincaré quantum groups, namely, that the Lorentz subalgebra and the whole Poincaré sector are unmodified with respect to the θ -Poincaré Hopf algebra, as stated here above and as will be clarified in Sec. III D. This property allows us to define linear Lorentz transformation and conservation laws following the standard recipe. From now on, we will mention the Lorentz sector, without specifying that it belongs to the θ -Poincaré or Poincaré algebra Hopf algebra.

B. A different philosophy to unveil \mathcal{P}_θ

The emergence of the θ -Poincaré symmetry structure does not rely on the particular procedure we adopted in the preceding subsection. For instance, we could have chosen to adopt the ‘‘Weyl system’’ procedure [42], as it has been done in Ref. [38] for the purpose of analyzing the symmetry structure of a scalar field theory on θ -Minkowski space-time. The Weyl map Ω associates to any function $f(\hat{x})$ of θ -Minkowski space-time an auxiliary commutative function $f^{(c)}(x)$. The easiest way of implementing this map is to consider the Fourier transform $\tilde{f}(p)$ of $f(\hat{x})$ and then apply on the Fourier modes the Weyl map, namely,

$$\begin{aligned}
f(\hat{x}) &= \Omega(f^{(c)}(x)) \equiv \Omega\left(\int d^4 p \tilde{f}(p) e^{ip\hat{x}}\right) \\
&= \int d^4 p \tilde{f}(p) :e^{ip\hat{x}}:,
\end{aligned} \tag{25}$$

³We thank the referee for suggesting to us to clarify this point.

⁴Formally, both the θ -Poincaré quantum group and the Poincaré algebra are Hopf algebra. But the latter one is usually referred to as ‘‘trivial’’ Hopf algebra, because of the standard cocommutativity in the coalgebra sector [40].

in which “ $\cdot \cdot \cdot$ ” denotes an ordering of the noncommutative coordinates associated to Ω . The inverse of the Weyl map, namely, Ω^{-1} , is also well defined and is called the Wigner map. Our Wigner map is expressed by the semiclassical limit of the measurement procedure involving coherent states $|z, t\rangle$, namely,

$$\Omega^{-1}(\dots) = \langle z, t | \dots | z, t \rangle. \quad (26)$$

This procedure also provides a physical picture in our context of the Wigner and Weyl maps. As we reminded above, because of the peculiar features of space-time noncommutativity, a Weyl map selects a particular normal ordering for the space-time coordinates. Suppose we choose the “semiclassical-state” ordering, which is defined in the $x_1 - x_2$ plane by the following action of the Weyl map on the Fourier-modes basis elements:

$$\begin{aligned} \Omega|_{x-y}(e^{ip_1x_1+ip_2x_2}) &= \Omega|_{x-y}(e^{ip_+\bar{z}+ip_-\bar{z}}) = e^{ip_+\hat{Z}^\dagger+ip_-\hat{Z}} \\ &= e^{ip_+\hat{Z}^\dagger} e^{ip_-\hat{Z}} e^{(p_--p_+/2)[\hat{Z}^\dagger, \hat{Z}]} \end{aligned} \quad (27)$$

On the whole θ -Minkowski space-time (24), in which $\theta_{03} = \theta_{12} = \theta = -\theta_{30} = -\theta_{21}$ and $\theta_{13} = \theta_{23} = \theta_{01} = \theta_{02} = 0$, we might consider the definition of semiclassical-state ordering and the related Weyl map by adopting the coordinates \hat{T} and \hat{T}^\dagger :

$$\begin{aligned} \Omega(e^{ip_\mu x^\mu}) &= e^{ip_+\hat{Z}^\dagger} e^{ip_-\hat{Z}} e^{(p_--p_+/2)[\hat{Z}^\dagger, \hat{Z}]} \\ &\quad \times e^{i\bar{p}_-\hat{T}^\dagger} e^{i\bar{p}_+\hat{T}} e^{(\bar{p}_--\bar{p}_+/2)[\hat{T}^\dagger, \hat{T}]} \end{aligned} \quad (28)$$

Now define the integration map \int on the noncommutative θ -Minkowski space-time as the map such that

$$\int \Omega(f(x))\Omega(g(x)) = \int d^4p \tilde{f}(p)\tilde{g}(-p)e^{-\theta(p_\mu p^\mu)^2}. \quad (29)$$

A quantum theory of noncommutative fields can now be constructed by following the same steps as in Ref. [43], namely, considering an expansion of the quantum field, fulfilling a Lorentz covariant equation of motion, on a noncommutative Fourier basis

$$\begin{aligned} \Psi_r(\hat{X}) &= \int d\mu_p [a_p \Omega(e^{-ip_\mu x^\mu}) + a_p^\dagger \Omega(e^{ip_\mu x^\mu})] \\ &= \int \frac{d^3\vec{p}}{2p_0} [a_{\vec{p}} \Omega(e^{-ip_\mu x^\mu}) + a_{\vec{p}}^\dagger \Omega(e^{ip_\mu x^\mu})], \end{aligned} \quad (30)$$

and then imposing braiding relations on the ladder operators by means of the bialgebra twisting element

$$\mathcal{F}_\theta = \exp(\frac{1}{2}\theta^{\mu\nu}P_\mu \otimes P_\nu).$$

Notice that in (30) we have introduced the notation for the Lorentz invariant measure $d\mu_p = d^4p \delta(p^2) = d^3\vec{p}/2p_0$. The requirement of compatibility of the covariant action of symmetries on tensor product of states with the tensor product of state on which symmetries have already acted yields indeed the braiding in the multiparticle states. This peculiar feature of noncommutative quantum field theory enjoying θ -Poincaré symmetries originates by the action of the twisting element, which we define here by means of $\mathcal{F}_\theta \triangleright (|p\rangle \otimes |q\rangle) = \mathcal{F}_\theta(p, q)|p\rangle \otimes |q\rangle$, i.e. through

$$\begin{aligned} a_p a_q &= \mathcal{F}_\theta^{-2}(q, p)a_q a_p, \\ a_p a_q^\dagger &= \mathcal{F}_\theta^{-2}(-q, p)a_q^\dagger a_p + 2p_0 \delta^4(p - q), \\ a_p^\dagger a_q^\dagger &= \mathcal{F}_\theta^{-2}a_q^\dagger a_p^\dagger. \end{aligned} \quad (31)$$

The vacuum state of the Fock space is defined by $a_p|0\rangle = 0$ and states of the Hilbert by $|p\rangle = a_p^\dagger|0\rangle$. The second quantization procedure hence defined can be applied to the geometric two-tensor field, as defined in Ref. [41] and perturbative-expanded as in Ref. [2]. The second relation in (31) is what we need in order to compute the graviton propagator in the noncommutative theory, namely,

$$\begin{aligned} D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha) &= \int d\mu_p d\mu_k \{ \langle 0 | (a_p a_k^\dagger \Omega(e^{-ip_\mu x^\mu}) \Omega(e^{ik_\mu y^\mu}) \\ &\quad + a_p^\dagger a_k \Omega(e^{ip_\mu x^\mu}) \Omega(e^{-ik_\mu y^\mu})) | 0 \rangle \theta(\hat{X}_0 - \hat{Y}_0) \\ &\quad + [\hat{X}_0 \leftrightarrow \hat{Y}_0 \text{ and } x \leftrightarrow y] \} \times \text{TS} \\ &= \int d\mu_p \times \text{TS} \times \Omega(e^{ip_\mu(x^\mu - y^\mu)}) \times \text{pole structure}. \end{aligned}$$

We emphasize that $D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha)$ differs from the expectation value (on the coherent states $|z, t\rangle$) of the propagator of the quantum theory, namely, $D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$. We would have obtained $D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$ if we had followed the same strategy as in Refs. [34,35]. In our notation, in terms of the Wigner map, this accounts for

$$\Omega^{-1}(D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha)) = D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$$

that leads to the graviton propagator

$$\begin{aligned} D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha) &= \int d\mu_p d\mu_k \{ \Omega^{-1}[\langle 0 | (a_p a_k^\dagger \Omega(e^{-ip_\mu x^\mu}) \Omega(e^{ik_\mu y^\mu}) \\ &\quad + a_p^\dagger a_k \Omega(e^{ip_\mu x^\mu}) \Omega(e^{-ik_\mu y^\mu})) | 0 \rangle \theta(\hat{X}_0 - \hat{Y}_0)] \\ &\quad + \Omega^{-1}[x \leftrightarrow y] \} \times \text{TS} \\ &= \int d^4p \frac{e^{-p^2/\Lambda^2}}{p^2} \times \text{TS}. \end{aligned} \quad (32)$$

The Fourier transform of the graviton propagator sketched in (31) by using the ordering introduced in (28) turns out to give the same value determined henceforth at the beginning of this section, in (17).

The procedure incorporated in this second section is more general than the one based on the expectation value on coherent states and must be, in general, considered as distinct. Nevertheless, this reduces to the one exposed in the preceding section whenever we consider (26) as a concrete definition for the Wigner map.

C. Uniqueness of the link between quantum groups and SRQG and falsifiability of the theory

In this section, we prove a simple theorem stating the uniqueness of the link between quantum groups and

SRQG. Specifically, we prove that the only nontrivial Hopf algebra connected to SRQG is the θ -Poincaré Hopf algebra and that this latter selects only one among the many possible theories [namely, the theory defined by the choice $\tilde{h}_2(z) = \tilde{h}_0(z) = \exp z$, with $z = -\square/\Lambda^2$] described in Ref. [2]. Thus, in what follows, we scrutinize the possibility of generalizing results previously exposed to a wider class of noncommutative space-times and prove the impossibility of achieving this goal if we decide not to relax the requirement of associativity for the noncommutative space-time algebra.

The natural place in order to seek for the generalization of previous results is represented by (20) and the implementation within it of the Baker-Campbell-Hausdorff (BCH) formula and of its inverse formula, the Zassenhaus formula. Suppose indeed we consider in (17) the integer function to be $H(k^2/\Lambda^2) = c_1 k^2/\Lambda^2 + c_2 (k^2/\Lambda^2)^2$. We address the search for a suitable Lie algebra reproducing this structure for $H(-\square/\Lambda^2)$ in terms of generic functions θ_{34} , depending⁵ on \hat{Z} and \hat{Z}^\dagger , and θ_{12} depending on \hat{T} and \hat{T}^\dagger . Commutation relations for the ladder operators now read

$$[\hat{Z}, \hat{Z}^\dagger] = \theta_{34}(\hat{Z}, \hat{Z}^\dagger), \quad [\hat{T}, \hat{T}^\dagger] = \theta_{12}(\hat{T}, \hat{T}^\dagger). \quad (33)$$

By maintaining unchanged the definition of \hat{Z} and \hat{T} , formulas (33) yield a space-time noncommutativity of the form

$$\begin{aligned} [\hat{X}^1, \hat{X}^2] &= i\tilde{\theta}_{12}(\hat{X}_1, \dots, \hat{X}_4), \\ [\hat{X}^3, \hat{X}^4] &= i\tilde{\theta}_{34}(\hat{X}_1, \dots, \hat{X}_4). \end{aligned} \quad (34)$$

Notice that in general both the θ -Minkowski type and κ -Minkowski [44] type of noncommutativity are present in the expansion of the functions $\tilde{\theta}_{12}$ and $\tilde{\theta}_{34}$. Such a copresence of space-time noncommutativities has been considered in the literature [45] in light of its relation with string-theory scenarios.⁶ But (34) is not sufficient in order to ensure the desired behavior for the Fourier trans-

⁵We recall that the BCH formula and its inverse have been developed by considering only Lie-algebra cases. Thus, θ_{34} and θ_{12} could be rigorously expanded only up to linear order in the generators of the algebra.

⁶Expansion of Eqs. (34) makes sense up to second order in a scale κ having the dimension of inverse energy. Following dimensional arguments, for a space-time noncommutativity of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}(\hat{X}^\alpha),$$

the only class of deformations of space-time having a classical limit for $\kappa \rightarrow 0$ are of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\kappa^2 \theta_0^{\mu\nu} + i\kappa \theta_{(1)\rho}^{\mu\nu} \hat{X}^\rho + \theta_{(1)\rho\sigma}^{\mu\nu} \hat{X}^\rho \hat{X}^\sigma,$$

with $\theta_0^{\mu\nu}$, $\theta_{(1)\rho}^{\mu\nu}$, and $\theta_{(1)\rho\sigma}^{\mu\nu}$ dimensionless quantities. Within this class of deformations of space-time, previous expansion describes (for open string first-quantized in $D = 10$) noncommutative coordinates on D -branes providing the localizations of the ends of the strings.

form of the integer function $H(k^2/\Lambda^2)$ appearing in the graviton propagator calculation. In other words, we cannot achieve the Fourier transform

$$H(k^2/\Lambda^2) = c_1 k^2/\Lambda^2 + c_2 (k^2/\Lambda^2)^2 \quad (35)$$

on a perturbed background of the form (5) if we still require the noncommutative algebra to be associative. We can prove this theorem by considering that two requirements should be fulfilled as necessary conditions in order to add a term like $c_2 (k^2/\Lambda^2)^2$ in (17). The first one reads

$$\frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}} = \frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}^\dagger} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T}} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T}^\dagger} = 0 \quad (36)$$

and ensures that momenta are not redefined at linear order in $\sqrt{\theta}$, i.e. for linear Planck mass corrections. The second condition is

$$\frac{\partial^2 \tilde{\theta}_{12}}{\partial \hat{Z} \partial \hat{Z}^\dagger} = \frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}^\dagger \partial \hat{Z}} = \frac{\partial^2 \tilde{\theta}_{34}}{\partial \hat{T}^\dagger \partial \hat{T}} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T} \partial \hat{T}^\dagger} = \theta \quad (37)$$

and ensures the existence of two terms summing in $H(p^2/\Lambda^2)$ within (17) that are $\theta(p_1^2 + p_2^2)^2$ from the $\hat{T} - \hat{T}^\dagger$ sector and $\theta(p_3^2 + p_4^2)^2$ from the $\hat{Z} - \hat{Z}^\dagger$ sector. But, once summed, these contributions are not sufficient to recreate a covariant $(p^2)^2$ term, which instead would come from the Fourier transform of \square^2 as it appears in the second term of (35). Therefore, we should consider now interaction between the two sectors $\hat{T} - \hat{T}^\dagger$ and $\hat{Z} - \hat{Z}^\dagger$, which would now determine the appearance of mixed terms in p_2^2 and p_3^2 , from one side, and p_4^2 and p_1^2 from the other side. In the Euclidean space-time, we now label operators within the $\hat{Z} - \hat{Z}^\dagger$ sector as

$$\hat{Z} = \hat{Z}_{34}, \quad \hat{Z}^\dagger = \hat{Z}_{34}^\dagger \quad (38)$$

and momenta as

$$p_+ = p_{34}, \quad p_- = p_{34}^*. \quad (39)$$

In the $\hat{T} - \hat{T}^\dagger$ sector, operators are now labeled as

$$\hat{T} = \hat{Z}_{12}, \quad \hat{T}^\dagger = \hat{Z}_{12}^\dagger, \quad (40)$$

while momenta are labeled as follows:

$$\tilde{p}_+ = p_{12}, \quad \tilde{p}_- = p_{12}^*. \quad (41)$$

We emphasize that, in order to obtain a dumping exponential phase term $\exp[-2\theta^2(p_1^2 + p_2^2)(p_3^2 + p_4^2)]$ multiplying the other dumping phase term $\exp\{-\theta^2[(p_1^2 + p_2^2)^2 + (p_3^2 + p_4^2)^2]\}$, and hence recreating a covariant exponential-dumping phase factor $\exp-\theta^2(p^2)^2$, new conditions must be fulfilled about the noncommutativity in the $\hat{X}_1 - \hat{X}_3$ plane and in the $\hat{X}_1 - \hat{X}_4$ plane, as well as in the $\hat{X}_2 - \hat{X}_3$ and $\hat{X}_2 - \hat{X}_4$ planes. These can be derived by looking at the exponential

$$\begin{aligned} &\exp i(\hat{X}_1 p_1 + \hat{X}_2 p_2 + \hat{X}_3 p_3 + \hat{X}_4 p_4) \\ &= \exp i(\hat{Z}_{34}^\dagger p_{34}^* + \hat{Z}_{34} p_{34} + \hat{Z}_{12}^\dagger p_{12}^* + \hat{Z}_{12} p_{12}) \end{aligned} \quad (42)$$

and at its decomposition by means of the Zassenhaus formula and then imposing that

$$e^{i(p_1\hat{X}_1+p_2\hat{X}_2+p_3\hat{X}_3+p_4\hat{X}_4)} = e^{i(p_1\hat{X}_1+p_2\hat{X}_2)} e^{i(p_3\hat{X}_3+p_4\hat{X}_4)} e^{(1/2)\Gamma[p_1,p_2,p_3,p_4]} e^{(-1/6)\{2\Phi[p_1,p_2,p_3,p_4]+\Sigma[p_1,p_2,p_3,p_4]\}} \\ \times e^{(-1/24)\{\Xi[p_1,p_2,p_3,p_4]+3\Psi[p_1,p_2,p_3,p_4]+3\Theta[p_1,p_2,p_3,p_4]\}}, \quad (43)$$

with

$$\begin{aligned} \Gamma[p_1, p_2, p_3, p_4] &= [(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)], \\ \Phi[p_1, p_2, p_3, p_4] &= [(p_3\hat{X}_3 + p_4\hat{X}_4), [(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)]], \\ \Sigma[p_1, p_2, p_3, p_4] &= [(p_1\hat{X}_1 + p_2\hat{X}_2), [(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)]], \\ \Xi[p_1, p_2, p_3, p_4] &= [[[(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)], (p_1\hat{X}_1 + p_2\hat{X}_2)], (p_1\hat{X}_1 + p_2\hat{X}_2)], \\ \Psi[p_1, p_2, p_3, p_4] &= [[[(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)], (p_1\hat{X}_1 + p_2\hat{X}_2)], (p_3\hat{X}_3 + p_4\hat{X}_4)], \\ \Theta[p_1, p_2, p_3, p_4] &= [[[(p_1\hat{X}_1 + p_2\hat{X}_2), (p_3\hat{X}_3 + p_4\hat{X}_4)], (p_3\hat{X}_3 + p_4\hat{X}_4)], (p_3\hat{X}_3 + p_4\hat{X}_4)]. \end{aligned}$$

For arbitrary values of p_μ , the requirement on the Lorentz invariance of the algebraic sector (and thus on the Lorentz invariance of the Fourier space) implies from enforcing $\Gamma[p_1, p_2, p_3, p_4] = 0$ that

$$[\hat{X}_1, \hat{X}_3] = [\hat{X}_1, \hat{X}_4] = [\hat{X}_2, \hat{X}_3] = [\hat{X}_2, \hat{X}_4] = 0. \quad (44)$$

Namely, Lorentz invariance requires that the only type of affordable noncommutativity is the one we considered above on the $\hat{X}_1 - \hat{X}_2$ plane and on the $\hat{X}_3 - \hat{X}_4$ plane. We would have reached the same conclusion from (44) by imposing Lorentz invariance on the Fourier space and hence the simultaneous vanishing of $\Phi[p_1, p_2, p_3, p_4]$ and $\Sigma[p_1, p_2, p_3, p_4]$. Notice that also (36) and (37), once expressed in terms of the \hat{X}^μ operator, would lead to the same type of inconsistencies. Finally, requirements in (44) impose the vanishing of $\Xi[p_1, p_2, p_3, p_4]$, $\Psi[p_1, p_2, p_3, p_4]$, and $\Theta[p_1, p_2, p_3, p_4]$. As a consequence, it manifests the impossibility of recovering a term which goes like $(k^2/\Lambda)^2$ for $H(k^2/\Lambda^2)$ in (17), if we start from a Lie-algebra type of noncommutativity. This result could be in part anticipated. We know indeed that only the twisted θ -Poincaré Hopf algebra at the same time preserves Lorentz symmetry, at least in the algebraic sector and in the Fourier space, and consistently realizes the associativity in the module algebra (of space-time coordinate functions).

Finally, the argument developed here above and based on the choice of the particular SRQG theory defined by (35) can be repeated for any entire function $H(z)$ and, thus, for any generic SRQG theory. This ends our proof about the uniqueness of the link between SRQG and nontrivial Hopf algebra, specifically the θ -Poincaré quantum group. It is not overwhelming to emphasize that the uniqueness, i.e. having fixed $H(z)$ to a unique function, traces back to the definition of a unique theory of SRQG among the many allowed in the framework of Ref. [2]. We also emphasize that such a result relies on the construction of the phase space with Heisenberg-type noncommutativity between space-time and momenta; thus, it is consistent with the associativity of the space-time coordinates considered in

previous sections.⁷ Extending the analysis to deformed phase space, and hence to nonassociative space-times, would not have allowed us to conclude with the same statement.

D. Poincaré and θ -Poincaré symmetries

In this section, we want to recall some basic facts about the Poincaré and the θ -Poincaré Hopf algebras and show explicitly their relation. We start by reviewing how the Poincaré Lie algebra can be recast as a Hopf algebra. Before doing that, we recall the Lie-multiplication rules for the Poincaré algebra \mathcal{P} , whose elements are denoted as P_μ (generators of translations) and $M_{\mu\nu}$ (generators of Lorentz transformations):

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} \\ &\quad + \eta_{\nu\beta}M_{\mu\alpha}), \\ [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu). \end{aligned} \quad (45)$$

The Poincaré Lie algebra \mathcal{P} can be easily promoted to a Hopf algebra if we introduce the same basic definitions. In particular, in order to deal with deformation of the Leibniz rule, it is convenient from an abstract algebraic point of view to introduce the ‘‘coproduct’’ map Δ as the application from elements of the algebra \mathcal{P} to elements of the tensor product $\mathcal{P} \otimes \mathcal{P}$, namely, $\Delta: \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$. If the module space, i.e. the space on which generators of the algebra act, is thought to be the space of functions on commutative space, it is of immediate evidence that $\Delta(1) = 1 \otimes 1$, and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ for each generator $Y = \{M_{\mu\nu}, P_\beta\}$ of the Poincaré Lie algebra. This last property, which plays a key role in allowing a description of the symmetries at the simple Lie-algebraic level—without any true need to resort to a full Hopf-algebra description—is

⁷A notable example of nonassociative space-time is the Snyder noncommutative space-time (see e.g. Ref. [46] and references therein).

actually connected with the commutativity of functions in Minkowski space-time \mathcal{M} and thus fully reflects the Leibnitz rule. Consider in fact two generic functions f and g . From $f \cdot g = g \cdot f$, one easily finds for each $U \in \mathcal{U}(\mathcal{P})$ that⁸ Δ is symmetric, i.e. $U_{(1)} \otimes U_{(2)} = U_{(2)} \otimes U_{(1)}$ for all U . Adopting math jargon, Δ is ‘‘cocommutative.’’ We say that a cocommutative coproduct is, in some sense, trivial in order to emphasize its simple structure with respect to the coproduct of the generators of the symmetry transformations in a noncommutative space-time. These latter coproducts may be instead ‘‘non-cocommutative.’’

The axiomatic definition of a Hopf algebra requires also the definition of a ‘‘counit’’ map $\epsilon: \mathcal{P} \rightarrow \mathbb{C}$, such that for any function $f(x)$ it results that the action $\int d^4x U f(x) = \epsilon(U) \int d^4x f(x)$ can be defined. It is straightforward to verify that $\epsilon(1) = 1$ and $\epsilon(Y) = 0$. We may also define the ‘‘unit map’’ $\eta: \mathbb{C} \rightarrow \mathcal{P}$ and the ‘‘multiplication map’’ $m: \mathcal{P} \otimes \mathcal{P} \rightarrow \mathcal{P}$. Another ingredient we might add to this construction concerns how to construct generators of inverse transformations. At this purpose, we can define $S(1) = 1$, $S(Y) = -Y$ for each generator $Y \in \mathcal{P}$, and $S(UU') = S(U')S(U)$ for each element of the corresponding enveloping algebra. We then obtain a map satisfying $U_{(1)}S(U_{(2)}) = S(U_{(1)})U_{(2)} = \epsilon(U)$. The map S so far introduced is called an ‘‘antipode.’’ This makes \mathcal{P} a Hopf algebra provided that some axioms are satisfied.⁹

The universal enveloping algebra of a Lie algebra is then equivalent to a Hopf algebra generated by the primitive elements Y :

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \epsilon(Y) = 0, \quad S(Y) = -Y.$$

This algebraic structure is usually referred to as a ‘‘trivial Hopf algebra.’’

The new symmetry structure, whose emergence in the framework of SRQG theories we are claiming, by looking

⁸ $\mathcal{U}(\mathcal{P})$ represents the universal enveloping algebra of the Poincaré Lie algebra \mathcal{P} . The universal enveloping algebra $\mathcal{U}(\mathcal{A})$ of an algebra \mathcal{A} is the noncommutative algebra generated by 1 and elements of \mathcal{A} , modulo the relations that specify the Lie-multiplication rules for the elements of \mathcal{A} .

⁹We refer to Ref. [40] for a detailed review of Hopf algebras and of the Hopf-algebra axioms that must be fulfilled. Here, we briefly mention that, for an algebra \mathcal{A} , the axioms to be fulfilled are the *algebra axioms*

$$m(m \otimes 1) = m(1 \otimes m) \quad (\text{associativity}),$$

$$m(1 \otimes \eta) = m(\eta \otimes 1) = 1 \quad (\text{unit}),$$

and the *coalgebra axioms*

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad (\text{coassociativity}),$$

$$(1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta = 1 \quad (\text{counit}),$$

which together specify a *bialgebra* $(A, m, \eta, \Delta, \epsilon)$, and finally the *antipode axioms*, which specify a *Hopf algebra*, namely,

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \eta \circ \epsilon.$$

at the tree level for the graviton propagator, is instead the θ -Poincaré algebra. This is a ‘‘nontrivial’’ Hopf algebra, in that the coproducts do not reflect anymore the Leibnitz rule; i.e. the module space of the algebra is that one of noncommutative functions on noncommutative θ -Minkowski space-time. The algebra is still defined by (45), but the coalgebra is now deformed in the tensor θ . Far from deriving the deformation of the coproducts in (46) and discussing the whole θ -Poincaré algebra, we just mention a few basic properties of \mathcal{P}_θ . Coproducts Δ_θ are obtained by an element of the bialgebra $\mathcal{P} \otimes \mathcal{P}$ called the twist element:

$$\mathcal{F}_\theta = e^{(i/2)\theta^{\mu\nu}P_\mu \otimes P_\nu}, \quad (46)$$

which satisfies

$$\mathcal{F}_\theta(\Delta_0 \otimes 1)\mathcal{F}_\theta = \mathcal{F}_\theta(1 \otimes \Delta_0)\mathcal{F}_\theta. \quad (47)$$

Given the generators $Y \in \mathcal{P}$, the twist element \mathcal{F} ‘‘modifies’’ the coproduct of $U(\mathcal{P})$ in the following way [47]:

$$\Delta_0(Y) \mapsto \Delta_\theta(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1},$$

$\Delta_0(Y)$ denoting the trivial coproduct in (46). Moreover, in the limit $\theta \rightarrow 0$ the coalgebraic structure of \mathcal{P} is recovered from the one of \mathcal{P}_θ , i.e. $\Delta_\theta(Y) \rightarrow \Delta_0(Y)$.

Finally, the generators of translations P_α being commutative among each other, it can be easily recovered that their coproduct is not deformed (in the math jargon, $\Delta_\theta = \Delta_0$ is ‘‘primitive’’ or also the subalgebra of translation is ‘‘cocommutative’’):

$$\Delta_\theta(P_\alpha) = \Delta_0(P_\alpha) = P_\alpha \otimes 1 + 1 \otimes P_\alpha, \quad (48)$$

while it is more laborious but nevertheless easy to check that

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= A d e^{(i/2)\theta^{\alpha\beta}P_\alpha \otimes P_\beta} \Delta_0(M_{\mu\nu}) \\ &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \\ &\quad \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)], \end{aligned} \quad (49)$$

which shows the modification in $\theta^{\mu\nu}$ of the coalgebraic structure of the θ -Poincaré Hopf algebra, reflecting the modification in θ of the Leibnitz rule.

E. SRQG and Noncommutative Gravity

In preparation for the conclusions, we want to address in this section a brief comparison of the model above with the theory of noncommutative geometry and gravity developed in Ref. [41] and with seminal works on the relation between differential calculi over a given noncommutative associative algebra and space-time metrics addressed in Refs. [48,49]. We first emphasize the differences between the model presented in this paper and the works in Refs. [41,48–50] and then conclude with a list of points to be investigated in forthcoming works in order to gain a clearer physical picture.

The most striking point we are confronted with is the inequivalence of our model with the ones addressed in Refs. [41,48,49]. This feature indeed is already evident at the level of the linearized equation of motion for the SRQG theories described in Refs. [1,2]. A first heuristic analysis based on the work reported in Ref. [51] reveals indeed that the linearized equation for the model here treated would involve a nonlocal operator $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ acting on the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$ in the form

$$G_{\mu\nu} + \kappa^2 H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha) R_{\rho\sigma} = 0, \quad (50)$$

in which $G_{\mu\nu}$ denotes the Einstein tensor and $\kappa^2 = 32\pi G_N$ in natural units. In (50), $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ acts as a total derivative only on the Ricci tensor, and therefore the associativity condition of an eventual star-product would not be satisfied. Thus it would be completely meaningless even try to make sense of $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ in terms of a star-product and of an underlying noncommutative of space-time, even in the situation in which the background has been fixed and gravity has been linearized at the first order, as, for instance, when considering $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Moreover is the matter of fact that Eq. (50) differs from the equation of motion derived in the model studied in Ref. [41], where the noncommutative Einstein equations read

$$\mathbf{Ric} - \frac{1}{2} \mathbf{g} \star \mathfrak{R} = 0. \quad (51)$$

In (51), the \star -product is defined in order to be consistent with the twist element that modifies the diffeomorphism algebra. The noncommutative metric tensor reads locally as $\mathbf{g} = \theta^j \otimes_\star \theta^i \star g_{ij}$, with θ^i basis one-forms and \otimes_\star the associative \star -tensor product associated to the deformed algebra of noncommutative tensor fields. The Ricci tensor \mathbf{Ric} and the Ricci scalar \mathfrak{R} are derived as the contraction of the curvature tensor that is defined in terms of the \star -covariant derivative ∇_u^\star (along any vector field u of the module algebra). In brief, Eq. (51) would read as a modification of the only Einstein tensor, while the equation of motion of the SRQG model we have studied involves higher derivatives applied to the square of the curvature tensor and their contractions. However, as this heuristic argument does not provide a solid proof, in order to be confident about the inequivalence of the two models, it would be appropriate to analyze some particular symmetry-reduced solutions, which we will do in the future to provide accurate equations of motion in the general curved case.

Another point of difference we should single out is the absence in our framework of a consistent interpretation of the nonlocality within the action (1) in terms of a twisted star-product. Following, for instance, a common procedure (see e.g. Ref. [52] and references therein), we can express any field theory on noncommutative space-time as a nonlocal field theory on a commutative space-time, provided that nonlocality is described in terms of a star-product. Thus, in principle, we can ask whether it is possible to do

the converse in our framework, recovering a star-product. But we should also consider a twist element which leaves undeformed the Lorentz sector of the Poincaré algebra, because of the particular dependence on the d'Alembertian covariant operator \square in the nonlocal function F . This feature represents a strong constraint for the theories studied in Ref. [2]. A twist element would naturally achieve this goal, but it is quite easy to see that from the particular form of F we would not be able to derive the associativity of the star-product nor the normalization condition for it (see e.g. Sec. II of Ref. [41]), both of them necessary requirements to recover a twist element. Therefore, we would be naturally lead to search for a generalization of our framework, and more in general of the theories presented in Ref. [2], in order to account for a consistent twist element. We emphasize that in this latter theoretical framework we would be able to address interesting conceptual questions. Indeed, although in the seminal works in Refs. [48,49] cases in which noncommutativity singled out a preferred metric were considered, in Ref. [41] any moving frame has been treated on equal footing, and it has been shown that there are infinitely many metrics compatible with a given noncommutative differential geometry. Moreover, as a consequence of the bicovariant differential calculus and of the framework single out in Ref. [41], torsion appears also in the vacuum. This scheme hence implies a deformation of the geodesic motion, and consequences for the equivalence principle should be also investigated in detail. Conversely, the compatibility of the metric and the validity of the equivalence principle are imposed from the beginning in Refs. [1,2] and not questioned.

IV. CONCLUSIONS

Moving from the work in Ref. [2] defining a class of SRQG theories, we have shown in this paper that is possible to define a unique SRQG theory provided with a nontrivial Hopf algebra of space-time symmetries. The associated phase space is Heisenberg type, and associativity must be preserved in the noncommutative theory. Specifically, the nontrivial Hopf algebra connected to SRQG is the twisted algebra of θ -Poincaré symmetry, which shows as a remarkable feature that one having a Poincaré algebra, and thus a Lorentz subalgebra, that are unmodified in the dimensionful parameter θ . For θ -Poincaré symmetry, deformation emerges in the coalgebra structure and in the other mathematical structures defining the concept of Hopf algebra, which is a bialgebra fulfilling certain consistency relations [40]. Therefore, Lorentz transformation and Lorentz covariance are defined in the standard way in this quantum-group symmetric SRQG theory, and locally it still makes sense to say that the Lorentzian theory is invariant under action of the generators of $\mathfrak{so}(3,1)$. We emphasize that the analysis we have developed at the tree level for the graviton propagator must be developed at high order in the perturbation

theory and generalized to n -point functions in the quantum theory, in order to show a full correspondence between the noncommutative field theory and the model of SRQG here investigated. Furthermore, the analysis of the noncommutative quantum theory provided with interactions, which can be achieved by following the lines of Refs. [34,53–55], and the development of possible intriguing features of the multiparticle states in the Fock space of the quantum theory are points strictly intertwined that may have a relation to the model we have focused on. Indeed, one might argue that the $\theta \sim 1/\Lambda^2$ modifications to the coalgebraic sector induce a θ modification to the many-particle states at the quantum level, and establishing firmly this point would open the path to the study of entanglement effects, as studied in the last one of Ref. [39] for the case of the κ -Poincaré algebra. Finally, a clear understanding of the noncommutative space-time reformulation of the models studied may be derived through a comparison of the interacting terms contained in the SRQG theory at the perturbative level with an appropriate expansion of a selected noncommutative theory of gravity, which may be described by Ref. [41] or some other model. Nevertheless, at the present stage of development of the theory, the knowledge of the two-point function is sufficient to derive an appropriate physical description at least of the gravitational potential.

The ones listed here above are all suggestive questions, which can be addressed only at the level of a full quantized noncommutative theory; we therefore leave them for future developments.

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APPENDIX. A: BCH FORMULAS

We summarize in this appendix some useful formulas that we have used in the above sections. We first consider a linear operator A , which is defined by means of

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \quad (\text{A1})$$

As a consequence, $\partial_\tau e^{\tau A} = A e^{\tau A} = e^{\tau A} A$. Let us consider another linear operator B , and let $B(\tau) = e^{\tau A} B e^{-\tau A}$. The *Sophus-Lie* formula then provides us with the following series representation for $B(\tau)$:

$$B(\tau) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} B_m, \quad (\text{A2})$$

in which $B_m = [A, B]_m := [A, [A, B]_{m-1}]$ and $B_0 := B$. The BCH formula is a particular case of the Sophus-Lie formula. Setting $\tau = 1$, one obtains indeed

$$e^A B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} B_m. \quad (\text{A3})$$

This latter expression can be remanipulated in the form

$$[B, e^{-A}] = e^{-A}([A, B] + [A, [A, B]]/2 + \dots) \quad (\text{A4})$$

or

$$[e^A, B] = ([A, B] + [A, [A, B]]/2 + \dots)e^A. \quad (\text{A5})$$

Furthermore, in addition to the BCH formula, there is another expression which is also referred to as the BCH formula, but which is due to Dynkin. This latter expression provides us with the multiplication law for two exponentials of linear operators within the assumptions $[A, [A, B]] = [B, [B, A]] = 0$, corresponding to a central algebra in our θ -Minkowski case. It follows that

$$e^A e^B = e^{A+B} e^{(1/2)[A,B]}, \quad (\text{A6})$$

and, reshuffling this latter expression, one obtains the Zassenhaus formula at the second order:

$$e^{A+B} = e^A e^B e^{(-1/2)[A,B]}. \quad (\text{A7})$$

As for practical reasons we were mostly interested in the Zassenhaus formula up to the fourth order, here below we furnish it for completeness (see e.g. [56] and references therein)

$$e^{A+B} = e^A e^B e^{(-1/2)[A,B]} e^{(1/3!)[2[B,[A,B]]+[A,[A,B]]} \\ \times e^{(-1/4!)[[[[A,B],A],A]+3[[[A,B],A],Y]+3[[[A,B],B],B]}. \quad (\text{A8})$$

We emphasize that the general BCH expansion is not a special case of the Hadamard lemma. Indeed the Dynkin formula is exact but not closed, and in general there is no explicit closed form for the BCH expansion, except in the “degenerate cases.” Nevertheless, those latter cases are the most interesting for the application to physics [57].

APPENDIX. B: $P^{(2)}$, $P^{(1)}$, $P^{(0-s)}$, $P^{(0-s\omega)}$ TENSORS

We furnish here below the expression for some quantities introduced in Sec. II, namely,

$$P_{\mu\nu\rho\sigma}^{(2)}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \\ P_{\mu\nu\rho\sigma}^{(1)}(k) = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}), \\ P_{\mu\nu\rho\sigma}^{(0-s)}(k) = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad P_{\mu\nu\rho\sigma}^{(0-\omega)}(k) = \omega_{\mu\nu}\omega_{\rho\sigma}, \\ P_{\mu\nu\rho\sigma}^{(0-s\omega)}(k) = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma}, \quad P_{\mu\nu\rho\sigma}^{(0-\omega s)}(k) = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma}, \\ \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \quad (\text{B1})$$

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