Self-gravitating Bjorken flow

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(Received 29 March 2012; published 4 June 2012)

I present a solution to the full Einstein-fluid equations representing a self-gravitating Bjorken flow. The motion and the geometry become inhomogeneous in the plane transversal to the flow and the energy density profile acquires, due to gravity, corrections in terms of proper time as compared to the original test hydrodynamics. The transverse distribution of energy density, for example, becomes $\epsilon(\tau, r)/\epsilon(\tau, 0) = \cosh^{-4}(3ar)$.

DOI: 10.1103/PhysRevD.85.124006 PACS numbers: 04.40.Nr, 04.20.Jb, 47.75.+f

Bjorken flow [1] represents the most fascinating application of relativistic hydrodynamics to an extremely complex physical system describing an average motion of partons resulting in a collision of heavy ions. The application of hydrodynamics to similar problems was pioneered by Landau [2] to describe the high-energy multiparticle collisions. Both in Bjorken and Landau descriptions it is assumed that after the collision of heavy ions the mean free path of the constituencies is short enough, so that the hydrodynamical description is meaningful. The difference between the two pictures is in the symmetry assumptions. In Bjorken hydrodynamics, one assumes the so-called boost invariance, so that the energy density only depends on proper time τ, while in the Landau picture, no such symmetry restriction is made and the density may be a function of all spatial coordinates. Needless to say, Bjorken flow is a particular and simpler version of Landau hydrodynamics; nevertheless, it is surprising that it works so well [3]. A different and renewed motivation in these studies comes from their relation to the AdS-CFT correspondence conjecture [4–6], because they serve as an input to understand the highly nontrivial behavior of quantum chromodynamics in a strong coupling regime.

The symmetries one imposes on the Bjorken flow are as follows: the boost symmetry along the beam, and translational and rotational invariance in the transverse plane. These symmetries allow one to parametrize the "future wedge" of the Minkowski spacetime in the following way:

$$ds^2 = -d\tau^2 + \tau^2 d\eta^2 + d\rho^2 + \rho^2 d\phi^2. \qquad (1)$$

Here τ is the proper time and η is historically called the rapidity; the rest are the usual cylindrical coordinates.

For further purposes I will write the metric in the following form:

$$ds^2 = -d\tau^2 + d\rho^2 + \tau\rho(\tau/\rho d\eta^2 + \rho/\tau d\phi^2). \qquad (2)$$

The spacelike Killing fields $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\eta}$ are the rotational and the boost Killing vectors, respectively, and the form of the line element (2) gives one an idea as to how to proceed in order to generalize this setup to a general relativistic flow.

Now, if one introduces an ideal fluid with the linear equation of state $p = 1/3\epsilon$ with the energy density and the pressure such that these only depend on the proper time τ, in other words, the fluid velocity has no tilt, it is quite easy to solve the hydrodynamical equations and to obtain that the fluid density scales as $\propto \tau^{-4/3}$. This is quite close to the experimental picture, *grosso modo*, within the range where the hydrodynamics makes sense.

On the other hand, the above picture is quite idealized, of course; some amount of viscosity should be added and some symmetries relaxed [7]. Nevertheless, it is quite surprising that the picture works so well.

While the Bjorken hydrodynamics deals exclusively with the flow on a given fixed flat background geometry (test hydrodynamics), and given that even slight relaxations of symmetry would lead to quite complicated nonlinear hydrodynamical equations, even the so-called Khalatnikov solution [8] of a 1 + 1 dimensional flow is quite a mess [9], it is yet another pleasant surprise that one may integrate an exact general relativistic solution which describes self-gravitating Bjorken flow. The main purpose of this paper is to present such a solution and to compare it to the original Bjorken test hydrodynamics. As a byproduct, I will also obtain solutions to the test hydrodynamics, without a tilt, on a class of cylindrical geometries.

I will stick as closely as possible to the original Bjorken picture and will assume that the fluid velocity has no tilt [see, however, a comment after Eq. (6)]. Nevertheless, since one must solve the coupled Einstein-fluid equations self-consistently, one cannot expect that the geometry would share all the above-mentioned symmetries. This is the essence of the Einstein theory; the matter influences the geometry which in turn changes its motion. We may keep the boost and the rotational Killing vectors intact; however, there is no reason why the line element should not depend on both τ and ρ coordinates. In fact, the form of the line element (2) indeed suggests the dependence on ρ. I will therefore assume the following geometry:

$$ds^2 = f(-dt^2 + dr^2) + g(qd\eta^2 + q^{-1}d\phi^2). \qquad (3)$$

Here f, g, and q are functions of both t and r, the "conformal" coordinates which are labeled differently to distinguish them from the proper time and the proper distance coordinates. Obviously, for the original Bjorken flow these functions are $f = 1$, $g = tr$, and $q = t/r$. Equation (3) seems a natural generalization of Bjorken geometry, while the attempts to use the homogeneous Kasner, or flat Friedmann Robertson Walker line element, as some authors do, fail to address the inherent symmetries of the problem.

I now specify the matter. The perfect fluid is assumed to have a linear equation of state, and because the fluid flow is irrotational, which is a must in this geometry, one may introduce the following velocity potential σ [10,11]:

$$u^\mu = \sigma^\mu/\sqrt{-\sigma_\alpha\sigma^\alpha}. \tag{4}$$

As will be seen later, the velocity potential is an extremely useful tool to solve the hydrodynamics.

Having done so, one may further define the kinetic scalar ("enthalpy") $X = -1/2\sigma_\alpha\sigma^\alpha$. The pressure and the energy density can then be expressed as follows [12]:

$$p = p(X), \qquad \epsilon = 2Xp' - p. \tag{5}$$

Here the prime, as usual, stands for the derivative of the function with respect to its argument.

If the equation of state is linear, $p = w\epsilon$, one may further write [11,12]

$$p(X) = X^{(w+1)/2w}. \tag{6}$$

We now assume that we have chosen our coordinates comoving with the fluid flow as in the original setting so that the velocity has only a zero component u_0. This means, in terms of the velocity potential, that σ is a function of t alone. In fact, if one even allowed a tilt, so that the velocity would "catch" a component in the transversal direction ($\frac{\partial\sigma}{\partial r} \neq 0$), as, for example, in [7], there would still be a way to introduce a new coordinate system comoving with the fluid and maintain the form of the metric [13]. This would not necessarily be true for the test hydrodynamics, and it may also spoil the separability of the metric functions which I will assume in the future.

The full Einstein equations for the line element (3) with the fluid specified above are found in [11]. It is instructive, however, to display the dynamical equation for the velocity potential $\nabla_\mu(p'\sigma^\mu) = 0$ [12]. This reads

$$\frac{1}{v_s^2}\ddot{\sigma} + \left[\dot{g}/g - \frac{1}{2}\left(\frac{1}{v_s^2} - 1\right)\dot{f}/f\right]\dot{\sigma} = 0, \tag{7}$$

where the velocity of sound v_s is given by

$$v_s^2 = \frac{p'(X)}{2Xp''(X) + p'(X)}, \tag{8}$$

and can be easily obtained from the relations (5) and the expression $v_s^2 = \frac{\partial p}{\partial \epsilon}$.

Note that the transversal degree of freedom of the metric q plays no explicit role in the dynamics of the fluid, but it does influence the flow via the full Einstein equations contributing to the longitudinal expansion f. However, both the longitudinal expansion f and the function g, which is proportional to the area of the isometry group orbits, do appear in the equation. These functions (p, g, and f) are determined by the Einstein equations.

Using the linear equation of state (6), so that $v_s^2 = w$, and assuming that all the functions of the metric are separable, $f = f_T f_R$, $g = g_T g_R$, and so on, one may easily integrate the dynamical equation (7) to get

$$\dot{\sigma} = b\frac{f_T^{(1-w)/2}}{g_T^w}, \tag{9}$$

where the lower index T indicates the time dependent part of the respective function and b is an arbitrary integration constant. The solution (9) represents, therefore, Bjorken flow on a generalized geometry given by the line element (3). Hence, given the background geometry (3) with separable functions, the velocity potential is given by the solution (9), and therefore all the kinematical fluid variables may be easily evaluated. For the Bjorken geometry and $w = 1/3$ we get $\sigma \propto t^{2/3}$, which leaves us with $\epsilon \propto t^{-4/3}$, and t is a proper time coordinate in this case. For the spatially flat Friedmann Robertson Walker geometry sourced by the same fluid (radiation) $f \propto g \propto t^2$, we get $\sigma \propto t$, $X \propto t^{-2}$, and $\epsilon \propto t^{-4}$, which becomes $\epsilon \propto \tau^{-2}$ in terms of proper time. One may play around with Eq. (7) further, but I will leave this for future work.

Equation (7), though important, is only part of the full Einstein equations, and it describes a test hydrodynamics on a given geometry. If one solves the rest of the equations with the following initial and boundary conditions compatible with the Bjorken geometry, $g \to tr$ as t and $r \to 0$, and $t \to 0$ as $\tau \to 0$, one can integrate the following solution to the Einstein equations:

$$\begin{aligned} f &= \sinh^4(at)\cosh^2(3ar), \\ g &= \sinh(at)\sinh(3ar)\cosh^{-2/3}(3ar), \\ q &= \sinh^3(at)\sinh(3ar). \end{aligned} \tag{10}$$

The velocity potential for this solution is given by

$$\sigma = (15a^2)^{1/4}\cosh(at), \tag{11}$$

while the energy density becomes

$$\epsilon = 15a^2\sinh^{-4}(at)\cosh^{-4}(3ar). \tag{12}$$

Here a is a free parameter specifying the density. In fact, this solution was derived some years ago by the present author and J. Senovilla [14] in the context of inhomogeneous cosmology, but it did not occur to us then that the solution describes the inhomogeneous self-gravitating Bjorken flow. Indeed, since the coordinate t is not the proper time, we find that the coordinate time $t \propto \tau^{1/3}$, and therefore the energy density scales as

$$\epsilon \propto \tau^{-4/3}, \tag{13}$$

to the lowest order in proper time. However, some corrections are due. Because we have chosen our coordinates in a way that r remains constant along the fluid lines, the proper time is given by the following expression:

$$\tau = \int_0^t \sqrt{f} dt = \frac{1}{3a}(at)^3 + \frac{1}{15a}(at)^5 + \mathcal{O}((at)^7). \tag{14}$$

The energy density ϵ then evolves as (just the first two terms)

$$\epsilon \propto \alpha(a)\tau^{-4/3} - \beta(a)\tau^{-2/3}. \tag{15}$$

Here α and β are both positive functions of the parameter a. I have assumed a constant r and have used the lowest order of the proper time expansion. Of course, having at hand the exact solution, there is no need for the series expansions; nevertheless, these are instructive in order to further elucidate the physics. As one can easily see from Eq. (15), the second term becomes dominant at late times, where the energy density appears to become negative. This is an *artifact* of the series expansions, as the exact energy density never becomes negative. On the other hand, while the first term scales as the energy density of the ideal fluid, the second term acts as if it were viscosity by enhancing the falloff of the evolving density. Of course, the fluid remains inviscid, and this effect is purely due to self-gravity. Assuming Stefan-Boltzmann's law ($\epsilon \sim T^4$) one can easily find the temperature distribution and then define the temperature contrast δT as

$$\delta T = \frac{T - T_B}{T_B}, \tag{16}$$

where T is the temperature found from the exact solution, while T_B represents the temperature of the test flow. This temperature contrast evolves as $\delta T \sim -\tau^{1/3}$. Another interesting physical quantity is the distribution of energy density in the transverse coordinate,

$$\epsilon(\tau, r)/\epsilon(\tau, 0) = \cosh^{-4}(3ar), \tag{17}$$

which is given by a neat, simple expression in terms of conformal distance r. It is assumed that this quantity is proportional to the distribution of the nucleus in the fireball; the actual numbers, of course, would depend on the colliding constituencies.

In closing, I have presented an exact solution to the Einstein-fluid equations which describes self-gravitating Bjorken flow. The gravity changes the energy density distribution and its evolution in proper time. From the exact expressions one may easily find all relevant kinematical and thermodynamical quantities. As a by-product, I have obtained some test hydrodynamical solutions on the expanding cylindrical geometry. It would be interesting in the future to study in more detail the test hydrodynamics with nonlinear equations of state, as well as to consider the solution (10) as an input to study the ADS/CFT correspondence in situations where the geometry is both inhomogeneous and evolving in time.

It is a pleasure to thank Manuel Valle and Raül Vera for valuable comments. This work is partially supported by the Basque government Grant No. GICO7/51-IT-221-07, the Spanish Science Ministry Grant No. FIS2010-15492, and the UFI 11/55 program of the University of the Basque Country UPV/EHU.

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