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Light-by-light scattering sum rules constraining meson transition form factors

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Relating the forward light-by-light scattering to energy weighted integrals of the $\gamma^*\gamma$ fusion cross sections, with one real photon (γ) and one virtual photon (γ^*), we find two new exact superconvergence relations. They complement the known superconvergence relation based on the extension of the Gerasimov-Drell-Hearn sum rule to the light-light system. We also find a set of sum rules for the low-energy photon-photon interaction. All of the new relations are verified here exactly at leading order in scalar and spinor QED. The superconvergence relations, applied to the $\gamma^*\gamma$ production of mesons, lead to intricate relations between the $\gamma\gamma$ decay widths or the $\gamma^*\gamma$ transition form factors for (pseudo)scalar, axial-vector, and tensor mesons. We discuss the phenomenological implications of these results for mesons in both the light-quark sector and the charm-quark sector.

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I. INTRODUCTION

Light-by-light (LbL) scattering is a prediction of the quantum theory [1,2] which, thus far, has not been directly observed, mainly due to smallness of the cross section. On the other hand, the process of $\gamma^*\gamma^*$ fusion (by quasireal photons γ or virtual photons γ^*) into leptons and hadrons has been observed at nearly all high-energy colliders; see e.g. [3–5] for reviews. The two phenomena—LbL scattering and $\gamma\gamma$ fusion—must be related by causality, similar to how the refraction index of light is related to its absorption in the Kramers-Kronig relation. The main goal of this work is to establish such relations and use them to investigate the structure of hadrons in the realm of QCD.

The electromagnetic interaction provides a clean probe, and the two-photon state allows the production of hadrons with nearly all quantum numbers (with $C = +$), in contrast to the well-studied single-photon scattering or production processes, which only access the vector states. When producing exclusive final states such as in the $\gamma^*\gamma^* \to$ meson process, one accesses meson transition form factors (FFs), which are some of the simplest observables where the approach to the asymptotic limit of QCD is studied along with the quark content of mesons described by distribution amplitudes (DAs). The nonperturbative dynamics of QCD also plays a profound role in these FFs at low momentum transfers. For example, the transition FFs of the η and η' mesons depend on the interplay of various symmetry breaking mechanisms in QCD, i.e. $U_A(1)$ symmetry breaking [6], and dynamical and explicit chiral symmetry breaking. In addition, the $\gamma^*\gamma^* \to$ meson transition FFs are important for providing and improving constraints on the light-by-light hadronic contribution to the anomalous magnetic moment of the muon, $(g-2)_\mu$. The hadronic contributions to $(g-2)_\mu$ are, at present, the major uncertainties in the search for new, beyond standard model, physics in this high-precision quantity [7].

In recent years, new experiments at high luminosity e^+e^- colliders such as *BABAR* and Belle have vastly expanded the field of $\gamma\gamma$ physics. The result of a measurement of the $\gamma^*\gamma \to \pi^0$ FF at large momentum transfers by the *BABAR* Collaboration [8] came as a surprise, as this form factor seems to rise much faster than the perturbative QCD predictions for momentum transfers up to 40 GeV^2. A $\gamma\gamma$ physics program is planned now by the BES-III Collaboration [9], which will allow one to provide high-statistics results at intermediate momentum transfers for a multitude of $\gamma^*\gamma^* \to$ hadron observables.

In this work we use the dispersion theory to relate the two phenomena of LbL scattering and $\gamma^*\gamma$ fusion, and express the low-energy LbL scattering as integrals over the $\gamma^*\gamma$-fusion cross sections, where one photon is real while the second may have arbitrary (spacelike) virtuality. These integrals, or "sum rules," lead to interesting constraints on $\gamma\gamma$ decay widths or $\gamma^*\gamma$ transition FFs of $q\bar{q}$ states, and more general meson states. The first sum rule of this type involves the helicity-difference cross section for real photons and reads as

$$\int_{s_0}^{\infty} \frac{ds}{s}[\sigma_2(s) - \sigma_0(s)] = 0, \tag{1}$$

where s is the total energy squared, s_0 is the first inelastic threshold for the $\gamma\gamma$ fusion process, and the subscripts 0 or 2 for the $\gamma\gamma$ cross sections indicate the total helicity of the state of two circularly polarized photons. This sum rule was originally[1] inferred [11,12] from the Gerasimov-Drell-Hearn (GDH) sum rule, using the fact that the photon has no anomalous moments.

[1]An earlier version of this sum rule had been proposed in Ref. [10], where a contribution from π^0 production appears on the right-hand side (rhs) of Eq. (1), while integration on the left-hand side starts at the 2π production threshold. That version would be fully compatible with Eq. (1), if it were not for the sign of the π^0 contribution obtained in [10].

Parametrizing the lowest energy LbL interaction by means of an effective Lagrangian (which contains operators of dimension eight at lowest order) as

$$\mathcal{L}^{(8)} = c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2, \quad (2)$$

with F and $\tilde{F}$ being the electromagnetic field strength and its dual, one finds sum rules for the LbL low-energy constants (LECs) [13]:

$$c_1 = \frac{1}{8\pi}\int_{s_0}^{\infty} ds \frac{\sigma_{\parallel}(s)}{s^2}, \qquad c_2 = \frac{1}{8\pi}\int_{s_0}^{\infty} ds \frac{\sigma_{\perp}(s)}{s^2}, \quad (3)$$

where the subscripts $\parallel$ or $\perp$ indicate if the colliding photons are polarized parallel or perpendicular to each other. While the GDH-type sum rule provides a stringent constraint on the polarized $\gamma\gamma$ fusion, the sum rules for the LECs allow one, in principle, to fully determine the low-energy LbL interaction through measuring the linearly polarized $\gamma\gamma$ fusion.

In this work we extend the GDH-type sum rule to the case where one of the colliding photons is virtual, with arbitrary (spacelike) virtuality. Furthermore, we find two additional sum rules, involving the longitudinally polarized $\gamma^*\gamma$ cross sections. All details of sum rule derivation are gathered in Sec. II. In Sec. III, all of the newly derived sum rules are verified at leading order in scalar and spinor QED. Next we apply these results to the $\gamma^*\gamma^*$ fusion to mesons. Using the available data, we quantitatively study the new sum rules derived in this paper for the case of production of light-quark mesons as well as mesons containing charm quarks, both by real photons in Sec. IV A, and by virtual photons in Sec. IV B. We demonstrate the intricate cancellations that must occur among the (pseudo) scalar, tensor, and axial-vector mesons in order to satisfy these sum rules. In the case of production of virtual photons, we use these relations to provide estimates of hitherto unmeasured $\gamma^*\gamma$ transition form factors of tensor mesons, such as $f_2(1285)$ and $a_2(1320)$. The conclusion and outlook are given in Sec. V.

Appendix A contains a review of the kinematical notations and $e^{\pm} + e^- \to e^{\pm} + e^- + X$ cross-section conventions; Appendix B contains expressions for the tree-level $\gamma^*\gamma^*$ cross sections for the case of scalar and spinor QED; Appendix C contains the general formalism for the $\gamma^*\gamma \to$ meson transitions with different quantum numbers (J^{PC}), i.e.: pseudoscalars (0^{-+}), scalars (0^{++}), axial vectors (1^{++}), and tensors (2^{++}).

II. DERIVATION OF SUM RULES FOR LIGHT-LIGHT SCATTERING

A. Forward scattering amplitudes

In the most general case we consider the forward scattering of virtual photons on virtual photons:

$$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \gamma^*(\lambda_1', q_1) + \gamma^*(\lambda_2', q_2), \quad (4)$$

where q_1, q_2 are photon four-momenta, and λ_1, λ_2 (λ_1', λ_2') are the helicities of the initial (final) virtual photons, which can take on the values ± 1 (transverse polarizations) and zero (longitudinal). The total helicity in the $\gamma^*\gamma^*$ c.m. system is given by $\Lambda = \lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$. To define the kinematics, we first introduce the photon virtualities ($Q_1^2 = -q_1^2$, $Q_2^2 = -q_2^2$), the Mandelstam invariants [$s = (q_1 + q_2)^2$, $u = (q_1 - q_2)^2$], and the following crossing-symmetric variable:

$$\nu \equiv \tfrac{1}{4}(s - u) = q_1 \cdot q_2, \quad (5)$$

such that $s = 2\nu - Q_1^2 - Q_2^2$, $u = -2\nu - Q_1^2 - Q_2^2$.

The $\gamma^*\gamma^* \to \gamma^*\gamma^*$ forward scattering amplitudes, denoted as $M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}$, are functions of ν, Q_1^2, Q_2^2. Parity invariance (P) and time-reversal invariance (T) imply the following relations among the matrix elements with different helicities:

$$P\colon M_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = M_{-\lambda_1'-\lambda_2',-\lambda_1-\lambda_2}, \quad (6)$$

$$T\colon M_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = M_{\lambda_1\lambda_2,\lambda_1'\lambda_2'}, \quad (7)$$

which leaves out only eight independent amplitudes [14]:

$$M_{++,++},\ M_{+-,+-},\ M_{++,--},\ M_{00,00},\ M_{+0,+0},\ M_{0+,0+},\ M_{++,00},\ M_{0+,-0}. \quad (8)$$

We next look at the constraint imposed by crossing symmetry, which requires that the amplitudes for the process (4) equal the amplitudes for the process where the photons with e.g. label 2 are crossed:

$$\gamma^*(\lambda_1, q_1) + \gamma^*(-\lambda_2', -q_2) \to \gamma^*(\lambda_1', q_1) + \gamma^*(-\lambda_2, -q_2). \quad (9)$$

As under photon crossing $\nu \to -\nu$, one obtains

$$M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(\nu, Q_1^2, Q_2^2) = M_{\lambda_1'-\lambda_2,\lambda_1-\lambda_2'}(-\nu, Q_1^2, Q_2^2). \quad (10)$$

It becomes convenient to introduce amplitudes which are either even or odd in ν (at fixed Q_1^2 and Q_2^2). One easily verifies that the following six amplitudes are *even* in ν:

$$\begin{array}{lll} (M_{++,++} + M_{+-,+-}), & M_{++,--}, & M_{00,00}, \\ M_{+0,+0}, & M_{0+,0+}, & (M_{++,00} + M_{0+,-0}), \end{array} \quad (11)$$

whereas the following two amplitudes are *odd* in ν:

$$(M_{++,++} - M_{+-,+-}), \qquad (M_{++,00} - M_{0+,-0}). \quad (12)$$

B. Fusion of two virtual photons

The optical theorem allows one to relate the absorptive part of the $\gamma^*\gamma^* \to \gamma^*\gamma^*$ forward scattering amplitudes to cross sections for the process $\gamma^*\gamma^* \to X$, where X stands for any possible final state. Denoting the absorptive part as

$$W_{\lambda_1'\lambda_2',\lambda_1\lambda_2} \equiv \mathrm{Abs}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}, \tag{13}$$

the optical theorem yields

$$W_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = \frac{1}{2}\int d\Gamma_X (2\pi)^4 \delta^4(q_1 + q_2 - p_X) \times \mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2; p_X)\mathcal{M}^*_{\lambda_1'\lambda_2'}(q_1, q_2; p_X), \tag{14}$$

where $\mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2; p_X)$ denotes the invariant amplitude for the process

$$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to X(p_X). \tag{15}$$

As a result, the absorptive parts are expressed in terms of eight independent $\gamma^*\gamma^* \to X$ cross sections (see Ref. [3] for details):

$$W_{++,++} + W_{+-,+-} \equiv 2\sqrt{X}(\sigma_0 + \sigma_2) = 2\sqrt{X}(\sigma_\parallel + \sigma_\perp) \equiv 4\sqrt{X}\sigma_{TT}, \tag{16a}$$

$$W_{++,++} - W_{+-,+-} \equiv 2\sqrt{X}(\sigma_0 - \sigma_2) \equiv 4\sqrt{X}\tau^a_{TT}, \tag{16b}$$

$$W_{++,--} \equiv 2\sqrt{X}(\sigma_\parallel - \sigma_\perp) \equiv 2\sqrt{X}\tau_{TT}, \tag{16c}$$

$$W_{00,00} \equiv 2\sqrt{X}\sigma_{LL}, \tag{16d}$$

$$W_{+0,+0} \equiv 2\sqrt{X}\sigma_{TL}, \tag{16e}$$

$$W_{0+,0+} \equiv 2\sqrt{X}\sigma_{LT}, \tag{16f}$$

$$W_{++,00} + W_{0+,-0} \equiv 4\sqrt{X}\tau_{TL}, \tag{16g}$$

$$W_{++,00} - W_{0+,-0} \equiv 4\sqrt{X}\tau^a_{TL}, \tag{16h}$$

where the virtual photon flux factor is defined through

$$X \equiv (q_1 \cdot q_2)^2 - q_1^2 q_2^2 = \nu^2 - Q_1^2 Q_2^2. \tag{17}$$

In Eq. (16), $\sigma_0(\sigma_2)$ are the $\gamma^*\gamma^* \to X$ cross sections for total helicity 0 (2), respectively, and $\sigma_\parallel(\sigma_\perp)$ are the cross sections for linear photon polarizations with both photon polarization directions parallel (perpendicular) to each other, respectively. The remaining cross sections (positive definite quantities σ) involve either one transverse (T) and one longitudinal (L) photon polarization, or two longitudinal photon polarizations, with σ_{LT} and σ_{TL} related as

$$\sigma_{LT}(\nu, Q_1^2, Q_2^2) = \sigma_{TL}(\nu, Q_2^2, Q_1^2). \tag{18}$$

The quantities τ_{TT}, τ^a_{TT}, τ_{TL}, τ^a_{TL} denote interference cross sections (which are not sign-definite) with either both photons transverse (TT), or for one transverse and one longitudinal photon (TL), where the superscript a indicates the combinations which are odd in ν.

C. Dispersion relations

The principle of (micro)causality is known to translate into exact statements about analytic properties of the scattering amplitude in the complex energy plane. In our case this principle translates into the statement of analyticity of the forward $\gamma^*\gamma^*$ scattering amplitude in the entire ν plane, except for the real axis where the branch cuts associated with particle production are located. Assuming that the threshold for particle production is $\nu_0 > 0$, one can write down the usual dispersion relations, in which the amplitude is given by integrals over the nonanalyticities, which in this case are branch cuts extending from $\pm\nu_0$ to $\pm\infty$. Finally, for amplitudes that are even or odd in ν we can write (for any fixed values of Q_1^2, $Q_2^2 > 0$)

$$f_{\mathrm{even}}(\nu) = \frac{2}{\pi}\int_{\nu_0}^\infty d\nu' \frac{\nu'}{\nu'^2 - \nu^2 - i0^+}\mathrm{Abs} f_{\mathrm{even}}(\nu'), \tag{19a}$$

$$f_{\mathrm{odd}}(\nu) = \frac{2\nu}{\pi}\int_{\nu_0}^\infty d\nu' \frac{1}{\nu'^2 - \nu^2 - i0^+}\mathrm{Abs} f_{\mathrm{odd}}(\nu'), \tag{19b}$$

where 0^+ is an infinitesimal positive number.

These dispersion relations are derived with the provision that the integrals converge. If they do not, subtractions must be made; e.g., the once-subtracted dispersion relation for the even amplitudes reads

$$f_{\mathrm{even}}(\nu) = f_{\mathrm{even}}(0) + \frac{2\nu^2}{\pi} \times \int_{\nu_0}^\infty d\nu' \frac{1}{\nu'(\nu'^2 - \nu^2 - i0^+)}\mathrm{Abs} f_{\mathrm{even}}(\nu'). \tag{20}$$

We are thus led to examine the high-energy behavior ($\nu \to \infty$ at fixed Q_1^2, Q_2^2) of the absorptive parts given by Eq. (16). In Ref. [14], a Regge pole model assumption for the high-energy asymptotics of the light-by-light forward amplitudes yielded:

$$\begin{aligned} &(W_{++,++} + W_{+-,+-}), \quad W_{+0,+0}, \quad W_{0+,0+}, \quad W_{00,00} \sim \nu^{\alpha_P(0)}, \\ &(W_{++,++} - W_{+-,+-}), \quad W_{++,--} \sim \nu^{\alpha_\pi(0)}, \quad (W_{++,00} + W_{0+,-0}), \\ &(W_{++,00} - W_{0+,-0}) \sim \nu^{\alpha_\pi(0)-1}, \end{aligned} \tag{21}$$

where $\alpha_P(0) \simeq 1.08$ is the intercept of the Pomeron trajectory, and $\alpha_\pi(0) \simeq -0.014$ is the intercept of the pion trajectory. This means that for all the even amplitudes, except $M_{++,00} + M_{0+,-0}$, one can only use the subtracted dispersion relation Eq. (20). We therefore need the information about these amplitudes at zero energy ν. Anticipating the discussion of the low-energy expansion of the LbL scattering, we can state that at $\nu = 0$ these amplitudes vanish when one of the photons is real [cf. Eq. (25)]. Using Eq. (16) then to substitute the cross sections in place of the absorptive parts, we obtain the following sum rules for the case of one real and one virtual photon (when the virtual photon flux factor becomes $X = \nu^2$):

$$M_{++,++}(\nu) + M_{+-,+-}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sigma_{\parallel}(\nu') + \sigma_{\perp}(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22a}$$

$$M_{++,--}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sigma_{\parallel}(\nu') - \sigma_{\perp}(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22b}$$

$$M_{0+,0+}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sigma_{LT}(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22c}$$

$$M_{+0,+0}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sigma_{TL}(\nu')}{\nu'^2 - \nu^2 - i0^+}. \tag{22d}$$

We cannot write such a subtracted sum rule for $M_{00,00}$, since it trivially vanishes when one of the photons is real. Instead, considering an unsubtracted dispersion relation, we find the following sum rule:

$$M_{00,00}(\nu) = \frac{4}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\nu' \sqrt{X'} \sigma_{LL}(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22e}$$

with $X' = \nu'^2 - Q_1^2 Q_2^2$. At least in perturbative QED calculations (cf. Appendix B), the above integral converges, which seems to validate this sum rule in a renormalizable, perturbative field theory. We emphasize, however, that this observation is in contradiction with the expectation of non-convergence from the Regge pole model shown above. A validation of this sum rule in nonperturbative field theory, particularly in QCD, is therefore an open issue.

For all the remaining amplitudes the asymptotic behavior of Eq. (22) justifies the use of unsubtracted dispersion relations which, upon substituting Eq. (16), lead to the following sum rules, valid for both photons virtual:

$$M_{++,++}(\nu) - M_{+-,+-}(\nu) = \frac{4\nu}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sqrt{X'}[\sigma_0(\nu') - \sigma_2(\nu')]}{\nu'^2 - \nu^2 - i0^+}, \tag{22f}$$

$$M_{++,00}(\nu) - M_{0+,-0}(\nu) = \frac{8\nu}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sqrt{X'} \tau_{TL}^a(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22g}$$

$$M_{++,00}(\nu) + M_{0+,-0}(\nu) = \frac{8}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\nu' \sqrt{X'} \tau_{TL}(\nu')}{\nu'^2 - \nu^2 - i0^+}, \tag{22h}$$

where the dependence on virtualities Q_1^2, Q_2^2 is tacitly assumed.

The above sum rules, relating all the forward $\gamma^*\gamma^*$ elastic scattering amplitudes to the energy integrals of the $\gamma^*\gamma^*$ fusion cross sections, should hold for any spacelike photon virtualities in the unsubtracted cases, and for one of the virtualities equal to zero in the subtracted cases. In the following we examine the low-energy expansion of these sum rules.

D. Low-energy expansion via the effective Lagrangian

To obtain more specific relations from the sum rules established in Eq. (22), we parametrize the low-energy (small ν) behavior of the $\gamma^*\gamma^* \to \gamma^*\gamma^*$ forward scattering amplitudes M. At lowest order in the energy, the self-interactions of the electromagnetic field are described by an effective Lagrangian (of fourth order in the photon energy and/or momentum, and fourth order in the electromagnetic field):

$$\mathcal{L}^{(8)} = c_1 (F_{\mu\nu} F^{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2, \tag{23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha A_\beta$, and where c_1, c_2 are two low-energy constants (LECs) which contain the structure dependent information. It is often referred to as the Euler-Heisenberg Lagrangian due to the seminal work [1].

At the next order in energy, one considers the terms involving two derivatives on the field tensors, corresponding with the sixth order in the photon energy and/or momentum. Writing down all such dimension-ten operators and reducing their number using the antisymmetry of the field tensors, the Bianchi identities, as well as adding or removing total derivative terms, we find that there are six independent terms at that order, which we choose as

$$\begin{aligned} \mathcal{L}^{(10)} = {} & c_3 (\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\lambda\nu}) F_{\lambda\rho} F^{\mu\rho} \\ & + c_4 (\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu}) F_{\lambda\rho} F^{\lambda\rho} \\ & + c_5 (\partial^\alpha F_{\alpha\nu})(\partial_\beta F^{\beta\nu}) F_{\lambda\rho} F^{\lambda\rho} \\ & + c_6 (\partial_\alpha \partial^\alpha F_{\mu\nu}) F^{\lambda\nu} F_{\lambda\rho} F^{\mu\rho} \\ & + c_7 (\partial_\alpha \partial^\alpha F_{\mu\nu}) F^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \\ & + c_8 (\partial^\alpha F_{\alpha\mu})(\partial_\beta F^{\beta\lambda}) F_{\rho\lambda} F^{\rho\mu}, \end{aligned} \tag{24}$$

where $c_3, \ldots, c_8$ are the new LECs arising at this order. Only c_3 and c_4 appear in the case of real photons.

We can now specify the low-energy limit of the light-by-light scattering amplitudes in terms of the LECs describing the low-energy self-interactions of the electromagnetic field:

$$M_{++,++} + M_{+-,+-} = Q_1^2 Q_2^2 [64(c_1 - c_2) + 4(Q_1^2 + Q_2^2)(-c_3 - 8c_4 - 4c_5 + 8c_7 - c_8) + \mathcal{O}(Q^4)] + 8\nu^2 [8(c_1 + c_2) + (Q_1^2 + Q_2^2)(-c_3 + 3c_6 + 4c_7) + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^4), \tag{25a}$$

$$M_{++,--} = Q_1^2 Q_2^2 [64c_2 + 4(Q_1^2 + Q_2^2)(-c_3 + 2c_6 - c_8) + \mathcal{O}(Q^4)] + 8\nu^2 [8(c_1 - c_2) + (Q_1^2 + Q_2^2)(c_6 + 4c_7) + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^4), \tag{25b}$$

$$M_{0+,0+} = Q_1^2 Q_2^2 [-32c_1 + 4Q_1^2 c_8 + 4(Q_1^2 + Q_2^2)(c_3 + 4c_4 + 2c_5 - 2c_6 - 4c_7) + \mathcal{O}(Q^4)] + \nu^2 [-4Q_1^2 c_8 + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^4), \tag{25c}$$

$$M_{+0,+0} = Q_1^2 Q_2^2 [-32c_1 + 4Q_2^2 c_8 + 4(Q_1^2 + Q_2^2)(c_3 + 4c_4 + 2c_5 - 2c_6 - 4c_7) + \mathcal{O}(Q^4)] + \nu^2 [-4Q_2^2 c_8 + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^4), \tag{25d}$$

$$M_{00,00} = Q_1^2 Q_2^2 [96c_1 + 4(Q_1^2 + Q_2^2)(-2c_3 - 4c_4 - 2c_5 + 6c_6 + 12c_7 - c_8) + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^2), \tag{25e}$$

$$M_{++,++} - M_{+-,+-} = 8\nu Q_1^2 Q_2^2 [-c_3 - 4c_5 + c_8 + \mathcal{O}(Q^2)] + \nu^3 [-64c_4 + \mathcal{O}(Q^2)] + \mathcal{O}(\nu^5), \tag{25f}$$

$$M_{++,00} - M_{0+,-0} = \nu Q_1 Q_2 [-64c_1 + (Q_1^2 + Q_2^2)(4c_3 - 16c_6 - 32c_7 + 4c_8) + \mathcal{O}(Q^4)] + \mathcal{O}(\nu^3), \tag{25g}$$

$$M_{++,00} + M_{0+,-0} = Q_1^3 Q_2^3 [4c_5 - 12c_8 + \mathcal{O}(Q^2)] + 4\nu^2 Q_1 Q_2 [2c_3 + 16c_4 + 4c_5 + c_8 + \mathcal{O}(Q^2)] + \mathcal{O}(\nu^4). \tag{25h}$$

These expressions can be treated as a simultaneous expansion in ν and the virtualities Q_i^2 of the left-hand side of the sum rules, Eq. (22). Concerning the Q dependence, it is important that the leading in ν term, in any of the amplitudes, is proportional to $Q_1 Q_2$ and hence vanishes for at least one real photon. The latter statement is valid for any values of virtualities, not just when they are small. For example, let us show for the amplitude $(M_{++,++} - M_{+-,+-})$ that its leading term in ν is proportional to the combination $Q_1^2 Q_2^2$, to all orders in Q_1 and Q_2.

Since all photons are transversely polarized the only nonvanishing structures involving polarization vectors of photons $\varepsilon(\lambda_i)$ are their mutual scalar products $\varepsilon(\lambda_i) \cdot \varepsilon(\lambda_j)$. Because of gauge invariance, the electromagnetic fields enter the Lagrangian through the field tensor $F_{\mu\nu}$, which contributes to the amplitude as $q_\mu \varepsilon_\nu - q_\nu \varepsilon_\mu$. Thus an arbitrary term in the effective Lagrangian contributes to $(M_{++,++} - M_{+-,+-})$ as

$$M_{++,++} - M_{+-,+-} \sim q_1^\mu q_2^\nu q_1^\lambda q_2^\rho T_{\mu\nu\lambda\rho}, \tag{26}$$

where the tensor $T_{\mu\nu\lambda\rho}$ is constructed from four-vectors q_i and the metric tensor. Since this amplitude is odd with respect to ν, it is required to be proportional to at least ν^1. Assuming that one factor ν comes from contraction of two of the q's in Eq. (26), we are left with $q_1^\mu q_2^\nu$. Now, if we suppose that q_1 is contracted with q_2, we obtain an extra power of ν, and such an amplitude vanishes when taking the limit $\nu \to 0$. Thus, both q_1 and q_2 must be contracted with another q_1 and q_2, respectively, giving a global factor $Q_1^2 Q_2^2$.

We are now in a position to examine the sum rules in Eq. (22) order by order in ν. For this we expand the rhs of Eq. (22) using $1/(\nu'^2 - \nu^2) = 1/\nu'^2 + \nu^2/\nu'^4 + \mathcal{O}(\nu^4)$. As a result we obtain from Eqs. (22f)–(22h) the following set of superconvergence relations, valid for at least one real photon (e.g., $Q_1 \geq 0$, $Q_2^2 = 0$):

$$0 = \int_{s_0}^\infty ds \frac{1}{(s + Q_1^2)} \tau_{TT}^a(s, Q_1^2, 0), \tag{27a}$$

$$0 = \int_{s_0}^\infty ds \frac{1}{(s + Q_1^2)^2} \left[\sigma_\parallel + \sigma_{LT} + \frac{(s + Q_1^2)}{Q_1 Q_2} \tau_{TL}^a \right]_{Q_2^2 = 0}, \tag{27b}$$

$$0 = \int_{s_0}^\infty ds \left[\frac{\tau_{TL}(s, Q_1^2, Q_2^2)}{Q_1 Q_2} \right]_{Q_2^2 = 0}, \tag{27c}$$

and the following set of sum rules for the LECs of the dimension-eight (Euler-Heisenberg) Lagrangian, valid when both photons are quasireal:

$$c_1 = \frac{1}{8\pi} \int_{s_0}^\infty \frac{ds}{s^2} \sigma_\parallel(s, 0, 0), \tag{28a}$$

$$= -\frac{1}{8\pi} \int_{s_0}^\infty \frac{ds}{s} \left[\frac{\tau_{TL}^a(s, Q_1^2, Q_2^2)}{Q_1 Q_2} \right]_{Q_1^2 = Q_2^2 = 0}, \tag{28b}$$

$$= \frac{1}{8\pi} \int_{s_0}^\infty ds \left[\frac{\sigma_{LL}(s, Q_1^2, Q_2^2)}{Q_1^2 Q_2^2} \right]_{Q_1^2 = Q_2^2 = 0}, \tag{28c}$$

$$c_2 = \frac{1}{8\pi} \int_{s_0}^\infty \frac{ds}{s^2} \sigma_\perp(s, 0, 0), \tag{28d}$$

where $s_0 = 2\nu_0 - Q_1^2 - Q_2^2$. We emphasize again that, unlike the other sum rules, the sum rule of Eq. (28c) is only shown to hold in perturbative field theory.

There are, as well, the sum rules for the LECs of the dimension-ten Lagrangian, most notably

$$c_4 = -\frac{1}{4\pi} \int_{s_0}^\infty \frac{ds}{s^3} \tau_{TT}^a(s, 0, 0), \tag{29}$$

but presently they are of far less importance and we do not write them out here explicitly.

Let us remark again that the relation of Eq. (27a), obtained by combining Eqs. (22f) and (25f), is essentially a GDH sum rule for the photon target; see [10–12]. For large virtuality Q_1^2, it leads to the sum rule for the photon structure function g_1^γ [15]: $\int_0^1 dx g_1^\gamma(x, Q^2) = 0$.

The sum rules in Eqs. (28a) and (28d), first established in [13], are obtained by combining Eqs. (22a) with (25a) and (22b) with (25b), respectively. All the other relations presented above are new. In the following section we verify these sum rules in QED at leading order in the fine-structure constant α.

III. SUM RULES IN PERTURBATION THEORY

We will subsequently discuss a pair production in scalar QED (e.g., Born approximation to $\gamma^*\gamma^* \to \pi^+\pi^-$) and in spinor QED ($\gamma^*\gamma^* \to q\bar{q}$, where q stands for a charged lepton or a quark).

A. Scalar QED

The response functions for the case of scalar QED at lowest order in the electromagnetic coupling can be found in Appendix B 1. We first study the three sum rules of Eqs. (27a)–(27c) for the case of one real or quasireal photon ($Q_2^2 \to 0$) and for arbitrary spacelike virtuality ($Q_1^2 \geq 0$) of the other photon. To better see the cancellation which must take place in these sum rules between contributions at low and higher energies, we show the integrands of the three sum rules in Figs. 1–3 multiplied by s. In this way, when plotted logarithmically, one can clearly see how the low- and high-energy contributions cancel each other. For the sum rule of Eq. (27b), we denote the integrand as

$$I = \frac{1}{(s+Q_1^2)^2}\left[\sigma_\parallel + \sigma_{LT} + \frac{(s+Q_1^2)}{Q_1 Q_2}\tau_{TL}^a\right]_{Q_2^2=0}. \quad (30)$$

All three sum rules of Eqs. (27a)–(27c) are exactly verified in scalar QED for arbitrary spacelike values of Q_1^2. One notices from Figs. 1–3 that for larger values of Q_1^2, the zero crossing of the integrands shifts to larger values of s, requiring higher energy contributions for the cancellation to take place. For the helicity-difference sum rule of Eq. (27a), one notices that at low energies σ_0 dominates while with increasing energies σ_2 overtakes.

Besides exactly verifying the sum rules which integrate to zero, we can also use the above derived sum rules to study the low-energy coefficients for light-by-light scattering in scalar QED. Using Eqs. (28a) and (28d), we obtain, for the tree-level contributions to the lowest order coefficients c_1 and c_2 in scalar QED,

$$c_1 = \frac{\alpha^2}{m^4}\frac{7}{1440}, \qquad c_2 = \frac{\alpha^2}{m^4}\frac{1}{1440}. \quad (31)$$

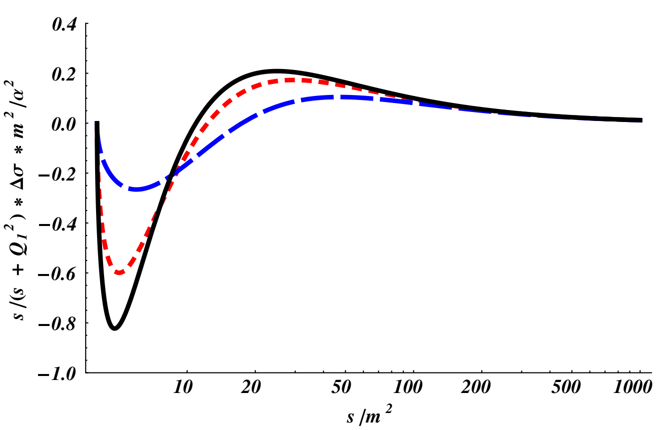

FIG. 1 (color online). The $\gamma^*\gamma \to \pi^+\pi^-$ tree-level result (scalar QED) for the integrand in the $\Delta\sigma \equiv \sigma_2 - \sigma_0$ sum rule of Eq. (27a), multiplied by s, where one of the photons is real. The different curves are for different virtualities for the other photon: $Q_1^2 = 0$ (solid black curve), $Q_1^2 = m^2$ (short-dashed red curve), and $Q_1^2 = 5m^2$ (long-dashed blue curve).

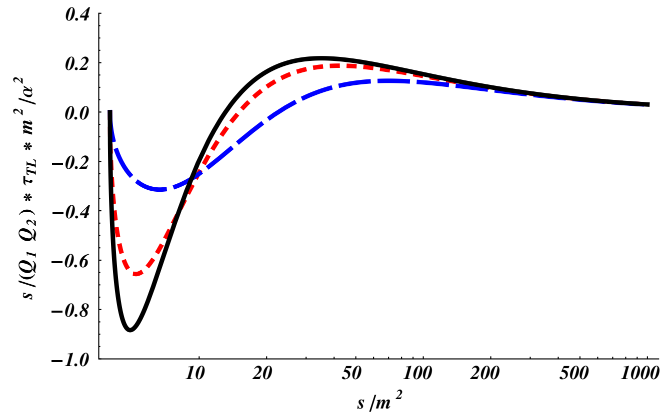

FIG. 2 (color online). The $\gamma^*\gamma \to \pi^+\pi^-$ tree-level result (scalar QED) for the integrand in the τ_{TL} sum rule of Eq. (27c) multiplied by s, where one of the photons is quasireal. The different curves are for different virtualities for the other photon: $Q_1^2 = 0$ (solid black curve), $Q_1^2 = m^2$ (short-dashed red curve), and $Q_1^2 = 5m^2$ (long-dashed blue curve).

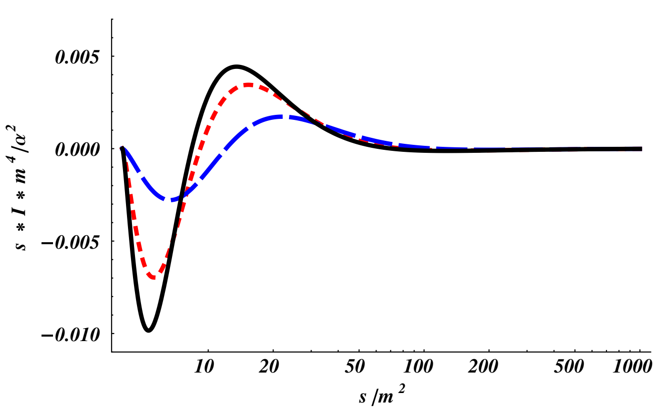

FIG. 3 (color online). The $\gamma^*\gamma \to \pi^+\pi^-$ tree-level result (scalar QED) for the integrand I in the sum rule of Eq. (27b) multiplied by s, with I given by Eq. (30), where one of the photons is quasireal. The different curves are for different virtualities for the other photon: $Q_1^2 = 0$ (solid black curve), $Q_1^2 = m^2$ (short-dashed red curve), and $Q_1^2 = 5m^2$ (long-dashed blue curve).

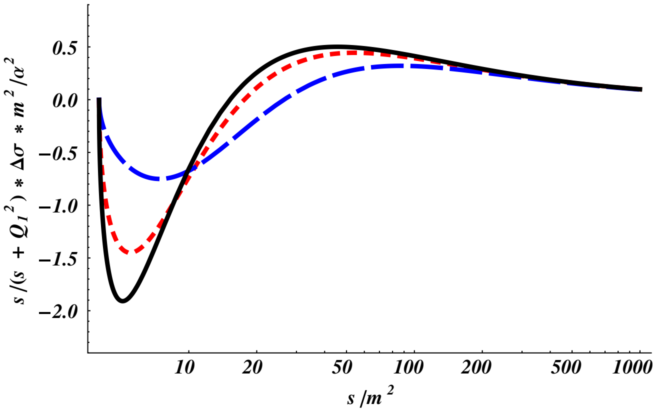

FIG. 4 (color online). The $\gamma^*\gamma \to e^+e^-$ tree-level result (spinor QED) for the integrand in the $\Delta\sigma \equiv \sigma_2 - \sigma_0$ sum rule of Eq. (27a), multiplied by s, where one of the photons is real. The different curves are for different virtualities for the other photon: $Q_1^2 = 0$ (solid black curve), $Q_1^2 = m^2$ (short-dashed red curve), and $Q_1^2 = 5m^2$ (long-dashed blue curve).

B. Spinor QED

The response functions for the case of spinor QED at lowest order in the electromagnetic coupling can be found in Appendix B 2. We again study the three sum rules of Eqs. (27a)–(27c) for the case of one real or quasireal photon ($Q_2^2 \to 0$) for different spacelike virtualities of the other photon. As the tree-level contribution to τ_{TL} in spinor QED vanishes for one quasireal photon, one notices that the sum rule of Eq. (27c) is trivially satisfied. For the sum rules involving the helicity difference of Eq. (27a), and involving the integrand I of Eq. (30), we show the corresponding integrands multiplied by s in Figs. 4 and 5 for the case of one real or quasireal photon and for different virtualities of the other photon. We again verify that the sum rules involve an exact cancellation between low- and high-energy contributions.

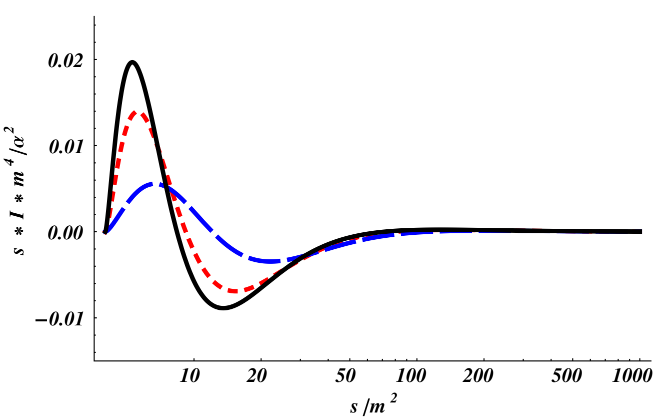

FIG. 5 (color online). The $\gamma^*\gamma \to e^+e^-$ tree-level result (spinor QED) for the integrand I in the sum rule of Eq. (27b) multiplied by s, with I given by Eq. (30), where one of the photons is quasireal. The different curves are for different virtualities for the other photon: $Q_1^2 = 0$ (solid black curve), $Q_1^2 = m^2$ (short-dashed red curve), and $Q_1^2 = 5m^2$ (long-dashed blue curve).

Using Eqs. (28a) and (28d), we obtain, for the tree-level contributions to the lowest order coefficients c_1 and c_2 for light-by-light scattering in spinor QED,

$$c_1 = \frac{\alpha^2}{m^4}\frac{1}{90}, \qquad c_2 = \frac{\alpha^2}{m^4}\frac{7}{360}. \tag{32}$$

In this case we also were able to verify the sum rule in Eq. (29), yielding

$$c_4 = -\frac{\alpha^2}{m^6}\frac{1}{315}, \tag{33}$$

in agreement with the result obtained in Ref. [16] for the low-energy photon-photon scattering.

A more detailed study of the LbL sum rules in field theory, including loop effects, production of vector bosons, etc., is the subject of our forthcoming publication [17].

IV. MESON PRODUCTION IN $\gamma\gamma$ COLLISION

In the previous section, the sum rules of Eqs. (27a)–(27c), integrating to zero, have been shown to hold exactly in perturbative calculations (e.g., in QED or QCD in the perturbative regime). However, as their derivation is general, their realization in QCD, in its nonperturbative regime, allows us to gain insight into the $\gamma^*\gamma \to$ hadrons cross sections. This was illustrated in Ref. [13] for the sum rule of Eq. (27a). In the remainder of this paper, we will elaborate on the discussion of Ref. [13] and extend it to the other sum rules presented above. The required nonperturbative input for the absorptive parts of the sum rules are the $\gamma^*\gamma \to$ hadrons response functions. In this paper, we will perform a first analysis by estimating the hadronic contributions to these response functions by the corresponding $\gamma^*\gamma^* \to M$ (with M a meson) production processes, which are described in terms of the $\gamma^*\gamma^* \to M$ transition form factors.

In Appendix C we detail the formalism and the available data for the $\gamma^*\gamma^* \to M$ transition FFs, and successively discuss the C-even pseudoscalar ($J^{PC} = 0^{-+}$), scalar ($J^{PC} = 0^{++}$), axial-vector ($J^{PC} = 1^{++}$), and tensor ($J^{PC} = 2^{++}$) mesons.

A. Real photons

We first consider the helicity sum rule of Eq. (27a) with two real photons producing a meson, as well as the sum rules of Eq. (28d) for the mesonic contributions to the low-energy constants c_1 and c_2 describing the forward light-by-light scattering amplitude. When producing mesons, the sum rules will hold separately for states of given intrinsic quantum numbers. Therefore, we will separately study the sum rule contributions for light-quark isovector mesons (Table I) and light-quark isoscalar mesons (Table II), as

TABLE I. $\gamma\gamma$ sum rule contributions of the light-quark isovector mesons based on the present PDG values [18] of the meson masses (m_M) and their 2γ decay widths $\Gamma_{\gamma\gamma}$. Fourth column: $\sigma_2 - \sigma_0$ sum rule of Eq. (27a). Fifth, sixth columns: c_1, c_2 sum rules of Eqs. (28a) and (28d), respectively.

	m_M (MeV)	$\Gamma_{\gamma\gamma}$ (keV)	$\int \frac{ds}{s}(\sigma_2 - \sigma_0)$ (nb)	c_1 (10^{-4} GeV^{-4})	c_2 (10^{-4} GeV^{-4})
π^0	134.9766 ± 0.0006	$(7.8 \pm 0.5) \times 10^{-3}$	-195 ± 13	0	10.94 ± 0.70
$a_0(980)$	980 ± 20	0.3 ± 0.1	-20 ± 8	0.021 ± 0.007	0
$a_2(1320)$	1318.3 ± 0.6	1.00 ± 0.06	134 ± 8	0.039 ± 0.002	0.039 ± 0.002
$a_2(1700)$	1732 ± 16	0.30 ± 0.05	18 ± 3	0.003 ± 0.001	0.003 ± 0.001
Sum			-63 ± 17	0.06 ± 0.01	10.98 ± 0.70

well as $c\bar{c}$ mesons (Table III). For the isoscalar mesons, one could, in principle, separate the contributions according to singlet or octet states [or, alternatively, according to $(u\bar{u} + d\bar{d})/\sqrt{2}$ or $s\bar{s}$ states]. The corresponding mesons involve mixings, however, which complicate such separation, as this mixing is not known well enough for some of the states. We will postpone such a separation for a future work and add all isoscalar meson contributions in the present work.

The pseudoscalar mesons contribute to the helicity-0 cross section only, given by Eq. (C4). The corresponding contributions to the helicity sum rule of Eq. (27a), as well as the c_1 and c_2 sum rules, are shown for the π^0 in Table I, for the η, η' in Table II, and for the $\eta_c(1S)$ state in Table III.

Besides the pseudoscalar mesons, scalar mesons can also only contribute to σ_0. We show the contributions of the $a_0(980)$ in Table I, of the $f_0(980)$ and $f_0'(1370)$ in Table II, and of the $\chi_{c0}(1P)$ state in Table III. For the scalar mesons, mainly the $f_0'(1370)$ state gives a sizable contribution due to its large 2γ decay width. An analogous analysis for the contribution of the $f_0(600)$ may yield a sizable, although largely uncertain, contribution. The uncertainty mainly comes from the relatively large range in the extracted values for the 2γ decay width $\Gamma_{\gamma\gamma}$, as well as the mass of the $f_0(600)$, as listed by the PDG [18]. Furthermore, given the large width of the $f_0(600)$, a strong interference with the $\gamma\gamma \to \pi\pi$ nonresonant background contribution is expected here. Estimating such a contribution goes beyond the narrow resonance analysis performed in this work and will require one to perform the sum rule analysis directly for multimeson channels ($\pi\pi, K\bar{K}, \ldots$). Such an analysis is certainly an interesting topic for future studies.

For the helicity sum rule, one notices that in order to compensate for the large negative contribution from the pseudoscalar mesons, and to a lesser extent from the scalar meson states, an equal strength is required in the helicity-2 cross section, σ_2. For light-quark mesons, the dominant feature of the helicity-2 cross section in the resonance region arises from the multiplet of tensor mesons $f_2(1270)$, $a_2(1320)$, and $f_2'(1525)$. For $c\bar{c}$ tensor mesons, the dominant tensor contribution is given by the $\chi_{c2}(1P)$ state.

Measurements at various e^+e^- colliders, notably the recent high-statistics measurements by the Belle Collaboration of the $\gamma\gamma$ cross sections to $\pi^+\pi^-$, $\pi^0\pi^0$, $\eta\pi^0$, and K^+K^- channels [19–21], have allowed us to accurately establish their parameters. For the light-quark mesons, the experimental analyses of decay angular distributions have found [22] that the tensor mesons are produced predominantly (around 95% or more) in a state of helicity $\Lambda = 2$. We will therefore assume in all of the following analyses that $\Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 0)) \approx 0$, and that $\Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 2)) \approx \Gamma_{\gamma\gamma}(\mathcal{T})$ in Tables I, II, and III. We show all tensor meson contributions to the helicity-difference sum rule, as well as the c_1, c_2 sum rules for which the 2γ decay widths are known.

TABLE II. $\gamma\gamma$ sum rule contributions of the light-quark isoscalar mesons based on the present PDG values [18] of the meson masses (m_M) and their 2γ decay widths $\Gamma_{\gamma\gamma}$. Fourth column: $\sigma_2 - \sigma_0$ sum rule of Eq. (27a). Fifth, sixth columns: c_1, c_2 sum rules of Eqs. (28a) and (28d), respectively.

	m_M (MeV)	$\Gamma_{\gamma\gamma}$ (keV)	$\int \frac{ds}{s}(\sigma_2 - \sigma_0)$ (nb)	c_1 (10^{-4} GeV^{-4})	c_2 (10^{-4} GeV^{-4})
η	547.853 ± 0.024	0.510 ± 0.026	-191 ± 10	0	0.65 ± 0.03
η'	957.78 ± 0.06	4.29 ± 0.14	-300 ± 10	0	0.33 ± 0.01
$f_0(980)$	980 ± 10	0.29 ± 0.07	-19 ± 5	0.020 ± 0.005	0
$f_0'(1370)$	1200–1500	3.8 ± 1.5	-91 ± 36	0.049 ± 0.019	0
$f_2(1270)$	1275.1 ± 1.2	3.03 ± 0.35	449 ± 52	0.141 ± 0.016	0.141 ± 0.016
$f_2'(1525)$	1525 ± 5	0.081 ± 0.009	7 ± 1	0.002 ± 0.000	0.002 ± 0.000
$f_2(1565)$	1562 ± 13	0.70 ± 0.14	56 ± 11	0.012 ± 0.002	0.012 ± 0.002
Sum			-89 ± 66	0.22 ± 0.03	1.14 ± 0.04

TABLE III. $\gamma\gamma$ sum rule contributions of the lowest $c\bar{c}$ mesons based on the present PDG values [18] of the meson masses (m_M) and their 2γ decay widths $\Gamma_{\gamma\gamma}$. Fourth column: the $\sigma_2 - \sigma_0$ sum rule of Eq. (27a), for which we also show the duality estimate of Eq. (37) for the continuum contribution above $D\bar{D}$ threshold, as well as the sum of resonances and continuum contributions. Fifth, sixth columns: c_1, c_2 sum rules of Eqs. (28a) and (28d), respectively.

	m_M (MeV)	$\Gamma_{\gamma\gamma}$ (keV)	$\int \frac{ds}{s}(\sigma_2 - \sigma_0)$ (nb)	c_1 (10^{-7} GeV^{-4})	c_2 (10^{-7} GeV^{-4})
$\eta_c(1S)$	2980.3 ± 1.2	6.7 ± 0.9	-15.6 ± 2.1	0	1.79 ± 0.24
$\chi_{c0}(1P)$	3414.75 ± 0.31	2.32 ± 0.13	-3.6 ± 0.2	0.31 ± 0.02	0
$\chi_{c2}(1P)$	3556.2 ± 0.09	0.50 ± 0.06	3.4 ± 0.4	0.14 ± 0.02	0.14 ± 0.02
Sum resonances			-15.8 ± 2.1	0.49 ± 0.03	1.97 ± 0.24
Duality estimate continuum ($\sqrt{s} \geq 2m_D$)			15.1		
Resonances + continuum			-0.7 ± 2.1		

For the isovector meson contributions to the helicity sum rule, shown in Table I, we conclude that the lowest isovector tensor meson composed of light quarks, $a_2(1320)$, compensates to around 70% the contribution of the π^0, which is entirely governed by the chiral anomaly. Including the $a_0(980)$ and $a_2(1700)$ meson contributions, one notices from Table I that the deviation from zero for the helicity sum rule is at the 3σ level. This is most likely due to additional, yet unmeasured, two-photon strength at higher energies in the tensor channel, contributing to σ_2. The perturbative calculation for the $\gamma\gamma \to q\bar{q}$ cross sections indeed shows that at higher energies σ_2 dominates over σ_0; see Fig. 4. Using a duality argument, one can then expect additional strength from tensor mesons at higher energies.

For the isoscalar states composed of light quarks, the cancellation is even more remarkable: the sum of $f_2(1270)$ and $f_2'(1525)$, within the experimental accuracy, nearly entirely compensates for the combined contribution of the η and η' mesons. Also here, one may expect the remaining strength to arise from tensor mesons at higher energies.

For the $c\bar{c}$ states, one notices that the known strength in the tensor channel from the $\chi_{c2}(1P)$ state only compensates for about 20% of the strength arising from the $\eta_c(1S)$ and $\chi_{c0}(1P)$ states. We can, however, expect a sizable contribution to this sum rule from states above the nearby $D\bar{D}$ threshold, which we denote by $s_D = 4m_D^2 \approx 14\ \mathrm{GeV}^2$, using the D-meson mass $m_D \approx 1.87$ GeV. So far, the helicity cross sections have not been measured above $D\bar{D}$ threshold. To estimate this continuum contribution to the helicity sum rule, which we denote by I_{cont}, we use a quark-hadron duality argument [23], which amounts to replacing the integral of the helicity-difference cross section for the $\gamma\gamma \to X$ process (with X any hadronic final state containing charm quarks) by the corresponding integral of the helicity-difference cross section for the perturbative $\gamma\gamma \to c\bar{c}$ process:

$$\begin{aligned} I_{\mathrm{cont}} &\equiv \int_{s_D}^{\infty} ds \frac{1}{s}[\sigma_2 - \sigma_0](\gamma\gamma \to X) \\ &\approx \int_{s_D}^{\infty} ds \frac{1}{s}[\sigma_2 - \sigma_0](\gamma\gamma \to c\bar{c}), \end{aligned} \tag{34}$$

where the perturbative cross section is given in Appendix B 2. The duality expressed by the approximate equality in Eq. (34) is meant to hold in a global sense, i.e. after integration over the energy of the helicity-difference cross section above the threshold s_D. As we have verified in Sec. III that the perturbative cross section satisfies the helicity sum rule exactly, i.e.

$$0 = \int_{4m_c^2}^{\infty} ds \frac{1}{s}[\sigma_2 - \sigma_0](\gamma\gamma \to c\bar{c}), \tag{35}$$

with m_c the charm-quark mass, we can reexpress Eq. (34) as

$$I_{\mathrm{cont}} \approx -\int_{4m_c^2}^{s_D} ds \frac{1}{s}[\sigma_2 - \sigma_0](\gamma\gamma \to c\bar{c}). \tag{36}$$

Using Eq. (B27) for the $\gamma\gamma \to c\bar{c}$ helicity-difference cross section, we finally obtain

$$\begin{aligned} I_{\mathrm{cont}} \approx -8\pi\alpha^2 \int_{4m_c^2}^{s_D} ds \frac{1}{s^2}\Bigg\{ -3\sqrt{1 - \frac{4m_c^2}{s}} \\ + 2\ln\left(\frac{\sqrt{s}}{2m_c}\left[1 + \sqrt{1 - \frac{4m_c^2}{s}}\right]\right)\Bigg\}. \end{aligned} \tag{37}$$

Using the PDG value $m_c \approx 1.27$ GeV [18], we show the duality estimate for $-I_{\mathrm{cont}}$ in Fig. 6, as a function of the integration limit s_D (solid red curve). Using the physical value of the $D\bar{D}$ threshold, $s_D \approx 14\ \mathrm{GeV}^2$, we obtain $I_{\mathrm{cont}} \approx 15.1$ nb. We notice that within the experimental uncertainty, this fully cancels the sum of the $\eta_c(1S)$, $\chi_{c0}(1P)$, and $\chi_{c2}(1P)$ resonance contributions to the $\sigma_2 - \sigma_0$ sum rule, as is shown in Table III. This cancellation quantitatively illustrates the interplay between resonances with hidden charm ($c\bar{c}$ states) and production of charmed mesons in order to satisfy the sum rule. It will be interesting to further test this experimentally by measuring the $\gamma\gamma$ production cross sections above $D\bar{D}$ threshold, where a plethora of new states (so-called XYZ states) have been found in recent years; see e.g. Ref. [24] for a review.

We have also computed the meson contributions to the forward light-by-light scattering coefficients c_1 and c_2 (fifth and sixth columns, respectively, in Tables I, II, and

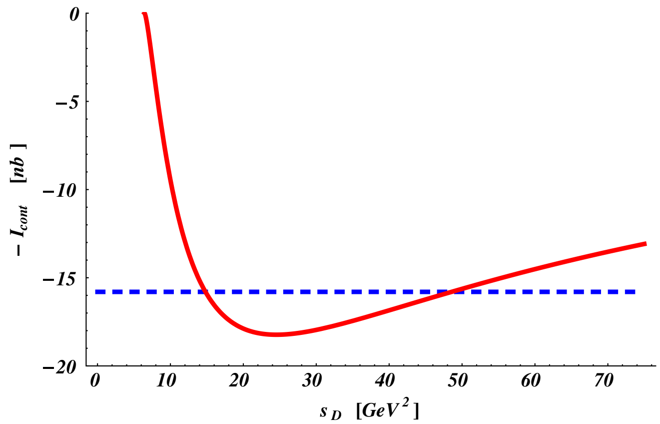

FIG. 6 (color online). Solid (red) curve: duality estimate for the negative of the continuum contribution of Eq. (37) to the helicity-difference sum rule for charm quarks as a function of the integration limit s_D, which represents the threshold for charmed meson production ($D\bar{D}$ threshold). For reference, the dashed (blue) horizontal curve indicates the sum of the $\eta_c(1S)$, $\chi_{c0}(1P)$, and $\chi_{c2}(1P)$ resonance contributions to the $\sigma_2 - \sigma_0$ sum rule, as listed in Table III. The intersection between both curves near the physical $D\bar{D}$ threshold, $s_D \approx 14$ GeV^2, indicates a perfect cancellation between these resonance contributions and the duality estimate for the continuum contribution.

III). The dimensionality of these coefficients requires them to scale with the meson mass m_M as $1/m_M^4$. Therefore, the higher mass mesons contribute very insignificantly to these coefficients. One notes that the coefficient c_1, which involves the cross section $\sigma_\parallel$, does not receive any contributions from pseudoscalar mesons and is dominated by the tensor mesons $a_2(1320)$ and $f_2(1270)$, with smaller contributions from the scalar states around 1 GeV. On the other hand, the coefficient c_2, which involves the cross section $\sigma_\perp$, is totally dominated by the contributions from pseudoscalar mesons, especially the light π^0, with contributions of η and η' at the 10% level of the π^0 contribution.

B. Virtual photons

We next discuss the sum rule of Eq. (27b) when both photons are quasireal. One immediately observes that pseudoscalar mesons do not contribute to this sum rule. However, scalar, axial-vector, and tensor mesons will contribute to this sum rule. The sum rule will therefore require a cancellation mechanism between scalar, axial-vector, and tensor mesons, which we will study subsequently. According to Eq. (C13), scalar mesons (with mass m_S) can only contribute to the $\sigma_\parallel$ term in the sum rule, and their contribution is given by

$$\int ds \frac{1}{s^2}[\sigma_\parallel]_{Q_1^2=Q_2^2=0} = 16\pi^2 \frac{\Gamma_{\gamma\gamma}(\mathcal{S})}{m_S^5}. \qquad (38)$$

In contrast, Eq. (C18) shows that axial-vector mesons (with mass m_A) can only contribute to the τ_{TL}^a term in the sum rule as

$$\int ds \frac{1}{s}\left[\frac{\tau_{TL}^a}{Q_1 Q_2}\right]_{Q_1^2=Q_2^2=0} = -8\pi^2 \frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A^5}, \qquad (39)$$

where we introduced the equivalent 2γ decay width $\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})$ of Eq. (C16).

The tensor mesons, in general, contribute to both terms of the sum rule of Eq. (27b). For the $\sigma_\parallel$ contribution, we will use the experimental observation that light tensor mesons are produced predominantly (around 95% or more) in a state of helicity $\Lambda = 2$, as discussed above. Neglecting, therefore, the much smaller σ_0 term, we obtain from Eq. (C28)

$$\int ds \frac{1}{s^2}[\sigma_\parallel]_{Q_1^2=Q_2^2=0} = \int ds \frac{1}{s^2}\frac{1}{2}[\sigma_2]_{Q_1^2=Q_2^2=0} = 8\pi^2 \frac{5\Gamma_{\gamma\gamma}(\mathcal{T})}{m_T^5}, \qquad (40)$$

with tensor meson mass m_T. For the τ_{TL}^a contribution to the sum rule of Eq. (27b), one sees from Eq. (C28) that it involves a helicity-1 amplitude for tensor meson production by quasireal photons, which unfortunately is not known experimentally for any tensor meson. It is reasonable to assume that for quasireal photons this amplitude is much smaller than the helicity-2 amplitude which is known to dominate in the real photon limit. We will therefore neglect the helicity-1 contribution in the following analysis.

One notes from Eqs. (38)–(40) that only axial-vector mesons give a negative contribution to the sum rule of Eq. (27b), whereas scalar and tensor mesons contribute positively. As the sum rule has to integrate to zero, one therefore obtains a cancellation mechanism between axial-vector mesons on one hand, and scalar and tensor mesons on the other. In Table IV, we show the contributions of the lowest-lying scalar, axial-vector, and tensor mesons, for which the 2γ widths are known experimentally. One sees from Table IV that the two lowest-lying axial-vector mesons $f_1(1285)$ and $f_1(1420)$ are entirely canceled, within error bars, by the contribution of the dominant tensor meson $f_2(1270)$. Using the experimentally known 2γ widths, the deviation of the (zero) sum rule value is at the 2σ level, which hints at a moderate contribution of either another higher mass axial-vector meson state or a nonresonant contribution with axial-vector quantum numbers.

At finite Q_1^2, for $Q_2^2 = 0$, the three sum rules of Eqs. (27a)–(27c) imply relations between the transition form factors for the contributing mesons. To date, experimental results for the $\gamma^*\gamma \to$ meson FFs exist for the pseudoscalar mesons π^0, η, η', and $\eta_c(1S)$, as well as for the axial-vector mesons $f_1(1285)$ and $f_1(1420)$. For other mesons, in particular, the tensor mesons, the corresponding form factors still wait to be extracted. We have seen from Table II that for real photons the dominant contributions to the helicity sum rule of Eq. (27a) come from η, η', and $f_2(1270)$ mesons, where the $f_2(1270)$

TABLE IV. Light isoscalar meson contributions to the sum rule of Eq. (27b) based on the present PDG values [18] of the meson masses (m_M) and their 2γ decay widths $\Gamma_{\gamma\gamma}$. For the axial-vector mesons, we quote the equivalent 2γ decay width $\tilde{\Gamma}_{\gamma\gamma}$ of Table VI. Fourth column: $\sigma_\parallel$ contribution; fifth column: τ^a_{TL} contribution; sixth column: total contribution to the sum rule of Eq. (27b).

	m_M (MeV)	$\Gamma_{\gamma\gamma}$ (keV)	$\int \frac{ds}{s^2}\sigma_\parallel(s)$ (nb/GeV^2)	$\int ds[\frac{1}{s}\frac{\tau^a_{TL}}{Q_1Q_2}]_{Q_i^2=0}$ (nb/GeV^2)	$\int ds[\frac{1}{s^2}\sigma_\parallel + \frac{1}{s}\frac{\tau^a_{TL}}{Q_1Q_2}]_{Q_i^2=0}$ (nb/GeV^2)
$f_1(1285)$	1281.8 ± 0.6	3.5 ± 0.8	0	-93 ± 21	-93 ± 21
$f_1(1420)$	1426.4 ± 0.9	3.2 ± 0.9	0	-50 ± 14	-50 ± 14
$f_0(980)$	980 ± 10	0.29 ± 0.07	20 ± 5	0	20 ± 5
$f_0'(1370)$	1200–1500	3.8 ± 1.5	48 ± 19	0	48 ± 19
$f_2(1270)$	1275.1 ± 1.2	3.03 ± 0.35	138 ± 16	$\gtrsim 0$	138 ± 16
$f_2'(1525)$	1525 ± 5	0.081 ± 0.009	1.5 ± 0.2	$\gtrsim 0$	1.5 ± 0.2
$f_2(1565)$	1562 ± 13	0.70 ± 0.14	12 ± 2	$\gtrsim 0$	12 ± 2
Sum					76 ± 36

contribution cancels to 90% the contribution from the η and η' mesons. We will therefore use the corresponding sum rule of Eq. (27a) at finite Q_1^2 to estimate the $\gamma^*\gamma \to f_2(1270)$ helicity-2 FF from the measured η and η' FFs, given by Eq. (C7). Assuming that the helicity sum rule of Eq. (27a) is saturated by the η, η', and $f_2(1270)$ mesons, we then obtain

$$\frac{5\Gamma_{\gamma\gamma}(f_2)}{m_{f_2}^3}\left[\frac{T_{f_2}^{(2)}(Q_1^2,0)}{T_{f_2}^{(2)}(0,0)}\right]^2 \simeq c_\eta \frac{1}{(1+Q_1^2/\Lambda_\eta^2)^2} + c_{\eta'}\frac{1}{(1+Q_1^2/\Lambda_{\eta'}^2)^2}, \quad (41)$$

where we have introduced the shorthand notation

$$c_P \equiv \frac{\Gamma_{\gamma\gamma}(\mathcal{P})}{m_P^3}. \quad (42)$$

For $Q_1^2 = 0$, the $f_2(1270)$ meson contribution cancels to 90% the $\eta + \eta'$ contributions to the helicity sum rule. We can therefore use

$$\frac{5\Gamma_{\gamma\gamma}(f_2)}{m_{f_2}^3} \simeq c_\eta + c'_\eta, \quad (43)$$

which allows us to express Eq. (41) as

$$\frac{T_{f_2}^{(2)}(Q_1^2,0)}{T_{f_2}^{(2)}(0,0)} \simeq \left[\frac{c_\eta}{c_\eta + c_{\eta'}}\frac{1}{(1+Q_1^2/\Lambda_\eta^2)^2} + \frac{c_{\eta'}}{c_\eta + c_{\eta'}}\frac{1}{(1+Q_1^2/\Lambda_{\eta'}^2)^2}\right]^{1/2}. \quad (44)$$

We can obtain a second estimate for the $T^{(2)}$ FF for the $f_2(1270)$ meson from the sum rule of Eq. (27b). We have seen from Table IV that for quasireal photons the dominant contributions to this sum rule come from $f_1(1285)$, $f_1(1420)$, and $f_2(1270)$ mesons, where the $f_2(1270)$ contribution cancels to 95% the contribution from the $f_1(1285)$ and $f_1(1420)$ mesons. We can then also use the corresponding sum rule of Eq. (27b) at finite Q_1^2 to estimate the $\gamma^*\gamma \to f_2(1270)$ helicity-2 FF from the measured $f_1(1285)$ and $f_1(1420)$ FFs, using Eqs. (C21) and (C24). Assuming that the helicity sum rule of Eq. (27b) is saturated by the $f_1(1285)$, $f_1(1420)$, and $f_2(1270)$ mesons, which we denote by f_1, f_1', and f_2, respectively, and retaining only the supposedly dominant $\Lambda = 2$ FF for the tensor mesons, we obtain

$$\frac{5\Gamma_{\gamma\gamma}(f_2)}{m_{f_2}^5}\frac{1}{\left(1+\frac{Q_1^2}{m_{f_2}^2}\right)}\left[\frac{T_{f_2}^{(2)}(Q_1^2,0)}{T_{f_2}^{(2)}(0,0)}\right]^2 \simeq c_{f_1}\frac{1}{(1+Q_1^2/\Lambda_{f_1}^2)^4} + c_{f_1'}\frac{1}{(1+Q_1^2/\Lambda_{f_1'}^2)^4}, \quad (45)$$

where

$$c_A \equiv \frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A^5}. \quad (46)$$

For $Q_1^2 = 0$, the $f_2(1270)$ meson contribution cancels to 95% the $f_1(1285) + f_1(1420)$ contributions to the sum rule of Eq. (27b), which implies

$$\frac{5\Gamma_{\gamma\gamma}(f_2)}{m_{f_2}^5} \simeq c_{f_1} + c_{f_1'}. \quad (47)$$

This allows us to obtain a second estimate for the $T^{(2)}$ FF for the $f_2(1270)$ meson as

$$\frac{T_{f_2}^{(2)}(Q_1^2,0)}{T_{f_2}^{(2)}(0,0)} \simeq \left(1+\frac{Q_1^2}{m_{f_2}^2}\right)^{1/2}\left[\frac{c_{f_1}}{c_{f_1}+c_{f_1'}}\frac{1}{(1+Q_1^2/\Lambda_{f_1}^2)^4} + \frac{c_{f_1'}}{c_{f_1}+c_{f_1'}}\frac{1}{(1+Q_1^2/\Lambda_{f_1'}^2)^4}\right]^{1/2}. \quad (48)$$

In Fig. 7 we show the two sum rule estimates of Eqs. (44) and (48) for the FF $T^{(2)}$ for the tensor meson $f_2(1270)$ using the known experimental information for either η, η' in Eq. (44), or $f_1(1285)$, $f_1(1420)$ in Eq. (48). When taking the ratio of both estimates, one sees that it is larger than 80% below 1 GeV^2 and about 65% around $Q^2 = 2$ GeV^2. It will be interesting to confront these estimates with a direct measurement of the $T^{(2)}$ FF for the $f_2(1270)$ tensor meson.

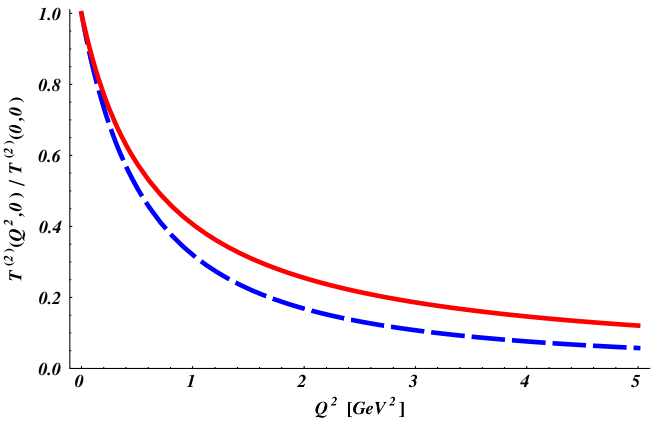

FIG. 7 (color online). Sum rule estimates for the form factor $T^{(2)}(Q^2, 0)/T^{(2)}(0, 0)$ with helicity $\Lambda = 2$ for the tensor meson $f_2(1270)$. Red solid curve: sum rule estimate from Eq. (44), using the experimental input from the η and η' FFs. Blue dashed curve: sum rule estimate from Eq. (48), using the experimental input from the $f_1(1285)$ and $f_1(1420)$ FFs.

In an analogous way, we can provide an estimate for the $a_2(1320)$ FF from the π^0 FF. We have seen from Table I that π^0 and $a_2(1320)$ provide the dominant isovector contributions to the helicity sum rule of Eq. (27a), where the $a_2(1320)$ contribution cancels to 70% the contribution from the π^0. We can therefore use the sum rule of Eq. (27a) for one virtual photon to estimate the helicity-2 FF $T^{(2)}$ for the $a_2(1320)$ meson in terms of the π^0 FF, given by Eq. (C7), as

$$\frac{T^{(2)}_{a_2}(Q_1^2, 0)}{T^{(2)}_{a_2}(0, 0)} \simeq \frac{1}{(1 + Q_1^2/\Lambda_\pi^2)}. \tag{49}$$

As, empirically, the $\gamma^*\gamma \to \pi^0$ FF is the best-known meson transition FF, it will be interesting to test the above prediction for the $a_2(1320)$ FF experimentally.

V. CONCLUSIONS AND OUTLOOK

We have studied the forward light-by-light scattering and derived three sum rules which involve energy weighted integrals of $\gamma^*\gamma$ fusion cross sections, measurable at e^+e^- colliders, which integrate to zero (superconvergence relations):

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s + Q_1^2)} [\sigma_0 - \sigma_2]_{Q_2^2=0},$$

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s + Q_1^2)^2} \left[\sigma_\parallel + \sigma_{LT} + \frac{(s + Q_1^2)}{Q_1 Q_2} \tau^a_{TL}\right]_{Q_2^2=0},$$

$$0 = \int_{s_0}^{\infty} ds \left[\frac{\tau_{TL}}{Q_1 Q_2}\right]_{Q_2^2=0}.$$

In these sum rules the $\gamma^*\gamma$ fusion cross sections are for one (quasi)real photon and a second virtual photon which can have arbitrary (spacelike) virtuality. The first of the sum rules generalizes the GDH sum rule for the helicity-difference $\gamma\gamma$ fusion cross section to the case of one real and one virtual photon. The two further sum rules are for $\gamma^*\gamma$ fusion cross sections which involve longitudinal photon amplitudes.

We have shown that these sum rules are exactly verified for the tree-level scalar and spinor QED cross sections. Verifications beyond the tree level in various field theories are underway [17].

We have performed a detailed quantitative study of the new sum rules for the case of the production of light-quark mesons as well as for the production of mesons in the charm-quark sector. Using the empirical information in evaluating the sum rules, we have found that the helicity-difference sum rule requires cancellations between different mesons, implying nonperturbative relations. For the light-quark isovector mesons, the π^0 contribution was found to be compensated to around 70% by the contribution of the lowest-lying isovector tensor meson $a_2(1320)$. For the isoscalar light-quark mesons, the η and η' contributions were found to be entirely compensated within the experimental accuracy by the two lowest-lying tensor mesons $f_2(1270)$ and $f_2'(1525)$. In the charm-quark sector, the situation is different, as it involves the narrow resonance contributions below $D\bar{D}$ threshold and the continuum contribution above $D\bar{D}$ threshold. For the narrow resonances, the η_c was found to give, by far, the dominant contribution. When using a duality estimate for the continuum contribution, we found that it entirely cancels the narrow resonance contributions, verifying the sum rule, and pointing to large tensor strength (helicity 2) in the cross sections above $D\bar{D}$ threshold. It will be interesting to test this property experimentally.

The helicity-difference sum rule has also been applied for the case of one real and one virtual photon. In this case the $\gamma^*\gamma$ fusion cross sections depend on the meson transition FFs. We have reviewed the general formalism and parametrization for the $\gamma^*\gamma \to$ meson transition FFs for (pseudo)scalar, axial-vector, and tensor mesons. Because for scalar and tensor mesons the $\gamma^*\gamma$ transition FFs have not yet been measured, a direct test of the sum rules for finite virtuality is not possible at present. However, we were able to show that the helicity-difference sum rule allows one to provide an estimate for the $f_2(1270)$ tensor FF in terms of the η and η' FFs, and for the $a_2(1320)$ tensor FF in terms of the π^0 FF. Since empirical information on pseudoscalar meson FFs is available, these relations provide predictions for tensor meson FFs which will be interesting to confront with experiment.

The second new sum rule derived in this paper, involving the $\sigma_\parallel$, σ_{LT}, and τ^a_{TL} $\gamma^*\gamma$ response functions, has also been tested for the case of quasireal photons. As pseudoscalar mesons cannot contribute to this sum rule, a cancellation between scalar and tensor mesons on one hand and axial-vector mesons on the other hand is at work. Using the existing empirical information for quasireal photons, the contribution of the two lowest-lying axial-vector mesons $f_1(1285)$ and $f_1(1420)$ was found to be entirely canceled, within error bars, by the contribution of the dominant tensor meson $f_2(1270)$. When applying this sum rule to the case of one virtual photon, it again allows one to relate the $f_2(1270)$ tensor FF, this time to the transition FFs for the $f_1(1285)$ and $f_1(1420)$ mesons, which have both been measured. The predictions from the two different sum rules for the $f_2(1270)$ FF were found to agree within 20% for a virtuality below 1 GeV^2, and within 35% up to about 2 GeV^2.

Besides the three superconvergence relations, we have also derived sum rules which express the coefficients in a low-energy expansion of the forward light-by-light scattering amplitude in terms of $\gamma^*\gamma \to X$ cross sections. These evaluations may be used as a cross-check for models of the nonforward light-by-light scattering which are applied to evaluate the hadronic LbL contribution to $(g-2)_\mu$.

On the experimental side, the ongoing $\gamma\gamma$ physics programs by the *BABAR* and Belle Collaborations, as well as the upcoming $\gamma\gamma$ physics program by the BES-III Collaboration, will allow us to further improve the data situation significantly. In particular, the extraction of the $\gamma^*\gamma$ response functions through their different azimuthal angular dependencies, and the measurements of multimeson final states ($\pi\pi$, $\pi\eta$, ...) promise to access, besides the pseudoscalar meson FFs, also the scalar, axial-vector, and tensor meson FFs, thus allowing direct tests of the sum rule predictions presented in this work.

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APPENDIX A: KINEMATICS AND CROSS SECTIONS OF THE $e^\pm + e^- \to e^\pm + e^- + X$ PROCESS

In this appendix we detail the kinematics of the $e^\pm + e^- \to e^\pm + e^- + X$ process and show how the cross sections accessible in experiments are expressed in terms of the $\gamma^*\gamma^* \to X$ cross sections, which enter in the sum rules discussed in the present work.

The kinematics of the process $e(p_1) + e(p_2) \to e(p_1') + e(p_2') + X$, with X the produced hadronic state, in the c.m. system of the colliding beams (which we denote by c.m. ee), is characterized by the four-vectors of the incoming leptons:

$$p_1(E, \vec{p}_1), \qquad p_2(E, -\vec{p}_1), \tag{A1}$$

with beam energy $E = \sqrt{s}/2$ and $s = (p_1 + p_2)^2$. The kinematics of the outgoing leptons can be related to the virtual photon four-momenta as

$$q_1 = p_1 - p_1', \qquad q_2 = p_2 - p_2'. \tag{A2}$$

The kinematics of the outgoing leptons then determines five kinematical quantities:

(i) the energies of both virtual photons:

$$\omega_1 \equiv q_1^0 = E - E_1', \qquad \omega_2 = q_2^0 \equiv E - E_2', \tag{A3}$$

with E_1' and E_2' the energies of both outgoing leptons;

(ii) the virtualities of both virtual photons:

$$\begin{aligned} Q_1^2 &\equiv -q_1^2 = 4EE_1'\sin^2\theta_1/2 + Q_{1,\min}^2, \\ Q_2^2 &\equiv -q_2^2 = 4EE_2'\sin^2\theta_2/2 + Q_{2,\min}^2, \end{aligned} \tag{A4}$$

where θ_1 and θ_2 are the (polar) angles of the scattered electrons relative to the respective beam directions, and where the minimal values of the virtualities are given by (in the limit where $E_1' \gg m$ and $E_2' \gg m$, with m the lepton mass)

$$Q_{1,\min}^2 \simeq m^2 \frac{\omega_1^2}{EE_1'}, \qquad Q_{2,\min}^2 \simeq m^2 \frac{\omega_2^2}{EE_2'}; \tag{A5}$$

(iii) the azimuthal angle ϕ between both lepton planes, which in the lepton c.m. frame can be obtained as

$$(\cos\phi)_{\text{c.m.}ee} \equiv -\frac{p_{1\perp}' \cdot p_{2\perp}'}{[(p_{1\perp}')^2 (p_{2\perp}')^2]^{1/2}}, \tag{A6}$$

where $p_{1\perp}'$ and $p_{2\perp}'$ denote the components of the outgoing lepton four-vectors which are perpendicular to the respective beam directions, and are defined in the lepton c.m. frame as

$$\begin{aligned} (p_{1\perp}')^\mu &= -R^{\mu\nu}(p_1, p_2)(p_1')_\nu, \\ (p_{2\perp}')^\mu &= -R^{\mu\nu}(p_1, p_2)(p_2')_\nu, \end{aligned} \tag{A7}$$

with

$$\begin{aligned} R^{\mu\nu}(p_1, p_2) = -g^{\mu\nu} + \frac{1}{[(p_1 \cdot p_2)^2 - m^4]} &\times \{(p_1 \cdot p_2)(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \\ &- m^2(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu)\}. \end{aligned} \tag{A8}$$

In the following it will also turn out to be useful to determine kinematical quantities in the c.m. system of the virtual photons (which we denote by c.m. $\gamma\gamma$). In particular, the azimuthal angle between both lepton planes, in the $\gamma\gamma$ c.m. frame, which we denote by $\tilde{\phi}$, is given by

$$\cos\tilde{\phi} \equiv -\frac{\tilde{p}_{1\perp} \cdot \tilde{p}_{2\perp}}{[(\tilde{p}_{1\perp})^2 (\tilde{p}_{2\perp})^2]^{1/2}}, \tag{A9}$$

where $\tilde{p}_{1\perp}$ and $\tilde{p}_{2\perp}$ denote the transverse components of the incoming lepton four-vectors in the $\gamma\gamma$ c.m. frame and are defined in a covariant way as

$$(\tilde{p}_{1\perp})^\mu = -R^{\mu\nu}(q_1, q_2)(p_1)_\nu,$$
$$(\tilde{p}_{2\perp})^\mu = -R^{\mu\nu}(q_1, q_2)(p_2)_\nu, \quad \text{(A10)}$$

with

$$R^{\mu\nu}(q_1, q_2) = -g^{\mu\nu} + \frac{1}{[(q_1\cdot q_2)^2 - q_1^2 q_2^2]}\{(q_1\cdot q_2) \times (q_1^\mu q_2^\nu + q_2^\mu q_1^\nu) - q_1^2 q_2^\mu q_2^\nu - q_2^2 q_1^\mu q_1^\nu\}. \quad \text{(A11)}$$

As the rhs of Eq. (A9) is expressed in a Lorentz invariant way, one can then evaluate all four-momenta in the lepton c.m. frame, to obtain the expression of $\cos\tilde{\phi}$ in terms of the lepton c.m. kinematics.

The cross section for the process $e(p_1) + e(p_2) \to e(p_1') + e(p_2') + X$, with X the produced hadronic state, can be expressed in terms of eight cross sections for the $\gamma^*\gamma^* \to X$ process, which were defined in Eq. (16), as

$$d\sigma = \frac{\alpha^2}{16\pi^4 Q_1^2 Q_2^2}\frac{2\sqrt{X}}{s(1-4m^2/s)}\cdot\frac{d^3\vec{p}_1'}{E_1'}\cdot\frac{d^3\vec{p}_2'}{E_2'}\{4\rho_1^{++}\rho_2^{++}\sigma_{TT} + \rho_1^{00}\rho_2^{00}\sigma_{LL} + 2\rho_1^{++}\rho_2^{00}\sigma_{TL} + 2\rho_1^{00}\rho_2^{++}\sigma_{LT} + 2(\rho_1^{++}-1)(\rho_2^{++}-1)(\cos 2\tilde{\phi})\tau_{TT} + 8\left[\frac{(\rho_1^{00}+1)(\rho_2^{00}+1)}{(\rho_1^{++}-1)(\rho_2^{++}-1)}\right]^{1/2}(\cos\tilde{\phi})\tau_{TL} + h_1 h_2 4[(\rho_1^{00}+1)(\rho_2^{00}+1)]^{1/2}\tau_{TT}^a + h_1 h_2 8[(\rho_1^{++}-1)(\rho_2^{++}-1)]^{1/2}(\cos\tilde{\phi})\tau_{TL}^a\}, \quad \text{(A12)}$$

where $h_1 = \pm 1$ and $h_2 = \pm 1$ are both lepton beam helicities, and where we have defined kinematical coefficients:

$$\rho_1^{++} = \frac{1}{2}\left\{1 - \frac{4m^2}{Q_1^2} + \frac{1}{X}(2p_1\cdot q_2 - \nu)^2\right\},$$
$$\rho_2^{++} = \frac{1}{2}\left\{1 - \frac{4m^2}{Q_2^2} + \frac{1}{X}(2p_2\cdot q_1 - \nu)^2\right\},$$
$$\rho_1^{00} = \frac{1}{X}(2p_1\cdot q_2 - \nu)^2 - 1, \quad \text{(A13)}$$
$$\rho_2^{00} = \frac{1}{X}(2p_2\cdot q_1 - \nu)^2 - 1.$$

APPENDIX B: TREE-LEVEL $\gamma^*\gamma^*$ CROSS SECTIONS IN QED

1. Scalar QED

The $\gamma^*\gamma^* \to \mathcal{S}\bar{\mathcal{S}}$ cross sections (with $\mathcal{S}$ an electrically charged structureless scalar particle) to lowest order in α are given by

$$\sigma_0 + \sigma_2 = \sigma_\parallel + \sigma_\perp = \alpha^2\frac{\pi}{2}\frac{s^2\nu^3}{X^3}\left\{\sqrt{a}\left[2 - a - \left(1 - \frac{2X}{s\nu}\right)^2\right] - (1-a)\left(3 - \frac{4X}{s\nu} + a\right)L\right\}, \quad \text{(B1)}$$

$$\sigma_\parallel - \sigma_\perp = \alpha^2\frac{\pi}{4}\frac{s^2\nu^3}{X^3}\left\{\sqrt{a}\left[1 - a + 2\left(1 - \frac{2X}{s\nu}\right)^2\right] - (1-a)\left(3 - \frac{8X}{s\nu} + a\right)L\right\}, \quad \text{(B2)}$$

$$\sigma_0 - \sigma_2 = \alpha^2 2\pi\frac{s\nu^2}{X^2}\left\{-\sqrt{a}\left(1 - \frac{X}{s\nu}\right) + (1-a)L\right\}, \quad \text{(B3)}$$

$$\sigma_{LL} = \alpha^2\pi Q_1^2 Q_2^2\frac{s^2\nu}{X^3}\left\{\sqrt{a}\left[2 + \frac{1}{1-a}\left(1 - \frac{X}{s\nu}\right)^2\right] - \left(3 + \frac{X}{s\nu}\right)\left(1 - \frac{X}{s\nu}\right)L\right\}, \quad \text{(B4)}$$

$$\sigma_{LT} = \alpha^2\frac{\pi}{2}Q_1^2\frac{s\nu(\nu - Q_2^2)^2}{X^3}\{-3\sqrt{a} + (3-a)L\}, \quad \text{(B5)}$$

$$\tau_{TL} = \alpha^2\frac{\pi}{2}Q_1 Q_2\frac{s\nu}{X^2}\left\{-\sqrt{a} + \left(1 - \frac{2X}{s\nu} + a\right)L\right\}, \quad \text{(B6)}$$

$$\tau_{TL}^a = \alpha^2\frac{\pi}{2}Q_1 Q_2\frac{s^2\nu^2}{X^3}\left\{\sqrt{a}\left(3 - \frac{4X}{s\nu}\right) - \left[1 - a + 2\left(1 - \frac{X}{s\nu}\right)^2\right]L\right\}, \quad \text{(B7)}$$

with

$$L \equiv \ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right), \qquad a \equiv \frac{X}{\nu^2}\left(1 - \frac{4m^2}{s}\right). \quad \text{(B8)}$$

In the limit where one of the virtual photons becomes real ($Q_2^2 = 0$) in case of the response functions involving only transverse photons, or becomes quasireal ($Q_2^2 \approx 0$) in case of the response functions involving a longitudinal photon, the above expressions simplify to

$$[\sigma_\parallel + \sigma_\perp]_{Q_2^2=0} = \alpha^2 4\pi\frac{s^2}{(s+Q_1^2)^3}\left\{\sqrt{1 - \frac{4m^2}{s}}\left(1 + \frac{4m^2}{s} + \frac{Q_1^4}{s^2}\right) - \frac{8m^2}{s}\left(1 - \frac{2m^2}{s} - \frac{Q_1^2}{s}\right)L\right\}, \quad \text{(B9)}$$

$$[\sigma_\parallel - \sigma_\perp]_{Q_2^2=0} = \alpha^2 4\pi \frac{s^2}{(s+Q_1^2)^3}\left\{\sqrt{1-\frac{4m^2}{s}}\left(\frac{2m^2}{s}+\frac{Q_1^4}{s^2}\right) + \frac{8m^2}{s}\left(\frac{m^2}{s}+\frac{Q_1^2}{s}\right)L\right\}, \tag{B10}$$

$$[\sigma_0 - \sigma_2]_{Q_2^2=0} = \alpha^2 4\pi \frac{s}{(s+Q_1^2)^2}\left\{-\sqrt{1-\frac{4m^2}{s}}\left(1-\frac{Q_1^2}{s}\right) + \frac{8m^2}{s}L\right\}, \tag{B11}$$

$$\left[\frac{1}{Q_1^2 Q_2^2}\sigma_{LL}\right]_{Q_2^2=0} = \alpha^2 8\pi \frac{s^2}{(s+Q_1^2)^5}\left\{\sqrt{1-\frac{4m^2}{s}} \times \left(8+\frac{s}{4m^2}\left(1-\frac{Q_1^2}{s}\right)^2\right) - \left(7+\frac{Q_1^2}{s}\right)\left(1-\frac{Q_1^2}{s}\right)L\right\}, \tag{B12}$$

$$\left[\frac{1}{Q_1^2}\sigma_{LT}\right]_{Q_2^2=0} = \alpha^2 4\pi \frac{s}{(s+Q_1^2)^3}\left\{-3\sqrt{1-\frac{4m^2}{s}} + 2\left(1+\frac{2m^2}{s}\right)L\right\}, \tag{B13}$$

$$\left[\frac{1}{Q_1 Q_2}\tau_{TL}\right]_{Q_2^2=0} = \alpha^2 4\pi \frac{s}{(s+Q_1^2)^3}\left\{-\sqrt{1-\frac{4m^2}{s}} + \left(1-\frac{Q_1^2}{s}-\frac{4m^2}{s}\right)L\right\}, \tag{B14}$$

$$\left[\frac{1}{Q_1 Q_2}\tau^a_{TL}\right]_{Q_2^2=0} = \alpha^2 8\pi \frac{s^2}{(s+Q_1^2)^4}\left\{\sqrt{1-\frac{4m^2}{s}}\left(1-\frac{2Q_1^2}{s}\right) - \left[\frac{1}{2}\left(1-\frac{Q_1^2}{s}\right)^2+\frac{4m^2}{s}\right]L\right\}, \tag{B15}$$

with

$$[L]_{Q_2^2=0} = \ln\left(\frac{\sqrt{s}}{2m}\left[1+\sqrt{1-\frac{4m^2}{s}}\right]\right). \tag{B16}$$

2. Spinor QED

The $\gamma^*\gamma^* \to q\bar{q}$ cross sections (with q an electrically charged structureless spin-1/2 particle) to lowest order in α are given by

$$\sigma_0+\sigma_2 = \sigma_\parallel + \sigma_\perp = \alpha^2\pi\frac{s^2\nu^3}{X^3}\left\{\sqrt{a}\left[-4\left(1-\frac{X}{s\nu}\right)^2-(1-a) + \frac{Q_1^2Q_2^2}{\nu^2}\left(2-\frac{1}{(1-a)}\frac{4X^2}{s^2\nu^2}\right)\right] + \left[3-a^2+2\left(1-\frac{2X}{s\nu}\right)^2-\frac{2Q_1^2Q_2^2}{\nu^2}(1+a)\right]L\right\}, \tag{B17}$$

$$\sigma_\parallel - \sigma_\perp = \alpha^2\frac{\pi}{2}\frac{s^2\nu^3}{X^3}\left\{\sqrt{a}\left[-(1-a)-2\left(1-\frac{2X}{s\nu}\right)^2\right] + \left[-(1-a)^2+4(1-a)\left(1-\frac{2X}{s\nu}\right) + \frac{Q_1^2Q_2^2}{\nu^2}\frac{8X^2}{s^2\nu^2}\right]L\right\}, \tag{B18}$$

$$\sigma_0-\sigma_2 = \alpha^2 4\pi\frac{s\nu^2}{X^2}\left\{\sqrt{a}\left[2-\frac{X}{s\nu}-\frac{Q_1^2Q_2^2}{\nu^2}\frac{1}{(1-a)}\frac{X}{s\nu}\right] - 2\left(1-\frac{X}{s\nu}\right)L\right\}, \tag{B19}$$

$$\sigma_{LL} = \alpha^2 2\pi Q_1^2Q_2^2\frac{s^2}{\nu X^2}\left\{\sqrt{a}\left[-2-\frac{(3-2a)}{(1-a)}\frac{Q_1^2Q_2^2}{X}\right] + \left(2+\frac{3Q_1^2Q_2^2}{X}\right)L\right\}, \tag{B20}$$

$$\sigma_{LT} = \alpha^2\pi Q_1^2\frac{s}{\nu X^2}\left\{\sqrt{a}\left[(\nu-Q_2^2)^2\left(2+\frac{3Q_1^2Q_2^2}{X}\right) - 2\nu Q_2^2 + Q_2^4\frac{(3-a)}{(1-a)}\right] + \left[(\nu-Q_2^2)^2\left(-2(1-a) - (3-a)\frac{Q_1^2Q_2^2}{X}\right) + 2\nu Q_2^2(1+a) - Q_2^4(3+a)\right]L\right\}, \tag{B21}$$

$$\tau_{TL} = \alpha^2 2\pi(Q_1Q_2)^3\frac{s}{\nu X^2}\left\{\frac{\sqrt{a}}{1-a}-L\right\}, \tag{B22}$$

$$\tau^a_{TL} = \alpha^2\pi Q_1Q_2\frac{s^2\nu^2}{X^3}\left\{-\sqrt{a}\left(3-\frac{4X}{s\nu}\right)+\left(3-\frac{4X}{s\nu}-a\right)L\right\}, \tag{B23}$$

with

$$L \equiv \ln\left(\frac{1+\sqrt{a}}{\sqrt{1-a}}\right), \qquad a \equiv \frac{X}{\nu^2}\left(1-\frac{4m^2}{s}\right). \tag{B24}$$

In the limit where one of the virtual photons becomes real ($Q_2^2 = 0$) in case of the response functions involving only transverse photons, or becomes quasireal ($Q_2^2 \approx 0$) in case of the response functions involving a longitudinal photon, the above expressions simplify to

$$[\sigma_\parallel + \sigma_\perp]_{Q_2^2=0} = \alpha^2 8\pi \frac{s^2}{(s+Q_1^2)^3}\Bigg\{\sqrt{1-\frac{4m^2}{s}} \times\Bigg[-\Big(1-\frac{Q_1^2}{s}\Big)^2 - \frac{4m^2}{s}\Bigg] + 2\Big(1+\frac{4m^2}{s}-\frac{8m^4}{s^2}+\frac{Q_1^4}{s^2}\Big)L\Bigg\}, \quad \text{(B25)}$$

$$[\sigma_\parallel - \sigma_\perp]_{Q_2^2=0} = -\alpha^2 8\pi \frac{s^2}{(s+Q_1^2)^3}\Bigg\{\sqrt{1-\frac{4m^2}{s}} \times\Big(\frac{2m^2}{s}+\frac{Q_1^4}{s^2}\Big) + \frac{8m^2}{s}\Big(\frac{m^2}{s}+\frac{Q_1^2}{s}\Big)L\Bigg\}, \quad \text{(B26)}$$

$$[\sigma_0 - \sigma_2]_{Q_2^2=0} = \alpha^2 8\pi \frac{s}{(s+Q_1^2)^2}\Bigg\{\sqrt{1-\frac{4m^2}{s}}\Big(3-\frac{Q_1^2}{s}\Big) - 2\Big(1-\frac{Q_1^2}{s}\Big)L\Bigg\}, \quad \text{(B27)}$$

$$\Bigg[\frac{1}{Q_1^2Q_2^2}\sigma_{LL}\Bigg]_{Q_2^2=0} = \alpha^2 128\pi \frac{s^2}{(s+Q_1^2)^5}\Bigg\{-\sqrt{1-\frac{4m^2}{s}}+L\Bigg\}, \quad \text{(B28)}$$

$$\Bigg[\frac{1}{Q_1^2}\sigma_{LT}\Bigg]_{Q_2^2=0} = \alpha^2 16\pi \frac{s}{(s+Q_1^2)^3}\Bigg\{\sqrt{1-\frac{4m^2}{s}}-\frac{4m^2}{s}L\Bigg\}, \quad \text{(B29)}$$

$$\Bigg[\frac{1}{Q_1Q_2}\tau_{TL}\Bigg]_{Q_2^2=0} = 0, \quad \text{(B30)}$$

$$\Bigg[\frac{1}{Q_1Q_2}\tau_{TL}^a\Bigg]_{Q_2^2=0} = \alpha^2 16\pi \frac{s^2}{(s+Q_1^2)^4}\Bigg\{-\sqrt{1-\frac{4m^2}{s}} \times\Big(1-\frac{2Q_1^2}{s}\Big) + \Big(-\frac{2Q_1^2}{s}+\frac{4m^2}{s}\Big)L\Bigg\}, \quad \text{(B31)}$$

with

$$[L]_{Q_2^2=0} = \ln\Bigg(\frac{\sqrt{s}}{2m}\Bigg[1+\sqrt{1-\frac{4m^2}{s}}\Bigg]\Bigg). \quad \text{(B32)}$$

APPENDIX C: $\gamma^*\gamma^* \to$ MESON TRANSITION FORM FACTORS

In this appendix we detail the formalism and the available data for the $\gamma^*\gamma^* \to$ meson transition FFs, and successively discuss the C-even pseudoscalar ($J^{PC} = 0^{-+}$), scalar ($J^{PC} = 0^{++}$), axial-vector ($J^{PC} = 1^{++}$), and tensor ($J^{PC} = 2^{++}$) mesons.

1. Pseudoscalar mesons

The process $\gamma^*(q_1, \lambda_1) + \gamma^*(q_2, \lambda_2) \to \mathcal{P}$, describing the transition from an initial state of two virtual photons, with four-momenta q_1, q_2 and helicities $\lambda_1, \lambda_2 = 0, \pm 1$, to a pseudoscalar meson $\mathcal{P} = \pi^0, \eta, \eta', \eta_c, \ldots (J^{PC} = 0^{-+})$ with mass m_P, is described by the matrix element

$$\mathcal{M}(\lambda_1, \lambda_2) = -ie^2\varepsilon_{\mu\nu\alpha\beta}\varepsilon^\mu(q_1, \lambda_1)\varepsilon^\nu(q_2, \lambda_2)q_1^\alpha q_2^\beta \times F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, Q_2^2), \quad \text{(C1)}$$

where $\varepsilon^\alpha(q_1, \lambda_1)$ and $\varepsilon^\beta(q_2, \lambda_2)$ are the polarization vectors of the virtual photons, and where the meson structure information is encoded in the FF $F_{\mathcal{P}\gamma^*\gamma^*}$, which is a function of the virtualities of both photons, satisfying $F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{\mathcal{P}\gamma^*\gamma^*}(Q_2^2, Q_1^2)$. From Eq. (C1), one can easily deduce that the only nonzero $\gamma^*\gamma^* \to \mathcal{P}$ helicity amplitudes, which we define in the rest frame of the produced meson, are given by

$$\mathcal{M}(\lambda_1 = +1, \lambda_2 = +1) = -\mathcal{M}(\lambda_1 = -1, \lambda_2 = -1) = -e^2\sqrt{X}F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, Q_2^2). \quad \text{(C2)}$$

The FF at $Q_1^2 = Q_2^2 = 0$, $F_{\mathcal{P}\gamma^*\gamma^*}(0, 0)$, describes the two-photon decay width of the pseudoscalar meson:

$$\Gamma_{\gamma\gamma}(\mathcal{P}) = \frac{\pi\alpha^2}{4}m_P^3|F_{\mathcal{P}\gamma^*\gamma^*}(0, 0)|^2, \quad \text{(C3)}$$

with m_P the pseudoscalar meson mass, and $\alpha = e^2/(4\pi) \simeq 1/137$.

In this paper, we study the sum rules involving cross sections for one real photon and one virtual photon. For one real photon ($Q_2^2 = 0$), the only nonvanishing cross sections in Eq. (16) are given by

$$[\sigma_0]_{Q_2^2=0} = [\sigma_\perp]_{Q_2^2=0} = 2[\sigma_{TT}]_{Q_2^2=0} = -[\tau_{TT}]_{Q_2^2=0} = \delta(s-m_P^2)16\pi^2\frac{\Gamma_{\gamma\gamma}(\mathcal{P})}{m_P}\Big(1+\frac{Q_1^2}{m_P^2}\Big) \times\Bigg[\frac{F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, 0)}{F_{\mathcal{P}\gamma^*\gamma^*}(0, 0)}\Bigg]^2. \quad \text{(C4)}$$

For massless quarks, the divergence of the isovector axial current, $A_3^\mu \equiv \frac{1}{\sqrt{2}}(\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d)$, does not vanish but exhibits an anomaly due to the triangle graphs which allow the π^0 to couple to two vector currents (Wess-Zumino-Witten anomaly). For the π^0 the chiral (isovector axial) anomaly predicts that its transition FF at $Q_1^2 = Q_2^2 = 0$ is given by

$$F_{\pi^0\gamma^*\gamma^*}(0, 0) = \frac{1}{4\pi^2 f_\pi}, \quad \text{(C5)}$$

where the pion decay constant f_π is defined through the isovector axial current matrix element:

$$\langle 0|A_3^\mu(0)|\pi^0(p)\rangle = i(\sqrt{2}f_\pi)p^\mu. \quad \text{(C6)}$$

When using the current empirical value of the pion decay constant $f_\pi \simeq 92.4$ MeV to evaluate the chiral anomaly prediction of Eq. (C5), one obtains the value $F_{M\gamma^*\gamma}(0) \simeq 0.274\ \mathrm{GeV}^{-1}$, which yields, through Eq. (C3), a 2γ decay width in very good agreement with the experimental value (see Table I).

The form factors $F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, 0)$ for one virtual photon and one real photon have been measured for π^0, η, η' by the CELLO [27], CLEO [28], and *BABAR* [8,29] Collaborations, and for $\eta_c(1S)$ by the *BABAR* Collaboration [30]. In the Q_1^2 range up to 10 GeV^2, a good parametrization of the data is obtained by the monopole form

$$\frac{F_{\mathcal{P}\gamma^*\gamma^*}(Q_1^2, 0)}{F_{\mathcal{P}\gamma^*\gamma^*}(0, 0)} = \frac{1}{1 + Q_1^2/\Lambda_P^2}, \tag{C7}$$

where Λ_P is the monopole mass parameter. In Table V we show the experimental extraction of Λ_P for the π^0, η, η', and $\eta_c(1S)$ mesons.

2. Scalar mesons

We next consider the process $\gamma^*(q_1, \lambda_1) + \gamma^*(q_2, \lambda_2) \to \mathcal{S}$, describing the transition from an initial state of two virtual photons, with four-momenta q_1, q_2 and helicities λ_1, $\lambda_2 = 0, \pm 1$, to a scalar meson $\mathcal{S}$ ($J^{PC} = 0^{++}$) with mass m_S. Scalar mesons can be produced either by two transverse photons or by two longitudinal photons [4,31]. Therefore, the $\gamma^*\gamma^* \to \mathcal{S}$ transition can be described by the matrix element

$$\begin{aligned}\mathcal{M}(\lambda_1, \lambda_2) = e^2 \varepsilon^\mu(q_1, \lambda_1)\varepsilon^\nu(q_2, \lambda_2)\left(\frac{\nu}{m_S}\right) \\ \times \Big\{ -R^{\mu\nu}(q_1, q_2) F^T_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, Q_2^2) \\ + \frac{\nu}{X}\left(q_1^\mu + \frac{Q_1^2}{\nu} q_2^\mu\right)\left(q_2^\nu + \frac{Q_2^2}{\nu} q_1^\nu\right) \\ \times F^L_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, Q_2^2)\Big\},\end{aligned} \tag{C8}$$

where we introduced the symmetric transverse tensor $R^{\mu\nu}$:

$$\begin{aligned}R^{\mu\nu}(q_1, q_2) \equiv -g^{\mu\nu} + \frac{1}{X}\{\nu(q_1^\mu q_2^\nu + q_2^\mu q_1^\nu) + Q_1^2 q_2^\mu q_2^\nu \\ + Q_2^2 q_1^\mu q_1^\nu\},\end{aligned} \tag{C9}$$

TABLE V. Experimental extraction of the monopole mass parameter in the $\gamma^*\gamma \to \mathcal{P}$ form factors, according to the fit of Eq. (C7). The measured value of Λ_P for $\mathcal{P} = \pi^0$, η, η' is from the CLEO Collaboration [28]. For the $\eta_c(1S)$ state, the measured value is from the *BABAR* Collaboration [30].

	Λ_P (MeV)
π^0	776 ± 22
η	774 ± 29
η'	859 ± 28
$\eta_c(1S)$	2920 ± 160

which projects onto both transverse photons, having the properties

$$q_{1\mu}R^{\mu\nu}(q_1, q_2) = 0, \qquad q_{1\nu}R^{\mu\nu}(q_1, q_2) = 0,$$
$$q_{2\mu}R^{\mu\nu}(q_1, q_2) = 0, \qquad q_{2\nu}R^{\mu\nu}(q_1, q_2) = 0.$$

In Eq. (C8), the scalar meson structure information is encoded in the form factors $F^T_{\mathcal{S}\gamma^*\gamma^*}$ and $F^L_{\mathcal{S}\gamma^*\gamma^*}$, which are functions of the virtualities of both photons, where the superscripts indicate the situation where either both photons are transverse (T) or longitudinal (L). Note that the prefactor ν/m_S in Eq. (C8) is chosen such that the FFs are dimensionless. Furthermore, both form factors are symmetric under the interchange of both virtualities:

$$F^{T,L}_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F^{T,L}_{\mathcal{S}\gamma^*\gamma^*}(Q_2^2, Q_1^2). \tag{C10}$$

From Eq. (C8), one can easily deduce that the only nonzero $\gamma^*\gamma^* \to \mathcal{S}$ helicity amplitudes are given by

$$\begin{aligned}\mathcal{M}(\lambda_1 = +1, \lambda_2 = +1) &= \mathcal{M}(\lambda_1 = -1, \lambda_2 = -1) \\ &= e^2 \frac{\nu}{m_S} F^T_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, Q_2^2), \\ \mathcal{M}(\lambda_1 = 0, \lambda_2 = 0) &= -e^2 \frac{Q_1 Q_2}{m_S} F^L_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, Q_2^2).\end{aligned} \tag{C11}$$

The transverse FF at $Q_1^2 = Q_2^2 = 0$, $F^T_{\mathcal{S}\gamma^*\gamma^*}(0, 0)$, describes the two-photon decay width of the scalar meson:

$$\Gamma_{\gamma\gamma}(\mathcal{S}) = \frac{\pi\alpha^2}{4} m_S |F^T_{\mathcal{S}\gamma^*\gamma^*}(0, 0)|^2. \tag{C12}$$

In this paper, we study the sum rules involving cross sections for one real photon and one virtual photon. For one real photon ($Q_2^2 = 0$), the only nonvanishing cross sections in Eq. (16) are given by

$$\begin{aligned}[\sigma_0]_{Q_2^2=0} = [\sigma_\parallel]_{Q_2^2=0} = 2[\sigma_{TT}]_{Q_2^2=0} = [\tau_{TT}]_{Q_2^2=0} \\ = \delta(s - m_S^2) 16\pi^2 \frac{\Gamma_{\gamma\gamma}(\mathcal{S})}{m_S}\left(1 + \frac{Q_1^2}{m_S^2}\right) \\ \times \left[\frac{F^T_{\mathcal{S}\gamma^*\gamma^*}(Q_1^2, 0)}{F^T_{\mathcal{S}\gamma^*\gamma^*}(0, 0)}\right]^2.\end{aligned} \tag{C13}$$

3. Axial-vector mesons

We next discuss the two-photon production of an axial-vector meson. Because of the symmetry under rotational invariance, spatial inversion as well as the Bose symmetry of a state of two real photons, the production of a spin-1 resonance by two real photons is forbidden, a statement known as the Landau-Yang theorem [32]. However, the production of an axial-vector meson by two photons is possible when one or both photons are virtual. The matrix element for the process $\gamma^*(q_1, \lambda_1) + \gamma^*(q_2, \lambda_2) \to \mathcal{A}$, describing the transition from an initial state of two virtual photons, with four-momenta q_1, q_2 and helicities $\lambda_1, \lambda_2 = 0, \pm 1$, to an axial-vector meson $\mathcal{A}$ ($J^{PC} = 1^{++}$) with mass m_A and helicity $\Lambda = \pm 1, 0$ (defined along the direction of $\vec{q}_1$), is described by three structures [4,31] and can be parametrized as

$$\begin{aligned}\mathcal{M}(\lambda_1, \lambda_2; \Lambda) = {} & e^2 \varepsilon_\mu(q_1, \lambda_1)\varepsilon_\nu(q_2, \lambda_2)\varepsilon^{\alpha *}(p_f, \Lambda) i\varepsilon_{\rho\sigma\tau\alpha}\Big\{ R^{\mu\rho}(q_1, q_2) R^{\nu\sigma}(q_1, q_2)(q_1 - q_2)^\tau \frac{\nu}{m_A^2} F^{(0)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, Q_2^2) \\ & + R^{\nu\rho}(q_1, q_2)\left(q_1^\mu + \frac{Q_1^2}{\nu} q_2^\mu\right) q_1^\sigma q_2^\tau \frac{1}{m_A^2} F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, Q_2^2) + R^{\mu\rho}(q_1, q_2)\left(q_2^\nu + \frac{Q_2^2}{\nu} q_1^\nu\right) \\ & \times q_2^\sigma q_1^\tau \frac{1}{m_A^2} F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_2^2, Q_1^2)\Big\}. \end{aligned} \tag{C14}$$

In Eq. (C14), the axial-vector meson structure information is encoded in the form factors $F^{(0)}_{\mathcal{A}\gamma^*\gamma^*}$ and $F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}$, where the superscript indicates the helicity state of the axial-vector meson. Note that only transverse photons give a nonzero transition to a state of helicity zero. The form factors are functions of the virtualities of both photons, and $F^{(0)}_{\mathcal{A}\gamma^*\gamma^*}$ is symmetric under the interchange $Q_1^2 \leftrightarrow Q_2^2$. In contrast, $F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}$ does not need to be symmetric under the interchange of both virtualities, as can be seen from Eq. (C14).

From Eq. (C14), one can easily deduce that the only nonzero $\gamma^*\gamma^* \to \mathcal{A}$ helicity amplitudes are given by

$$\begin{aligned} & \mathcal{M}(\lambda_1 = +1, \lambda_2 = +1; \Lambda = 0) = -\mathcal{M}(\lambda_1 = -1, \lambda_2 = -1; \Lambda = 0) = e^2(Q_1^2 - Q_2^2)\frac{\nu}{m_A^3} F^{(0,T)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, Q_2^2), \\ & \mathcal{M}(\lambda_1 = 0, \lambda_2 = +1; \Lambda = -1) = -e^2 Q_1\left(\frac{X}{\nu m_A^2}\right) F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, Q_2^2), \\ & \mathcal{M}(\lambda_1 = -1, \lambda_2 = 0; \Lambda = -1) = -e^2 Q_2\left(\frac{X}{\nu m_A^2}\right) F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_2^2, Q_1^2). \end{aligned} \tag{C15}$$

Note that the helicity amplitude with two transverse photons vanishes when both photons are real, in accordance with the Landau-Yang theorem.

The matrix element $F^{(1)}_{\mathcal{A}\gamma^*\gamma}(0,0)$ allows one to define an equivalent two-photon decay width for an axial-vector meson to decay in one quasireal longitudinal photon and a (transverse) real photon as[2]

$$\begin{aligned}\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A}) & \equiv \lim_{Q_1^2 \to 0} \frac{m_A^2}{Q_1^2} \frac{1}{2} \Gamma(\mathcal{A} \to \gamma_L^* \gamma_T) \\ & = \frac{\pi \alpha^2}{4} m_A \frac{1}{3} [F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)]^2, \end{aligned} \tag{C16}$$

where we have introduced the decay width $\Gamma(\mathcal{A} \to \gamma_L^* \gamma_T)$ for an axial-vector meson to decay into a virtual longitudinal photon, with virtuality Q_1^2, and a real transverse photon ($Q_2^2 = 0$), as

$$\begin{aligned}\Gamma(\mathcal{A} \to \gamma_L^* \gamma_T) & \\ & = \frac{\pi \alpha^2}{2} m_A \frac{1}{3} \frac{Q_1^2}{m_A^2}\left(1 + \frac{Q_1^2}{m_A^2}\right)^3 [F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, 0)]^2. \end{aligned} \tag{C17}$$

In this paper, we study the sum rules involving cross sections for one real photon and one virtual photon. For one quasireal photon ($Q_2^2 \to 0$), we can obtain from the above helicity amplitudes and using Eq. (14) the axial-vector meson contributions to the response functions of Eq. (16) as

$$\begin{aligned} & [\sigma_0]_{Q_2^2=0} = [\sigma_\perp]_{Q_2^2=0} = 2[\sigma_{TT}]_{Q_2^2=0} = -[\tau_{TT}]_{Q_2^2=0} = \delta(s - m_A^2) 4\pi^3 \alpha^2 \frac{Q_1^4}{m_A^4}\left(1 + \frac{Q_1^2}{m_A^2}\right)[F^{(0)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, 0)]^2, \\ & [\sigma_{LT}]_{Q_2^2=0} = \delta(s - m_A^2) 16\pi^2 \frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A} \frac{Q_1^2}{m_A^2}\left(1 + \frac{Q_1^2}{m_A^2}\right)\left[\frac{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, 0)}{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)}\right]^2, \\ & [\tau_{TL}]_{Q_2^2=0} = -[\tau^a_{TL}]_{Q_2^2=0} = \delta(s - m_A^2) 8\pi^2 \frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A} \frac{Q_1 Q_2}{m_A^2}\left(1 + \frac{Q_1^2}{m_A^2}\right)\left[\frac{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2, 0)}{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)} \cdot \frac{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0, Q_1^2)}{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)}\right]. \end{aligned} \tag{C18}$$

[2]In defining the equivalent two-photon decay width for an axial-vector meson, we follow the convention of Ref. [31], which is also followed in experimental analyses [25,26]. Note, however, that the definition for $\tilde{\Gamma}_{\gamma\gamma}$ adopted here is one-half of that used in Ref. [33].

Extracting the FFs $F^{(1)}$ and $F^{(0)}$ separately from experiment requires the measurements of σ_{LT} and σ_{TT}, respectively. As experiments to date have not achieved this separation, one is, so far, only sensitive to the quantity $\sigma_{TT} + \varepsilon_1 \sigma_{LT}$, where ε_1 is a kinematical parameter (so-called virtual photon polarization parameter) defined as $\varepsilon_1 \equiv \rho_1^{00}/2\rho_1^{++}$; see Appendix A. Note that in high-energy collider experiments, one typically has $\varepsilon_1 \approx 1$. From Eq. (C18) one then obtains for this experimentally accessible combination

$$\left[\sigma_{LT}\left(1+\frac{1}{\varepsilon_1}\frac{\sigma_{TT}}{\sigma_{LT}}\right)\right]_{Q_2^2=0} = \delta(s-m_A^2)16\pi^2\frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A}\frac{Q_1^2}{m_A^2}\left(1+\frac{Q_1^2}{m_A^2}\right)\left(\left[\frac{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2,0)}{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)}\right]^2 + \frac{1}{\varepsilon_1}\frac{Q_1^2}{2m_A^2}\left[\frac{F^{(0)}_{\mathcal{A}\gamma^*\gamma^*}(Q_1^2,0)}{F^{(1)}_{\mathcal{A}\gamma^*\gamma^*}(0,0)}\right]^2\right). \tag{C19}$$

We can compare the above general formalism for the two-photon production of an axial-vector meson with the description of Ref. [33], which is commonly used in the literature, and is based on a nonrelativistic quark model calculation leading to only one independent amplitude for the $\gamma^*\gamma^* \to \mathcal{A}$ process as

$$\mathcal{M}(\lambda_1,\lambda_2;\Lambda) = e^2\varepsilon^\mu(q_1,\lambda_1)\varepsilon^\nu(q_2,\lambda_2)\varepsilon^{\alpha*}(p_f,\Lambda)i\varepsilon_{\mu\nu\tau\alpha}(-Q_1^2q_2^\tau + Q_2^2q_1^\tau)A(Q_1^2,Q_2^2), \tag{C20}$$

where the independent form factor A satisfies $A(Q_1^2,Q_2^2) = A(Q_2^2,Q_1^2)$. In such a nonrelativistic quark model limit, we can recover Eq. (C20) from Eq. (C14) through the identifications

$$\begin{aligned} &F^{(0)}(Q_1^2,Q_2^2) = m_A^2A(Q_1^2,Q_2^2), \qquad F^{(1)}(Q_1^2,Q_2^2) = -\frac{\nu}{X}(\nu+Q_2^2)m_A^2A(Q_1^2,Q_2^2),\\ &F^{(1)}(Q_2^2,Q_1^2) = -\frac{\nu}{X}(\nu+Q_1^2)m_A^2A(Q_1^2,Q_2^2), \end{aligned} \tag{C21}$$

in which $2\nu = m_A^2 + Q_1^2 + Q_2^2$. In such a model, the experimentally measured two-photon cross-section combination of Eq. (C19), where $Q_2^2 = 0$, is proportional to

$$\left[\sigma_{LT}\left(1+\frac{1}{\varepsilon_1}\frac{\sigma_{TT}}{\sigma_{LT}}\right)\right]_{Q_2^2=0} = \delta(s-m_A^2)16\pi^2\frac{3\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})}{m_A}\frac{Q_1^2}{m_A^2}\left(1+\frac{Q_1^2}{m_A^2}\right)\left(1+\frac{1}{\varepsilon_1}\frac{Q_1^2}{2m_A^2}\right)\left[\frac{A(Q_1^2,0)}{A(0,0)}\right]^2. \tag{C22}$$

To apply this formula to experimental results where the axial-vector meson has a finite width, one commonly replaces the delta function in Eq. (C22) by a Breit-Wigner form, yielding

$$\left[\sigma_{LT}\left(1+\frac{1}{\varepsilon_1}\frac{\sigma_{TT}}{\sigma_{LT}}\right)\right]_{Q_2^2=0} = 48\pi\frac{\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A})\Gamma_{\mathrm{total}}}{(s-m_A^2)^2+m_A^2\Gamma_{\mathrm{total}}^2}\frac{Q_1^2}{m_A^2}\left(1+\frac{Q_1^2}{m_A^2}\right)\left(1+\frac{1}{\varepsilon_1}\frac{Q_1^2}{2m_A^2}\right)\left[\frac{A(Q_1^2,0)}{A(0,0)}\right]^2, \tag{C23}$$

where Γ_{total} is the total decay width of the axial-vector meson.

Phenomenologically, the two-photon production cross sections have been measured for the two lowest-lying axial-vector mesons: $f_1(1285)$ and $f_1(1420)$. The most recent measurements were performed by the L3 Collaboration [25,26]. In those works, the nonrelativistic quark model expression of Eq. (C23) in terms of a single FF A has been assumed, and the resulting FF has been parametrized by a dipole:

$$\frac{A(Q_1^2,0)}{A(0,0)} = \frac{1}{(1+Q_1^2/\Lambda_A^2)^2}, \tag{C24}$$

where Λ_A is the dipole mass. By fitting the resulting expression of Eq. (C23) to experiment (for which $\varepsilon_1 \approx 1$, and for a Q_1^2 range which extends up to 6 GeV^2), one can then extract the parameters $\tilde{\Gamma}_{\gamma\gamma}$ and Λ_1. Table VI shows the present experimental status of the equivalent 2γ decay widths of the axial-vector mesons $f_1(1285)$ and $f_1(1420)$, which we use in this work.

TABLE VI. Present values [18] of the $f_1(1285)$ meson and $f_1(1420)$ meson masses m_A, their equivalent 2γ decay widths $\tilde{\Gamma}_{\gamma\gamma}$, defined according to Eq. (C16), as well as their dipole masses Λ_A entering the FF of Eq. (C24). For $\tilde{\Gamma}_{\gamma\gamma}$, we use the experimental results from the L3 Collaboration: $f_1(1285)$ from Ref. [25] and $f_1(1420)$ from Ref. [26]. Note that for the $f_1(1420)$ state, only the branching ratio $\tilde{\Gamma}_{\gamma\gamma} \times \Gamma_{K\bar{K}\pi}/\Gamma_{\mathrm{total}}$ is measured so far, which we use as a lower limit on $\tilde{\Gamma}_{\gamma\gamma}$.

	m_A (MeV)	$\tilde{\Gamma}_{\gamma\gamma}$ (keV)	Λ_A (MeV)
$f_1(1285)$	1281.8 ± 0.6	3.5 ± 0.8	1040 ± 78
$f_1(1420)$	1426.4 ± 0.9	3.2 ± 0.9	926 ± 78

4. Tensor mesons

The process $\gamma^*(q_1,\lambda_1) + \gamma^*(q_2,\lambda_2) \to \mathcal{T}(\Lambda)$, describing the transition from an initial state of two virtual photons to a tensor meson $\mathcal{T}$ ($J^{PC} = 2^{++}$) with mass m_T and helicity $\Lambda = \pm 2, \pm 1, 0$ (defined along the direction of $\vec{q}_1$), is described by five independent structures [4,31] and can be parametrized as

$$\mathcal{M}(\lambda_1, \lambda_2; \Lambda) = e^2 \varepsilon_\mu(q_1, \lambda_1)\varepsilon_\nu(q_2, \lambda_2)\varepsilon^*_{\alpha\beta}(p_f, \Lambda)\bigg\{\bigg[R^{\mu\alpha}(q_1, q_2)R^{\nu\beta}(q_1, q_2) + \frac{s}{8X}R^{\mu\nu}(q_1, q_2)(q_1 - q_2)^\alpha(q_1 - q_2)^\beta\bigg]$$
$$\times \frac{\nu}{m_T}T^{(2)}(Q_1^2, Q_2^2) + R^{\nu\alpha}(q_1, q_2)(q_1 - q_2)^\beta\bigg(q_1^\mu + \frac{Q_1^2}{\nu}q_2^\mu\bigg)\frac{1}{m_T}T^{(1)}(Q_1^2, Q_2^2) + R^{\mu\alpha}(q_1, q_2)(q_2 - q_1)^\beta$$
$$\times \bigg(q_2^\nu + \frac{Q_2^2}{\nu}q_1^\nu\bigg)\frac{1}{m_T}T^{(1)}(Q_2^2, Q_1^2) + R^{\mu\nu}(q_1, q_2)(q_1 - q_2)^\alpha(q_1 - q_2)^\beta\frac{1}{m_T}T^{(0,T)}(Q_1^2, Q_2^2)$$
$$+ \bigg(q_1^\mu + \frac{Q_1^2}{\nu}q_2^\mu\bigg)\bigg(q_2^\nu + \frac{Q_2^2}{\nu}q_1^\nu\bigg)(q_1 - q_2)^\alpha(q_1 - q_2)^\beta\frac{1}{m_T^3}T^{(0,L)}(Q_1^2, Q_2^2)\bigg\}, \tag{C25}$$

where $\varepsilon_{\alpha\beta}(p_f, \Lambda)$ is the polarization tensor for the tensor meson with four-momentum p_f and helicity Λ. Furthermore in Eq. (C25) $T^{(\Lambda)}$ are the $\gamma^*\gamma^* \to \mathcal{T}$ transition form factors, for tensor meson helicity Λ. For the case of helicity zero, there are two form factors depending on whether both photons are transverse (superscript T) or longitudinal (superscript L).

From Eq. (C25), we can easily calculate the different helicity amplitudes as

$$\begin{aligned}
&\mathcal{M}(\lambda_1 = +1, \lambda_2 = -1; \Lambda = +2) = \mathcal{M}(\lambda_1 = -1, \lambda_2 = +1; \Lambda = -2) = e^2\frac{\nu}{m_T}T^{(2)}(Q_1^2, Q_2^2),\\
&\mathcal{M}(\lambda_1 = 0, \lambda_2 = +1; \Lambda = -1) = -e^2 Q_1\frac{1}{\sqrt{2}}\bigg(\frac{2X}{\nu m_T^2}\bigg)T^{(1)}(Q_1^2, Q_2^2),\\
&\mathcal{M}(\lambda_1 = -1, \lambda_2 = 0; \Lambda = -1) = e^2 Q_2\frac{1}{\sqrt{2}}\bigg(\frac{2X}{\nu m_T^2}\bigg)T^{(1)}(Q_2^2, Q_1^2),\\
&\mathcal{M}(\lambda_1 = +1, \lambda_2 = +1; \Lambda = 0) = \mathcal{M}(\lambda_1 = -1, \lambda_2 = -1; \Lambda = 0) = -e^2\sqrt{\frac{2}{3}}\bigg(\frac{4X}{m_T^3}\bigg)T^{(0,T)}(Q_1^2, Q_2^2),\\
&\mathcal{M}(\lambda_1 = 0, \lambda_2 = 0; \Lambda = 0) = -e^2 Q_1 Q_2\sqrt{\frac{2}{3}}\bigg(\frac{4X^2}{\nu^2 m_T^5}\bigg)T^{(0,L)}(Q_1^2, Q_2^2).
\end{aligned} \tag{C26}$$

The transverse FFs $T^{(2)}$ and $T^{(0,T)}$ at $Q_1^2 = Q_2^2 = 0$ describe the two-photon decay widths of the tensor meson with helicities $\Lambda = 2$ and $\Lambda = 0$, respectively:

$$\Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 2)) = \frac{\pi\alpha^2}{4}m_T\frac{1}{5}|T^{(2)}(0, 0)|^2, \qquad \Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 0)) = \frac{\pi\alpha^2}{4}m_T\frac{2}{15}|T^{(0,T)}(0, 0)|^2. \tag{C27}$$

In this work, we study the sum rules involving cross sections for one real photon and one virtual photon. For one quasireal photon ($Q_2^2 \to 0$), we can obtain, from the above helicity amplitudes and using Eq. (14), the tensor meson contributions to the response functions of Eq. (16) as

$$\begin{aligned}
&[\sigma_2]_{Q_2^2=0} = \delta(s - m_T^2)16\pi^2\frac{5\Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 2))}{m_T}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)\bigg[\frac{T^{(2)}(Q_1^2, 0)}{T^{(2)}(0, 0)}\bigg]^2,\\
&[\sigma_0]_{Q_2^2=0} = \delta(s - m_T^2)16\pi^2\frac{5\Gamma_{\gamma\gamma}(\mathcal{T}(\Lambda = 0))}{m_T}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)^3\bigg[\frac{T^{(0,T)}(Q_1^2, 0)}{T^{(0,T)}(0, 0)}\bigg]^2, \qquad [\sigma_\parallel]_{Q_2^2=0} = \bigg[\frac{1}{2}\sigma_2 + \sigma_0\bigg]_{Q_2^2=0},\\
&[\sigma_\perp]_{Q_2^2=0} = \bigg[\frac{1}{2}\sigma_2\bigg]_{Q_2^2=0}, \qquad [\sigma_{LT}]_{Q_2^2=0} = \delta(s - m_T^2)8\pi^3\alpha^2\frac{Q_1^2}{m_T^2}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)[T^{(1)}(Q_1^2, 0)]^2,\\
&\bigg[\frac{1}{Q_1 Q_2}\tau_{TL}\bigg]_{Q_2^2=0} = \delta(s - m_T^2)8\pi^3\alpha^2\frac{1}{m_T^2}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)\bigg\{\frac{2}{3}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)^2 T^{(0,T)}(Q_1^2, 0)T^{(0,L)}(Q_1^2, 0) - \frac{1}{2}T^{(1)}(Q_1^2, 0)T^{(1)}(0, Q_1^2)\bigg\},\\
&\bigg[\frac{1}{Q_1 Q_2}\tau_{TL}^a\bigg]_{Q_2^2=0} = \delta(s - m_T^2)8\pi^3\alpha^2\frac{1}{m_T^2}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)\bigg\{\frac{2}{3}\bigg(1 + \frac{Q_1^2}{m_T^2}\bigg)^2 T^{(0,T)}(Q_1^2, 0)T^{(0,L)}(Q_1^2, 0) + \frac{1}{2}T^{(1)}(Q_1^2, 0)T^{(1)}(0, Q_1^2)\bigg\},\\
&[\sigma_{LL}]_{Q_2^2=0} = 0.
\end{aligned} \tag{C28}$$

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