Derivation of transient relativistic fluid dynamics from the Boltzmann equation

G. S. Denicol,[1] H. Niemi,[2,3] E. Molnár,[2,4] and D. H. Rischke[1,2]

[1]*Institut für Theoretische Physik, Johann Wolfgang Goethe-Universität, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany*
[2]*Frankfurt Institute for Advanced Studies, Ruth-Moufang-Str. 1, D-60438 Frankfurt am Main, Germany*
[3]*Department of Physics, University of Jyväskylä, P.O. Box 35 (YFL) FI-40014, Finland*
[4]*MTA Wigner Research Centre for Physics, H-1525 Budapest, P.O. Box 49, Hungary*
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In this work we present a general derivation of relativistic fluid dynamics from the Boltzmann equation using the method of moments. The main difference between our approach and the traditional 14-moment approximation is that we will not close the fluid-dynamical equations of motion by truncating the expansion of the distribution function. Instead, we keep all terms in the moment expansion. The reduction of the degrees of freedom is done by identifying the microscopic time scales of the Boltzmann equation and considering only the slowest ones. In addition, the equations of motion for the dissipative quantities are truncated according to a systematic power-counting scheme in Knudsen and inverse Reynolds number. We conclude that the equations of motion can be closed in terms of only 14 dynamical variables, as long as we only keep terms of second order in Knudsen and/or inverse Reynolds number. We show that, even though the equations of motion are closed in terms of these 14 fields, the transport coefficients carry information about all the moments of the distribution function. In this way, we can show that the particle-diffusion and shear-viscosity coefficients agree with the values given by the Chapman-Enskog expansion.

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I. INTRODUCTION

Relativistic fluid dynamics is an effective theory to describe the long-distance, longtime dynamics of macroscopic systems, with important applications in relativistic heavy-ion collisions and astrophysics [1]. Relativistic fluid dynamics describes the conservation of (net) particle number and energy momentum,

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0. \tag{1}$$

In general, these five equations contain 14 unknown fields, the four components of the particle 4-current N^μ, and the ten components of the (symmetric) energy-momentum tensor $T^{\mu\nu}$. Thus, these equations are not closed and one needs to specify nine additional equations of motion to solve them. The coefficients in the equations of motion (equation of state, transport coefficients, etc.) must be determined by matching fluid dynamics to the underlying microscopic theory. In the case of dilute gases, this is the Boltzmann equation.

There are two widespread methods to provide additional equations of motion from the Boltzmann equation: the Chapman-Enskog expansion and the method of moments. In the Chapman-Enskog expansion [2], the corrections to the single-particle distribution function in local equilibrium are assumed to be functions of the five traditional fluid-dynamical variables, temperature, chemical potential, and the three components of the fluid-velocity field, as well as gradients thereof. The corrections are systematically arranged in terms of an expansion in powers of the Knudsen number, given by the ratio of the mean-free path of the particles and a characteristic macroscopic length scale. As is well known, the first-order truncation of the expansion leads to Navier-Stokes theory. Keeping second- and higher-order terms one obtains the Burnett and super-Burnett equations, respectively [3]. However, it has been shown that the Burnett equations suffer from the so-called Bobylev instability [4]. In the relativistic case, even the first-order equations, i.e., the relativistic generalization of the Navier-Stokes equations, are unstable [5].

Therefore, the relativistic extension of Chapman-Enskog theory should not be applied to derive the equations of relativistic fluid dynamics from kinetic theory. On the other hand, the method of moments [6] avoids the above mentioned problems. The method of moments was first developed by Grad [7] for nonrelativistic systems. In Grad's original work, the single-particle distribution function is expanded around its local equilibrium value in terms of a complete set of Hermite polynomials [8]. This expansion is truncated and the distribution function is finally expressed in terms of 13 fluid-dynamical variables: the velocity field, the temperature, the chemical potential, the heat-conduction current, and the shear-stress tensor. In this case the heat-conduction current and shear-stress tensor become independent dynamical variables that satisfy partial differential equations that describe their relaxation towards their respective Navier-Stokes values. Grad's method is usually considered to be independent of the Chapman-Enskog expansion. However, we emphasize that Burnett-type equations can be obtained as the solution of Grad's equations in the longtime limit [9,10].

Nevertheless, Grad's method has one major drawback: unlike the Chapman-Enskog expansion it lacks a small parameter, such as the Knudsen number, in which one can do power counting and thus systematically improve the approximation [11]. This deficiency, together with the bad performance of Grad's method in comparison to microscopic calculations [12], have led researchers to abandon this approach for some time. However, recently a lot of effort has been made to reformulate the method of moments into a more reliable tool to describe nonequilibrium phenomena for large Knudsen numbers [12]. For instance, in Ref. [13] Grad's equations were regularized to have a wider domain of validity in Knudsen number and then shown to be in good agreement with microscopic calculations. Such approaches, however, were only formulated for nonrelativistic systems.

The generalization of Grad's method of moments to relativistic systems has been pursued by several authors [14]. The most widely employed approach is due to Israel and Stewart [15,16]. Here, the distribution function is expanded around its local equilibrium value in terms of a series of (reducible) Lorentz tensors formed of particle 4-momentum k^μ, i.e., $1, k^\mu, k^\mu k^\nu, \ldots$. In Israel and Stewart's 14-moment approximation one truncates the expansion at second order in momentum, i.e., one only keeps the tensors 1, k^μ, and $k^\mu k^\nu$, with 14 unknown coefficients (the trace of $k^\mu k^\nu$ is equal to m^2, the rest mass of the particles) to describe the distribution function. The coefficients of the *truncated* expansion can then be uniquely related to the 14 components of the particle 4-current, N^μ, and the energy-momentum tensor, $T^{\mu\nu}$, the so-called *matching procedure*. While particle and energy-momentum conservation (1) are obtained from the zeroth and the first moment of the Boltzmann equation, the additional nine equations of motion follow from the second moment of the Boltzmann equation. However, Israel and Stewart's theory shares the same problems of Grad's original approach: it lacks a parameter in which one can do systematic power counting of corrections to the local equilibrium distribution function.

It was recently confirmed that, at least for some special problems, the Israel-Stewart equations [15,16] are not in good agreement with the numerical solution of the Boltzmann equation [17,18]. Initial attempts to improve Israel and Stewart's theory were already made in Refs. [19–22], but Israel and Stewart's 14-moment approximation was still used. In this paper we demonstrate that Israel-Stewart theory, as well as all previous attempts to improve it, are actually incomplete. The reason is that the 14-moment approximation neglects infinitely many terms of first order in the Knudsen number. In our approach all terms of the moment expansion are included and the exact equations of motion for these moments are derived. These exact equations still contain the degrees of freedom and microscopic time scales of the Boltzmann equation. We prove that, in order to derive a causal dynamical equation for a given dissipative current, it is necessary to resolve at least the slowest corresponding microscopic time scale arising from the Boltzmann equation, in agreement with the results of Ref. [10]. Unlike in Israel-Stewart theory, the truncation of the resulting equations of motion in terms of only 14 dynamical variables is then implemented by a systematic power-counting scheme in Knudsen number, Kn, and in the ratios, $\mathrm{R}_\Pi^{-1} \equiv |\Pi|/P_0$, $\mathrm{R}_n^{-1} \equiv |n^\mu|/n_0$, $\mathrm{R}_\pi^{-1} \equiv |\pi^{\mu\nu}|/P_0$, where Π is the bulk viscous pressure, n^μ is the particle-diffusion current, $\pi^{\mu\nu}$ is the shear-stress tensor, and P_0 and n_0 are the pressure and the particle density in local equilibrium, respectively. The ratio R_π^{-1} is related to the inverse Reynolds number in nonrelativistic situations. We shall in somewhat loose terminology refer to all of them as "inverse Reynolds numbers" in the following. The resulting fluid-dynamical equations and coefficients are different from the ones obtained via the 14-moment approximation. We calculate the numerical values of the coefficients for a massless classical gas. We show that our values for the heat-conductivity and shear-viscosity coefficient agree with the ones calculated via Chapman-Enskog theory [6].

This paper is organized as follows. In Sec. II we review how fluid-dynamical variables are extracted from the Boltzmann equation. In Sec. III we demonstrate how to expand the single-particle distribution function in terms of a *complete, orthogonal* basis in momentum space. In contrast to Israel and Stewart's nonorthogonal basis $1, k^\mu, k^\mu k^\nu, \ldots$, our approach uses *irreducible tensors* in 4-momentum k^μ, and is thus orthogonal. The coefficients of the irreducible tensors in the expansion of the single-particle distribution function are orthogonal polynomials in the rest-frame energy and moments of the correction to the equilibrium distribution function. Section IV derives an infinite set of equations for these moments, which is still completely equivalent to the Boltzmann equation. In Sec. V we introduce our power-counting scheme in terms of Knudsen and inverse Reynolds numbers. Then, by diagonalizing the linear part of the set of moment equations, we demonstrate how to identify the slowest microscopic time scale of the Boltzmann equation for each dissipative current. We shall derive dynamical equations for the slowest modes, but approximate faster modes by their asymptotic solution for long times. This will then lead, in Sec. VI, to the complete set of fluid-dynamical equations that contains *all terms up to second order in Knudsen and inverse Reynolds numbers*, i.e., $\mathcal{O}(\mathrm{Kn}^2, \mathrm{R}_i^{-1}\mathrm{R}_j^{-1}, \mathrm{Kn}\mathrm{R}_i^{-1})$. In Sec. VII we first demonstrate the validity of our approach by restricting the calculation to the 14-moment approximation and recovering the results of Ref. [21] for the transport coefficients for the case of an ultrarelativistic, classical gas with constant cross section. We then show how to successively improve the expression for the transport coefficients by extending the number of moments to $14 + 9 \times n$. We explicitly study the cases $n = 1$, 2, and 3. We end this work with a discussion and conclusions in Sec. VIII. Various appendices contain intermediate steps of our calculations. We use natural units $\hbar = c = k_B = 1$. The metric tensor is $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$.

II. FLUID-DYNAMICAL VARIABLES FROM THE BOLTZMANN EQUATION

We start with the relativistic Boltzmann equation,

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f], \tag{2}$$

where $k^\mu = (k^0, \mathbf{k})$, with $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ and m being the mass of the particles. For the collision term, we consider only elastic two-to-two collisions with incoming momenta k, k', and outgoing momenta p, p',

$$C[f] = \frac{1}{\nu}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'}), \tag{3}$$

where ν is a symmetry factor ($= 2$ for identical particles), $W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}$ is the Lorentz-invariant transition rate, and $dK \equiv g d^3\mathbf{k}/[(2\pi)^3 k^0]$ is the Lorentz-invariant momentum-space volume, with g being the number of internal degrees of freedom. We introduced the notation $\tilde{f}_{\mathbf{k}} \equiv 1 - a f_{\mathbf{k}}$, where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ for a classical gas.

In kinetic theory, the conserved particle current N^μ and the energy-momentum tensor $T^{\mu\nu}$ are expressed as moments of the single-particle distribution function,

$$N^\mu = \langle k^\mu \rangle, \qquad T^{\mu\nu} = \langle k^\mu k^\nu \rangle, \tag{4}$$

where we adopted the following notation:

$$\langle \cdots \rangle \equiv \int dK (\cdots) f_{\mathbf{k}}. \tag{5}$$

The particle current and the energy-momentum tensor can be tensor-decomposed with respect to the fluid 4-velocity u^μ. To this end, we have to specify the rest frame of the fluid. From a mathematical point of view, the velocity can be defined in numerous ways. From the physical perspective, there are, however, two natural choices. The *Landau frame*[23] in which the velocity is defined by the flow of the total energy

$$u_\mu T^{\mu\nu} \equiv \varepsilon u^\nu, \tag{6}$$

and the *Eckart frame* [24] in which the velocity is specified by the flow of particles,

$$N^\mu \equiv n u^\mu. \tag{7}$$

In other words, in the Landau picture the velocity field is fixed to always eliminate any diffusion of energy while in the Eckart picture it is defined to eliminate any diffusion of particles. Note that if the system has more than one type of particle (or conserved charge), the Eckart frame must be defined by choosing one of these particle types (or charge types). We remark that other definitions of the velocity field were investigated by Stewart in Ref. [16].

In this paper, we work in the Landau frame [23]. Next, we divide the momentum of the particles k^μ into two parts: one parallel and one orthogonal to u^μ,

$$k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}, \tag{8}$$

where we defined the scalar $E_{\mathbf{k}} \equiv u_\mu k^\mu \equiv u \cdot k$ and used the notation $A^{\langle\mu\rangle} = \Delta^\mu_\nu A^\nu$, with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ being the projection operator onto the 3-space orthogonal to u^μ.

Then, the tensor decomposition of N^μ and $T^{\mu\nu}$ reads

$$N^\mu = n u^\mu + n^\mu,$$
$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P_0 + \Pi) + \pi^{\mu\nu}, \tag{9}$$

where the particle density n, the particle-diffusion current n^μ, the energy density ε, the shear-stress tensor $\pi^{\mu\nu}$, and the sum of thermodynamic pressure, P_0, and bulk viscous pressure, Π, are defined by

$$n \equiv \langle E_{\mathbf{k}} \rangle, \qquad n^\mu \equiv \langle k^{\langle\mu\rangle} \rangle, \qquad \varepsilon \equiv \langle E^2_{\mathbf{k}} \rangle,$$
$$\pi^{\mu\nu} \equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle, \qquad P_0 + \Pi \equiv -\tfrac{1}{3}\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle, \tag{10}$$

where $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$ and $\Delta^{\mu\nu}_{\alpha\beta} \equiv [\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\nu_\alpha \Delta^\mu_\beta - (2/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}]/2$ denotes a projector onto that part of a rank-2 tensor, which is symmetric, orthogonal to u^μ, and traceless.

Next, we introduce the local-equilibrium distribution function as $f_{0\mathbf{k}} = [\exp(\beta_0 E_{\mathbf{k}} - \alpha_0) + a]^{-1}$, where β_0 and α_0 are the inverse temperature and the ratio of the chemical potential to temperature, respectively. The fluid 4-velocity has already been defined. The values of α_0 and β_0 are defined in terms of a fictitious equilibrium state, constructed from n and ε. First, we introduce the thermodynamic entropy density in thermodynamic equilibrium,

$$s_0 \equiv s_0(n, \varepsilon). \tag{11}$$

Then, α_0 and β_0 are defined by the following thermodynamic relations:

$$\beta_0 = \frac{\partial s_0}{\partial \varepsilon}\bigg|_n, \qquad \alpha_0 = \frac{\partial s_0}{\partial n}\bigg|_\varepsilon, \tag{12}$$

while the thermodynamic pressure P_0 is extracted using Euler's relation,

$$\beta_0 P_0 = s_0 + \alpha_0 n - \beta_0 \varepsilon. \tag{13}$$

In practice, this prescription leads to the following matching conditions:

$$n \equiv n_0 = \langle E_{\mathbf{k}} \rangle_0, \qquad \varepsilon \equiv \varepsilon_0 = \langle E^2_{\mathbf{k}} \rangle_0, \tag{14}$$

where

$$\langle \cdots \rangle_0 \equiv \int dK (\cdots) f_{0\mathbf{k}}. \tag{15}$$

Note that this definition of the thermodynamic variables is not unique and other possibilities in constructing such an equilibrium state were discussed in Ref. [16] and, recently, in Refs. [25].

The separation between thermodynamic pressure and bulk viscous pressure is achieved as $P_0 = -\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0 / 3$ and $\Pi = -\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_\delta / 3$, where

$$\langle \cdots \rangle_\delta = \langle \cdots \rangle - \langle \cdots \rangle_0. \tag{16}$$

The fluid-dynamical conservation laws (1) are equations of motion for n, ε, and u^μ; hence, one needs nine additional equations to determine the dissipative corrections Π, n^μ, and $\pi^{\mu\nu}$. In the following, we shall use the method of moments to derive these equations.

III. EXPANSION OF THE SINGLE-PARTICLE DISTRIBUTION FUNCTION IN TERMS OF IRREDUCIBLE TENSORS

In this section, we expand the single-particle distribution $f_{\mathbf{k}}$ in terms of irreducible tensors. It is convenient to factorize the local-equilibrium distribution function $f_{0\mathbf{k}}$ from $f_{\mathbf{k}}$,

$$f_{\mathbf{k}} = f_{0\mathbf{k}}(1 + \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}}), \tag{17}$$

where $\phi_{\mathbf{k}}$ represents the deviation from local equilibrium and is a function of x^μ and k^μ, which is ultimately determined by the solution of the Boltzmann equation (2).

The next step is to expand $\phi_{\mathbf{k}}$ in terms of a complete basis of tensors formed of k^μ and $E_{\mathbf{k}}$. As mentioned in the introduction, Israel and Stewart chose the following basis to expand $\phi_{\mathbf{k}}$: $1, k^\mu, k^\mu k^\nu, k^\mu k^\nu k^\lambda, \ldots,$ and then truncated the expansion after the second-rank tensor $k^\mu k^\nu$, that is $\phi_{\mathbf{k}} = \epsilon_{\mathbf{k}} + \epsilon^\mu_{\mathbf{k}} k_\mu + \epsilon^{\mu\nu}_{\mathbf{k}} k_\mu k_\nu$, where $\epsilon_{\mathbf{k}}$, $\epsilon^\mu_{\mathbf{k}}$, $\epsilon^{\mu\nu}_{\mathbf{k}}$ are the expansion coefficients [15]. Note that these tensors are *not irreducible* with respect to Lorentz transformations Λ^μ_ν that leave the fluid 4-velocity u^μ invariant, $\Lambda^\mu_\nu u^\nu = u^\mu$. As a consequence, they are also not orthogonal, see Chapter VI, Sec. 2a of Ref. [6]. Therefore, the expansion coefficients cannot be straightforwardly obtained: in a nonorthogonal basis, this requires in general the inversion of an infinite-dimensional matrix. Also, this implies that the *exact* form of the expansion coefficients cannot be obtained once the expansion is *truncated*. Therefore, the approach of Israel and Stewart does not provide the complete expressions for the expansion coefficients.

In order to avoid such problems, we expand $\phi_{\mathbf{k}}$ using the *irreducible* tensors,

$$1, \qquad k^{\langle\mu\rangle}, \qquad k^{\langle\mu}k^{\nu\rangle}, \qquad k^{\langle\mu}k^\nu k^{\lambda\rangle}, \ldots, \tag{18}$$

as a basis. It should be emphasized that these tensors form a *complete and orthogonal* set, analogous to the spherical harmonics [26]. These irreducible tensors are defined by using the symmetrized and, for $m > 1$ traceless, projection orthogonal to u^μ as

$$A^{\langle\mu_1\cdots\mu_m\rangle} \equiv \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} A^{\nu_1\cdots\nu_m}, \tag{19}$$

where the projectors $\Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m}$ are defined in Ref. [6], see Appendix F for details. In order to obtain the irreducible tensors (18), we apply the projection (19) to $A^{\nu_1\cdots\nu_m} \equiv k^{\nu_1}\cdots k^{\nu_m}$. The tensors (18) satisfy an orthogonality condition,

$$\int dK F_{\mathbf{k}} k^{\langle\mu_1}\cdots k^{\mu_m\rangle} k_{\langle\nu_1}\cdots k_{\nu_n\rangle} = \frac{m!\delta_{mn}}{(2m+1)!!} \Delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} \int dK F_{\mathbf{k}} (\Delta^{\alpha\beta} k_\alpha k_\beta)^m, \tag{20}$$

where $n, m = 0, 1, 2, \ldots$, $F_{\mathbf{k}}$ is an arbitrary function of $E_{\mathbf{k}}$ and δ_{mn} denotes the Kronecker delta. Using the basis (18), $\phi_{\mathbf{k}}$ can be expanded as

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}. \tag{21}$$

The index ℓ indicates the rank of the tensor $\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}}$ and $\ell = 0$ corresponds to the scalar λ. The coefficients $\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}}$ are complicated functions of $E_{\mathbf{k}}$ and are further expanded in terms of an orthogonal basis of functions $P^{(\ell)}_{\mathbf{k}n}$,

$$\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}} = \sum_{n=0}^{N_\ell} c^{\langle\mu_1\cdots\mu_\ell\rangle}_n P^{(\ell)}_{\mathbf{k}n}, \tag{22}$$

where N_ℓ is the number of functions $P^{(\ell)}_{\mathbf{k}n}$ considered to describe the ℓth rank tensor $\lambda^{\langle\mu_1\cdots\mu_\ell\rangle}_{\mathbf{k}}$. In principle, N_ℓ should be infinite; however, in practice, the expansion (22) must be truncated and N_ℓ characterizes the truncation order. The function $P^{(\ell)}_{\mathbf{k}n}$ are chosen to be polynomials of order n in energy, $E_{\mathbf{k}}$,

$$P^{(\ell)}_{\mathbf{k}n} = \sum_{r=0}^{n} a^{(\ell)}_{nr} E^r_{\mathbf{k}}, \tag{23}$$

which are constructed to satisfy the orthonormality condition

$$\int dK \omega^{(\ell)} P^{(\ell)}_{\mathbf{k}m} P^{(\ell)}_{\mathbf{k}n} = \delta_{mn}, \tag{24}$$

where $\omega^{(\ell)}$ is defined as

$$\omega^{(\ell)} \equiv \frac{W^{(\ell)}}{(2\ell+1)!!} (\Delta^{\alpha\beta} k_\alpha k_\beta)^\ell f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \tag{25}$$

The coefficients $a^{(\ell)}_{nr}$ and the normalization constants $W^{(\ell)}$ can be found via Gram-Schmidt orthogonalization using the orthonormality condition (24), see Appendix E for details. We note that, in the limit of massless, classical particles, the polynomials $P^{(\ell)}_{\mathbf{k}n}$ correspond to the associated Laguerre polynomials.

Since the expansion (21) employs an orthogonal basis, the expansion coefficients in Eq. (22) can be immediately determined using Eqs. (20) and (24). For $n \le N_\ell$ they are given by

$$c_n^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{W^{(\ell)}}{\ell!}\langle P_{\mathbf{k}n}^{(\ell)} k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\rangle_\delta. \tag{26}$$

For the sake of later convenience, these expansion coefficients are reexpressed as linear combinations of irreducible moments of $\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}}$,

$$\rho_n^{\mu_1\cdots\mu_\ell} \equiv \langle E_{\mathbf{k}}^n k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\rangle_\delta, \tag{27}$$

such that

$$\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} = \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell}, \tag{28}$$

where we defined the energy-dependent coefficients

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} \equiv \frac{W^{(\ell)}}{\ell!}\sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)}. \tag{29}$$

Consequently, the distribution function itself can be expressed as a series in the irreducible moments (27) of $\delta f_{\mathbf{k}}$,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty}\sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}. \tag{30}$$

We remark that our choice of matching conditions and the definition of the velocity field imply that

$$\rho_1 = \rho_2 = \rho_1^\mu = 0. \tag{31}$$

Note that, if we were working in the Eckart frame a different set of moments would vanish,

$$\rho_1 = \rho_2 = \rho_0^\mu = 0. \tag{32}$$

IV. GENERAL EQUATIONS OF MOTION

The time-evolution equations for the moments $\rho_r^{\mu_1\cdots\mu_\ell}$ can be obtained directly from the Boltzmann equation by applying the comoving derivative to the definition (27), together with the symmetrized traceless projection,

$$\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell}\frac{d}{d\tau}\int dK E_{\mathbf{k}}^r k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle}\delta f_{\mathbf{k}}, \tag{33}$$

where $\dot{A} \equiv u^\mu \partial_\mu A \equiv dA/d\tau$ and $\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell} \dot{\rho}_r^{\nu_1\cdots\nu_\ell}$. Using the Boltzmann equation (2) in the form

$$\delta\dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1}k_\nu\nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1}k_\nu\nabla^\nu\delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1}C[f], \tag{34}$$

where $\nabla_\mu = \Delta_\mu^\nu\partial_\nu$, and substituting this expression into Eq. (33), one can obtain the *exact* equations for the comoving derivatives of $\rho_r^{\mu_1\cdots\mu_l}$.

Using the power-counting scheme developed in Sec. V, we will show that, in order to derive the equations of motion for relativistic fluid dynamics, it is sufficient to know the time-evolution equations for the moments (27) up to rank 2, i.e., for ρ_r, ρ_r^μ, and $\rho_r^{\mu\nu}$. Similar equations could also be derived for higher-rank irreducible moments, if needed. Thus, using Eqs. (33) and (34), we obtain

$$\dot{\rho}_r - C_{r-1} = \alpha_r^{(0)}\theta - \frac{G_{2r}}{D_{20}}\Pi\theta + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{G_{3r}}{D_{20}}\partial_\mu n^\mu + (r-1)\rho_{r-2}^{\mu\nu}\sigma_{\mu\nu} + r\rho_{r-1}^\mu\dot{u}_\mu - \nabla_\mu\rho_{r-1}^\mu - \frac{1}{3}[(r+2)\rho_r - (r-1)m^2\rho_{r-2}]\theta, \tag{35}$$

$$\dot{\rho}_r^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} = \alpha_r^{(1)}I^\mu + \rho_r^\nu\omega_\nu^\mu + \frac{1}{3}[(r-1)m^2\rho_{r-2}^\mu - (r+3)\rho_r^\mu]\theta - \Delta_\lambda^\mu\nabla_\nu\rho_{r-1}^{\lambda\nu} + r\rho_{r-1}^{\mu\nu}\dot{u}_\nu + \frac{1}{5}[(2r-2)m^2\rho_{r-2}^\nu - (2r+3)\rho_r^\nu]\sigma_\nu^\mu + \frac{1}{3}[m^2 r\rho_{r-1} - (r+3)\rho_{r+1}]\dot{u}^\mu + \frac{\beta_0 J_{r+2,1}}{\varepsilon_0 + P_0}(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta_\nu^\mu\partial_\lambda\pi^{\lambda\nu}) - \frac{1}{3}\nabla^\mu(m^2\rho_{r-1} - \rho_{r+1}) + (r-1)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\lambda\nu}, \tag{36}$$

$$\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)}\sigma^{\mu\nu} - \tfrac{2}{7}[(2r+5)\rho_r^{\lambda\langle\mu} - m^2 2(r-1)\rho_{r-2}^{\lambda\langle\mu}]\sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu}\omega_\lambda^{\nu\rangle} + \tfrac{2}{15}[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}]\sigma^{\mu\nu} + \tfrac{2}{5}\nabla^{\langle\mu}(\rho_{r+1}^{\nu\rangle} - m^2\rho_{r-1}^{\nu\rangle}) - \tfrac{2}{5}[(r+5)\rho_{r+1}^{\langle\mu} - rm^2\rho_{r-1}^{\langle\mu}]\dot{u}^{\nu\rangle} - \tfrac{1}{3}[(r+4)\rho_r^{\mu\nu} - m^2(r-1)\rho_{r-2}^{\mu\nu}]\theta + (r-1)\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda, \tag{37}$$

where we introduced the generalized irreducible collision terms

$$C_r^{\langle\mu_1\cdots\mu_\ell\rangle} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}C[f]. \tag{38}$$

We further defined the shear tensor $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$, the expansion scalar $\theta \equiv \nabla_\mu u^\mu$, the vorticity tensor $\omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ and introduced $I^\mu = \nabla^\mu\alpha_0$. All comoving derivatives of α_0 and β_0 that appeared during the derivation of the above equations were replaced using the *exact* equations obtained from the conservation laws of particle number, energy, and momentum,

$$\dot{\alpha}_0 = \frac{1}{D_{20}}\{-J_{30}(n_0\theta + \partial_\mu n^\mu) + J_{20}[(\varepsilon_0 + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}]\}, \tag{39}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}}\{-J_{20}(n_0\theta + \partial_\mu n^\mu) + J_{10}[(\varepsilon_0 + P_0 + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu}]\}, \quad (40)$$

$$\dot{u}^\mu = \frac{1}{\varepsilon_0 + P_0}(\nabla^\mu P_0 - \Pi \dot{u}^\mu + \nabla^\mu \Pi - \Delta^\mu_\alpha \partial_\beta \pi^{\alpha\beta}). \quad (41)$$

The coefficients $\alpha_r^{(0)}$, $\alpha_r^{(1)}$, and $\alpha_r^{(2)}$ are functions of temperature and chemical potential and have the general form,

$$\alpha_r^{(0)} = (1 - r)I_{r1} - I_{r0} - \frac{1}{D_{20}}[G_{2r}(\varepsilon_0 + P_0) - G_{3r}n_0], \quad (42)$$

$$\alpha_r^{(1)} = J_{r+1,1} - \frac{n_0}{\varepsilon_0 + P_0}J_{r+2,1}, \quad (43)$$

$$\alpha_r^{(2)} = I_{r+2,1} + (r - 1)I_{r+2,2}, \quad (44)$$

where we defined the thermodynamic functions

$$I_{nq}(\alpha_0, \beta_0) = \frac{1}{(2q + 1)!!}\langle E_{\mathbf{k}}^{n-2q}(-\Delta^{\alpha\beta}k_\alpha k_\beta)^q\rangle_0, \quad J_{nq} = \frac{\partial I_{nq}}{\partial \alpha_0}\bigg|_{\beta_0}, \quad (45)$$

$$G_{nm} = J_{n0}J_{m0} - J_{n-1,0}J_{m+1,0}, \quad D_{nq} = J_{n+1,q}J_{n-1,q} - J_{nq}^2. \quad (46)$$

The dissipative quantities appearing in the conservation laws can be (exactly) identified with the moments

$$\rho_0 = -\frac{3}{m^2}\Pi, \qquad \rho_0^\mu = n^\mu, \qquad \rho_0^{\mu\nu} = \pi^{\mu\nu}. \quad (47)$$

We note that the derivation of these general equations of motion is independent of the form of the expansion of the single-particle distribution we introduced in the previous section.

V. POWER COUNTING AND THE REDUCTION OF DYNAMICAL VARIABLES

So far, we have derived a general expansion of the distribution function in terms of the irreducible moments of $\delta f_{\mathbf{k}}$, as well as exact equations of motion for these moments. There is an infinite number of equations (labeled by the index r), and the equations for the moments up to rank 2, Eqs. (35)–(37), contain moments of rank higher than two. In general, one would have to solve this infinite set of coupled equations in order to determine the time evolution of the system. However, in the fluid-dynamical limit, it is expected that the macroscopic dynamics of a given system simplifies, and therefore it can be described by the conserved currents N^μ and $T^{\mu\nu}$ alone.

From the kinetic point of view, it is usually assumed that the validity of the fluid-dynamical limit can be quantified by the Knudsen number,

$$\mathrm{Kn} \equiv \frac{\ell_{\mathrm{micr}}}{L_{\mathrm{macr}}}, \quad (48)$$

where ℓ_{micr} and L_{macr} are typical microscopic and macroscopic length or time scales of the system, respectively. The relevant macroscopic scales are usually estimated from the gradients of fluid-dynamical quantities, while the microscopic scales are of the order of the mean-free path or time between collisions. It is generally assumed that when there is a clear separation of the microscopic and macroscopic scales, i.e., when $\mathrm{Kn} \ll 1$, the microscopic details can be safely integrated out and the dynamics of the system can be described using only a few macroscopic fields.

Furthermore, we also expect fluid dynamics to be valid near local thermal equilibrium, i.e., when $\delta f_{\mathbf{k}} \ll f_{0\mathbf{k}}$. We can quantify the deviation from equilibrium in terms of the macroscopic variables by defining a set of ratios of dissipative quantities to the equilibrium pressure or density. These can be understood as generalizations of the inverse Reynolds number and will be denoted as

$$\mathrm{R}_\Pi^{-1} \equiv \frac{|\Pi|}{P_0}, \qquad \mathrm{R}_n^{-1} \equiv \frac{|n^\mu|}{n_0}, \qquad \mathrm{R}_\pi^{-1} \equiv \frac{|\pi^{\mu\nu}|}{P_0}. \quad (49)$$

Since the nonequilibrium moments are integrals of $\delta f_{\mathbf{k}}$ while the equilibrium pressure and particle density are integrals over the equilibrium distribution function $f_{0\mathbf{k}}$, these ratios quantify the deviations from equilibrium.

With this in mind, it is clear that these two measures, the Knudsen number and the inverse Reynolds number, can be used to quantify the proximity of the system to the fluid-dynamical limit. In general, these two measures are independent of each other, e.g. a system can be initialized in such way that the Knudsen number is large, but the inverse Reynolds number is small or vice versa. When deriving transient fluid dynamics, one should not *a priori* assume that $\mathrm{Kn} \sim \mathrm{R}_i^{-1}$; while the Reynolds and Knudsen numbers are certainly related, their relation is in principle dynamical and is precisely what we aim to find. Only for asymptotically long times, the solutions of the dynamical equations yield $\mathrm{Kn} \sim \mathrm{R}_i^{-1}$, as will be discussed in more detail below.

In the traditional 14-moment approximation introduced by Israel and Stewart [15], the fluid-dynamical limit is implemented by a truncation of the expansion of the distribution function, which corresponds neither to a truncation in Knudsen nor in inverse Reynolds number. In this sense, the domain of validity of the equations of motion obtained via the traditional 14-moment approximation is not clear, because it is not possible to determine the order of the terms that were neglected. In order to obtain a closed set of macroscopic equations with a clear domain of validity in both Kn and R_i^{-1}, another truncation procedure is necessary. The derivation of this is the main purpose of this section.

First, we rewrite the collision terms $C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle}$ by linearizing the collision operator $C[f]$ in the deviations from the equilibrium distribution functions. We then use the moment expansion (30) to obtain

$$C_{r-1}^{\langle\mu_1\ldots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in } \delta f), \tag{50}$$

where

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle\nu_1}\cdots k^{\nu_\ell\rangle} (\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\nu_1}\cdots k_{\nu_\ell\rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle\nu_1}\cdots k'_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\nu_1}\cdots p_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle\nu_1}\cdots p'_{\nu_\ell\rangle}). \tag{51}$$

The details of the derivation are relegated to Appendix A. The coefficient $\mathcal{A}_{rn}^{(\ell)}$ is the (rn) element of an $(N_\ell+1)\times(N_\ell+1)$ matrix $\mathcal{A}^{(\ell)}$ and contains all the information of the underlying microscopic theory. We remark that, for $\ell=0$, the second and third rows and columns $(r,n=1,2)$ and, for $\ell=1$, the second row and column $(r,n=1)$ are zero, because the moments ρ_1, ρ_2, and ρ_1^μ vanish due to the definition of the velocity field and the matching conditions, Eqs. ((6) and (14). Therefore, in order to invert $\mathcal{A}^{(\ell)}$, for $\ell=0$, we have to exclude the second and third rows and columns and, for $\ell=1$, the second row and column.

As already mentioned, fluid dynamics is expected to emerge when the microscopic degrees of freedom are integrated out, and the system can be described solely by the conserved currents. The exact equations of motion (35)–(37) contain infinitely many degrees of freedom, given by the irreducible moments of the distribution function, and also infinitely many microscopic time scales, related to the coefficients $\mathcal{A}_{rn}^{(\ell)}$. The slowest microscopic time scale should dominate the dynamics at long times, i.e., in the transient fluid-dynamical limit. In order to extract the relevant relaxation scales, we have to determine the normal modes of Eqs. (35)–(37), i.e., we diagonalize the part that is linear in the irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$. These are the linear terms on the left-hand sides arising from Eq. (50) and the first terms on the right-hand sides. The nonlinear terms from Eq. (50) as well as the remaining terms on the right-hand sides, which are nonlinear in the moments or are gradients of moments, are not considered in the diagonalization procedure. Identifying and separating the microscopic time scales of the Boltzmann equation is also the basic step for obtaining general relations between the irreducible moments and the dissipative currents and, as we shall see, closing the equations of motion in terms of N^μ and $T^{\mu\nu}$.

For this purpose, we shall introduce the matrix $\Omega^{(\ell)}$ that diagonalizes $\mathcal{A}^{(\ell)}$,

$$(\Omega^{-1})^{(\ell)}\mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}(\chi_0^{(\ell)},\ldots,\chi_j^{(\ell)},\ldots), \tag{52}$$

where $\chi_j^{(\ell)}$ are the eigenvalues of $\mathcal{A}^{(\ell)}$. Above, $(\Omega^{-1})^{(\ell)}$ is defined as the matrix inverse of $\Omega^{(\ell)}$. We further define the tensors $X_i^{\mu_1\cdots\mu_\ell}$ as

$$X_i^{\mu_1\cdots\mu_\ell} \equiv \sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell}. \tag{53}$$

These are the eigenmodes of the linearized Boltzmann equation. Multiplying Eq. (50) with $(\Omega^{-1})^{(\ell)}$ from the left and using Eqs. (52) and (53) we obtain

$$\sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} C_{j-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\chi_i^{(\ell)} X_i^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in } \delta f), \tag{54}$$

where we do not sum over the index i on the right-hand side of the equation. Then we multiply Eqs. (35)–(37) with $(\Omega^{-1})_{ir}^{(\ell)}$ and sum over r. Using Eq. (54), we obtain the equations of motion for the variables $X_i^{\mu_1\cdots\mu_\ell}$,

$$\dot{X}_i + \chi_i^{(0)} X_i = \beta_i^{(0)}\theta + (\text{higher-order terms}),$$
$$\dot{X}_i^{\langle\mu\rangle} + \chi_i^{(1)} X_i^\mu = \beta_i^{(1)} I^\mu + (\text{higher-order terms}),$$
$$\dot{X}_i^{\langle\mu\nu\rangle} + \chi_i^{(2)} X_i^{\mu\nu} = \beta_i^{(2)}\sigma^{\mu\nu} + (\text{higher-order terms}), \tag{55}$$

where we introduced the coefficients

$$\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} (\Omega^{-1})_{ij}^{(0)} \alpha_j^{(0)}, \qquad \beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} (\Omega^{-1})_{ij}^{(1)} \alpha_j^{(1)},$$
$$\beta_i^{(2)} = 2\sum_{j=0}^{N_2} (\Omega^{-1})_{ij}^{(2)} \alpha_j^{(2)}. \tag{56}$$

With "higher-order terms" in Eqs. (55) we refer to the terms nonlinear in δf from Eq. (54) as well as to the nonlinear and gradient terms on the right-hand sides of Eqs. (35)–(37). As expected, the equations of motion for the tensors $X_i^{\mu_1\cdots\mu_\ell}$ decouple in the linear regime. Without loss of generality, we order the tensors $X_r^{\mu_1\cdots\mu_\ell}$ according to increasing $\chi_r^{(\ell)}$, e.g., in such a way that $\chi_r^{(\ell)} < \chi_{r+1}^{(\ell)}$, $\forall\ \ell$.

By diagonalizing Eqs. (35)–(37) we were able to identify the microscopic time scales of the Boltzmann equation given by the inverse of the coefficients $\chi_r^{(\ell)}$. It is clear that, if the nonlinear terms in Eqs. (55) are small enough, each tensor $X_r^{\mu_1\cdots\mu_\ell}$ relaxes independently to its respective asymptotic value, given by the first term on the right-hand sides of Eqs. (55) [divided by the corresponding $\chi_r^{(\ell)}$], on a time scale $\sim 1/\chi_r^{(\ell)}$. We will refer to these asymptotic solutions as Navier-Stokes values. By neglecting all these relaxation scales, i.e., taking the limit $\chi_r^{(\ell)} \to \infty$ with $\beta_r^{(\ell)}/\chi_r^{(\ell)}$ fixed, all irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$ become proportional to gradients of α_0, β_0, and u^μ, and we obtain a Chapman-Enskog-type solution, which at first order in the Knudsen number results in the relativistic Navier-Stokes equations of fluid dynamics. As already mentioned in the introduction, this type of solution is unstable and acausal, hence it cannot serve as a proper description of relativistic fluids.

The solution for this problem was also mentioned in the Introduction. To obtain causal and stable equations one must take into account the characteristic times within which the bulk viscous pressure, the particle-diffusion current, and the shear-stress tensor relax towards their asymptotic Navier-Stokes values. As shown in Ref. [10], in the fluid-dynamical limit these are given by the slowest microscopic time scales of the underlying microscopic theory, i.e., the fast relaxation scales are not expected to contribute.

In practice, this is implemented by assuming that only the slowest modes with rank 2 and smaller, X_0, X_0^μ, and $X_0^{\mu\nu}$, remain in the transient regime and satisfy the partial differential equations (55),

$$\begin{aligned}\dot{X}_0 + \chi_0^{(0)} X_0 &= \beta_0^{(0)}\theta + (\text{higher-order terms}),\\ \dot{X}_0^{\langle\mu\rangle} + \chi_0^{(1)} X_0^\mu &= \beta_0^{(1)} I^\mu + (\text{higher-order terms}),\\ \dot{X}_0^{\langle\mu\nu\rangle} + \chi_0^{(2)} X_0^{\mu\nu} &= \beta_0^{(2)}\sigma^{\mu\nu} + (\text{higher-order terms}),\end{aligned} \tag{57}$$

while the modes described by faster relaxation scales, i.e., X_r, X_r^μ, and $X_r^{\mu\nu}$, for any r *larger than* 0, will be approximated by their asymptotic solution, constructed by iterating their respective $\mathcal{O}(\text{Kn})$ solutions, $\beta_r^{(0)}\theta/\chi_r^{(0)}$, $\beta_r^{(1)} I^\mu/\chi_r^{(1)}$, and $\beta_r^{(2)}\sigma^{\mu\nu}/\chi_r^{(2)}$. Up to first order in Knudsen number, this leads to

$$\begin{aligned}X_r &\simeq \frac{\beta_r^{(0)}}{\chi_r^{(0)}}\theta + (\text{higher-order terms}),\\ X_r^\mu &\simeq \frac{\beta_r^{(1)}}{\chi_r^{(1)}} I^\mu + (\text{higher-order terms}),\\ X_r^{\mu\nu} &\simeq \frac{\beta_r^{(2)}}{\chi_r^{(2)}}\sigma^{\mu\nu} + (\text{higher-order terms}),\end{aligned} \tag{58}$$

where the terms denoted as (higher-order terms) can be organized in powers of the Knudsen number and the slow modes, X_0, X_0^μ, and $X_0^{\mu\nu}$, which cannot be approximated by their asymptotic solution. Note that, while this approximation is similar to the Chapman-Enskog expansion, Eqs. (57) go beyond the Chapman-Enskog expansion by including the transient dynamics.

Note that, for $r \geq 1$, X_r, X_r^μ, and $X_r^{\mu\nu}$ are of first order in Knudsen number, $\mathcal{O}(\text{Kn})$. The reason is that the gradient terms θ, I^μ, and $\sigma^{\mu\nu}$ are proportional to L_{macr}^{-1}, while $1/\chi_r^{(\ell)}$ is proportional to ℓ_{micr}. The coefficients $\beta_r^{(\ell)}$ are simply functions of the thermodynamic variables α_0, β_0, and thus of order $\mathcal{O}(1)$.

Furthermore, in order to obtain the traditional equations of fluid dynamics given in terms of the conserved currents, there should not appear any tensor $X_r^{\mu\nu\lambda\ldots}$ with rank higher than 2. Neglecting such tensors can be justified by proving that they have asymptotic solutions that are at least $\mathcal{O}(\text{Kn}^2, \text{KnR}_i^{-1})$, i.e., beyond the order we consider here.

Equations (58) enable us to approximate the irreducible moments that do not appear in the conserved currents in terms of those that do occur, namely, the particle-diffusion current, the bulk viscous pressure, and the shear-stress tensor. We now show how to do this. We first invert Eq. (53),

$$\rho_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}, \tag{59}$$

then, using Eqs. (58), we obtain

$$\begin{aligned}\rho_i &\simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0}\Omega_{ij}^{(0)}\frac{\beta_j^{(0)}}{\chi_j^{(0)}}\theta = \Omega_{i0}^{(0)} X_0 + \mathcal{O}(\text{Kn}),\\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1}\Omega_{ij}^{(1)}\frac{\beta_j^{(1)}}{\chi_j^{(1)}} I^\mu = \Omega_{i0}^{(1)} X_0^\mu + \mathcal{O}(\text{Kn}),\\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2}\Omega_{ij}^{(2)}\frac{\beta_j^{(2)}}{\chi_j^{(2)}}\sigma^{\mu\nu} = \Omega_{i0}^{(2)} X_0^{\mu\nu} + \mathcal{O}(\text{Kn}).\end{aligned} \tag{60}$$

Here, we indicated that the contribution from the modes X_r, X_r^μ, and $X_r^{\mu\nu}$ for $r \geq 1$ is of order $\mathcal{O}(\text{Kn})$.

Taking $i = 0$ in the above equations and, without loss of generality, setting $\Omega_{00}^{(\ell)} = 1$, we obtain from Eqs. (47) the relations

$$\begin{aligned}X_0 &\simeq -\frac{3}{m^2}\Pi - \sum_{j=3}^{N_0}\Omega_{0j}^{(0)}\frac{\beta_j^{(0)}}{\chi_j^{(0)}}\theta,\\ X_0^\mu &\simeq n^\mu - \sum_{j=2}^{N_1}\Omega_{0j}^{(1)}\frac{\beta_j^{(1)}}{\chi_j^{(1)}} I^\mu,\\ X_0^{\mu\nu} &\simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2}\Omega_{0j}^{(2)}\frac{\beta_j^{(2)}}{\chi_j^{(2)}}\sigma^{\mu\nu}.\end{aligned} \tag{61}$$

Substituting Eqs. (61) into Eqs. (60),

$$\begin{aligned}\frac{m^2}{3}\rho_i &\simeq -\Omega_{i0}^{(0)}\Pi - (\zeta_i - \Omega_{i0}^{(0)}\zeta_0)\theta = -\Omega_{i0}^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}),\\ \rho_i^{\mu} &\simeq \Omega_{i0}^{(1)}n^{\mu} + (\kappa_i - \Omega_{i0}^{(1)}\kappa_0)I^{\mu} = \Omega_{i0}^{(1)}n^{\mu} + \mathcal{O}(\mathrm{Kn}),\\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2(\eta_i - \Omega_{i0}^{(2)}\eta_0)\sigma^{\mu\nu}\\ &= \Omega_{i0}^{(2)}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}),\\ \rho_i^{\mu\nu\lambda\cdots} &\simeq \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\mathrm{R}_i^{-1}). \end{aligned} \tag{62}$$

To obtain Eqs. (62), we further used that $X_r^{\mu_1\cdots\mu_\ell} \sim \mathcal{O}(\mathrm{Kn}^2, \mathrm{Kn}\mathrm{R}_i^{-1})$ for $\ell \geq 3$, and defined the transport coefficients

$$\zeta_i = \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)}\alpha_r^{(0)}, \qquad \kappa_i = \sum_{r=0,\neq 1}^{N_1}\tau_{ir}^{(1)}\alpha_r^{(1)},$$
$$\eta_i = \sum_{r=0}^{N_2}\tau_{ir}^{(2)}\alpha_r^{(2)}, \tag{63}$$

where we introduced the inverse of $\mathcal{A}^{(\ell)}$, $\tau^{(\ell)} \equiv (\mathcal{A}^{-1})^{(\ell)}$ and used the relation,

$$\tau_{in}^{(\ell)} = \sum_{m=0}^{N_\ell}\Omega_{im}^{(\ell)}\frac{1}{\chi_m^{(\ell)}}(\Omega^{-1})_{mn}^{(\ell)}. \tag{64}$$

In the next subsection, we shall identify the coefficients ζ_0, κ_0, and η_0 as the bulk-viscosity, particle-diffusion, and shear-viscosity coefficients, respectively.

So far we have proved that, by taking into account only the slowest relaxation time scales, *all* irreducible moments $\rho_i^{\mu\nu\lambda\cdots}$ of the deviation of the single-particle distribution function from the equilibrium one can be related, *up to first order in Knudsen number*, $\mathcal{O}(\mathrm{Kn})$, to the dissipative currents, Π, n^μ, and $\pi^{\mu\nu}$. This demonstrates that in this limit, it is possible to reduce the number of dynamical variables in Eqs. (35)–(37) to quantities appearing in the conserved currents. This will be explicitly shown in the next section.

We remark that similar relations between the irreducible moments and the dissipative currents can also be obtained with the 14-moment approximation, but with a different set of proportionality coefficients. However, in the traditional 14-moment approximation such relations are obtained by explicitly truncating the moment expansion (30) and, as a result, they are not of a definite order in powers of Knudsen number. This is the reason why the 14-moment approximation does not give rise to equations of motion with a definite domain of validity in Knudsen and inverse Reynolds numbers.

Note, however, that the relations (62) are only valid for the moments $\rho_r^{\mu\nu\lambda\cdots}$ with positive r. This is not a problem since similar relations can also be obtained for the irreducible moments with negative r. We expect the expansion (30) to be complete and, therefore, any moment that does not appear in this expansion must be linearly related to those that do appear. This means that, using the moment expansion, Eq. (30), it is possible to express the moments with negative r in terms of the ones with positive r. Substituting Eq. (30) into Eq. (27) and using Eq. (20), we obtain

$$\rho_{-r}^{\nu_1\cdots\nu_\ell} = \sum_{n=0}^{N_\ell}\mathcal{F}_{rn}^{(\ell)}\rho_n^{\nu_1\cdots\nu_\ell}, \tag{65}$$

where we defined the following thermodynamic integral:

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!}\int dK f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}E_{\mathbf{k}}^{-r}\mathcal{H}_{\mathbf{k}n}^{(\ell)}(\Delta^{\alpha\beta}k_\alpha k_\beta)^\ell. \tag{66}$$

Therefore, Eqs. (62) lead to

$$\rho_{-r} = -\frac{3}{m^2}\gamma_r^{(0)}\Pi + \mathcal{O}(\mathrm{Kn}), \qquad \rho_{-r}^{\mu} = \gamma_r^{(1)}n^\mu + \mathcal{O}(\mathrm{Kn}),$$
$$\rho_{-r}^{\mu\nu} = \gamma_r^{(2)}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}), \qquad \rho_{-r}^{\mu\nu\cdots} = \mathcal{O}(\mathrm{Kn}^3), \tag{67}$$

where we introduced the coefficients

$$\gamma_r^{(0)} = \sum_{n=0,\neq 1,2}^{N_0}\mathcal{F}_{rn}^{(0)}\Omega_{n0}^{(0)}, \qquad \gamma_r^{(1)} = \sum_{n=0,\neq 1}^{N_1}\mathcal{F}_{rn}^{(1)}\Omega_{n0}^{(1)},$$
$$\gamma_r^{(2)} = \sum_{n=0}^{N_2}\mathcal{F}_{rn}^{(2)}\Omega_{n0}^{(2)}. \tag{68}$$

VI. COMPLETE FLUID-DYNAMICAL EQUATIONS TO SECOND ORDER

Now we are ready to close Eqs. (35)–(37) in terms of the dissipative currents appearing in N^μ and $T^{\mu\nu}$ and derive the fluid-dynamical equations of motion. For this purpose, it is convenient to use the inverse of $\mathcal{A}^{(\ell)}$, $\tau^{(\ell)} = (\mathcal{A}^{-1})^{(\ell)}$, which naturally satisfies $\tau^{(\ell)}\mathcal{A}^{(\ell)} = \mathbb{1}$. Hence, it is straightforward to rewrite Eq. (50) as

$$\sum_{j=0}^{N_\ell}\tau_{ij}^{(\ell)}C_{j-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\rho_i^{\mu_1\cdots\mu_\ell} + (\text{terms nonlinear in } \delta f). \tag{69}$$

Then we multiply Eqs. (35)–(37) by $\tau_{nr}^{(\ell)}$, sum over r, and substitute Eq. (69). Next, we use Eqs. (62) and (67) to replace all irreducible moments $\rho_i^{\mu_1\cdots\mu_\ell}$ appearing in the equations by the fluid-dynamical variables. Additionally, all covariant time derivatives of α_0, β_0, and u^μ are replaced by spatial gradients of fluid-dynamical variables using the conservation laws in the form shown in Eqs. (39)–(41). The resulting equations of motion are formally given as

$$\begin{aligned}\tau_\Pi\dot{\Pi} + \Pi &= -\zeta\theta + \mathcal{J} + \mathcal{K} + \mathcal{R},\\ \tau_n\dot{n}^{\langle\mu\rangle} + n^\mu &= \kappa I^\mu + \mathcal{J}^\mu + \mathcal{K}^\mu + \mathcal{R}^\mu,\\ \tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{R}^{\mu\nu}.\end{aligned} \tag{70}$$

We remark that in order to derive these equations of motion, it is necessary to use Eq. (52) in the following form:

$$\sum_{j=0}^{N_\ell} \tau_{ij}^{(\ell)} \Omega_{jm}^{(\ell)} = \Omega_{im}^{(\ell)} \frac{1}{\chi_m^{(\ell)}}. \tag{71}$$

In the above equations of motion all nonlinear terms and couplings to other currents were collected in the tensors $\mathcal{J}$, $\mathcal{K}$, $\mathcal{R}$, $\mathcal{J}^\mu$, $\mathcal{K}^\mu$, $\mathcal{R}^\mu$, $\mathcal{J}^{\mu\nu}$, $\mathcal{K}^{\mu\nu}$, and $\mathcal{R}^{\mu\nu}$. The tensors $\mathcal{J}$, $\mathcal{J}^\mu$, and $\mathcal{J}^{\mu\nu}$ contain all terms of first order in Knudsen and inverse Reynolds numbers,

$$\begin{aligned}
\mathcal{J} &= -\ell_{\Pi n} \nabla \cdot n - \tau_{\Pi n} n \cdot F - \delta_{\Pi\Pi} \Pi \theta - \lambda_{\Pi n} n \cdot I + \lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}, \\
\mathcal{J}^\mu &= -n_\nu \omega^{\nu\mu} - \delta_{nn} n^\mu \theta - \ell_{n\Pi} \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu} \nabla_\lambda \pi_\nu^\lambda + \tau_{n\Pi} \Pi F^\mu - \tau_{n\pi} \pi^{\mu\nu} F_\nu \\
&\quad - \lambda_{nn} n_\nu \sigma^{\mu\nu} + \lambda_{n\Pi} \Pi I^\mu - \lambda_{n\pi} \pi^{\mu\nu} I_\nu, \\
\mathcal{J}^{\mu\nu} &= 2\pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\lambda\langle\mu} \sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} - \tau_{\pi n} n^{\langle\mu} F^{\nu\rangle} + \ell_{\pi n} \nabla^{\langle\mu} n^{\nu\rangle} + \lambda_{\pi n} n^{\langle\mu} I^{\nu\rangle},
\end{aligned} \tag{72}$$

where we defined $F^\mu = \nabla^\mu P_0$. In principle, one could replace this quantity by the acceleration $\dot{u}^\mu$ using Eq. (41). The tensors $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$ contain all terms of second order in Knudsen number,

$$\begin{aligned}
\mathcal{K} &= \tilde{\zeta}_1 \omega_{\mu\nu} \omega^{\mu\nu} + \tilde{\zeta}_2 \sigma_{\mu\nu} \sigma^{\mu\nu} + \tilde{\zeta}_3 \theta^2 + \tilde{\zeta}_4 I \cdot I + \tilde{\zeta}_5 F \cdot F + \tilde{\zeta}_6 I \cdot F + \tilde{\zeta}_7 \nabla \cdot I + \tilde{\zeta}_8 \nabla \cdot F, \\
\mathcal{K}^\mu &= \tilde{\kappa}_1 \sigma^{\mu\nu} I_\nu + \tilde{\kappa}_2 \sigma^{\mu\nu} F_\nu + \tilde{\kappa}_3 I^\mu \theta + \tilde{\kappa}_4 F^\mu \theta + \tilde{\kappa}_5 \omega^{\mu\nu} I_\nu + \tilde{\kappa}_6 \Delta_\lambda^\mu \partial_\nu \sigma^{\lambda\nu} + \tilde{\kappa}_7 \nabla^\mu \theta, \\
\mathcal{K}^{\mu\nu} &= \tilde{\eta}_1 \omega_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \tilde{\eta}_2 \theta \sigma^{\mu\nu} + \tilde{\eta}_3 \sigma^{\lambda\langle\mu} \sigma_\lambda^{\nu\rangle} + \tilde{\eta}_4 \sigma_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \tilde{\eta}_5 I^{\langle\mu} I^{\nu\rangle} + \tilde{\eta}_6 F^{\langle\mu} F^{\nu\rangle} \\
&\quad + \tilde{\eta}_7 I^{\langle\mu} F^{\nu\rangle} + \tilde{\eta}_8 \nabla^{\langle\mu} I^{\nu\rangle} + \tilde{\eta}_9 \nabla^{\langle\mu} F^{\nu\rangle}.
\end{aligned} \tag{73}$$

It is important to remark that among the terms of $\mathcal{O}(\mathrm{Kn}^2)$ is a term $\omega_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda}$. Such a term was believed not to exist in a derivation of fluid dynamics from the Boltzmann equation and was therefore speculated to be of quantum nature [27]. From our derivation of fluid dynamics, one can see that this is not the case: it simply emerges from a proper truncation of the single-particle distribution function. The tensors $\mathcal{R}$, $\mathcal{R}^\mu$, and $\mathcal{R}^{\mu\nu}$ contain all terms of second order in inverse Reynolds number,

$$\begin{aligned}
\mathcal{R} &= \varphi_1 \Pi^2 + \varphi_2 n \cdot n + \varphi_3 \pi_{\mu\nu} \pi^{\mu\nu}, \\
\mathcal{R}^\mu &= \varphi_4 n_\nu \pi^{\mu\nu} + \varphi_5 \Pi n^\mu, \\
\mathcal{R}^{\mu\nu} &= \varphi_6 \Pi \pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu} \pi_\lambda^{\nu\rangle} + \varphi_8 n^{\langle\mu} n^{\nu\rangle}.
\end{aligned} \tag{74}$$

In Eq. (70), terms of order $\mathcal{O}(\mathrm{Kn}^3)$, $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{R}_j^{-1}\mathrm{R}_k^{-1})$, $\mathcal{O}(\mathrm{Kn}^2\mathrm{R}_i^{-1})$, and $\mathcal{O}(\mathrm{Kn}\mathrm{R}_i^{-1}\mathrm{R}_j^{-1})$ were omitted.

Note that we have obtained equations of motion that are closed in terms of 14 dynamical variables. We remark that this was accomplished without making use of the 14-moment approximation. This means that the reduction of degrees of freedom was not obtained by a direct truncation of the moment expansion, but by a separation of the microscopic time scales and the power-counting scheme itself. The information about all other moments are actually included in the transport coefficients; as will be shown later. If we also neglect the terms of second order in inverse Reynolds number we recover the equations of motion that are of the same form as those derived via the 14-moment approximation [21]. However, even in this case, the coefficients in Eqs. (72) and relaxation times are not the same as those calculated from the 14-moment approximation of Israel and Stewart.

The resulting equations of motion (70) contain a large number of transport coefficients. In particular, the viscosity coefficients and relaxation times of the dissipative currents were found to be

$$\begin{gathered}
\tau_\Pi = \frac{1}{\chi_0^{(0)}}, \qquad \tau_n = \frac{1}{\chi_0^{(1)}}, \qquad \tau_\pi = \frac{1}{\chi_0^{(2)}}, \\
\zeta = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{0r}^{(0)} \alpha_r^{(0)}, \qquad \kappa = \sum_{r=0,\neq 1}^{N_1} \tau_{0r}^{(1)} \alpha_r^{(1)}, \\
\eta = \sum_{r=0}^{N_2} \tau_{0r}^{(2)} \alpha_r^{(2)}.
\end{gathered} \tag{75}$$

Note that in general these transport coefficients depend not only on one moment of the distribution function but on all moments of corresponding rank ℓ. As in Chapman-Enskog theory, the viscosity coefficients can only be obtained by inverting $\mathcal{A}^{(\ell)}$. However, to obtain the transient dynamics of the fluid, characterized by the relaxation times, it is also necessary to find the eigenvalues and eigenvectors of $\mathcal{A}^{(\ell)}$.

In practice, the expansion (22) is always truncated at some point and the matrices $\mathcal{A}^{(\ell)}$, $\Omega^{(\ell)}$, and $\tau^{(\ell)}$ will actually be finite. The truncation of this expansion was already introduced as an upper limit, N_ℓ, in the corresponding summations. In principle, one should only truncate the expansion (22) when the values of all relevant transport coefficients have converged. Note that different transport coefficients may require a different number of moments to converge.

VII. APPLICATIONS

In this section, we compute the transport coefficients for several cases. First, we considered the lowest possible truncation scheme for Eq. (22) with $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$. In this case, the distribution function is expanded in terms of 14 moments and is actually equivalent to the one obtained via Israel-Stewart's 14-moment ansatz. Second, we consider the next simplest case and take $N_0 = 3$, $N_1 = 2$, and $N_2 = 1$. Then, the distribution function is characterized by 23 moments, and consequently we shall refer to this case as 23-moment approximation. Finally, we include 32 and 41 moments and verify that the numerical values for the transport coefficients converge.

We also compute the transport coefficients of the terms appearing in $\mathcal{J}$, $\mathcal{J}^{\mu}$, and $\mathcal{J}^{\mu\nu}$ that are displayed in Appendix C. These transport coefficients were also calculated in previous derivations of fluid dynamics from the Boltzmann equation. We shall explicitly point out the corrections to the previous results introduced by our novel approach. Note, however, that we are using a linear approximation to the collision term. Nonlinear contributions could in principle also enter the transport coefficients in the equations of motion (70), but will not be calculated here. Such an investigation will be left for future work. For this reason we also do not compute any coefficient of the terms of $\mathcal{O}(\mathrm{R}_i^{-1}\mathrm{R}_j^{-1})$, i.e., entering $\mathcal{R}$, $\mathcal{R}^{\mu}$, and $\mathcal{R}^{\mu\nu}$, since all of them originate exclusively from nonlinear contributions to the collision term.

A. 14-moment approximation

The 14-moment approximation is recovered by truncating Eq. (22) at $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$. For this specific truncation $\mathcal{A}^{(\ell)}$ is nothing but a number [because for $\mathcal{A}^{(0)}$ we have to exclude the second and third rows and columns and for $\mathcal{A}^{(1)}$ the second row and column], and thus

$$\tau^{(\ell)} = \frac{1}{\mathcal{A}^{(\ell)}}, \qquad \Omega^{(\ell)} = 1, \qquad \chi^{(\ell)} = \mathcal{A}^{(\ell)}. \tag{76}$$

Then, the equations of motion and transport coefficients reduce to those derived in Ref. [21].

For a classical gas of hard spheres with total cross section σ, in the massless limit, the integrals $\mathcal{A}^{(1)} = \mathcal{A}_{00}^{(1)}$ and $\mathcal{A}^{(2)} = \mathcal{A}_{00}^{(2)}$ can be computed and have the simple form

$$\mathcal{A}^{(1)} = \frac{4}{9\lambda_{\mathrm{mfp}}}, \tag{77}$$

$$\mathcal{A}^{(2)} = \frac{3}{5\lambda_{\mathrm{mfp}}}, \tag{78}$$

where we defined the mean free path $\lambda_{\mathrm{mfp}} = 1/(n_0\sigma)$. The details of this calculation are shown in Appendix B. The coefficients in the ultrarelativistic limit, $m\beta_0 \to 0$, can then be calculated analytically. The coefficients of order $\mathcal{O}(\mathrm{Kn}\mathrm{R}_i^{-1})$ are collected for the shear stress and particle diffusion in Tables I and II. Note that, in this limit, the bulk viscous pressure vanishes, and thus we do not need to compute $\mathcal{A}_{00}^{(0)}$.

B. Next correction: 23-moment approximation and beyond

In order to better understand our formulas, Eqs. (75), we would like to compute the first correction to the expressions in Tables I and II. For this purpose, we consider $N_0 = 3$, $N_1 = 2$, and $N_2 = 1$. Then, $\mathcal{A}^{(\ell)}$, $\Omega^{(\ell)}$, and $\tau^{(\ell)}$ are, after removing trivial rows and columns, 2×2 matrices that can be computed from the collision integral Eq. (51). We obtain the elements of $\mathcal{A}^{(1,2)}$, its inverse $\tau^{(1,2)}$, and $\Omega^{(1,2)}$ as

$$\mathcal{A}^{(1)} = \frac{1}{3\lambda_{\mathrm{mfp}}}\begin{pmatrix} 2 & \beta_0^2/30 \\ -4\beta_0^{-2} & 1 \end{pmatrix}, \quad \mathcal{A}^{(2)} = \frac{1}{\lambda_{\mathrm{mfp}}}\begin{pmatrix} 9/10 & -\beta_0/20 \\ 4/(3\beta_0) & 1/3 \end{pmatrix}, \tag{79}$$

$$\tau^{(1)} = \frac{3}{8}\lambda_{\mathrm{mfp}}\begin{pmatrix} 15/4 & -\beta_0^2/8 \\ 15\beta_0^{-2} & 15/2 \end{pmatrix}, \quad \tau^{(2)} = \frac{1}{11}\lambda_{\mathrm{mfp}}\begin{pmatrix} 10 & 3\beta_0/2 \\ -40\beta_0^{-1} & 27 \end{pmatrix}, \tag{80}$$

$$\Omega^{(1)} = \begin{pmatrix} 1 & 1 \\ -(15+\sqrt{105})\beta_0^{-2} & (-15+\sqrt{105})\beta_0^{-2} \end{pmatrix}, \quad \Omega^{(2)} = \begin{pmatrix} 1 & 1 \\ 8\beta_0^{-1} & 10/3\beta_0^{-1} \end{pmatrix}; \tag{81}$$

see Appendix B for details. The eigenvectors of $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are

TABLE I. The coefficients for the particle diffusion for a classical gas with constant cross section in the ultrarelativistic limit, in the 14-moment approximation.

κ	$\tau_n[\lambda_{\mathrm{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
$3/(16\sigma)$	9/4	1	3/5	$\beta_0/20$	$\beta_0/20$	0

TABLE II. The coefficients for the shear stress for a classical gas with constant cross section in the ultrarelativistic limit, in the 14-moment approximation.

η	$\tau_\pi[\lambda_{\mathrm{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
$4/(3\sigma\beta_0)$	5/3	10/7	0	4/3	0	0

TABLE III. The coefficients for the particle diffusion for a classical gas with constant cross section in the ultrarelativistic limit, in the 23-moment approximation.

κ	$\tau_n[\lambda_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
$21/(128\sigma)$	2.59	1.00	0.96	$0.054\beta_0$	$0.118\beta_0$	$0.0295\beta_0/P_0$

TABLE IV. The coefficients for the shear stress for a classical gas with constant cross section in the ultrarelativistic limit, in the 23-moment approximation.

η	$\tau_\pi[\lambda_{\text{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
$14/(11\sigma\beta_0)$	2	134/77	$0.344\beta_0^{-1}$	4/3	$-0.689\beta_0^{-1}$	$-0.689/n_0$

TABLE V. The coefficients for the particle diffusion for a classical gas with constant cross section in the ultrarelativistic limit, in the 14, 23, 32, and 41-moment approximation.

Number of moments	κ	$\tau_n[\lambda_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	9/4	1	3/5	$\beta_0/20$	$\beta_0/20$	0
23	$21/(128\sigma)$	2.59	1.0	0.96	$0.054\beta_0$	$0.118\beta_0$	$0.0295\beta_0/P_0$
32	$0.1605/\sigma$	2.57	1.0	0.93	$0.052\beta_0$	$0.119\beta_0$	$0.0297\beta_0/P_0$
41	$0.1596/\sigma$	2.57	1.0	0.92	$0.052\beta_0$	$0.119\beta_0$	$0.0297\beta_0/P_0$

$$\chi_0^{(1)} = \frac{1}{2\lambda_{\text{mfp}}}\left(1-\sqrt{\frac{7}{135}}\right), \quad \chi_1^{(1)} = \frac{1}{2\lambda_{\text{mfp}}}\left(1+\sqrt{\frac{7}{135}}\right), \quad (82)$$

$$\chi_0^{(2)} = \frac{1}{2\lambda_{\text{mfp}}}, \qquad \chi_1^{(2)} = \frac{11}{15\lambda_{\text{mfp}}}. \quad (83)$$

Using the formulas derived in this paper, Eqs. (75), we calculate the corrected values for the particle-number diffusion coefficient and diffusion-relaxation time and for the shear-viscosity and shear-relaxation time,

$$\kappa = \frac{21}{128} n_0 \lambda_{\text{mfp}} \simeq 0.164 n_0 \lambda_{\text{mfp}}, \quad (84)$$

$$\tau_n = \frac{90}{45-\sqrt{105}} \lambda_{\text{mfp}} \simeq 2.5897 \lambda_{\text{mfp}}, \quad (85)$$

$$\eta = \frac{14}{11} P_0 \lambda_{\text{mfp}} \simeq 1.2727 P_0 \lambda_{\text{mfp}}, \quad (86)$$

$$\tau_\pi = 2\lambda_{\text{mfp}}, \quad (87)$$

where we used that, in the massless and classical limits,

$$\alpha_0^{(1)} = \frac{1}{12} n_0, \qquad \alpha_2^{(1)} = -\frac{1}{\beta_0} P_0,$$
$$\alpha_0^{(2)} = \frac{4}{5} P_0, \qquad \alpha_1^{(2)} = \frac{4}{\beta_0} P_0. \quad (88)$$

As before, the coefficients in the ultrarelativistic limit, $m\beta_0 \to 0$, can then be calculated analytically. The coefficients of order $\mathcal{O}(\text{KnR}_i^{-1})$ are collected for the shear-stress and particle diffusion in Tables III and IV.

To obtain these expressions we used the results from Appendix D and that, in the massless/classical limits, $D_{20} = 3P_0^2$. Note that most of the transport coefficients were corrected by the inclusion of more moments in the computation. The coefficients related to the shear-stress tensor were less affected by the additional moments, when compared to the particle-diffusion coefficients. This might explain the poor agreement between the Israel-Stewart theory and numerical solutions of the Boltzmann equation in Refs. [18] regarding heat flow and fugacity.

We further checked the convergence of this approach by taking 32 and 41 moments. In this case, the matrices $\mathcal{A}^{(1,2)}$, $\tau^{(1,2)}$, and $\Omega^{(1,2)}$ were computed numerically. There is a clear tendency of convergence as we increase the number of moments. For the particular case of classical particles with constant cross sections, 32 moments seems sufficient. See Tables V and VI for the results.

VIII. DISCUSSION AND CONCLUSIONS

A. Knudsen number and the reduction of dynamical variables

It is important to mention that the terms $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$, which are of second order in Knudsen number, lead to several problems. The terms that contain second-order spatial derivatives of u^μ, α_0, and P_0, e.g., $\nabla_\mu I^\mu$, $\nabla_\mu F^\mu$, $\nabla^{\langle\mu} I^{\nu\rangle}$, $\nabla^{\langle\mu} F^{\nu\rangle}$, $\Delta^\mu_\alpha \partial_\nu \sigma^{\alpha\nu}$, and $\nabla^\mu\theta$, are especially problematic since they change the boundary conditions of the equations. In relativistic systems these derivatives, even though they are spacelike, also contain time derivatives and thus require initial values. This means that, by including them, one would have to specify not only the initial spatial

TABLE VI. The coefficients for the shear stress for a classical gas with constant cross section in the ultrarelativistic limit, in the 14, 23, 32, and 41-moment approximation.

Number of moments	η	$\tau_\pi[\lambda_{\mathrm{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta_0)$	5/3	10/7	0	4/3	0	0
23	$14/(11\sigma\beta_0)$	2	134/77	$0.344\beta_0^{-1}$	4/3	$-0.689/\beta_0$	$-0.689/n_0$
32	$1.268/(\sigma\beta_0)$	2	1.69	$0.254\beta_0^{-1}$	4/3	$-0.687/\beta_0$	$-0.687/n_0$
41	$1.267/(\sigma\beta_0)$	2	1.69	$0.244\beta_0^{-1}$	4/3	$-0.685/\beta_0$	$-0.685/n_0$

distribution of the fluid-dynamical variables but also the spatial distribution of their time derivatives. In practice, this implies that we would be increasing the number of fluid-dynamical degrees of freedom.

There is an even more serious problem. By including terms of order higher than one in Knudsen number, the transport equations become parabolic. In a relativistic theory, this comes with disastrous consequences since the solutions are acausal and consequently unstable [5]. For this reason, if one wants to include terms of higher order in Knudsen number, it is mandatory to include also second-order comoving time derivatives of the dissipative quantities. Or, equivalently, one could promote the moments ρ_3, ρ_2^μ, $\rho_1^{\mu\nu}$ or further ones to dynamical variables. For this reason we do not compute the transport coefficients for these higher-order terms in this paper.

In practice, a way around this would be to replace e.g. the $\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle}$ term in $\mathcal{K}^{\mu\nu}$ using the asymptotic (Navier-Stokes) solution by $(1/2\eta)\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle}$, and thus effectively rendering it a term contributing to $\mathcal{J}^{\mu\nu}$. This should be a reasonable approximation if one is sufficiently close to the asymptotic solution. This would then change the coefficient of the respective term in $\mathcal{J}^{\mu\nu}$. In principle, this could be done to all terms in $\mathcal{K}$, $\mathcal{K}^\mu$, and $\mathcal{K}^{\mu\nu}$, except for the ones containing exclusively powers and/or gradients of F^μ and $\omega^{\mu\nu}$. In the same spirit, using the asymptotic solutions one could also shuffle some of the terms in $\mathcal{J}$, $\mathcal{J}^\mu$, and $\mathcal{J}^{\mu\nu}$ (those not containing F^μ, $\omega^{\mu\nu}$, and gradients of dissipative currents) into terms contributing to $\mathcal{R}$, $\mathcal{R}^\mu$, and $\mathcal{R}^{\mu\nu}$ (or vice versa). How this changes the actual transient dynamics remains to be investigated in the future.

B. Navier-Stokes limit

Note that one of the main features of transient theories of fluid dynamics is the relaxation of the dissipative currents towards their Navier-Stokes values, on time scales given by the transport coefficients τ_Π, τ_n, and τ_π. From the Boltzmann equation, Navier-Stokes theory is obtained by means of the Chapman-Enskog expansion that describes an asymptotic solution of the single-particle distribution function. It is already clear from the previous section that the equations of motion derived in this paper approach Navier-Stokes-type solutions at asymptotically long times, in which the dissipative currents are solely expressed in terms of gradients of fluid-dynamical variables.

It is interesting to investigate, however, if our equations approach the correct Navier-Stokes theory, i.e., if the viscosity coefficients obtained via our method are equivalent to the ones obtained via Chapman-Enskog theory. It should be noted that this is not the case for Grad's and Israel and Stewart's theories [6,15,21]. The viscosity coefficients computed by these theories do not coincide with those extracted from the Chapman-Enskog theory. We remark that, after taking into account the first corrections to the shear-viscosity coefficient, see Eq. (86) and Table VI, our result approached the solution obtained using Chapman-Enskog theory, $\eta_{NS} = 1.2654/(\beta_0\sigma)$ [6]. In principle there is no reason for the method of moments to attain a different Navier-Stokes limit than Chapman-Enskog theory. We can show that, if the same basis of irreducible tensors $k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}$ and polynomials $P_{n\mathbf{k}}^{(\ell)}$ is used in both calculations, they both yield the same result, even order by order.

C. "Nonhydrodynamic" modes and the microscopic origin of the relaxation time

One of the features of the theory derived in this paper (and also of Grad's and Israel-Stewart's theories) is the appearance of so-called nonhydrodynamics modes, i.e., modes that do not vanish in the limit of zero wave number. Such modes do not exist in Navier-Stokes theory or its extensions via the Chapman-Enskog expansion. For this reason, these modes are usually not associated with fluid-dynamical behavior, hence the label "nonhydrodynamic".

The nonhydrodynamic modes describe the relaxation of the dissipative currents towards their respective Navier-Stokes solutions and can be directly related to the respective relaxation times. For the case of the shear nonhydrodynamic mode, $\omega_{\mathrm{shear}}^{\mathrm{nonhydro}}(\mathbf{k})$, it can be shown that in the limit of $\mathbf{k}\to 0$ the mode is given by $\omega_{\mathrm{shear}}^{\mathrm{nonhydro}}(\mathbf{0}) = -i/\tau_\pi$ [5]. In Chapman-Enskog theory the transient dynamics of the system is neglected, e.g., it is assumed that in the absence of spacelike gradients, timelike gradients vanish as well, and it is natural that such modes do not exist.

The appearance of nonhydrodynamic modes in a fluid-dynamical theory seems to counteract the prevalent belief that fluid dynamics effectively describes the asymptotic longtime and long-distance behavior of the microscopic theory. Recently, a microscopic formula for the relaxation time of dissipative currents was obtained in the framework of linear response theory [10]. In that paper, the relaxation time was shown to be intrinsically related to the slowest microscopic time scale of the system, i.e., to the singularity of the retarded Green's function closest to the origin in the complex-plane. Thus, the nonhydrodynamic modes in Israel and Stewart's theory and in the equations derived in this paper belong to a description at long, but not asymptotically long, times.

This means that the theory derived in this paper (as well as Israel and Stewart's theory) attempts to describe the dynamics of the dissipative currents at time scales of the order of the (slowest) microscopic times scale (which is of the order of the mean free path). Such findings challenge the point of view that a fluid-dynamical description can only be formulated around zero frequency and wave number and that the inclusion of relaxation times can only be understood as a regularization method to control the instabilities of the gradient expansion. In fact, the relaxation times correspond to microscopic time scales, independent of any macroscopic scale related to the gradients of fluid-dynamical variables. Note that the expressions presented in Ref. [10] and in this paper for η and τ_π are equivalent.

D. Conclusions

In this work we have presented a general and consistent derivation of relativistic fluid dynamics from the Boltzmann equation using the method of moments. First, a general expansion of the single-particle distribution function in terms of its moments was introduced in Sec. III. We constructed an orthonormal basis that allowed us to expand and obtain exact relations between the expansion parameters and irreducible moments of the deviations of the distribution function from equilibrium. We then proceeded to derive exact equations for these moments.

The main difference of our approach to previous work is that we did not close the fluid-dynamical equations of motion by truncating the expansion of the distribution function. Instead, we kept all terms in the moment expansion and truncated the exact equations of motion according to a power-counting scheme in Knudsen and inverse Reynolds number. Contrary to many calculations, we did not assume that the inverse Reynolds and Knudsen numbers are of the same order. As a matter of fact, in order to obtain relaxation-type equations, we had to explicitly include the slowest microscopic time scales, which are shown to be the characteristic times within which dissipative currents relax towards their asymptotic Navier-Stokes solutions. Thus, Navier-Stokes theory, or the Chapman-Enskog expansion, is already included in our formulation as an asymptotic limit of the dynamical equations derived in this paper.

We concluded that the equations of motion can be closed in terms of only 14 dynamical variables, as long as we only keep terms of second order in Knudsen and/or inverse Reynolds number. Even though the equations of motion are closed in terms of these 14 fields, the transport coefficients carry information about all moments of the distribution function (all the different relaxation scales of the irreducible moments). The bulk-viscosity, particle-diffusion, and shear-viscosity coefficients agree with the values obtained via Chapman-Enskog theory.

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APPENDIX A: DERIVATION OF THE COLLISION TERMS

In this appendix, we derive Eqs. (50) and (51). The first step is to linearize the collision operator,

$$C[f]=\frac{1}{\nu}\int dK'dPdP'W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}(f_{\mathbf{p}}f_{\mathbf{p}'}\tilde{f}_{\mathbf{k}}\tilde{f}_{\mathbf{k}'}-f_{\mathbf{k}}f_{\mathbf{k}'}\tilde{f}_{\mathbf{p}}\tilde{f}_{\mathbf{p}'}), \tag{A1}$$

in the deviations from the equilibrium distribution functions. In the main text, the deviations from the local-equilibrium distribution function were parametrized as

$$\delta f_{\mathbf{p}}=f_{\mathbf{p}}-f_{0\mathbf{p}}=f_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}}\phi_{\mathbf{p}}. \tag{A2}$$

Then, only keeping terms of first order in ϕ, we can prove that

$$f_{\mathbf{p}}f_{\mathbf{p}'}=f_{0\mathbf{p}}f_{0\mathbf{p}'}(1+\tilde{f}_{0\mathbf{p}'}\phi_{\mathbf{p}'}+\tilde{f}_{0\mathbf{p}}\phi_{\mathbf{p}})+\mathcal{O}(\phi^2), \tag{A3}$$

$$\tilde{f}_{\mathbf{p}}\tilde{f}_{\mathbf{p}'}=\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}(1-af_{0\mathbf{p}'}\phi_{\mathbf{p}'}-af_{0\mathbf{p}}\phi_{\mathbf{p}})+\mathcal{O}(\phi^2). \tag{A4}$$

Substituting Eqs. (A3) and (A4) into Eq. (A1), we obtain,

$$C[f]=\frac{1}{\nu}\int dK'dPdP'W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'} \times(\phi_{\mathbf{p}}+\phi_{\mathbf{p}'}-\phi_{\mathbf{k}}-\phi_{\mathbf{k}'})+\mathcal{O}(\phi^2), \tag{A5}$$

where we also used the equalities

$$\tilde{f}_{0\mathbf{p}}=f_{0\mathbf{p}}\exp(\beta_0E_{\mathbf{p}}-\alpha_0), \tag{A6}$$

$$f_{0\mathbf{p}}f_{0\mathbf{p}'}\tilde{f}_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}'}=f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}. \tag{A7}$$

Inserting Eq. (A5) in the expression for the irreducible collision term (38), we obtain

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{1}{\nu}\int dKdK'dPdP'W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}(\phi_{\mathbf{p}}+\phi_{\mathbf{p}'}-\phi_{\mathbf{k}}-\phi_{\mathbf{k}'}) + \mathcal{O}(\phi^2). \quad \text{(A8)}$$

The next step is to substitute the moment expansion of the single-particle distribution function, Eqs. (21) and (28), into Eq. (A8), expressing it in the following form:

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{m=0}^{\infty}\sum_{n=0}^{N_m}(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}\rho_n^{\nu_1\cdots\nu_m} + \mathcal{O}(\phi^2), \quad \text{(A9)}$$

where we defined the tensor

$$(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} \equiv \frac{1}{\nu}\int dKdK'dPdP'W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}f_{0\mathbf{k}}f_{0\mathbf{k}'}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p}'}E_{\mathbf{k}}^{r-1}k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}(\mathcal{H}^{(m)}_{\mathbf{k}n}k_{\langle\nu_1}\cdots k_{\nu_m\rangle} + \mathcal{H}^{(m)}_{\mathbf{k}'n}k'_{\langle\nu_1}\cdots k'_{\nu_m\rangle} - \mathcal{H}^{(m)}_{\mathbf{p}n}p_{\langle\nu_1}\cdots p_{\nu_m\rangle} - \mathcal{H}^{(m)}_{\mathbf{p}'n}p'_{\langle\nu_1}\cdots p'_{\nu_m\rangle}). \quad \text{(A10)}$$

The integral $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ is a tensor of rank $m+\ell$, which is symmetric under permutations of μ-type indices and symmetric under permutations of ν-type indices, and which depends only on equilibrium distribution functions. The latter contain only the fluid 4-velocity u^μ. Therefore, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ must be constructed from tensor structures made of u^μ and the metric tensor $g^{\mu\nu}$. Also, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ was constructed to be orthogonal to u^μ and to satisfy the following property,

$$\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_m}_{\beta_1\cdots\beta_m}(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} = (\mathcal{A}_{rn})^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_m}. \quad \text{(A11)}$$

Since $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ is orthogonal to u^μ, it can only be constructed from combinations of projection operators, $\Delta^{\mu\nu}$. This already constrains $m+\ell$ to be an even number, since it is impossible to construct odd-ranked tensors solely from $\Delta^{\mu\nu}$s. This means that both ℓ and m are either even or odd. Therefore, the following type of terms could appear in $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$:

(i) Terms where all μ-type indices pair up on projectors $\Delta^{\mu_i\mu_j}$ and all ν-type indices on projectors $\Delta_{\nu_p\nu_q}$, e.g.

$$\Delta^{\mu_1\mu_2}\cdots\Delta^{\mu_i\mu_j}\cdots\Delta^{\mu_{\ell-1}\mu_\ell}\Delta_{\nu_1\nu_2}\cdots\Delta_{\nu_p\nu_q}\cdots\Delta_{\nu_{m-1}\nu_m}. \quad \text{(A12)}$$

All possible permutations of the μ-type indices among themselves and ν-type indices among themselves are allowed.

(ii) Terms where at least one μ-type index pairs with a ν-type index on a projector, e.g.

$$\Delta^{\mu_1}_{\nu_1}\Delta^{\mu_2\mu_3}\cdots\Delta^{\mu_i\mu_j}\cdots\Delta^{\mu_{\ell-1}\mu_\ell} \times\Delta_{\nu_2\nu_3}\cdots\Delta_{\nu_p\nu_q}\cdots\Delta_{\nu_{m-1}\nu_m}. \quad \text{(A13)}$$

Again, all possible permutations of the μ-type and ν-type indices are allowed. If there is an odd number of projectors of the type $\Delta^{\mu_i}_{\nu_p}$, both ℓ and m must be odd. If there is an even number, both ℓ and m must be even, too. Without loss of generality, suppose that $\ell > m$. For $\ell + m$ to be even, ℓ must be $m+2$, $m+4,\ldots$. Then one could pair all ν-type indices with μ-type indices on projectors of the form $\Delta^{\mu_i}_{\nu_p}$, with some projectors left over that carry only μ-type indices, e.g. $\Delta^{\mu_j\mu_k}$.

(iii) If $\ell = m$, all μ-type indices could be paired up with ν-type indices on projectors of the form $\Delta^{\mu_i}_{\nu_p}$, with no leftover projectors like what was explained at the end of (ii),

$$\Delta^{\mu_1}_{\nu_1}\cdots\Delta^{\mu_\ell}_{\nu_\ell}. \quad \text{(A14)}$$

Again, all permutations of the μ-type indices among themselves and ν-type indices among themselves are allowed.

Note that terms of the type (i) and (ii) by themselves do not satisfy the property (A11). This happens because any term that contains at least one projector of the type $\Delta^{\mu_i\mu_j}$ or $\Delta_{\nu_p\nu_q}$ vanishes when contracted with $\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_m}_{\beta_1\cdots\beta_m}$. Therefore, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ cannot be solely constructed from terms of type (i) and (ii), because otherwise it would vanish trivially, and property (A11) would not be satisfied. There must at least be one term of type (iii). However, this implies that $m=\ell$. This does not imply that terms of type (i) and (ii) do not appear; they do occur, but in such a way that Eq. (A11) is satisfied. In summary, $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m}$ has the form

$$(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_m} = \delta_{\ell m}\{\mathcal{A}^{(\ell)}_{rn}\Delta^{(\mu_1}_{(\nu_1}\cdots\Delta^{\mu_\ell)}_{\nu_\ell)} + [\text{terms of type (i) and (ii)}]\}, \quad \text{(A15)}$$

where the parentheses denote the symmetrization of all Lorentz indices. Contracting Eq. (A15) with $\Delta^{\alpha_1\cdots\alpha_\ell}_{\mu_1\cdots\mu_\ell}\Delta^{\nu_1\cdots\nu_\ell}_{\beta_1\cdots\beta_\ell}$ and using Eq. (A11), we prove that

$$(\mathcal{A}_{rn})^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_m} = \delta_{\ell m}\mathcal{A}^{(\ell)}_{rn}\Delta^{\alpha_1\cdots\alpha_\ell}_{\beta_1\cdots\beta_\ell}. \quad \text{(A16)}$$

Finally, substituting Eq. (A16) into Eq. (A9) we derive Eq. (50), introduced in the main text of the paper,

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{N_\ell}\mathcal{A}^{(\ell)}_{rn}\rho_n^{\mu_1\cdots\mu_\ell}. \quad \text{(A17)}$$

The coefficients $\mathcal{A}^{(\ell)}_{rn}$ can be obtained from the following projection of $(\mathcal{A}_{rn})^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$:

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\Delta_{\mu_1\cdots\mu_\ell}^{\mu_1\cdots\mu_\ell}} \Delta_{\mu_1\cdots\mu_\ell}^{\nu_1\cdots\nu_\ell} (\mathcal{A}_{rn})_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell},$$
$$= \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} E_{\mathbf{k}}^{r-1} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} (\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle\mu_1} \cdots k'_{\mu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\mu_1} \cdots p_{\mu_\ell\rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle\mu_1} \cdots p'_{\mu_\ell\rangle}), \tag{A18}$$

where we used that $\Delta_{\mu_1\cdots\mu_\ell}^{\mu_1\cdots\mu_\ell} = 2\ell + 1$.

APPEMDIX B: CALCULATION OF THE COLLISION INTEGRALS

In this appendix, we calculate the collision integrals, Eq. (51), for a classical gas, i.e., $\tilde{f}_{0\mathbf{k}} = 1$, of hard spheres in the ultrarelativistic limit, $m\beta_0 \ll 1$. Then, Eq. (51) becomes

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^{r-1} k^{\langle\nu_1} \cdots k^{\nu_\ell\rangle} (\mathcal{H}_{\mathbf{k}n}^{(\ell)} k_{\langle\nu_1} \cdots k_{\nu_\ell\rangle} + \mathcal{H}_{\mathbf{k}'n}^{(\ell)} k'_{\langle\nu_1} \cdots k'_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}n}^{(\ell)} p_{\langle\nu_1} \cdots p_{\nu_\ell\rangle} - \mathcal{H}_{\mathbf{p}'n}^{(\ell)} p'_{\langle\nu_1} \cdots p'_{\nu_\ell\rangle}). \tag{B1}$$

The functions $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ were defined in the main text, see Eq. (29). The transition rate $W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'}$ is written in terms of the differential cross section $\sigma(s, \Theta)$ as

$$W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} = s\sigma(s, \Theta_s)(2\pi)^6 \delta^{(4)}(k^\mu + k'^\mu - p^\mu - p'^\mu). \tag{B2}$$

The variable s and Θ_s are defined as

$$s = (k + k')^2, \qquad \cos\Theta_s = \frac{(k - k')\cdot(p - p')}{(k - k')^2}. \tag{B3}$$

We further define the total cross section as the integral

$$\sigma_T(s) = \frac{2\pi}{\nu} \int d\Theta_s \sin\Theta_s \sigma(s, \Theta_s). \tag{B4}$$

In order to calculate $\mathcal{A}_{rn}^{(\ell)}$ it is convenient to first define the tensors $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$

$$X^n_{\mu\nu\gamma_1\cdots\gamma_m} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^n k_\mu k_\nu (k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m} - p_{\gamma_1} \cdots p_{\gamma_m} - p'_{\gamma_1} \cdots p'_{\gamma_m}). \tag{B5}$$

The collision integrals $\mathcal{A}_{rn}^{(\ell)}$ can always be expressed as linear combinations of contractions/projections of $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$. For the purpose of this paper, we shall only need $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$ for $m = 2$ and 3. For now we concentrate on calculating these integrals. We separate $X^n_{\mu\nu\gamma_1\cdots\gamma_m}$ as

$$X^n_{\mu\nu\gamma_1\cdots\gamma_m} = A^n_{\mu\nu\gamma_1\cdots\gamma_m} + B^n_{\mu\nu\gamma_1\cdots\gamma_m}, \tag{B6}$$

with

$$A^n_{\mu\nu\gamma_1\cdots\gamma_m} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^n k_\mu k_\nu (k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m}),$$
$$B^n_{\mu\nu\gamma_1\cdots\gamma_m} = -\frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^n k_\mu k_\nu (p_{\gamma_1} \cdots p_{\gamma_m} + p'_{\gamma_1} \cdots p'_{\gamma_m}). \tag{B7}$$

The $dPdP'$ integration in the first tensor, $A^n_{\mu\nu\gamma_1\cdots\gamma_m}$, can be immediately performed and written in terms of the total cross section, $\sigma_T(s)$, as

$$A^n_{\mu\nu\gamma_1\cdots\gamma_m} = \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^n k_\mu k_\nu (k_{\gamma_1} \cdots k_{\gamma_m} + k'_{\gamma_1} \cdots k'_{\gamma_m}) \frac{s}{2} \sigma_T(s). \tag{B8}$$

The calculation of the second tensor, $B^n_{\mu\nu\gamma_1\ldots\gamma_m}$, is cumbersome. First, we write it in the general form

$$B^n_{\mu\nu\gamma_1\cdots\gamma_m} = -\int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E_{\mathbf{k}}^n k_\mu k_\nu \Theta_{\gamma_1\cdots\gamma_m}, \tag{B9}$$

where we introduced the tensor

$$\Theta_{\gamma_1\cdots\gamma_m} = \frac{2}{\nu}\int dP dP' W_{\mathbf{kk}'\to\mathbf{pp}'} p_{\gamma_1}\cdots p_{\gamma_m}. \tag{B10}$$

The integral $\Theta_{\gamma_1\cdots\gamma_m}$ is an mth rank tensor. Strictly speaking, for isotropic cross sections, this tensor can only depend on the normalized total momentum of the collision $\tilde{P}_T^\mu \equiv s^{-1/2}(k^\mu + k'^\mu) \equiv s^{-1/2}P_T^\mu$. Thus, the tensor structure of $\Theta_{\gamma_1\cdots\gamma_m}$ must be constructed by combinations of $\tilde{P}_T^\mu$ and the projection operator orthogonal to $\tilde{P}_T^\mu$, $\Delta_P^{\mu\nu} = g^{\mu\nu} - \tilde{P}_T^\mu \tilde{P}_T^\nu$. In general,

$$\Theta_{\gamma_1\cdots\gamma_m} = \sum_{q=0}^{[m/2]} (-1)^q a_{mq}\mathcal{C}_{mq} C^q_{\gamma_1\cdots\gamma_m}, \tag{B11}$$

where we defined

$$\begin{aligned} a_{mq} &= \frac{m!}{(m-2q)!2q!}(2q-1)!!, \\ C^q_{\gamma_1\cdots\gamma_m} &= \Delta_P^{(\gamma_1\gamma_2}\cdots\Delta_P^{\gamma_{2q-1}\gamma_{2q}}\tilde{P}_T^{\gamma_{2q+1}}\cdots\tilde{P}_T^{\gamma_m)}, \\ \mathcal{C}_{mq} &= \frac{2}{\nu(2q+1)!!}\int dP dP' W_{\mathbf{kk}'\to\mathbf{pp}'} \\ &\quad \times (\tilde{P}_T^\mu p_\mu)^{m-2q}(-\Delta_P^{\alpha\beta}p_\alpha p_\beta)^q. \end{aligned} \tag{B12}$$

The parentheses denote the symmetrization of the tensor. For example,

$$\begin{aligned} \Theta_{\gamma_1\gamma_2} &= \mathcal{C}_{20}\tilde{P}_{T\gamma_1}\tilde{P}_{T\gamma_2} - \mathcal{C}_{21}\Delta_{P\gamma_1\gamma_2}, \\ \Theta_{\gamma_1\gamma_2\gamma_3} &= \mathcal{C}_{30}\tilde{P}_{T\gamma_1}\tilde{P}_{T\gamma_2}\tilde{P}_{T\gamma_3} - \mathcal{C}_{31}(\Delta_{P\gamma_1\gamma_2}\tilde{P}_{T\gamma_3} \\ &\quad + \Delta_{P\gamma_1\gamma_3}\tilde{P}_{T\gamma_2} + \Delta_{P\gamma_2\gamma_3}\tilde{P}_{T\gamma_1}). \end{aligned} \tag{B13}$$

The integrals $\mathcal{C}_{nq}$ are scalars and can be computed in any frame. It is most convenient to calculate them in the center-of-momentum frame, where $\tilde{P}_T^\mu = (1,0,0,0)$ and $\Delta_P^{\mu\nu} = \mathrm{diag}(0,-1,-1,-1)$. Then, it is straightforward to prove that

$$\begin{aligned} \mathcal{C}_{nq} &= \frac{\sigma_T(s)}{2^n(2q+1)!!}s^{(n-2q+1)/2}(s-4m^2)^{(2q+1)/2} \\ &\underset{m\to 0}{=} \frac{\sigma_T(s)}{2^n(2q+1)!!}s^{(n+2)/2}. \end{aligned} \tag{B14}$$

In the massless limit, the tensors $X^n_{\mu\nu\gamma_1\gamma_2}$ and $X^n_{\mu\nu\gamma_1\gamma_2\gamma_3}$ become

$$\begin{aligned} X^n_{\mu\nu\gamma_1\gamma_2} &= \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \sigma_T(s) k^\lambda k'_\lambda \left(k_{\gamma_1}k_{\gamma_2} + k'_{\gamma_1}k'_{\gamma_2} - \frac{2}{3}P_{T\gamma_1}P_{T\gamma_2} + \frac{1}{6}s g_{\gamma_1\gamma_2}\right), \\ X^n_{\mu\nu\gamma_1\gamma_2\gamma_3} &= \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} E^n_{\mathbf{k}} k_\mu k_\nu \sigma_T(s) k^\lambda k'_\lambda \bigg[k_{\gamma_1}k_{\gamma_2}k_{\gamma_3} + k'_{\gamma_1}k'_{\gamma_2}k'_{\gamma_3} - \frac{1}{2}P_{T\gamma_1}P_{T\gamma_2}P_{T\gamma_3} \\ &\quad + \frac{1}{6}k^\beta k'_\beta(g_{\gamma_1\gamma_2}P_{T\gamma_3} + g_{\gamma_1\gamma_3}P_{T\gamma_2} + g_{\gamma_2\gamma_3}P_{T\gamma_1})\bigg], \end{aligned} \tag{B15}$$

where we used that, in the massless limit, $s = 2k^\lambda k'_\lambda$.

1. Particle-diffusion current

For the collision integrals related to the particle-number diffusion current, we need the following two contractions:

$$\begin{aligned} \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}X^n_{\mu\nu\gamma_1\gamma_2} &= -\sigma_T(I_{10}I_{n+5,1} - 4I_{21}I_{n+4,1} \\ &\quad - I_{31}I_{n+3,1}), \\ \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}u^{\gamma_3}X^n_{\mu\nu\gamma_1\gamma_2\gamma_3} &= -\frac{\sigma_T}{2}(3I_{10}I_{n+6,1} - 11I_{21}I_{n+5,1} \\ &\quad - 5I_{31}I_{n+4,1} - 3I_{41}I_{n+3,1}). \end{aligned} \tag{B16}$$

To obtain the above relations, we used Eq. (20) and the definitions (45). In the massless and classical limits the integrals $I_{nq} = J_{nq}$ can be calculated analytically

$$I_{nq} = g\frac{e^{\alpha_0}}{(2q+1)!!}\frac{1}{2\pi^2}\frac{(n+1)!}{\beta_0^{n+2}} = \frac{(n+1)!}{(2q+1)!!}\frac{P_0}{2\beta_0^{n-2}}. \tag{B17}$$

Then,

$$\begin{aligned} \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}X^{-2}_{\mu\nu\gamma_1\gamma_2} &= \frac{4}{3}n_0\sigma_T\frac{P_0}{\beta_0}, \\ \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}X^{0}_{\mu\nu\gamma_1\gamma_2} &= -24n_0\sigma_T\frac{P_0}{\beta_0^3}, \\ \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}u^{\gamma_3}X^{-2}_{\mu\nu\gamma_1\gamma_2\gamma_3} &= 12n_0\sigma_T\frac{P_0}{\beta_0^2}, \\ \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}u^{\gamma_3}X^{0}_{\mu\nu\gamma_1\gamma_2\gamma_3} &= -280n_0\sigma_T\frac{P_0}{\beta_0^4}. \end{aligned} \tag{B18}$$

As a consistency check, we confirmed that $\Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}X^{-1}_{\mu\nu\gamma_1\gamma_2} = \Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}u^{\gamma_3}X^{-1}_{\mu\nu\gamma_1\gamma_2\gamma_3} = 0$.

The components of $\mathcal{A}^{(1)}$ change according to the number of moments included. In the 14-moment approximation, using Eqs. (23) and (29), we obtain

$$\mathcal{A}^{(1)}_{00} = \frac{W^{(1)}}{3}a^{(1)}_{10}a^{(1)}_{11}\Delta^{\mu\gamma_1}u^\nu u^{\gamma_2}X^{-2}_{\mu\nu\gamma_1\gamma_2} = \frac{4}{9}n_0\sigma_T. \tag{B19}$$

In the 23-moment approximation, e.g. considering three polynomials in the expansion (22), for $\ell = 1$,

$$\mathcal{A}_{r0}^{(1)} = \frac{W^{(1)}}{3}[(a_{10}^{(1)}a_{11}^{(1)} + a_{20}^{(1)}a_{21}^{(1)})\Delta^{\mu\gamma_1}u^{\nu}u^{\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{r-2} + a_{20}^{(1)}a_{22}^{(1)}\Delta^{\mu\gamma_1}u^{\nu}u^{\gamma_2}u^{\gamma_3}X_{\mu\nu\gamma_1\gamma_2\gamma_3}^{r-2}],$$

$$\mathcal{A}_{r2}^{(1)} = \frac{W^{(1)}}{3}(a_{22}^{(1)}a_{21}^{(1)}\Delta^{\mu\gamma_1}u^{\nu}u^{\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{r-2} + a_{22}^{(1)}a_{22}^{(1)}\Delta^{\mu\gamma_1}u^{\nu}u^{\gamma_2}u^{\gamma_3}X_{\mu\nu\gamma_1\gamma_2\gamma_3}^{r-2}). \tag{B20}$$

Then, using the results from Appendix E for the coefficients $a_{nq}^{(\ell)}$ together with Eqs. (B17) and (B18), we obtain

$$\mathcal{A}_{00}^{(1)} = \frac{2}{3}n_0\sigma_T, \qquad \mathcal{A}_{02}^{(1)} = \frac{\beta_0^2}{90}n_0\sigma_T,$$
$$\mathcal{A}_{20}^{(1)} = -\frac{4}{3\beta_0^2}n_0\sigma_T, \qquad \mathcal{A}_{22}^{(1)} = \frac{1}{3}n_0\sigma_T. \tag{B21}$$

2. Shear-stress tensor

For the collision integrals related to the shear-stress tensor, we need the following two contractions:

$$\Delta^{\mu\nu\gamma_1\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{n} = \frac{10}{3}\sigma_T(I_{10}I_{n+5,2} + 4I_{21}I_{n+4,2}),$$
$$\Delta^{\mu\nu\gamma_1\gamma_2}u^{\gamma_3}X_{\mu\nu\gamma_1\gamma_2\gamma_3}^{n} = 5\sigma_T(I_{10}I_{n+6,2} - I_{21}I_{n+5,2} + 2I_{31}I_{n+4,2}). \tag{B22}$$

In order to obtain the above relations, we used Eq. (20) and the definitions (45). Using Eq. (B17),

$$\Delta^{\mu\nu\alpha\beta}X_{\mu\nu\alpha\beta}^{-1} = 24\sigma_T\frac{P_0^2}{\beta_0},$$
$$\Delta^{\mu\nu\alpha\beta}X_{\mu\nu\alpha\beta}^{0} = \frac{400}{3}\sigma_T\frac{P_0^2}{\beta_0^2},$$
$$\Delta^{\mu\nu\alpha\beta}u^{\gamma_1}X_{\mu\nu\alpha\beta\gamma_1}^{-1} = 132\sigma_T\frac{P_0^2}{\beta_0^2},$$
$$\Delta^{\mu\nu\alpha\beta}u^{\gamma_1}X_{\mu\nu\alpha\beta\gamma_1}^{0} = 880\sigma_T\frac{P_0^2}{\beta_0^3}. \tag{B23}$$

The components of $\mathcal{A}^{(2)}$ change according to the number of moments included. In the 14-moment approximation, using Eqs. (23) and (29), we obtain

$$\mathcal{A}_{00}^{(2)} = \frac{W^{(2)}}{10}\Delta^{\mu\nu\gamma_1\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{-1} = \frac{3}{5}n_0\sigma_T, \tag{B24}$$

where we used Eqs. (B17) and (B23), together with the results from Appendix E.

In the 23-moment approximation, e.g. considering two polynomials in the expansion (22), for $\ell = 2$,

$$\mathcal{A}_{r0}^{(2)} = \frac{W^{(2)}}{10}(1 + a_{10}^{(2)}a_{10}^{(2)})\Delta^{\mu\nu\gamma_1\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{r-1} + \frac{W^{(2)}}{10}a_{10}^{(2)}a_{11}^{(2)}\Delta^{\mu\nu\gamma_1\gamma_2}u^{\gamma_3}X_{\mu\nu\gamma_1\gamma_2\gamma_3}^{r-1},$$

$$\mathcal{A}_{r1}^{(2)} = \frac{W^{(2)}}{10}a_{11}^{(2)}a_{10}^{(2)}\Delta^{\mu\nu\gamma_1\gamma_2}X_{\mu\nu\gamma_1\gamma_2}^{r-1} + \frac{W^{(2)}}{10}a_{11}^{(2)}a_{11}^{(2)}\Delta^{\mu\nu\gamma_1\gamma_2}u^{\gamma_3}X_{\mu\nu\gamma_1\gamma_2\gamma_3}^{r-1}. \tag{B25}$$

Then, using once more the results from Appendix E and Eqs. (B17) and (B23), we obtain

$$\mathcal{A}_{00}^{(2)} = \frac{9}{10}n_0\sigma_T, \qquad \mathcal{A}_{01}^{(2)} = -\frac{1}{20}\beta_0 n_0\sigma_T,$$
$$\mathcal{A}_{10}^{(2)} = \frac{4}{3\beta_0}n_0\sigma_T, \qquad \mathcal{A}_{11}^{(2)} = \frac{1}{3}n_0\sigma_T. \tag{B26}$$

We did not calculate the coefficients related to the bulk viscous pressure, since this quantity vanishes in the massless limit. Also, if the mass was taken to be finite, some of the steps taken in this appendix would not be possible.

APPENDIX C: TRANSPORT COEFFICIENTS

In this appendix we list all the transport coefficients of fluid dynamics calculated in this paper. The transport coefficients for the bulk viscous pressure are

$$\ell_{\Pi n} = -\frac{m^2}{3}\left(\gamma_1^{(1)}\tau_{00}^{(0)} - \sum_{r=0,\neq 1,2}^{N_0}\tau_{0r}^{(0)}\frac{G_{3r}}{D_{20}} + \sum_{r=0}^{N_0-3}\tau_{0,r+3}^{(0)}\Omega_{r+2,0}^{(1)}\right), \tag{C1}$$

$$\tau_{\Pi n} = \frac{m^2}{3(\varepsilon_0 + P_0)}\left[\tau_{00}^{(0)}\frac{\partial\gamma_1^{(1)}}{\partial\ln\beta_0} - \sum_{r=0,\neq 1,2}^{N_0}\tau_{0r}^{(0)}\frac{G_{3r}}{D_{20}} + \sum_{r=0}^{N_0-3}\tau_{0,r+3}^{(0)}\beta_0\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \sum_{r=0}^{N_0-3}(r+3)\tau_{0,r+3}^{(0)}\Omega_{r+2,0}^{(1)}\right], \tag{C2}$$

$$\delta_{\Pi\Pi} = \frac{2}{3}\tau_{00}^{(0)} + \frac{m^2}{3}\gamma_2^{(0)}\tau_{00}^{(0)} - \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{N_0}\tau_{0r}^{(0)}\frac{G_{2r}}{D_{20}} + \frac{1}{3}\sum_{r=0}^{N_0-3}(r+5)\tau_{0,r+3}^{(0)}\Omega_{r+3,0}^{(0)} - \frac{m^2}{3}\sum_{r=0}^{N_0-5}(r+4)\tau_{0,r+5}^{(0)}\Omega_{r+3,0}^{(0)} + \frac{(\varepsilon_0 + P_0)J_{10} - n_0J_{20}}{D_{20}}\sum_{r=3}^{N_0}\tau_{0r}^{(0)}\frac{\partial\Omega_{r0}^{(0)}}{\partial\alpha_0} + \frac{(\varepsilon_0 + P_0)J_{20} - n_0J_{30}}{D_{20}}\sum_{r=3}^{N_0}\tau_{0r}^{(0)}\frac{\partial\Omega_{r0}^{(0)}}{\partial\beta_0}, \tag{C3}$$

$$\lambda_{\Pi n} = -\frac{m^2}{3}\bigg(\tau_{00}^{(0)}\frac{\partial\gamma_1^{(1)}}{\partial\alpha_0} + \tau_{00}^{(0)}\frac{1}{h_0}\frac{\partial\gamma_1^{(1)}}{\partial\beta_0} + \sum_{r=0}^{N_0-3}\tau_{0,r+3}^{(0)}\frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \sum_{r=0}^{N_0-3}\tau_{0,r+3}^{(0)}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0}\bigg), \quad \text{(C4)}$$

$$\lambda_{\Pi\pi} = -\frac{m^2}{3}\bigg[-\gamma_2^{(2)}\tau_{00}^{(0)} + \sum_{r=0,\neq 1,2}^{N_0}\tau_{0r}^{(0)}\frac{G_{2r}}{D_{20}} + \sum_{r=0}^{N_0-3}(r+2)\tau_{0,r+3}^{(0)}\Omega_{r+1,0}^{(2)}\bigg], \quad \text{(C5)}$$

where $h_0 = (\varepsilon_0 + P_0)/n_0$ is the enthalpy per particle. The transport coefficients for the particle-diffusion current are

$$\delta_{nn} = \tau_{00}^{(1)} + \frac{1}{3}m^2\gamma_2^{(1)}\tau_{00}^{(1)} - \frac{1}{3}m^2\sum_{r=0}^{N_1-2}(r+1)\tau_{0,r+2}^{(1)}\Omega_{r0}^{(1)} + \frac{1}{3}\sum_{r=2}^{N_1}(r+3)\tau_{0r}^{(1)}\Omega_{r0}^{(1)} - \sum_{r=2}^{N_1}\tau_{0r}^{(1)}\bigg[\frac{n_0}{D_{20}}\bigg(J_{20}\frac{\partial\Omega_{r0}^{(1)}}{\partial\beta_0} + J_{30}\frac{\partial\Omega_{r0}^{(1)}}{\partial\alpha_0}\bigg) - \frac{\varepsilon_0+P_0}{D_{20}}\bigg(J_{10}\frac{\partial\Omega_{r0}^{(1)}}{\partial\beta_0} + J_{20}\frac{\partial\Omega_{r0}^{(1)}}{\partial\alpha_0}\bigg)\bigg], \quad \text{(C6)}$$

$$\ell_{n\Pi} = \frac{1}{h_0}\tau_{00}^{(1)} - \gamma_1^{(0)}\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\beta_0 J_{r+4,1}}{\varepsilon_0+P_0} + \frac{1}{m^2}\sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\Omega_{r+3,0}^{(0)} - \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\Omega_{r+3,0}^{(0)}, \quad \text{(C7)}$$

$$\tau_{n\Pi} = \frac{1}{\varepsilon_0+P_0}\bigg[\frac{1}{h_0}\tau_{00}^{(1)} - \tau_{00}^{(1)}\frac{\partial\gamma_1^{(0)}}{\partial\ln\beta_0} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\beta_0 J_{r+4,1}}{\varepsilon_0+P_0} + \frac{1}{m^2}\sum_{r=0}^{N_1-2}(r+5)\tau_{0,r+2}^{(1)}\Omega_{r+3,0}^{(0)} + \frac{1}{m^2}\sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\ln\beta_0} - \sum_{r=0}^{N_1-4}(r+4)\tau_{0,r+4}^{(1)}\Omega_{r+3,0}^{(0)} - \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\ln\beta_0}\bigg], \quad \text{(C8)}$$

$$\ell_{n\pi} = -\gamma_1^{(2)}\tau_{00}^{(1)} + \frac{1}{h_0}\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\beta_0 J_{r+4,1}}{\varepsilon_0+P_0} - \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\Omega_{r+1,0}^{(2)}, \quad \text{(C9)}$$

$$\tau_{n\pi} = \frac{1}{\varepsilon_0+P_0}\bigg[\frac{1}{h_0}\tau_{00}^{(1)} - \tau_{00}^{(1)}\frac{\partial\gamma_1^{(2)}}{\partial\ln\beta_0} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\beta_0 J_{r+4,1}}{\varepsilon_0+P_0} - \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\ln\beta_0} - \sum_{r=0}^{N_1-2}(r+2)\tau_{0,r+2}^{(1)}\Omega_{r+1,0}^{(2)}\bigg], \quad \text{(C10)}$$

$$\lambda_{nn} = \frac{3}{5}\tau_{00}^{(1)} + \frac{2}{5}m^2\gamma_2^{(1)}\tau_{00}^{(1)} - \frac{2}{5}m^2\sum_{r=0,r\neq 1}^{N_1-2}(r+1)\tau_{0,r+2}^{(1)}\Omega_{r0}^{(1)} + \frac{1}{5}\sum_{r=2}^{N_1}(2r+3)\tau_{0r}^{(1)}\Omega_{r0}^{(1)}, \quad \text{(C11)}$$

$$\lambda_{n\Pi} = \tau_{00}^{(1)}\bigg(\frac{1}{h_0}\frac{\partial\gamma_1^{(0)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(0)}}{\partial\alpha_0}\bigg) - \frac{1}{m^2}\sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\bigg(\frac{1}{h_0}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\beta_0} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\alpha_0}\bigg) + \sum_{r=0}^{N_1-4}\tau_{0,r+4}^{(1)}\bigg(\frac{1}{h_0}\frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\beta_0} + \frac{\partial\Omega_{r+3,0}^{(0)}}{\partial\alpha_0}\bigg), \quad \text{(C12)}$$

$$\lambda_{n\pi} = \bigg(\frac{1}{h_0}\frac{\partial\gamma_1^{(2)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(2)}}{\partial\alpha_0}\bigg)\tau_{00}^{(1)} + \sum_{r=0}^{N_1-2}\tau_{0,r+2}^{(1)}\bigg(\frac{1}{h_0}\frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\beta_0} + \frac{\partial\Omega_{r+1,0}^{(2)}}{\partial\alpha_0}\bigg). \quad \text{(C13)}$$

The transport coefficients for the shear-stress tensor are

$$\delta_{\pi\pi} = \frac{1}{3}m^2\gamma_2^{(2)}\tau_{00}^{(2)} + \frac{1}{3}\sum_{r=0}^{N_2}(r+4)\tau_{0r}^{(2)}\Omega_{r0}^{(2)} - \frac{1}{3}m^2\sum_{r=0}^{N_2-2}(r+1)\tau_{0,r+2}^{(2)}\Omega_{r0}^{(2)} + \sum_{r=0}^{N_2}\tau_{0r}^{(2)}\bigg[\frac{(\varepsilon_0+P_0)J_{10} - n_0 J_{20}}{D_{20}}\frac{\partial\Omega_{r0}^{(2)}}{\partial\beta_0} + \frac{(\varepsilon_0+P_0)J_{20} - n_0 J_{30}}{D_{20}}\frac{\partial\Omega_{r0}^{(2)}}{\partial\alpha_0}\bigg], \quad \text{(C14)}$$

$$\tau_{\pi\pi} = \frac{2}{7}\sum_{r=0}^{N_2}(2r+5)\tau_{0r}^{(2)}\Omega_{r0}^{(2)} + \frac{4}{7}m^2\gamma_2^{(2)}\tau_{00}^{(2)} - \frac{4}{7}m^2\sum_{r=0}^{N_2-2}(r+1)\tau_{0,r+2}^{(2)}\Omega_{r0}^{(2)}, \quad \text{(C15)}$$

$$\lambda_{\pi\Pi} = \frac{6}{5}\tau_{00}^{(2)} + \frac{2}{5}m^2\gamma_2^{(0)}\tau_{00}^{(2)} + \frac{2}{5m^2}\sum_{r=0}^{N_2-1}(r+5)\tau_{0,r+1}^{(2)}\Omega_{r+3,0}^{(0)} + \frac{2}{5}\sum_{r=3}^{N_2}(2r+3)\tau_{0r}^{(2)}\Omega_{r0}^{(0)} - \frac{2}{5}m^2\sum_{r=0,\neq 1,2}^{N_2-2}(r+1)\tau_{0,r+2}^{(2)}\Omega_{r0}^{(0)}, \tag{C16}$$

$$\tau_{\pi n} = \frac{1}{\varepsilon_0 + P_0}\bigg[-\frac{2}{5}m^2\tau_{00}^{(2)}\frac{\partial\gamma_1^{(1)}}{\partial\ln\beta_0} + \frac{2}{5}\sum_{r=0}^{N_2-1}(r+6)\tau_{0,r+1}^{(2)}\Omega_{r+2,0}^{(1)} - \frac{2}{5}m^2\sum_{r=0,\neq 1}^{N_2-1}(r+1)\tau_{0,r+1}^{(2)}\Omega_{r0}^{(1)} + \frac{2}{5}\sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\ln\beta_0} - \frac{2}{5}m^2\sum_{r=0}^{N_2-3}\tau_{0,r+3}^{(2)}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\ln\beta_0}\bigg], \tag{C17}$$

$$\ell_{\pi n} = -\frac{2}{5}m^2\gamma_1^{(1)}\tau_{00}^{(2)} + \frac{2}{5}\sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\Omega_{r+2,0}^{(1)} - \frac{2}{5}m^2\sum_{r=0,\neq 1}^{N_2-1}\tau_{0,r+1}^{(2)}\Omega_{r0}^{(1)}, \tag{C18}$$

$$\lambda_{\pi n} = -\frac{2}{5}m^2\tau_{00}^{(2)}\left(\frac{1}{h_0}\frac{\partial\gamma_1^{(1)}}{\partial\beta_0} + \frac{\partial\gamma_1^{(1)}}{\partial\alpha_0}\right) + \frac{2}{5}\sum_{r=0}^{N_2-1}\tau_{0,r+1}^{(2)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0}\right) - \frac{2}{5}m^2\sum_{r=0}^{N_2-3}\tau_{0,r+3}^{(2)}\left(\frac{1}{h_0}\frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\beta_0} + \frac{\partial\Omega_{r+2,0}^{(1)}}{\partial\alpha_0}\right). \tag{C19}$$

APPENDIX D: CALCULATIONS

In this appendix we compute the quantity $\gamma_1^{(2)}$ in the 14-moment approximation and the 23-moment approximation. This variable was defined in the main text and is given by

$$\gamma_1^{(2)} = \sum_{n=0}^{N_2}\mathcal{F}_{rn}^{(2)}\Omega_{n0}^{(2)}. \tag{D1}$$

The first step is to compute the thermodynamic integral $\mathcal{F}_{rn}^{(\ell)}$,

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!}\int dK f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}E_{\mathbf{k}}^{-r}\mathcal{H}_{\mathbf{k}n}^{(\ell)}(\Delta^{\alpha\beta}k_\alpha k_\beta)^\ell. \tag{D2}$$

1. 14-moment approximation

In this case, $N_1 = 1$ and $N_2 = 0$, and

$$\gamma_1^{(2)} = \mathcal{F}_{10}^{(2)}. \tag{D3}$$

Also, in the 14-moment approximation,

$$\mathcal{H}_{\mathbf{k}0}^{(2)} \equiv \frac{W^{(2)}}{2!}a_{00}^{(2)}P_{\mathbf{k}0}^{(2)} = \frac{W^{(2)}}{2!}. \tag{D4}$$

In the massless/classical limits

$$\mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{\beta_0^2}{8P_0}, \tag{D5}$$

and finally

$$\gamma_1^{(2)} = \frac{\beta_0^2}{4P_0}\frac{1}{5!!}\int dK f_{0\mathbf{k}}E_{\mathbf{k}}^{-1}(\Delta^{\alpha\beta}k_\alpha k_\beta)^2 = \frac{\beta_0}{5}. \tag{D6}$$

2. 23-moment approximation

In this case, $N_1 = 2$ and $N_2 = 1$, and

$$\gamma_1^{(2)} = \mathcal{F}_{10}^{(2)} + \Omega_{10}^{(2)}\mathcal{F}_{11}^{(2)}. \tag{D7}$$

Also, in the 23-moment approximation,

$$\mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{W^{(2)}}{2!}(1 + a_{10}^{(2)}P_{\mathbf{k}1}^{(2)}) = \frac{W^{(2)}}{2!}[1 + (a_{10}^{(2)})^2 + a_{10}^{(2)}a_{11}^{(2)}E_{\mathbf{k}}],$$
$$\mathcal{H}_{\mathbf{k}1}^{(2)} = \frac{W^{(2)}}{2!}a_{11}^{(2)}P_{\mathbf{k}1}^{(2)} = \frac{W_{(2)}}{2!}[a_{10}^{(2)}a_{11}^{(2)} + (a_{11}^{(2)})^2E_{\mathbf{k}}]. \tag{D8}$$

We know that

$$W^{(2)} = \frac{\beta_0^2}{4P_0}, \qquad (a_{11}^{(2)})^2 = \frac{\beta_0^2}{6}, \qquad \frac{a_{10}^{(2)}}{a_{11}^{(2)}} = -\frac{6}{\beta_0}. \tag{D9}$$

Thus,

$$\mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{\beta_0^2}{8P_0}(7 - \beta_0E_{\mathbf{k}}), \quad \mathcal{H}_{\mathbf{k}1}^{(2)} = \frac{\beta_0^3}{8P_0}\left(-1 + \frac{1}{6}\beta_0E_{\mathbf{k}}\right), \tag{D10}$$

and

$$\mathcal{F}_{10}^{(2)} = \frac{\beta_0^2}{4P_0}\frac{1}{5!!}\int dK f_{0\mathbf{k}}E_{\mathbf{k}}^{-1}(7 - \beta_0E_{\mathbf{k}})(\Delta^{\alpha\beta}k_\alpha k_\beta)^2 = \frac{2}{5}\beta_0,$$
$$\mathcal{F}_{11}^{(2)} = \frac{\beta_0^3}{4P_0}\frac{1}{5!!}\int dK f_{0\mathbf{k}}E_{\mathbf{k}}^{-1}\left(-1 + \frac{1}{6}\beta_0E_{\mathbf{k}}\right)(\Delta^{\alpha\beta}k_\alpha k_\beta)^2 = -\frac{\beta_0^2}{30}. \tag{D11}$$

Substituting $\Omega^{(2)}$ from Eq. (79) we obtain

$$\gamma_1^{(2)} = \frac{2}{15}\beta_0 = 0.133\beta_0. \tag{D12}$$

APPENDIX E: ORTHOGONAL POLYNOMIALS

In this appendix, we construct the set of orthogonal polynomials used in the main text. These will be polynomials in energy, $E_{\mathbf{k}} = u_\mu k^\mu$, i.e., orthogonal polynomials generated by the set 1, $E_{\mathbf{k}}$, $E_{\mathbf{k}}^2, \ldots$. We construct this orthogonal set using the Gram-Schmidt orthogonalization method. First we introduce

$$\omega^{(\ell)} \equiv \frac{W^{(\ell)}}{(2\ell+1)!!}(\Delta^{\alpha\beta}k_\alpha k_\beta)^\ell f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}, \tag{E1}$$

where $f_{0\mathbf{k}}$ is the equilibrium distribution function as defined in the main text. The weight $W^{(\ell)}$ will be determined such that the orthogonal polynomial $P_{\mathbf{k}n}^{(\ell)}$ of order $n = 0$ and index ℓ is normalized,

$$\int dK \omega^{(\ell)} P_{\mathbf{k}0}^{(\ell)} P_{\mathbf{k}0}^{(\ell)} = 1. \tag{E2}$$

Without loss of generality, the polynomials of order 0 are set to 1 for all values of ℓ,

$$P_{\mathbf{k}0}^{(\ell)} \equiv a_{00}^{(\ell)} = 1. \tag{E3}$$

Then the normalization parameter $W^{(\ell)}$ is obtained from Eq. (E2),

$$W^{(\ell)} = (-1)^\ell \frac{1}{J_{2\ell,\ell}}. \tag{E4}$$

The thermodynamic functions J_{nq} were defined in the main text, see Eq. (45).

The polynomials are parametrized as

$$P_{\mathbf{k}n}^{(\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)} E_{\mathbf{k}}^r. \tag{E5}$$

We construct the polynomials in sequence according to the parametrization (E5) starting from $n = 0$, Eq. (E3), using the orthonormality condition (24). The orthogonality/normalization condition implies that, for a polynomial of order i, $P_{\mathbf{k}i}^{(\ell)}$,

$$\int dK \omega^{(\ell)} P_{\mathbf{k}i}^{(\ell)} P_{\mathbf{k}j}^{(\ell)} = \delta_{ij}, \tag{E6}$$

for all $j \le i$. Substituting Eq. (E5), we obtain the following equation for the coefficients $a_{ij}^{(\ell)}$:

$$\sum_{j=0}^{i} \mathcal{D}_{kj}^{(\ell i)} \frac{a_{ij}^{(\ell)}}{a_{ii}^{(\ell)}} = \frac{J_{2\ell,\ell}}{(a_{ii}^{(\ell)})^2}\delta_{ki}, \tag{E7}$$

where $k = 0, \ldots, i$, and we defined the $(i+1)\times(i+1)$ matrix $\mathcal{D}_{kj}^{(\ell i)} \equiv J_{k+j+2\ell,\ell}$. The solution of Eq. (E7) is

$$(a_{ii}^{(\ell)})^2 = (\mathcal{D}^{-1})_{ii}^{(\ell i)} J_{2\ell,\ell}, \qquad \frac{a_{ij}^{(\ell)}}{a_{ii}^{(\ell)}} = \frac{(\mathcal{D}^{-1})_{ji}^{(\ell i)}}{(\mathcal{D}^{-1})_{ii}^{(\ell i)}}, \tag{E8}$$

where $(\mathcal{D}^{-1})^{(\ell i)}$ is the inverse of $\mathcal{D}^{(\ell i)}$. For example, for any polynomial of order 1, the coefficients are

$$(a_{11}^{(\ell)})^2 = (\mathcal{D}^{-1})_{11}^{(\ell 1)} J_{2\ell,\ell} = \frac{(J_{2\ell,\ell})^2}{J_{2\ell+2,\ell}J_{2\ell,\ell} - (J_{2\ell+1,\ell})^2},$$
$$\frac{a_{10}^{(\ell)}}{a_{11}^{(\ell)}} = \frac{(\mathcal{D}^{-1})_{01}^{(\ell 1)}}{(\mathcal{D}^{-1})_{11}^{(\ell 1)}} = -\frac{J_{2\ell+1,\ell}}{J_{2\ell,\ell}}. \tag{E9}$$

APPENDIX F: IRREDUCIBLE TENSORS

In this appendix, we give some practical relations concerning the irreducible tensors $k^{\langle\mu_1}k^{\mu_2}\cdots k^{\mu_\ell\rangle}$ introduced in the main text. The definition of these tensors is

$$k^{\langle\mu_1}k^{\mu_2}\cdots k^{\mu_\ell\rangle} = \Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell} k^{\nu_1}\cdots k^{\nu_\ell}. \tag{F1}$$

The projection operator $\Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell}$ is symmetric and traceless in the indexes μ and ν

$$\Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell} = \Delta_{(\nu_1\cdots\nu_\ell)}^{(\mu_1\cdots\mu_\ell)},$$
$$g_{\mu_i\mu_j}\Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell} = g^{\nu_i\nu_j}\Delta_{\nu_1\cdots\nu_\ell}^{\mu_1\cdots\mu_\ell} = 0, \quad \forall\ 1 \le i, j \le \ell. \tag{F2}$$

The parentheses on the indices denotes symmetrization of the tensor. These projections are constructed in Ref. [6] and can be obtained from

$$\Delta^{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell} = \sum_{k=0}^{[\ell/2]} C(\ell,k)\Phi_{(\ell k)}^{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell},$$
$$C(\ell,k) = (-1)^k \frac{(\ell!)^2}{(2\ell)!}\frac{(2\ell-2k)!}{k!(\ell-k)!(\ell-2k)!}, \tag{F3}$$

where in the last summation the symbol $[\ell/2]$ denotes the largest integer not exceeding $\ell/2$ and

$$\Phi_{(\ell k)}^{\mu_1\cdots\mu_\ell\nu_1\cdots\nu_\ell} = (\ell-2k)!\left(\frac{2^k k!}{\ell!}\right)^2 \sum_{\wp_\mu\wp_\nu} \Delta^{\mu_1\mu_2}\cdots$$
$$\times \Delta^{\mu_{2k-1}\mu_{2k}}\Delta^{\nu_1\nu_2}\cdots\Delta^{\nu_{2k-1}\nu_{2k}}$$
$$\times \Delta^{\mu_{2k+1}\nu_{2k+1}}\cdots\Delta^{\mu_\ell\nu_\ell}. \tag{F4}$$

This summation is supposed to run over all *distinct* permutations of μ-type and ν-type indices (we do not permute the indices μ with ν). For $\ell = 2$ this recipe gives the usual double symmetric and traceless projection operator $\Delta_{\alpha\beta}^{\mu\nu}$ commonly employed in relativistic fluid dynamics. As mentioned in the main text, this set of tensors is useful because they form an orthogonal basis; see Eq. (20).

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