Extended analytic QCD model with perturbative QCD behavior at high momenta

César Ayala,[1] Carlos Contreras,[1] and Gorazd Cvetič[1,2]

[1]*Department of Physics, Universidad Técnica Federico Santa María (UTFSM), Valparaíso, Chile*
[2]*Centro Científico-Tecnológico de Valparaíso, UTFSM, Chile*

In contrast to perturbative QCD, the analytic QCD models have running coupling whose analytic properties correctly mirror those of spacelike observables. The discontinuity (spectral) function of such running coupling is expected to agree with the perturbative case at large timelike momenta; however, at low timelike momenta, it is not known. In the latter regime, we parametrize the unknown behavior of the spectral function as a sum of (two) delta functions; while the onset of the perturbative behavior of the spectral function is set to be 1.0–1.5 GeV. This is in close analogy with the "minimal hadronic ansatz" used in the literature for modeling spectral functions of correlators. For the running coupling itself, we impose the condition that it basically merges with the perturbative coupling at high spacelike momenta. In addition, we require that the well-measured nonstrange semihadronic $(V + A)$ tau decay ratio value be reproduced by the model. We thus obtain a QCD framework which is basically indistinguishable from perturbative QCD at high momenta ($Q > 1$ GeV), and at low momenta, it respects the basic analyticity properties of spacelike observables as dictated by the general principles of the local quantum field theories.

I. INTRODUCTION

The general principles of the local quantum field theories imply [1,2] that the spacelike observables $\mathcal{D}(Q^2)$ are analytic functions of Q^2 in the complex Q^2-plane, with the nonanalyticity allowed only on the timelike semiaxis $Q^2 < 0$. Here, $-q^2 \equiv Q^2$ where q is the typical momentum of the considered process, e.g., momentum of the exchanged photon in deep inelastic scattering. However, in perturbative QCD (pQCD), the running coupling $a_{\rm pt}(Q^2) \equiv \alpha_s(Q^2)/\pi$ in most renormalization schemes has (Landau) singularities in the complex Q^2-plane close to the origin ($|Q^2| \lesssim 1$ GeV^2) which do not reflect the aforementioned analytic properties. This fact represents a serious, and often underestimated, conceptual and practical problem. The (spacelike) observables evaluated in pQCD, as truncated power series of $a_{\rm pt}(\kappa Q^2)$ (with $\kappa \sim 1$), do not possess the aforementioned analyticity properties; furthermore, for low values of $|Q^2|$ (~ 1 GeV^2), they are numerically unreliable due to the vicinity of the scale κQ^2 to the unphysical (Landau) singularities. This numerical unreliability reflects itself in a strong dependence on the renormalization scale ($\leftrightarrow \kappa$) and scheme.

On the other hand, studies using Dyson-Schwinger equations [3,4] and lattice calculations [5] indicate that the QCD running coupling is finite ("conformal") at $Q^2 = 0$ and has, at least at positive Q^2 values, no Landau singularities.

The problem of Landau poles in the QCD coupling was first addressed in a systematic manner about 15 years ago by the authors of Refs. [6–8], who constructed and used an analytic QCD coupling parameter $\mathcal{A}_1^{\rm (MA)}(Q^2)$ closely based on the perturbative coupling parameter $a_{\rm pt}(Q^2)$: in the dispersive integral expression for $a_{\rm pt}(Q^2)$, they

removed the integration over the offending spacelike discontinuity cut (i.e., at $-Q^2 = \sigma < 0$), while keeping the discontinuity (spectral) function $\rho_1^{\rm (pt)}(\sigma) \equiv {\rm Im}\, a_{\rm pt}(Q^2 = -\sigma - i\epsilon)$ unchanged on the timelike discontinuity cut, i.e., at $\sigma \geq 0$. Therefore, we can call this the minimal analytic (MA) model. The authors of Refs. [6–8] called it the analytic perturbation approach (APT) and provided an analogous method of construction of analytic analogs $\mathcal{A}_n^{\rm (MA)}(Q^2)$ of the powers $a_{\rm pt}(Q^2)^n$ for $n = 2, 3, \ldots$. A method of construction of analytic analogs $\mathcal{A}_\nu^{\rm (MA)}(Q^2)$ of noninteger powers $a_{\rm pt}(Q^2)^\nu$ for MA (APT) was developed by the authors of Ref. [9]. Yet another analytic QCD model, based on the minimal analytization of the beta function $da_{\rm pt}(Q^2)/d\ln Q^2$, was constructed and used in Ref. [10].

On the other hand, a more general approach of constructing analytic QCD coupling $\mathcal{A}_1(Q^2)$, based on a given choice of the discontinuity function $\rho_1(\sigma) \equiv {\rm Im}\, \mathcal{A}_1(Q^2 = -\sigma - i\epsilon)$ (for $\sigma > 0$), was emphasized in Refs. [11,12] and (in a more specific context) in Ref. [13]. The spacelike coupling $\mathcal{A}_1(Q^2)$ is then constructed from $\rho_1(\sigma)$ by the usual dispersion relation

$$\mathcal{A}_1(Q^2) = \frac{1}{\pi} \int_0^{+\infty} d\sigma \frac{\rho_1(\sigma)}{(\sigma + Q^2)}. \qquad (1)$$

From our standpoint, it is this analytic coupling $\mathcal{A}_1$ (or equivalently: the spectral function ρ_1) which defines the analytic QCD (anQCD) model. The construction of analytic analogs $\mathcal{A}_n(Q^2)$ of integer powers $a_{\rm pt}(Q^2)^n$, applicable to any such analytic QCD model, was performed in Refs. [11,12], using the relations between the logarithmic derivatives $\tilde{a}_{\rm pt,n}(Q^2)$ ($\propto d^{n-1} a_{\rm pt}(Q^2)/d(\ln Q^2)^{n-1}$) and powers $a_{\rm pt}(Q^2)^k$

$$a_{\text{pt}}^n = \tilde{a}_{\text{pt},n} + \sum_{m \geq 1} \tilde{k}_m(n) \tilde{a}_{\text{pt},n+m}, \qquad (2)$$

and[1] imposing the condition of analogy on the logarithmic derivatives of a_{pt} and of $\mathcal{A}_1$: $\tilde{a}_{\text{pt},n+m} \mapsto \tilde{\mathcal{A}}_{n+m}$. This condition was shown to be imperative, in order to keep the renormalization scale and scheme dependence of the resulting truncated analytic series (in terms of $\tilde{\mathcal{A}}_n$, or $\mathcal{A}_n$) of physical observables under control. The construction of the power analogs $\mathcal{A}_n$ of a_{pt}^n in general analytic QCD models was obtained thus from $\mathcal{A}_1$ via the relations analogous to Eq. (2)[2]

$$\mathcal{A}_n = \tilde{\mathcal{A}}_n + \sum_{m \geq 1} \tilde{k}_m(n) \tilde{\mathcal{A}}_{n+m}. \qquad (3)$$

The extension of this construction to noninteger power analogs $\mathcal{A}_\nu(Q^2)$, for general analytic QCD models, was performed in Ref. [17].

So, from our standpoint, it remains an outstanding problem to obtain or construct the most acceptable analytic coupling $\mathcal{A}_1(Q^2)$, or equivalently, the spectral function $\rho_1(\sigma) = \text{Im} \mathcal{A}_1(Q^2 = -\sigma - i\epsilon)$ (for $\sigma > 0$). It is reasonable to assume that at large σ (> 1 GeV^2), we have $\rho_1(\sigma) = \rho_1^{(\text{pt})}(\sigma)$, i.e., the spectral function agrees with the pQCD result. On the other hand, at low $\sigma \lesssim 1$ GeV^2, the exact behavior of $\rho_1(\sigma)$ is unknown.

The construction of $\mathcal{A}_1(Q^2)$ can be performed in two different ways. One way is to construct first the beta function $\beta(\mathcal{A}_1) = d\mathcal{A}_1/d \ln Q^2$ as function of $\mathcal{A}_1$. This approach is convenient if we take the position that $\beta(x)$ is an analytic function of x at $x = 0$. In such a case, it tuns out that we obtain perturbative QCD, i.e., $\mathcal{A}_n = \mathcal{A}_1^n$. However, in this case, after ensuring additionally the analyticity of $\mathcal{A}_1(Q^2)$ as a function of Q^2, it turns out to be very difficult to reproduce the correct measured value of the tau lepton (nonstrange) semihadronic $(V + A)$ decay ratio $r_\tau \approx 0.20$, cf. [18]. In fact, as shown in Ref. [18], the large enough value r_τ can be obtained in the perturbative analytic QCD frameworks only at an (unacceptable?) price of choosing a renormalization scheme with increasingly large β_j coefficients, which makes the analytic perturbation series of observables convergent only when up to four terms are included, and the fifth (N^{4}LO) term in the expansion shows an explosive increase.

Another way is to construct first the discontinuity (spectral) function $\rho_1(\sigma) \equiv \text{Im} \mathcal{A}_1(Q^2 = -\sigma - i\epsilon)$ (for $\sigma > 0$). This approach leads, in general, to nonperturbative analytic QCD, i.e., $\mathcal{A}_n$ turns out to be different from

$\mathcal{A}_1^n$. We can follow here analogous ideas used in the construction of the spectral functions of spacelike observables (correlators) in the literature, e.g., Refs. [19–22]. In these references, analytization is applied directly to a considered (spacelike) observable $\mathcal{D}(Q^2)$ itself. Some of the new nonperturbative parameters introduced there were thus specific to the chosen observable. On the other hand, we take here the standpoint that it is the (universal) QCD coupling that needs analytization, and that the additional nonperturbative contributions for a considered observable, not contained in the analytized leading-twist contribution, are accounted for by a procedure containing other universal parameters. Such parameters can be vacuum expectation values of higher-dimensional operators, and the aforementioned additional nonperturbative contributions are represented by higher-twist terms of the operator product expansion (OPE) [23].

In our approach, at a large enough threshold value $\sigma_0 = M_0^2$ (~ 1 GeV^2), we have the onset of the perturbative behavior for the discontinuity function of the coupling

$$\rho_1(\sigma) = \rho_1^{(\text{pt})}(\sigma), \quad \text{for } \sigma \geq M_0^2. \qquad (4)$$

On the other hand, in the regime $0 < \sigma < M_0^2$, the behavior is nonperturbative and unknown in detail and could be parametrized as a sum (with different weights) of delta functions

$$\rho_1(\sigma) = \sum_{j=1}^{n} F_n^2 \delta(\sigma - M_n^2), \quad \text{for } 0 < \sigma < M_0^2. \quad (5)$$

As has been argued in Refs. [24,25], introduction of a sufficient number of positive delta functions in the discontinuity function $\rho(\sigma) = \text{Im} f(Q^2 = -\sigma - i\epsilon)$ can approximate sufficiently well any positive Stieltjes function $f(Q^2)$. The analytic coupling $\mathcal{A}_1(Q^2)$ is a positive Stieltjes function [25]. On the other hand, application of such type of approximations has been applied to the spectral functions of certain current correlators [21], under the name of the "minimal hadronic ansatz."

In Ref. [26], we constructed in this way a simple one-delta analytic QCD model, by introducing one delta function in the low-σ regime. The model contains three free parameters,[3] which were fixed by the condition (two requirements) $\mathcal{A}_1 - a_{\text{pt}} \sim (\Lambda^2/Q^2)^3$ at large $|Q^2| > \Lambda^2$ (where $\Lambda^2 = \Lambda_{\text{QCD}}^2 \sim 0.1$ GeV^2), and the (one) requirement of reproducing the correct value of the semihadronic tau decay ratio r_τ.

It may be regarded as overly optimistic to approximate the unknown low-σ regime by a single delta function. In the present work, we go beyond the one-delta approximation and investigate how to parametrize the low-σ regime in a reasonable manner with two positive delta functions.

---

[1]The recurrence relations leading to the above relations, within the context of the MA (APT) model of Refs. [6–8], were given in Refs. [14,15].

[2]The construction of higher-power analogs $\mathcal{A}_n$ as linear operations on $\mathcal{A}_1$ (not as: $\mathcal{A}_1^n$) incorporates a nice functional property: its compatibility with linear integral transformations, such as Fourier or Laplace [16].

[3]...in addition to the scale Λ_{QCD}, which was fixed by the condition of reproducing the world average value of $a_{\text{pt}}(M_Z^2) = 0.119/\pi$, in the $\overline{\text{MS}}$ scheme.

Since such an extension introduces in the model several new parameters, we will fix those parameters by specific reasonable conditions, which will be similar in spirit to those of the one-delta case. Specifically, the model can be made even closer to pQCD, $\mathcal{A}_1 - a_{\text{pt}} \sim (\Lambda^2/Q^2)^5$ at large $|Q^2| > \Lambda^2$, while reproducing the correct value of r_τ.

We do not introduce even more delta functions in the low-σ regime, because in such a case, we would need values of more low-energy QCD observables to fix at least some of the additional parameters. Most of the inclusive low-energy QCD observables, with the remarkable exception of the $(V + A)$ ratio r_τ, either have large experimental uncertainties or large theoretical uncertainties due to large higher-twist contributions or both. On the other hand, if we fix the parameters of the model (with more than two deltas) by simply imposing a further increase in the power index n of the difference $\mathcal{A}_1 - a_{\text{pt}} \sim (\Lambda^2/Q^2)^n$ $(n > 6)$, and without imposing the requirement of the reproduction of any additional low-energy observable value, some of the deltas become negative, indicating numerical instabilities.

In Sec. II, we describe the model and impose the conditions which will fix the unknown parameters. In Sec. III, we explain how to evaluate, in any anQCD model, the leading-twist contribution of the spacelike observables and of the related timelike observables, among the latter being r_τ. In Sec. IV, we present the numerical determination of the model parameters and other numerical results. In Sec. V, we summarize the results of this work and outline the prospects of further applications.

II. DESCRIPTION OF THE TWO-DELTA MODEL

As outlined in the introduction, we construct the two-delta anQCD model by starting with an ansatz for the discontinuity function $\rho_1(\sigma) \equiv \text{Im}\mathcal{A}_1(Q^2 = -\sigma - i\epsilon)$ (for $\sigma > 0$) which agrees with the perturbative counterpart $\rho_1^{(\text{pt})}(\sigma) \equiv \text{Im}a_{\text{pt}}(Q^2 = -\sigma - i\epsilon)$ at sufficiently high scales $\sigma \geq M_0^2$ $(M_0^2 \sim 1 \text{ GeV}^2)$; and in the low-scale regime $0 < \sigma < M_0^2$ its otherwise unknown behavior is parametrized as a linear combination of (two) delta functions

$$\rho_1^{(2d)}(\sigma; c_2) = \pi \sum_{j=1}^{2} f_j^2 \Lambda^2 \delta(\sigma - M_j^2) + \Theta(\sigma - M_0^2)$$
$$\times \rho_1^{(\text{pt})}(\sigma; c_2) \quad (6)$$

$$= \pi \sum_{j=1}^{2} f_j^2 \delta(s - s_j) + \Theta(s - s_0) \times r_1^{(\text{pt})}(s), \quad (7)$$

where we denoted $s = \sigma/\Lambda^2$, $s_j = M_j^2/\Lambda^2$ $(j = 0, 1, 2)$, and $r_1^{(\text{pt})}(s; c_2) = \rho_1^{(\text{pt})}(\sigma; c_2) = \text{Im}a_{\text{pt}}(Q^2 = -\sigma - i\epsilon; c_2)$. Here, Λ^2 $(\lesssim 10^{-1} \text{ GeV}^2)$ is the Lambert scale appearing in the following expression for a_{pt}:

$$a_{\text{pt}}(Q^2; c_2) = -\frac{1}{c_1} \frac{1}{[1 - c_2/c_1^2 + W_{\mp 1}(z)]}, \quad (8)$$

where $Q^2 = |Q^2| \exp(i\phi)$, the branches W_{-1} and W_{+1} of the Lambert function refer to the case $0 \leq \phi < +\pi$ and $-\pi < \phi < 0$, respectively,[4] and

$$z = -\frac{1}{c_1 e} \left(\frac{|Q^2|}{\Lambda^2}\right)^{-\beta_0/c_1} \exp(-i\beta_0\phi/c_1), \quad (9)$$

where the aforementioned Lambert scale Λ^2 appears. The explicit expression (8) is the solution of the (perturbative) renormalization group equation (RGE) of the form

$$\frac{\partial a_{\text{pt}}(Q^2; c_2)}{\partial \ln Q^2} = -\beta_0 a_{\text{pt}}^2 \frac{[1 + (c_1 - (c_2/c_1))a_{\text{pt}}]}{[1 - (c_2/c_1)a_{\text{pt}}]}. \quad (10)$$

Here, $\beta_0 = (1/4)(11 - 2n_f/3)$ and $c_1 = \beta_1/\beta_0 = (1/4)(102 - 38n_f/3)/(11 - 2n_f/3)$ are the universal constants, while $c_2 = \beta_2/\beta_0$ is the free three-loop renormalization scheme parameter. The expansion of the beta function $\beta(a_{\text{pt}}) = da_{\text{pt}}/d\ln Q^2$ in general gives

$$\beta(a_{\text{pt}}) = -\beta_0 a_{\text{pt}}^2 (1 + c_1 a_{\text{pt}} + c_2 a_{\text{pt}}^2 + c_3 a_{\text{pt}}^3 + \ldots), \quad (11)$$

where c_j $(j \geq 2)$ are general renormalization scheme parameters. The choice of the beta function on the right-hand side of Eq. (10) gives $c_j = c_2^{j-1}/c_1^{j-2}$ $(j \geq 3)$, which means that the three-loop scheme parameter c_2 can be chosen freely in this form, while the higher-loop scheme parameters are then fixed. The specific "effective three-loop" perturbative beta function of the Padé form of Eq. (10) was chosen for convenience, because it gives an explicit solution (8), in terms of the branches of the Lambert function W [28–30], and, at the same time, it allows us to vary the renormalization scheme at the three-loop level (c_2). In the following, it will turn out to be convenient to vary the scheme parameter c_2 in the explicit solution, the latter being used for $\rho_1^{(\text{pt})}(\sigma) \equiv \text{Im}a_{\text{pt}}(Q^2 = -\sigma - i\epsilon)$ appearing in the discontinuity function (6) and (7) of the anQCD model.

The Lambert function $W = W(z)$ is defined via the inverse relation $z = W \exp(W)$, cf. Fig. 1(a). The two branches $W_{\mp 1}(z)$ of the Lambert function are related via complex-conjugation $W_{+1}(z^*) = W_{-1}(z)^*$, and the point $z = -1/e$ is the branching point of these functions. In the interval $-1/e < z < 0$, $W_{-1}(z)$ is a decreasing function of z, cf. Fig. 1(b). When $z \to -0$, the scale Q^2 tends to $Q^2 \to +\infty$, cf. Eq. (9), and $W_{-1}(z) \to -\infty$, this reflecting the asymptotic freedom of $a_{\text{pt}}(Q^2)$ of Eq. (8). In our considered case of low-energy QCD (i.e., with number of quark flavors $n_f = 3$), the solution (8) has unphysical (Landau) singularities along the positive Q^2 axis, for any c_2. An extension of such beta function to the effective four- and five-loop case, such that the solution is explicit and involving Lambert functions, was made in Ref. [31]. For

---

[4]In MATHEMATICA [27], the functions $W_n(z)$ are implemented by the command ProductLog[n, z].
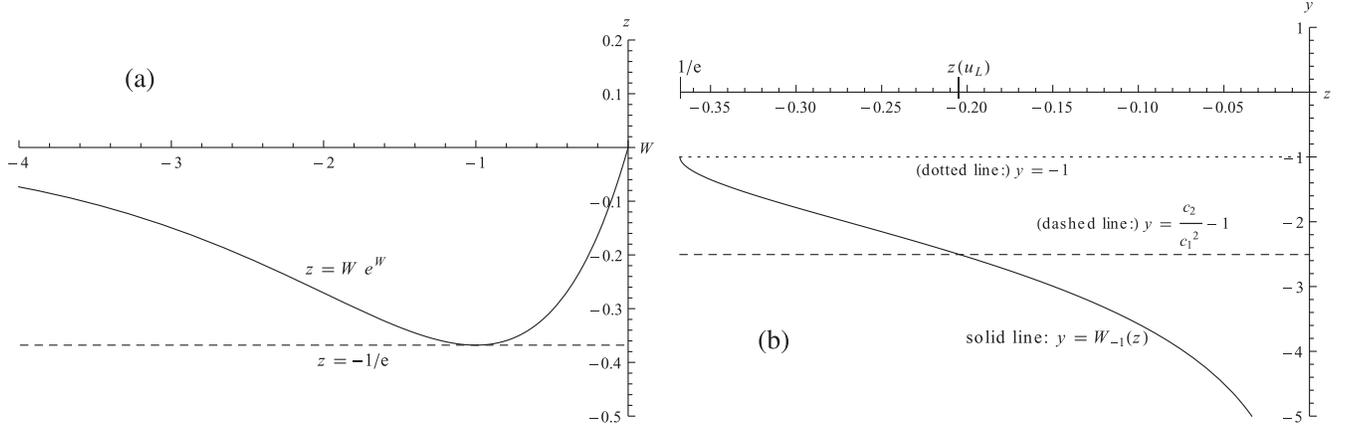
FIG. 1.   (a) The defining relation $z = We^W$ for the Lambert function $W(z)$, for $-1/e < z < 0$; (b) The branch $W_{-1}(z)$ for the same z-interval; when $c_2 < 0$, the denominator of Eq. (8) becomes zero at a $z(u_L)$ in this interval.

more details on the Lambert functions, we refer to Refs. [28–32].

The aforementioned branching point of nonanalyticity $z(s_L) = -1/e$ corresponds, according to Eq. (9), to the scale $Q^2(s_L) = \Lambda^2 s_L$ with $s_L = c_1^{-c_1/\beta_0}$ ($s_L = 0.6347$ when $n_f = 3$), and the interval $Q^2 \in (0, \Lambda^2 s_L)$ represents the interval of the unphysical (Landau) singularities of $a_{pt}(Q^2)$ of Eq. (8). If the scheme parameter c_2 is chosen to be negative (this will be our case), then there is an additional pole-type Landau singularity at a somewhat higher scale $Q^2(u_L) = \Lambda^2 u_L$ ($\Leftrightarrow z = z(u_L) = -u_L^{-\beta_0/c_1}/(c_1 e)$) at which the denominator of Eq. (8) becomes zero, cf. Fig. 1(b), i.e., when

$$-1 + c_2/c_1^2 = W_{-1}\left(\frac{-1}{c_1 e}|u_L|^{-\beta_0/c_1} + i\epsilon\right). \quad (12)$$

When $n_f = 3$ and $c_2 = -4.76$ (this will be our central choice of the scheme later), we get $u_L = 1.0095$ ($> s_L$). For this case, the coupling a_{pt} is presented in Fig. 2(a) as a function of z (for $-1/e < z < 0$, i.e., $s_L\Lambda^2 < Q^2 < \infty$) and in Fig. 2(b) as a function of $t = -\ln(-z) = 1.266 \ln(Q^2/\Lambda^2) + 1.575$.

It can be checked that, as a result of application of the Cauchy theorem to the function $a_{pt}(Q'^2)/(Q'^2 - Q^2)$ in the complex-Q'^2 plane, the following dispersion relation for a_{pt} holds:

$$a_{pt}(Q^2; c_2) = \frac{1}{\pi}\int_{\sigma=-Q_{min}^2-\eta'}^{\infty} d\sigma \frac{\text{Im}\,a_{pt}(-\sigma - i\epsilon; c_2)}{(\sigma + Q^2)}$$

$$= \frac{1}{\pi}\int_{s=-u_{min}-\eta}^{\infty} ds \frac{r_1^{(pt)}(s; c_2)}{(s + Q^2/\Lambda^2)} \quad (\eta, \eta' \to 0),$$

(13)

where the integration covers the entire cut, i.e., starting at a sufficiently low negative value $\sigma_{min} = -Q_{min}^2$ ($Q_{min}^2 \lesssim 1 \text{ GeV}^2$). The perturbative discontinuity function is denoted as $r_1^{(pt)}(s; c_2) = \text{Im}\,a_{pt}(Q^2 = -s\Lambda^2 - i\epsilon; c_2)$. Since the cut of the coupling $a_{pt}(Q'^2, c_2)$, Eq. (8) with $c_2 < 0$, includes also the pole $Q_L'^2 = u_L\Lambda^2$ of the coupling, the contour of integration in the complex (Q'^2/Λ^2)-plane is of the type as presented in Fig. 3 (with the outer radius going to infinity). Therefore, the dispersive relation (13) obtains a slightly generalized form
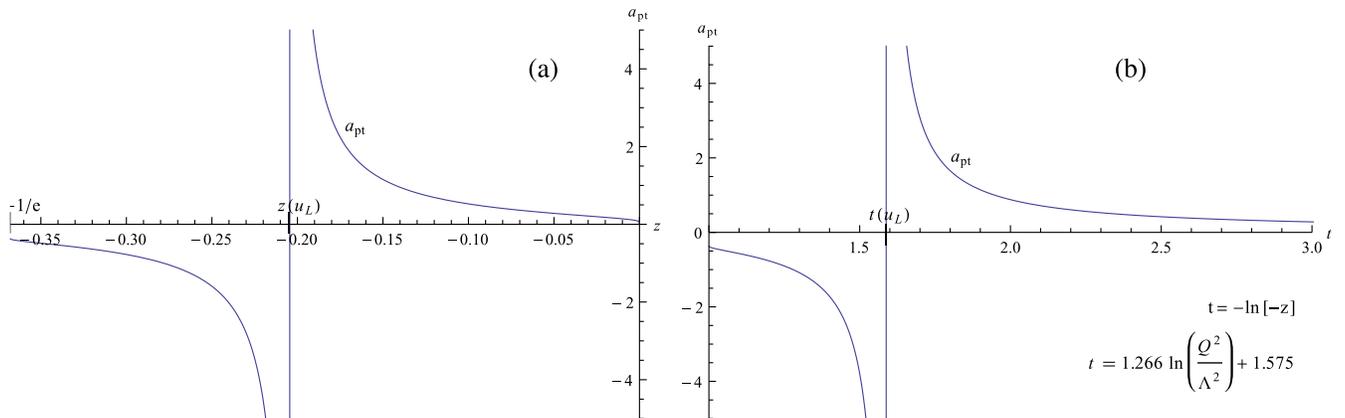
FIG. 2 (color online).   (a) The perturbative coupling a_{pt} of Eq. (8) as a function of z, for $-1/e < z < 0$; (b) as a function of $t = -\ln(-z)$. The curves are for the case of $n_f = 3$ and $c_2 = -4.76$ ($\Rightarrow t = 1.266 \ln(Q^2/\Lambda^2) + 1.575$).
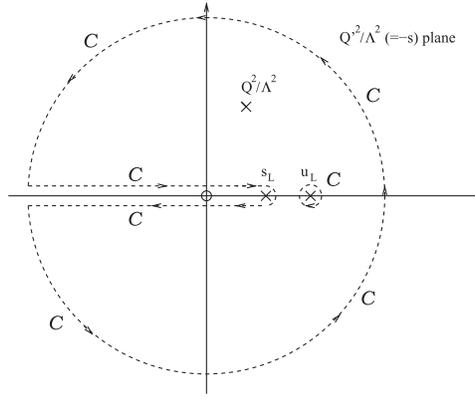
FIG. 3.   The path of contour integration in (Q'^2/Λ^2)-plane leading to the expression (14).

$$a_{\text{pt}}(Q^2) = \frac{1}{\pi} \int_{s=-s_{\text{L}}-\eta}^{\infty} ds \frac{r_1^{(\text{pt})}(s;c_2)}{(s + Q^2/\Lambda^2)} + \frac{\Delta u}{2\pi}$$

$$\times \int_{\Phi=-\pi}^{\pi} d\Phi e^{i\Phi} \frac{a_{\text{pt}}((u_{\text{L}} + \Delta u e^{i\Phi})\Lambda^2; c_2)}{[(Q^2/\Lambda^2) - u_{\text{L}} - \Delta u e^{i\Phi}]} \quad (14)$$

$$= \frac{1}{\pi} \int_{s=-s_{\text{L}}-\eta}^{\infty} ds \frac{r_1^{(\text{pt})}(s;c_2)}{(s + Q^2/\Lambda^2)} + \frac{\text{Res}_{(z=u_{\text{L}})} a_{\text{pt}}(z\Lambda^2; c_2)}{(-u_{\text{L}} + Q^2/\Lambda^2)}.$$

$$(15)$$

In Eq. (14), $\Delta u \Lambda^2$ is a sufficiently small (but otherwise arbitrary) radius of integration around the point $u_{\text{L}} \Lambda^2$ in the complex Q'^2-plane, cf. Fig. 3. In Eq. (15), this integration is expressed by the residue of the function $a_{\text{pt}}(z\Lambda^2)$ at this point.

The perturbative discontinuity function $r_1^{(\text{pt})}(s;c_2) = \text{Im} a_{\text{pt}}(Q^2 = -s\Lambda^2 - i\epsilon; c_2)$, which is nonzero for $-s_{\text{L}} < s < +\infty$, has the specific form

$$r_1^{(\text{pt})}(s; c_2) = \text{Im}\left[\frac{(-1)}{c_1} \frac{1}{[1 - (c_2/c_1^2) + W_{+1}(\frac{-1}{c_1 e}|s|^{-\beta_0/c_1} \exp(i\beta_0\pi/c_1))]} \right] \quad (s > 0), \quad (16)$$

$$= \text{Im}\left[\frac{(-1)}{c_1} \frac{1}{[1 - (c_2/c_1^2) + W_{+1}(\frac{-1}{c_1 e}|s|^{-\beta_0/c_1} - i\epsilon)]} \right] \quad (s < 0). \quad (17)$$

The analytic (spacelike) coupling $\mathcal{A}_1(Q^2; c_2)$ of the two-delta anQCD model is constructed on the basis of the discontinuity function (7) [cf. Eq. (16)] using the dispersion relation (1). This gives

$$\mathcal{A}_1(Q^2; c_2) = \sum_{j=1}^{2} \frac{f_j^2}{(u + s_j)} + \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{r_1^{(\text{pt})}(s;c_2)}{(s + u)}, \quad (18)$$

where $u = Q^2/\Lambda^2$.

In the presented two-delta anQCD model, we will consider the first three quark flavors to be massless and will consider that the momenta of the $n_f = 3$ regime in the anQCD model reach up to the threshold $|Q^2| = (2m_c)^2$ (≈ 6.45 GeV^2). Further, the anQCD model will be constructed in such a way as to practically merge with pQCD at such sufficiently high momenta [in the considered renormalization scheme as fixed by Eq. (10)]. Therefore, we will consider that the value of the Lambert scale Λ^2 used in our analytic coupling $\mathcal{A}_1(Q^2; c_2)$ coincides with the perturbative Lambert scale Λ_{pt}^2, the latter being determined by the condition $a_{\text{pt}}^{(\overline{\text{MS}})}(M_Z^2) =$

$0.1184/\pi$, i.e., by the central value of the world average [33]. Therefore, Λ^2 is determined by RGE-evolving this a_{pt} down to $a_{\text{pt}}^{(\overline{\text{MS}})}((2m_c)^2; n_f = 3)$, using the four-loop polynomial form of $\beta^{(\overline{\text{MS}})}(a_{\text{pt}})$, and the three-loop matching conditions [34] at quark thresholds $\mu^2 = (2m_q)^2$ ($q = b, c$), and then changing from the $\overline{\text{MS}}$ scheme to the scheme "c_2" [$\equiv (c_2, c_3 = c_2^2/c_1, \ldots)$] defined by the beta function of Eq. (10) (as explained, e.g., in Refs. [18,26]).

The conditions we impose to fix the parameters are the following:

(1) The analytic coupling $\mathcal{A}_1(Q^2; c_2)$ acquires the aforementioned pQCD value of the scale Λ^2 of $a_{\text{pt}}(Q^2; c_2; n_f = 3)$ at $Q^2 = (2m_c)^2$

$$\Lambda^2 = \Lambda_{\text{pt}}^2(n_f = 3). \quad (19)$$

(2) While in general we expect $\mathcal{A}_1(Q^2; c_2)$ to differ from $a_{\text{pt}}(Q^2; c_2)$ at $Q^2 > \Lambda^2$ by $\sim (\Lambda^2/Q^2)^1$, we impose the condition

$$\mathcal{A}_1(Q^2; c_2) - a_{\text{pt}}(Q^2; c_2) \sim (\Lambda^2/Q^2)^{n_{\max}}$$
$$\text{with} \quad n_{\max} = 5. \tag{20}$$

The condition (20) represents in practice four conditions. In the following few lines, we will describe how to formulate these four conditions.

When we subtract from the perturbative coupling (15) the analytic coupling (18), we obtain

$$a_{\text{pt}}(Q^2; c_2) - \mathcal{A}_1(Q^2; c_2) = \frac{\text{Res}_{(z=u_{\text{L}})} a_{\text{pt}}(z\Lambda^2)}{(u - u_{\text{L}})} - \sum_{j=1}^{2} \frac{f_j^2}{(u + s_j)}$$
$$+ \frac{1}{\pi} \int_{s_{\text{L}} - \eta}^{s_0} ds \frac{r_1^{(\text{pt})}(s; c_2)}{(s + u)}. \tag{21}$$

Expanding the left-hand side in powers of $(1/u) = (\Lambda^2/Q^2)$, the imposition of the condition (20) gives us the conditions that the terms of $\sim (\Lambda^2/Q^2)^{1+k}$ ($k = 0, 1, 2, 3$) in this expansion give us zero, i.e., we have the following four conditions:

$$\frac{1}{\pi} \int_{s_{\text{L}} - \eta}^{s_0} ds s^k r_1^{(\text{pt})}(s; c_2) + (-u_{\text{L}})^k \text{Res}_{(z=u_{\text{L}})} a_{\text{pt}}(z\Lambda^2)$$
$$= s_1^k f_1^2 + s_2^k f_2^2 \quad (k = 0, 1, 2, 3). \tag{22}$$

Altogether, Eqs. (19) and (22) represent five conditions. Once the scheme c_2 parameter is chosen, we have altogether six parameters in the model: f_1^2, f_2^2, s_1, s_2, s_0, and the scale Λ. Therefore, yet another condition will have to be imposed, possibly involving the correct reproduction of a low-energy observable. The best candidate for this appears to be the canonical $(V + A)$ nonstrange and massless semihadronic τ-lepton decay ratio r_τ, [35–37].[5] When we remove the (measured) strangeness-changing contribution, the color and Cabibbo-Kobayashi-Maskawa quark-mixing matrix factors and the electroweak effects, and the chirality-violating higher-twist (quark mass) contributions, the following value is obtained (cf. [38–40], and Appendix B of Ref. [18] for details):

$$r_\tau(\triangle S = 0, m_q = 0)_{\text{exp}} = 0.203 \pm 0.004. \tag{23}$$

Numerical analyses of the measured data indicate that the chirality-conserving higher-twist effects, such as gluon-condensate contributions, are negligible in the case of the considered $V + A$ decay channel. Although such analyses have been performed within pQCD + OPE approach, we will assume that they remain valid when an analysis is performed with the presented anQCD two-delta model +OPE. This assumption appears to be reasonable because

---

[5]The "canonical" means that the normalization is used such that $(r_\tau)_{\text{pt}} = a_{\text{pt}} + \mathcal{O}(a_{\text{pt}}^2)$.

the considered anQCD coupling $\mathcal{A}_1(Q^2)$ is very close to the pQCD coupling $a_{\text{pt}}(Q^2)$ (in the considered scheme) at momenta $|Q^2| \gtrsim 1$ GeV^2. Therefore, in the calculation of the discussed r_τ, Eq. (23), in the presented anQCD model, the (chirality-conserving) higher-twist contributions will be ignored.

III. CALCULATION OF ADLER FUNCTION AND r_τ IN ANALYTIC QCD

The calculation of r_τ is then performed in the same way as presented in Refs. [18,26], i.e., by performing explicitly the integration corresponding to the leading-β_0 (LB) resummation for r_τ and adding the three known beyond-LB (bLB) terms (i.e., including the exact contributions of $\sim \mathcal{A}_4$).

In this section, we will present only the main points of calculation of spacelike (such as Adler function) and time-like quantities (such as r_τ) in anQCD models. For details, we refer to Refs. [11,12,26], and especially Appendices C and D of Ref. [18].

The basic idea of the approach in the evaluation of spacelike observables $\mathcal{D}(Q^2)$ in general anQCD model is to reorganize first the perturbation series $\mathcal{D}(Q^2)_{\text{pt}}$

$$\mathcal{D}(Q^2)_{\text{pt}} = a_{\text{pt}} + d_1 a_{\text{pt}}^2 + d_2 a_{\text{pt}}^3 + \ldots, \tag{24}$$

into the modified perturbation series (mpt)

$$\mathcal{D}(Q^2)_{\text{mpt}} = a_{\text{pt}} + \tilde{d}_1 \tilde{a}_{\text{pt},2} + \tilde{d}_2 \tilde{a}_{\text{pt},3} + \ldots, \tag{25}$$

where $\tilde{d}_k \tilde{a}_{\text{pt},k+1}$ are the logarithmic derivatives of a_{pt}

$$\tilde{a}_{\text{pt},k+1}(Q^2) \equiv \frac{(-1)^k}{\beta_0^k k!} \frac{\partial^k a_{\text{pt}}(Q^2)}{\partial (\ln Q^2)^k}, \quad (k = 0, 1, 2, \ldots). \tag{26}$$

They are related with the powers of $a_{\text{pt}} \equiv \alpha_s/\pi$ in the following way [using RGEs in pQCD]:

$$\tilde{a}_{\text{pt},2} = a_{\text{pt}}^2 + c_1 a_{\text{pt}}^3 + c_2 a_{\text{pt}}^4 + \ldots, \tag{27}$$

$$\tilde{a}_{\text{pt},3} = a_{\text{pt}}^3 + \tfrac{5}{2} c_1 a_{\text{pt}}^4 + \ldots, \quad \tilde{a}_{\text{pt},4} = a_{\text{pt}}^4 + \ldots, \quad \text{etc.} \tag{28}$$

This, in turn, means that the powers of a_{pt} are linear combinations of logarithmic derivatives

$$a_{\text{pt}}^2 = \tilde{a}_{\text{pt},2} - c_1 \tilde{a}_{\text{pt},3} + (\tfrac{5}{2} c_1^2 - c_2) \tilde{a}_{\text{pt},4} + \ldots, \tag{29}$$

$$a_{\text{pt}}^3 = \tilde{a}_{\text{pt},3} - \tfrac{5}{2} c_1 \tilde{a}_{\text{pt},4} + \ldots, \quad a_{\text{pt}}^4 = \tilde{a}_{\text{pt},4} + \ldots, \quad \text{etc.,} \tag{30}$$

which allows us to relate the mpt coefficients with the usual perturbation series (pt) coefficients

$$\tilde{d}_1 = d_1, \qquad \tilde{d}_2 = d_2 - c_1 d_1, \tag{31}$$

$$\tilde{d}_3 = d_3 - \tfrac{5}{2} c_1 d_2 + (\tfrac{5}{2} c_1^2 - c_2) d_1, \qquad \text{etc.} \tag{32}$$

In Refs. [11,12], it was shown that it is imperative to construct first the analogs of the logarithmic derivatives of $a_{\rm pt}$ in the following way[6]:

$$\left(\frac{\partial^k a_{\rm pt}(Q^2)}{\partial(\ln Q^2)^k}\right)_{\rm an} = \frac{\partial^k}{\partial(\ln Q^2)^k}(a_{\rm pt}(Q^2))_{\rm an} = \frac{\partial^k \mathcal{A}_1(Q^2)}{\partial(\ln Q^2)^k}$$
$$(k = 0, 1, 2, \ldots). \qquad (33)$$

This means that the mpt expansion (25) becomes in anQCD the corresponding "modified analytic series" (man)

$$\mathcal{D}(Q^2)_{\rm man} = \mathcal{A}_1 + \tilde{d}_1 \tilde{\mathcal{A}}_2 + \tilde{d}_2 \tilde{\mathcal{A}}_3 + \ldots, \qquad (34)$$

and its truncated version is

$$\mathcal{D}(Q^2)^{[N]}_{\rm man} = \mathcal{A}_1 + \tilde{d}_1 \tilde{\mathcal{A}}_2 + \ldots + \tilde{d}_{N-1} \tilde{\mathcal{A}}_N. \qquad (35)$$

Here, we denoted by $\tilde{\mathcal{A}}_{k+1}$ the logarithmic derivatives of $\mathcal{A}_1$

$$\tilde{\mathcal{A}}_{k+1}(\mu^2) = \frac{(-1)^k}{\beta_0^k k!} \frac{\partial^k \mathcal{A}_1(\mu^2)}{\partial(\ln\mu^2)^k}, \quad (k = 1, 2, \ldots). \quad (36)$$

The expressions (34)–(36) are the basis of the evaluation of massless spacelike observables in any anQCD.[7]

The quantity r_τ is, on the other hand, a timelike observable. However, it is expressed via a contour integration [41]

$$r_\tau = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) d_{\rm Adl}(Q^2 = m_\tau^2 e^{i\phi}), \qquad (37)$$

which involves the (spacelike, massless, $n_f = 3$) Adler function $d_{\rm Adl}(Q^2) = a_{\rm pt}(Q^2) + \mathcal{O}(a_{\rm pt}^2)$. The perturbation expansion of $d_{\rm Adl}$ is known up to $\sim a_{\rm pt}^4$ [42–44]

$$d_{\rm Adl}(Q^2)^{[4]}_{\rm pt} = a_{\rm pt} + \sum_{n=1}^{3} (d_{\rm Adl})_n a_{\rm pt}^{n+1}. \qquad (38)$$

On the other hand, the leading-β_0 parts $(d_{\rm Adl})_n^{\rm (LB)} = c_{n,n}^{(1)} \beta_0^n$ $[= (\tilde{d}_{\rm Adl})_n^{\rm (LB)}]$ of all the coefficients $(d_{\rm Adl})_n$ are known[8], [45,46] and the resummation of these contributions can be performed by using the approach of Neubert of integration

with characteristic functions [47]—this can be performed in any anQCD without ambiguities (since no Landau singularities)

$$(d_{\rm Adl})^{\rm (LB)}_{\rm man}(Q^2) \equiv \mathcal{A}_1(Q^2) + c_{1,1}^{(1)} \beta_0 \tilde{\mathcal{A}}_2(Q^2) + \ldots$$
$$+ c_{n,n}^{(1)} \beta_0^n \tilde{\mathcal{A}}_{n+1}(Q^2) + \ldots \qquad (39)$$

$$= \int_0^\infty \frac{dt}{t} F_{\rm Adl}(t) \mathcal{A}_1(t Q^2 e^{-5/3}). \qquad (40)$$

Here, $F_{\rm Adl}(t)$ is the characteristic function of the Adler function, whose explicit expression was obtained in Ref. [47].

On the other hand, the full coefficients $(d_{\rm Adl})_n$ and $(\tilde{d}_{\rm Adl})_n$ are known only up to $n = 3$ [42–44]. Therefore, the full Adler function can be evaluated in anQCD by adding to the LB contribution the three known bLB terms

$$(d_{\rm Adl})^{\rm (LB+bLB)}_{\rm man}(Q^2)^{[4]} = \int_0^\infty \frac{dt}{t} F^{\mathcal{E}}_{\rm Adl}(t) \mathcal{A}_1(t Q^2 e^{-5/3})$$
$$+ \sum_{n=1}^{3} (T_{\rm Adl})_n \tilde{\mathcal{A}}_{n+1}, \qquad (41)$$

where

$$(T_{\rm Adl})_n = (\tilde{d}_{\rm Adl})_n - c_{nn}^{(1)} \beta_0^n \qquad (42)$$

are the complete bLB parts ($\sim \beta_0^{n-1}$) of the coefficients $(\tilde{d}_{\rm Adl})_n$.

Using the expression (41) in the contour integration (37) gives for r_τ

$$r_\tau^{\rm (LB+bLB),[4]} = r_\tau^{\rm (LB)} + \sum_{n=1}^{3} (T_{\rm Adl})_n I(\tilde{\mathcal{A}}_{n+1}), \qquad (43)$$

where $I(\tilde{\mathcal{A}}_{n+1})$ are the contour integrals given by

$$I(\tilde{\mathcal{A}}_{n+1}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi (1 + e^{i\phi})^3 (1 - e^{i\phi}) \tilde{\mathcal{A}}_{n+1}(m_\tau^2 e^{i\phi})$$
$$(n = 1, 2, 3), \qquad (44)$$

and the LB part in (43) is a well-defined (in anQCD models) integral of the form

$$r_\tau^{\rm (LB)} = \frac{1}{\pi} \int_0^\infty \frac{dt}{t} \tilde{F}_r(t) \rho_1(t m_\tau^2 e^{-5/3}), \qquad (45)$$

where $\tilde{F}_r(t)$ was calculated in [18] from the Minkowskian characteristic function $F_r^{\mathcal{M}}(t)$ of [48]. For details on $\tilde{F}_r(t)$ we refer to Appendix D of [18].[9]

Thus, the more explicit form of the r_τ-reproduction condition, mentioned in the previous section, is

---

[6]If the analytization is performed in any other way, the renormalization scale and scheme dependence of the resulting truncated analytic series of any observable $\mathcal{D}(Q^2)$ will in general increase (instead of decrease) when the number of terms in the series increases, cf. [11,12].

[7]If masses are involved in the evaluated physical quantity, perturbation series contains noninteger powers $a_{\rm pt}^\nu$ (and possibly derivatives thereof with respect to ν, i.e., $a_{\rm pt}^\nu \ln^k a_{\rm pt}$). The evaluation of such quantities in anQCD models is then based on the procedure presented in Ref. [17].

[8]We have $(d_{\rm Adl})_n = c_{n,n}^{(1)} \beta_0^n + \mathcal{O}(\beta_0^{n-1})$ and $(\tilde{d}_{\rm Adl})_n = c_{n,n}^{(1)} \beta_0^n + \mathcal{O}(\beta_0^{n-1})$. The expansions in powers of β_0 are obtained when $(d_{\rm Adl})_n$ and $(\tilde{d}_{\rm Adl})_n$ are written in powers of n_f ($= -6\beta_0 + 33/2$) and then reorganized in powers of β_0.

[9]In (43) and (44) we used in the Adler function the renormalization scale $\mu^2 = Q^2$ ($\equiv m_\tau^2 e^{i\phi}$); this scale, of course, can be varied, cf. [18,26] for details.

TABLE I. The parameters of the considered two-delta anQCD model, for the three chosen values of the pQCD-onset scale M_0: 1.0;1.25;1.50 GeV. In addition, the results of the 1d model of Ref. [26] in the scheme $c_2 = 0$ are given (the last line). See the text for details.

M_0 [GeV]	$c_2 = \beta_2/\beta_0$	Λ [GeV]	s_0	s_1	f_1^2	s_2	f_2^2	$\mathcal{A}_1(0)$
1.00	-7.15	0.193	26.86	19.473	0.3637	0.3594	0.7808	2.29
1.25	-4.76	0.260	23.06	16.837	0.2713	0.8077	0.5409	0.776
1.50	-2.10	0.363	17.09	12.523	0.1815	0.7796	0.3462	0.544
0.886 (1d)	0	0.472	3.525	0.4755	0.2086	$\cdots$	$\cdots$	0.544

$$(r_\tau^{(\text{LB+bLB})[4]} =) r_\tau^{(\text{LB})} + \sum_{n=1}^{3} (T_{\text{Adl}})_n I(\tilde{\mathcal{A}}_{n+1}) = 0.203. \quad (46)$$

The six conditions, (19), (22), and (46), then determine the six parameters of the model: f_j^2, f_2^2, s_1, s_2, s_0 and the scale Λ. This procedure can be performed once we have chosen a value of the scheme parameter c_2 of Eq. (10).

IV. NUMERICAL RESULTS

In this section, we present the numerical results for the parameters of the model, obtained from the imposition of the aforementioned six conditions. The additional (implicit) conditions which we choose are that the weights f_1^2 and f_2^2 are positive. This is based on the fact that the discontinuity function $\rho_1(\sigma)$ is positive in any reasonable scheme of pQCD. Furthermore, the condition of positivity of $\rho_1(\sigma)$ can be expressed also via the condition that the Minkowskian coupling

$$\mathfrak{A}_1(\sigma) = \frac{1}{\pi} \int_\sigma^\infty d\sigma' \frac{\rho_1(\sigma')}{\sigma'} \quad (47)$$

is a monotonously decreasing function of scale σ.

If we choose for the scheme parameter $c_2 = 0$, we obtain from the six conditions, as a result, that the pQCD-onset mass $M_0 = \sqrt{s_0}\Lambda$ is relatively high, $M_0 \approx 1.7$ GeV. Increasing c_2 to positive values increases M_0 further; e.g., for $c_2 = c_2^{\overline{\text{MS}}}/2$ (≈ 2.24), we get $M_0 \approx 2.0$ GeV, etc. On the other hand, decreasing c_2 to negative values, we obtain smaller M_0. We believe that the effective pQCD-onset scale M_0 should be significantly smaller than the mass of the τ lepton. This turned out to be so in the one-delta anQCD model of Ref. [26], with $c_2 = 0$, where $M_0 \approx 1$ GeV was obtained. In general, we do not want to parametrize (via delta functions) relatively small deviations from pQCD, i.e., those at $\sqrt{\sigma} > 1.5$ GeV. On the other hand, by lowering the value of c_2, we encounter at $c_2 < -8$ ($M_0 < 0.9$ GeV) negative values of s_2, implying that the analyticity is lost.

Therefore, we will adjust the scheme parameter c_2 in such a way as to get $M_0 = 1.25 \pm 0.25$ GeV. The results are given in Table I. We believe that all three choices of M_0 will give almost the same predictions for various physical observables. The reason for this lies in the renormalization scheme independence of pQCD results; and our model,

although nonperturbative and analytic at low momenta $|Q| \lesssim 1$ GeV, is practically indistinguishable from pQCD at all higher momenta. It is interesting that the value of the coupling $\mathcal{A}_1$ at $Q^2 = 0$, obtained in this model for $M_0 = 1.25$ GeV ($\mathcal{A}_1(0) \approx 0.8$), is not far from the value in Ref. [3] ($\mathcal{A}_1(0) \approx 8.9/N_c/\pi \approx 0.9$), which was obtained from an analysis using Dyson-Schwinger equations for the ghost and gluon sector under an assumption of regularity of the ghost-gluon vertex.

In Table I, we included, in the last line, also the results of the one-delta (1d) model of Ref. [26], obtained in an analogous way, in the scheme $c_2 = c_3 = \ldots = 0$.[10] In that model, though, the smaller number of parameters led to less stringent conditions (20), namely, with $n_{\max} = 3$ (cf. also a similar model in Ref. [50]).

In Fig. 4(a), we present, for the resulting "central" choice of $M_0 = 1.25$ GeV ($c_2 = -4.76$), the corresponding pQCD discontinuity function $\rho_1^{(\text{pt})}(\sigma) = \text{Im} a_{\text{pt}}(-\sigma - i\epsilon)$ of the underlying perturbative coupling a_{pt} (8), in the regime of low $|\sigma|$ and including the unphysical (Landau) regime of negative-σ cut. We can note that in the latter regime, there is an additional, polelike singularity at $\sigma = -u_L\Lambda^2$ (≈ -0.261 GeV^2); while the "continuous" part of $\rho_1^{(\text{pt})}(\sigma)$ ends a bit earlier, at $\sigma = -s_L\Lambda^2 \approx -0.207$ GeV^2.[11] In Fig. 4(b), $\rho_1(\sigma)$ of the considered two-delta anQCD model is presented, cf. Eq. (6).

In Figs. 5(a) and 5(b), we present, for the aforementioned central case $M_0 = 1.25$ GeV, the resulting space-like coupling $\mathcal{A}_1(Q^2)$ at positive Q^2. The higher-order couplings $\tilde{\mathcal{A}}_k(Q^2)$ ($k = 2, 3$), cf Eq. (36), are also presented; due to a strong hierarchy (even at low Q^2), they are rescaled, for better visibility, by factors 4 and 16, respectively. All the corresponding pQCD quantities ($a_{\text{pt}}, 4\tilde{a}_{\text{pt},2}, 16\tilde{a}_{\text{pt},3}$) are presented as dotted curves. For better visibility at low Q^2, Fig. 5(b) is presented with Q^2 on logarithmic scale. It is clearly visible that the model practically agrees

---

[10]In Ref. [26], the parameters differ a little from those in Table I, because the world average value taken there was from the year 2008, $a_{\text{pt}}(M_Z^2; \overline{\text{MS}}) = 0.1190/\pi$ [49], and because we imposed there the condition $\mathcal{A}_1((3m_c)^2) = a_{\text{pt}}((3m_c)^2; n_f = 3)$ instead of the (numerically similar) condition (19).

[11]Note: $s_L = c_1^{-c_1/\beta_0}$, which is approximately 0.635 when $n_f = 3$; at $Q^2 = s_L\Lambda^2$, the Lambert function $W_{-1}(z(Q^2))$ is equal to -1, and this is the branching point for $a_{\text{pt}}(Q^2)$, cf. [28,31].
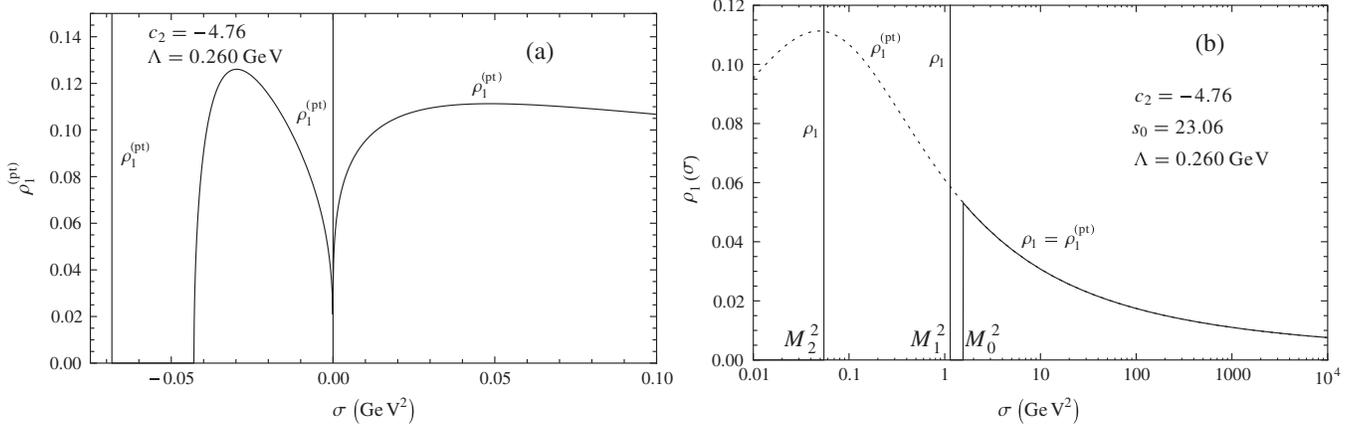
FIG. 4.   The discontinuity function $\rho_1(\sigma)$: (a) for the underlying pQCD coupling $a_{pt}(Q^2)$, where the Landau pole at $\sigma = -u_L\Lambda^2$ (≈ -0.068 GeV^2) and the branching point at $\sigma = -s_L\Lambda^2$ (≈ -0.043 GeV^2) are visible; (b) for the considered two-delta anQCD model, where the unknown region $0 < \sigma < M_0^2$ is parametrized by two delta functions, at $\sigma = M_1^2$ and $\sigma = M_2^2$. The parameters used correspond to the central case in Table I ($M_0 = 1.25$ GeV).

with the corresponding pQCD model at $Q^2 > 1$ GeV^2 and that for $Q^2 < 1$ GeV^2, the model differs from pQCD significantly, due to the imposition of the analyticity.

In Fig. 6(a), we present the difference between the perturbative and the analytic coupling ($a_{pt}(Q^2) - \mathcal{A}_1(Q^2)$) at positive Q^2, for the central choice of parameter ($M_0 = 1.25$ GeV) of Table I. In the figure, we keep, formally, $n_f = 3$ even for high Q^2. We see that the difference vanishes fast when Q^2 increases. For comparison, in Fig. 6(b), we show the difference ($a_{pt} - \mathcal{A}_1$) in the case of the one-delta anQCD model of Ref. [26], with parameters as given in the last line of Table I.

From Fig. 6(a), we can deduce that the difference ($a_{pt} - \mathcal{A}_1$) behaves as $\propto (\Lambda^2/Q^2)^{n_{\rm eff}}$, where the numerical value of the effective power index $n_{\rm eff}$ is somewhat less

than 5; it is somewhere between 4 and 5. This can be understood in the following way: the values of s_0 and s_1 in the model are relatively large ($\sim 10^1$), and therefore the coefficients $\mathcal{K}_n$ (for $n = 5, 6, \dots$) in the expansion

$$a_{pt}(Q^2; c_2) - \mathcal{A}_1(Q^2; c_2) = \mathcal{K}_5(\Lambda^2/Q^2)^5 \\ + \mathcal{K}_6(\Lambda^2/Q^2)^6 + \dots \quad (48)$$

are large ($\sim s_0^{n-1}$, s_1^{n-1}) and are increasing when n increases.

Finally, in Table II, we compare the convergence of the series for r_τ, Eq. (43), in the three cases of the two-delta model and in the one-delta model of Table I. We see that the convergence is good in general, and appears to be better when the underlying pQCD scheme parameter c_2 is more negative.
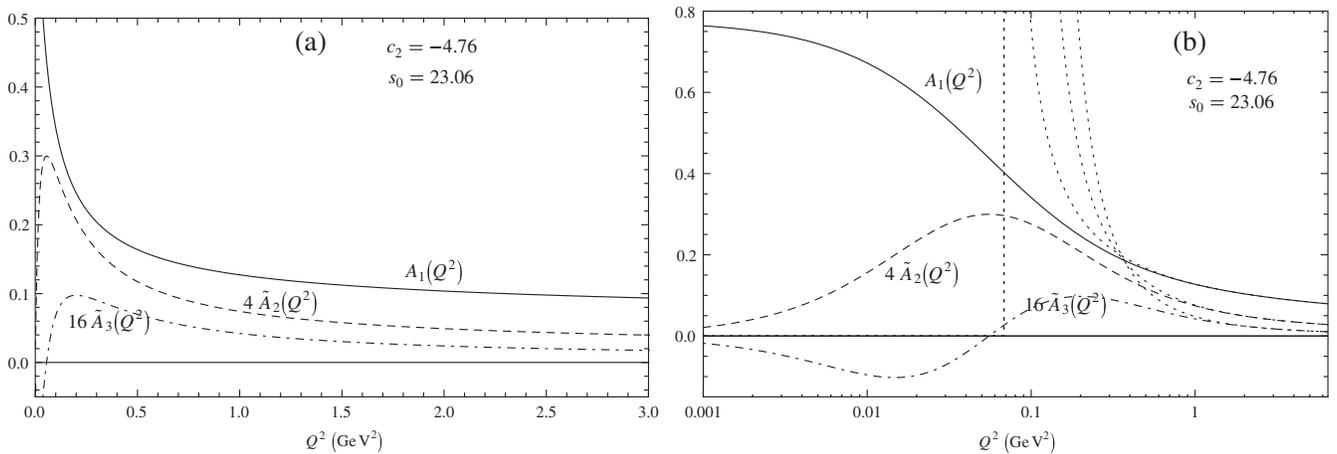
FIG. 5.   The analytic couplings $\mathcal{A}_1$ (full line), $4 \times \tilde{\mathcal{A}}_2$ (dashed curve), and $16 \times \tilde{\mathcal{A}}_3$ (dash-dotted curve) for positive Q^2, in the considered two-delta model, for the central parameter choice of Table I ($M_0 = 1.25$ GeV): (a) linear scale is used for Q^2; (b) logarithmic scale is used for Q^2, for better visibility at low values of Q^2. For comparison, the corresponding pQCD couplings ($a_{pt}, 4 \times \tilde{a}_{pt,2}, 16 \times \tilde{a}_{pt,3}$) are included, as dotted curves (in the same renormalization scheme, with $c_2 = -4.76$); the vertical dotted line is the Landau pole at $Q^2 = u_L\Lambda^2$ (≈ 0.068 GeV^2).
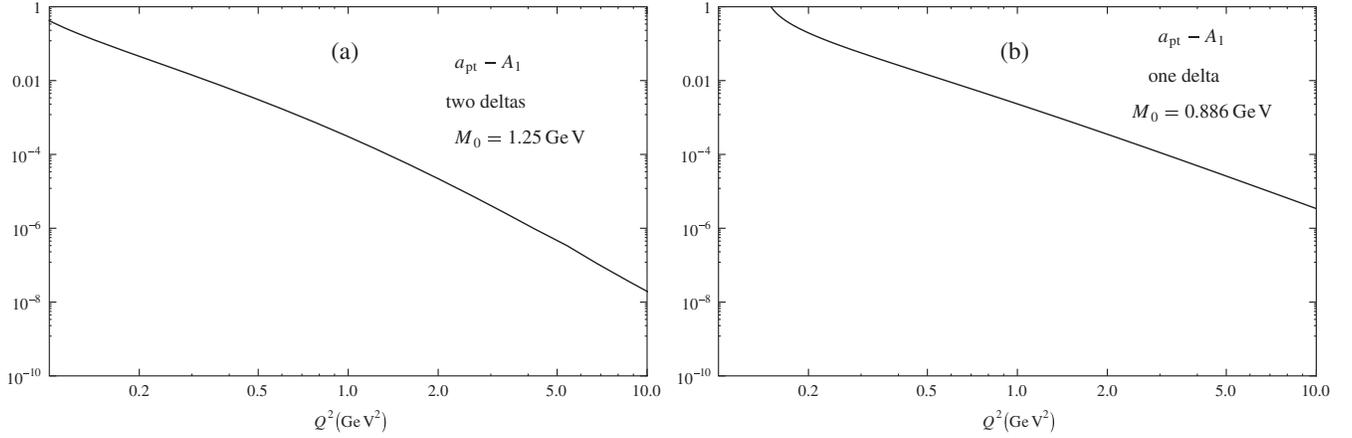
FIG. 6. The difference between the perturbative and the analytic coupling, as a function of positive Q^2, (a) for the considered two-delta anQCD model (with $M_0 = 1.25$ GeV); (b) for the one-delta anQCD parameter of Ref. [26], but with the input parameters somewhat modified (see the text for details).

There is one more aspect of the presented model which we should address. We recall in more detail how we matched the presented anQCD model indirectly to the $\overline{\text{MS}}$ scheme, via the condition (19):

(1) The Lambert scale of the model, at $n_f = 3$, was chosen to coincide with the Lambert scale of the underlying pQCD model in the same scheme determined by the beta function of Eq. (10) at $n_f = 3$;

(2) The latter scale was fixed in such a way that the change of the scheme parameters $(c_2, c_2^2/c_1, c_2^3/c_1^2, \ldots)$ to the four-loop $\overline{\text{MS}}$ scheme parameters $(c_2^{\overline{\text{MS}}}, c_3^{\overline{\text{MS}}})$, at $n_f = 3$ and $\mu^2 = (2m_c)^2$, gave the value of $a_{\text{pt}}^{(\overline{\text{MS}})}((2m_c)^2; n_f = 3)$ which corresponds to $a_{\text{pt}}^{(\overline{\text{MS}})}(M_Z^2) = 0.1184/\pi$, the central value of the world average [33]. The RGE running between $\mu^2 = M_Z^2$ and $\mu^2 = (2m_c)^2$, in $\overline{\text{MS}}$, was performed in the usual way, using the four-loop polynomial form of $\beta^{(\overline{\text{MS}})}(a_{\text{pt}})$ and the three-loop matching conditions [34] at quark thresholds $\mu^2 = (2m_q)^2$ ($q = b, c$).

As pointed out in Ref. [51], the aforementioned (three-loop) matching in principle introduces, indirectly, an element of nonanalyticity in the described framework, at[12] the scale $Q^2 = (2m_c)^2$ and (to a much lesser degree) at $Q^2 = (2m_b)^2$. This is so because the matching introduces nonanalyticity (even: discontinuity) in the running coupling $a_{\text{pt}}^{(\overline{\text{MS}})}(Q^2)$ at those threshold scales. It would be more convenient for the presented low-energy anQCD model to be matched [at energies $Q^2 \sim (2m_c)^2$] to a scheme which introduces the quark mass threshold effects in the running in a gradual (analytic) way. One such scheme is the pinch technique, Refs. [52]. The pinch

technique effective charge (i.e., running coupling) was presented for supersymmetric QCD at one-loop in Ref. [51]. Specifically, in nonsupersymmetric QCD with three massless flavors, the relations of Ref. [51] [especially their Eq. (A3)] imply [for $|Q_0^2|, |Q^2| \lesssim (2m_c)^2$]

$$a_{\text{PT}}(Q^2) = a_{\text{PT}}(Q_0^2) + a_{\text{PT}}^2(Q_0^2)[-\tfrac{9}{4}\ln(Q^2/Q_0^2) + \tfrac{1}{6}L_{1/2}(Q^2/m_c^2) - \tfrac{1}{6}L_{1/2}(Q_0^2/m_c^2)], \quad (49)$$

where the quark threshold function $L_{1/2}$ is

$$L_{1/2}(Q^2/m^2) = (3 - \beta^2)[\beta\text{ArcTanh}(1/\beta) - 1] + 2, \quad (50)$$

where $\beta = \sqrt{1 + 4m^2/Q^2}$ and the (complex) momenta Q^2 are not on the cut $Q^2 < -4m^2$. In the case of decoupling $(|Q^2| \ll m^2)$, this function acquires the value of $(5/3)$.

We can estimate the practical errors introduced in the calculations in our model due to the aforementioned matching to $\overline{\text{MS}}$, by estimating the errors that the actual nondecoupling of the charm quark mass introduced in our calculation of r_τ (note that in our calculation, we considered c quark to be completely decoupled, in the spirit of $\overline{\text{MS}}$). The relation (49) would be transcribed in our model, approximately, in the following way:

$$\mathcal{A}_1(Q^2)_{\text{thr}} \approx \mathcal{A}_1(Q_0^2)_{\text{thr}} + \tilde{\mathcal{A}}_2(\bar{Q}^2)[-\tfrac{9}{4}\ln(Q^2/Q_0^2) + \tfrac{1}{6}L_{1/2}(Q^2/m_c^2) - \tfrac{1}{6}L_{1/2}(Q_0^2/m_c^2)], \quad (51)$$

where $\tilde{\mathcal{A}}_2(\bar{Q}^2)$ is the logarithmic derivative of A_1 defined via Eq. (36), and the scale $\bar{Q}^2$ is taken to be the geometric mean[13] of Q_0^2 and Q^2: $\bar{Q}^2 = \sqrt{Q_0^2 Q^2}$. Further, the subscript

---

[12]We consider, throughout, the first three flavors to be massless.

[13]Formally, at one-loop level, it would be equivalent to take the scale $\bar{Q}^2 = Q_0^2$ or $\bar{Q}^2 = Q^2$; numerically, though, the geometric mean of these two scales (i.e., the arithmetic mean of the logarithms of these scales) is better.

TABLE II. The four terms in the truncated analytic expansion (43) in two-delta anQCD model for the three choices of the pQCD-onset scale M_0 and for the one-delta anQCD model, as specified in Table I.

M_0 [GeV]	c_2	Λ [GeV]	r_τ: LB	NLB	N^2LB	N^3LB	sum (sum)
1.00	−7.15	0.193	0.2023	0.0012	−0.0010	0.0005	0.2030
1.25	−4.76	0.260	0.2052	0.0013	−0.0053	0.0019	0.2030
1.50	−2.10	0.363	0.2096	0.0013	−0.0110	0.0030	0.2030
0.886	0	0.472	0.2149	0.0014	−0.0176	0.0043	0.2030

"thr" indicates that the nondecoupling effect $m_c \neq \infty$ is taken into account, in a first approximation. The corresponding relation in the model without the analytic threshold effects is

$$\mathcal{A}_1(Q^2) \approx \mathcal{A}_1(Q_0^2) + \tilde{\mathcal{A}}_2(\bar{Q}^2)[-\tfrac{9}{4}\ln(Q^2/Q_0^2)]. \quad (52)$$

$$\delta r_\tau^{(\text{LO})}(m_c \neq \infty) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}d\phi(1+e^{i\phi})^3(1-e^{i\phi})[\mathcal{A}_1(Q^2)_{\text{thr}} - \mathcal{A}_1(Q^2)]|_{Q^2=m_\tau^2 e^{i\phi}}$$

$$\approx \frac{1}{2\pi}\int_{-\pi}^{+\pi}d\phi(1+e^{i\phi})^3(1-e^{i\phi})\left[(\mathcal{A}_1(Q_0^2)_{\text{thr}} - \mathcal{A}_1(Q_0^2))\right.$$

$$\left. + \frac{1}{6}\tilde{\mathcal{A}}_2(\sqrt{Q_0^2 Q^2})(L_{1/2}(Q^2/m_c^2) - L_{1/2}(Q_0^2/m_c^2))\right]|_{Q^2=m_\tau^2 e^{i\phi}}. \quad (53)$$

The scale Q_0^2 here should have a low value ($|Q_0^2| \ll m_\tau^2$) so that the charm quark can be assumed to basically decouple ($m_c = \infty$) at such scale. Namely, in such a case, the first (unknown) term in the integrand ($\mathcal{A}_1(Q_0^2)_{\text{thr}} - \mathcal{A}_1(Q_0^2)$) can be taken to be zero, and the remaining term would indicate correctly the effects of nondecoupling of m_c. On the other hand, $|Q_0|^2$ cannot be taken to be too small, as then the $\sim\tilde{\mathcal{A}}_2$ term in the relations (51) and (52) would dominate over the $\mathcal{A}_1(Q_0^2)$ term, due to a very large value of the logarithm $\ln(Q^2/Q_0^2)$. Therefore, we will choose $Q_0^2 = \kappa Q^2$ ($= \kappa m_\tau^2 \exp(i\phi)$), with $\kappa < 1$ varying in a specific interval were both aforementioned restrictions are reasonably fulfilled

$$\delta r_\tau^{(\text{LO})}(m_c \neq \infty) \approx \frac{1}{12\pi}\int_{-\pi}^{+\pi}d\phi(1+e^{i\phi})^3(1-e^{i\phi})$$

$$\times \tilde{\mathcal{A}}_2(\sqrt{Q_0^2 Q^2})(L_{1/2}(Q^2/m_c^2)$$

$$- L_{1/2}(Q_0^2/m_c^2))|_{Q^2=m_\tau^2 e^{i\phi};Q_0^2=\kappa Q^2}. \quad (54)$$

The results of these estimates, for the central value of the parameters of the presented two-delta anQCD model (i.e., the case $M_0 = 1.25$ GeV of Table I) are presented in Table III. We see from the Table that the effects of the introduction of analytic threshold effects (i.e., the effects of the nondecoupling of m_c) in the calculation of r_τ are $\delta r_\tau \lesssim 10^{-3}$. This is to be compared with the theoretical value $r_\tau^{(\text{LO})} \approx 0.123$ in the anQCD model, and the full value $r_\tau = 0.203$ in the model, and with the experimental uncertainties $\delta r_\tau = \pm 0.004$ [Eq. (23)]. We conclude that, although

An estimate of the error in the calculation of r_τ, due to the nondecoupling of m_c, in the calculation of r_τ can be obtained by using the relation (37) in the leading order (LO) when $d_{\text{Adl}}(Q^2) = \mathcal{A}_1(Q^2)$ there and subtracting for $\mathcal{A}_1(Q^2)$ the two relations (51) and (52) on the contour

these effects are appreciable, they are somewhat lower than the present experimental uncertainty of r_τ.

V. SUMMARY

In this work, we presented an analytic QCD model which, at high squared momenta $|Q^2|$, becomes to a high degree indistinguishable from perturbative QCD; nonetheless, at low $|Q^2|$, the spacelike couplings $\mathcal{A}_n(Q^2)$ of the model mirror correctly the analytic properties of the (to-be-evaluated) spacelike observables $\mathcal{D}(Q^2)$ as dictated by the general principles of the local quantum field theories. The model reproduces correctly the experimental value of the τ lepton (nonstrange, $V + A$) semihadronic decay ratio r_τ, i.e., the only low-momentum QCD observable which is well-measured and whose higher-twist effects are small and appear to be under control. The difference between the analytic $\mathcal{A}_1(Q^2)$ and its perturbative counterpart $a_{\text{pt}}(Q^2)$ [$\equiv \alpha_s(Q^2)/\pi$] is formally $\sim(\Lambda^2/Q^2)^5$ at $|Q^2| > \Lambda^2$, where $\Lambda^2 \sim 0.1$ GeV^2 is the QCD (or: light meson) scale. The starting point was the construction of the discontinuity function $\rho_1(\sigma)$ for the analytic coupling $\mathcal{A}_1(Q^2)$. This $\rho_1(\sigma)$ is assumed to coincide with its pQCD counterpart for σ above a pQCD-onset scale $M_0^2 \sim 1$ GeV^2. On the other hand, the unknown behavior of $\rho_1(\sigma)$ in the low-σ regime is parametrized as a linear combination of two delta functions. The underlying pQCD scheme parameter c_2 is adjusted so that the pQCD-onset scale M_0 is either 1.0, 1.25, or 1.50 GeV. We believe that these three variants of the two-delta anQCD model represent almost the same

TABLE III. The estimates of nondecoupling of m_c in the calculation of r_τ, Eq. (54), for different value of the parameter κ. In the third column, we include a measure of the effect of nondecoupling of m_c at $Q_0^2 = \kappa m_\tau^2$ [assumed to be negligible in Eq. (54)]; and in the fourth column, the ratio of $\frac{9}{4} \ln(Q^2/Q_0^2)\tilde{\mathcal{A}}_2(\bar{Q}^2)$ with $\mathcal{A}_1(Q_0^2)$ [assumed to be small in Eqs. (51) and (52)], for $Q^2 = m_\tau^2$ and $Q_0^2 = \kappa m_\tau^2$.

κ	$\delta r_\tau^{(\text{LO})}(m_c \neq \infty)$	$(L_{1/2}(\kappa m_\tau^2/m_c^2) - L_{1/2}(0))/L_{1/2}(0)$	$-\frac{9}{4}\ln(\kappa)\tilde{\mathcal{A}}_2(\sqrt{\kappa}m_\tau^2)/\mathcal{A}_1(\kappa m_\tau^2)$
0.15	6.1×10^{-4}	0.034	0.42
0.20	4.6×10^{-4}	0.045	0.36
0.30	3.1×10^{-4}	0.066	0.28
0.40	2.3×10^{-4}	0.087	0.21

physics, since they reproduce the same value of the key low-momentum observable $r_\tau(V + A)$, and, at the same time, they differ in the value of the underlying pQCD (i.e., high-momentum) renormalization scheme parameter c_2, the latter being the only free input parameter of the model. If the value of c_2 is further adjusted so that M_0 is below 1.0 GeV, the analyticity gets lost at $M_0 \approx 0.9$ GeV. The model is an extension of the one-delta anQCD model of Ref. [26] where $c_2 = 0$ was used.

The main motivation behind the construction of the considered anQCD model is that with it, we can eventually evaluate low-momentum QCD quantities whose higher-twist contributions are appreciable. An example are the separate vector (V) and axial vector (A) channel of r_τ. Such evaluations would involve the considered anQCD model together with the operator product expansion. This application (anQCD + OPE) is consistent, due to the very suppressed difference $\mathcal{A}_1(Q^2) - a_{\text{pt}}(Q^2) \sim (\Lambda^2/Q^2)^5$ in the ultraviolet regime, the latter implying that the higher-twist terms in OPE, of dimension $d \leq 8$, will still be of infrared origin, in accordance with the philosophy of the Institute of Experimental and Theoretical Physics group [23].

[1] N. N. Bogoliubov and D. V. Shirkov, *Introduction to The Theory of Quantum Fields* (Wiley, New York, 1980).

[2] R. Oehme, Int. J. Mod. Phys. A **10,** 1995 (1995).

[3] C. Lerche and L. von Smekal, Phys. Rev. D **65,** 125006 (2002).

[4] R. Alkofer, C. S. Fischer, and F. J. Llanes-Estrada, Phys. Lett. B **611,** 279 (2005); **670,** 460(E) (2009); A. C. Aguilar, D. Binosi, and J. Papavassiliou, Phys. Rev. D **78,** 025010 (2008); A. C. Aguilar, D. Binosi, J. Papavassiliou, and J. Rodriguez-Quintero, Phys. Rev. D **80,** 085018 (2009).

[5] A. Cucchieri and T. Mendes, Phys. Rev. Lett. **100,** 241601 (2008); I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Phys. Lett. B **676,** 69 (2009).

[6] D. V. Shirkov and I. L. Solovtsov, arXiv:hep-ph/9604363; Phys. Rev. Lett. **79,** 1209 (1997).

[7] K. A. Milton, I. L. Solovtsov, and O. P. Solovtsova, Phys. Lett. B **415,** 104 (1997).

[8] D. V. Shirkov, Theor. Math. Phys. **127,** 409 (2001); Eur. Phys. J. C **22,** 331 (2001).

[9] A. P. Bakulev, S. V. Mikhailov, and N. G. Stefanis, Phys. Rev. D **72,** 074014 (2005); **72,** 119908(E) (2005); **75,** 056005 (2007); **77,** 079901(E) (2008); J. High Energy Phys. 06 (2010) 085.

[10] A. V. Nesterenko, Phys. Rev. D **62,** 094028 (2000); **64,** 116009 (2001); Int. J. Mod. Phys. A **18,** 5475 (2003); A. C. Aguilar, A. V. Nesterenko, and J. Papavassiliou, J. Phys. G **31,** 997 (2005).

[11] G. Cvetič and C. Valenzuela, J. Phys. G **32,** L27 (2006).

[12] G. Cvetič and C. Valenzuela, Phys. Rev. D **74,** 114030 (2006).

[13] A. V. Nesterenko and J. Papavassiliou, Phys. Rev. D **71,** 016009 (2005).

[14] D. V. Shirkov, Nucl. Phys. B, Proc. Suppl. **162,** 33 (2006).

[15] D. V. Shirkov and I. L. Solovtsov, Theor. Math. Phys. **150,** 132 (2007).

[16] D. V. Shirkov, Teor. Mat. Fiz. **119,** 55 (1999) [Theor. Math. Phys. **119,** 438 (1999)]; Lett. Math. Phys. **48,** 135 (1999).

[17] G. Cvetič and A. V. Kotikov, J. Phys. G **39,** 065005 (2012).

[18] G. Cvetič, R. Kögerler, and C. Valenzuela, J. Phys. G **37,** 075001 (2010); Phys. Rev. D **82,** 114004 (2010).

[19] A. V. Nesterenko and J. Papavassiliou, J. Phys. G **32,** 1025 (2006); A. V. Nesterenko, in *Ninth Workshop on Non-Perturbative Quantum Chromodynamics, Paris, 2007*, econf C0706044, 25 (2007); *Eleventh Workshop on Non-Perturbative Quantum Chromodynamics, Paris, 2011*, eonf C1106064, 23 (2011).

[20] S. Peris, M. Perrottet, and E. de Rafael, J. High Energy Phys. 05 (1998) 011.

[21] E. de Rafael, Nucl. Phys. B, Proc. Suppl. **96**, 316 (2001); arXiv:1111.6162.

[22] B. A. Magradze, Few Body Syst. **48**, 143 (2010).

[23] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[24] S. Peris, Phys. Rev. D **74**, 054013 (2006).

[25] G. Cvetič and H. E. Martínez, J. Phys. G **36**, 125006 (2009).

[26] C. Contreras, G. Cvetič, O. Espinosa, and H. E. Martínez, Phys. Rev. D **82**, 074005 (2010).

[27] MATHEMATICA 8.0.4, Wolfram Co.

[28] E. Gardi, G. Grunberg, and M. Karliner, J. High Energy Phys. 07 (1998) 007.

[29] D. S. Kourashev, arXiv:hep-ph/9912410; D. S. Kurashev and B. A. Magradze, Teor. Mat. Fiz. **135**, 95 (2003); [Theor. Math. Phys. **135**, 531 (2003)].

[30] B. A. Magradze, Few Body Syst. **40**, 71 (2006).

[31] G. Cvetič and I. Kondrashuk, J. High Energy Phys. 12 (2011) 019.

[32] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Adv. Comput. Math. **5**, 329 (1996).

[33] K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

[34] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Phys. Rev. Lett. **79**, 2184 (1997).

[35] R. Barate *et al.* (ALEPH Collaboration), Eur. Phys. J. C **4**, 409 (1998).

[36] S. Schael *et al.* (ALEPH Collaboration), Phys. Rep. **421**, 191 (2005).

[37] K. Ackerstaff *et al.* (OPAL Collaboration), Eur. Phys. J. C **7**, 571 (1999).

[38] M. Davier, S. Descotes-Genon, A. Höcker, B. Malaescu, and Z. Zhang, Eur. Phys. J. C **56**, 305 (2008).

[39] B. L. Ioffe, Prog. Part. Nucl. Phys. **56**, 232 (2006).

[40] K. Maltman and T. Yavin, Phys. Rev. D **78**, 094020 (2008).

[41] E. Braaten, Phys. Rev. Lett. **60**, 1606 (1988); S. Narison and A. Pich, Phys. Lett. B **211**, 183 (1988); E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373**, 581 (1992); A. Pich and J. Prades, J. High Energy Phys. 06 (1998) 013.

[42] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **85B**, 277 (1979); M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979); W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **44**, 560 (1980).

[43] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. B **259**, 144 (1991); L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560 (1991); **66**, 2416 (E) (1991).

[44] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **101**, 012002 (2008).

[45] D. J. Broadhurst, Z. Phys. C **58**, 339 (1993).

[46] M. Beneke, Phys. Lett. B **307**, 154 (1993); Nucl. Phys. **B405**, 424 (1993).

[47] M. Neubert, Phys. Rev. D **51**, 5924 (1995).

[48] M. Neubert, arXiv:hep-ph/9502264.

[49] C. Amsler *et al.* (Particle Data Group), Phys. Lett. B **667**, 1 (2008).

[50] A. I. Alekseev, Few Body Syst. **40**, 57 (2006).

[51] M. Binger and S. J. Brodsky, Phys. Rev. D **69**, 095007 (2004).

[52] J. M. Cornwall, Phys. Rev. D **26**, 1453 (1982); G. Degrassi and A. Sirlin, Phys. Rev. D **46**, 3104 (1992); N. J. Watson, Nucl. Phys. **B494**, 388 (1997); J. Papavassiliou, E. de Rafael, and N. J. Watson, Nucl. Phys. **B503**, 79 (1997); D. Binosi, J. Phys. G **30**, 1021 (2004).