PHYSICAL REVIEW D **85,** 114039 (2012)

More on the relation between the two physically inequivalent decompositions of the nucleon spin and momentum

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(Received 12 April 2012; published 22 June 2012)

In a series of papers, we have established the existence of two gauge-invariant decompositions of the nucleon spin, which are physically nonequivalent. The orbital angular momenta of quarks and gluons appearing in these two decompositions are gauge-invariant dynamical orbital angular momenta and "generalized" canonical orbital angular momenta with gauge-invariance, respectively. The key quantity, which characterizes the difference between these two types of orbital angular momenta is what-we-call the *potential angular momentum*. We argue that the physical meaning of the potential angular momentum in the nucleon can be made more transparent, by investigating a related but much simpler example from electrodynamics. We also make clear several remaining issues in the spin and momentum decomposition problem of the nucleon. We clarify the relationship between the evolution equations of orbital angular momenta corresponding to the two different decompositions above. We also try to answer the question whether the two different decompositions of the nucleon momentum really lead to different evolution equations, thereby predicting conflicting asymptotic values for the quark and gluon momentum fractions in the nucleon.

DOI: 10.1103/PhysRevD.85.114039 PACS numbers: 12.38.−t, 03.50.De, 12.20.−m, 14.20.Dh

I. INTRODUCTION

The nucleon spin puzzle raised by the EMC measurement in 1988 is still one of the fundamental unsolved problems in QCD [1,2]. The current status and homework of the nucleon spin problem can very briefly be summarized as follows. (For recent reviews, see, for example, [3,4].) First, the intrinsic quark spin contribution (or the quark polarization in the nucleon) was fairly precisely determined to be around $1/3$ [5–7]. Second, gluon polarization is likely to be small, although with large uncertainties [8–11]. So, what carries the remaining $2/3$ of the nucleon spin? That is a fundamental question of QCD which we want to answer. To answer this question unambiguously, we cannot avoid to clarify the following issues. What is a precise definition of each term of the decomposition based on quantum chromodynamics (QCD)? How can we extract individual term by means of direct measurements? Let us call it the nucleon spin decomposition problem [12–17].

The recent papers by Chen *et al.* [18,19] arose much controversy on the feasibility as well as the observability of the complete decomposition of the nucleon spin [20–42]. In the previous papers [23,24], we have established the existence of two physically nonequivalent decompositions of the nucleon spin, both of which are gauge-invariant. The quark and gluon intrinsic spin parts of these two decompositions are nothing different in these two decompositions. The difference appears in the orbital parts. The quark and gluon orbital angular momenta appearing in one decomposition is gauge-invariant *dynamical* (or *mechanical*) orbital angular momentum (OAM), while those appearing in another decomposition is generalized *canonical* OAM having gauge-invariance. The key quantity, which characterizes the difference between these two types of OAMs, is what-we-call the *potential angular momentum* [23]. Understanding its physical meaning is therefore of vital importance to make clear why there exist two decompositions at all and in what essential respects they are different. One of the purposes of the present paper is to clarify the physical meaning of this potential angular momentum term in a clearest fashion with the help of a plainer example from electrodynamics, i.e. through an analysis of a system of charged particles and photons, analogous to a system of color-charged quarks and gluons.

We also try to clarify several other issues left in the decomposition problem of the nucleon spin momentum. It is known that there also exist two different gauge-invariant decompositions of the nucleon momentum into the contributions of quarks and gluons. On the basis of a gauge-invariant decomposition of the nucleon momentum, which is different from the standardly-known one, Chen *et al.* threw doubt on a common wisdom of deep inelastic scattering physics that the gluons carry about half of the nucleon momentum in the asymptotic limit [19]. To verify the validity of this claim is of fundamental importance, since it challenges our common knowledge on one of the basics of perturbative QCD.

Also important to understand is a puzzling observation on the scale dependencies of the quark and gluon OAMs. In view of the physical inequivalence of the two types of OAMs, i.e. the dynamical OAMs and the (generalized) canonical OAMs, one might expect that they obey different evolution equations. However, the past researches indicate

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that they do obey the same evolution equation at least at the 1-loop level [43–50]. The reason of this somewhat mysterious observation need explanation.

The plan of the paper is as follows. To make the paper self-contained, we briefly summarize, in Sec. II, the current status of the nucleon spin decomposition problem from our own viewpoint. Next in Sec. III, we clarify the physical meaning of the potential angular momentum, which characterizes the difference between the two types of OAMs, and consequently the difference between the two inequivalent decompositions of the nucleon spin. Section V is devoted to the discussion on the relation between the two different decompositions of the nucleon momentum. It will be shown that the two decompositions lead to the same evolution equation at least for the longitudinal momentum fractions of quarks and gluons, thereby predicting the same asymptotic limits for them. The relation between the evolution equations for the quark and gluon OAMs corresponding to the two different decompositions of the nucleon spin is discussed in Sec. V. Then, we summarize what we have found in Sect. VI.

II. BRIEF REVIEW OF THE NUCLEON SPIN DECOMPOSITION PROBLEM—WHERE ARE WE NOW?

As is widely known, there have been two popular decompositions of the nucleon spin. One is the Jaffe-Manohar decomposition [12], and the other is the Ji decomposition [13,14]. Only the intrinsic quark spin part is common in these popular decompositions and the other parts are all different. A disadvantage of the Jaffe-Manohar decomposition is that each term is not separately gauge invariant except for the quark spin part. On the other hand, each term of the Ji decomposition is separately gauge invariant. Unfortunately, further gauge-invariant decomposition of J^g into its spin and orbital parts is given up in this widely-known decomposition. Especially annoying fact was that the sum of the gluon spin and OAM in the Jaffe-Manohar decomposition does not coincide with the total angular momentum of gluons in the Ji decomposition. Undoubtedly, this observation is inseparably connected with the fact that the quark OAMs in the two decompositions are also different.

In fact, first pay attention to the difference of the quark OAM parts in the two decompositions. What appears in the Jaffe-Manohar decomposition is the so-called canonical OAM, which is not gauge-invariant. On the other hand, what appears in the Ji decomposition is the so-called dynamical (or mechanical) OAM, which is manifestly gauge-invariant [51]. As is well-known, the gauge principle in physics dictates that observables must be gauge-invariant. Because of this reason, the observability of the canonical OAM has been questioned for a long time. On the other hand, Ji showed that the gauge-invariant dynamical quark OAM can be extracted from the combined analysis of unpolarized generalized parton distributions and the longitudinally polarized parton distributions [13,14].

Some years ago, however, Chen *et al.* proposed a new gauge-invariant decomposition of nucleon spin [18,19]. The basic idea is to decompose the gluon field $\boldsymbol{A}$ into two parts, i.e. the physical part $\boldsymbol{A}_{\rm phys}$ and the pure-gauge part $\boldsymbol{A}_{\rm pure}$, which is a generalization of the decomposition of the photon field $\boldsymbol{A}$ in QED into the transverse component $\boldsymbol{A}_\perp$ and the longitudinal component $\boldsymbol{A}_\parallel$. In addition to general conditions of decomposition, by imposing one plausible theoretical constraint, Chen *et al.* proposed a new decomposition of the nucleon spin. A prominent feature of their decomposition is that each term is separately gauge-invariant, while allowing the decomposition of the total gluon angular momentum into its spin and orbital parts. Another noteworthy feature of this decomposition is that it reduces to the gauge-variant decomposition of Jaffe and Manohar in a particular gauge, $\boldsymbol{A}_{\rm pure} = 0$, $\boldsymbol{A} = \boldsymbol{A}_{\rm phys}$ [12].

Chen *et al.*'s papers [18,19] arose much controversy on the feasibility as well as the observability of the complete decomposition of the nucleon spin [20–38]. We believe that we have arrived at one satisfactory solution to the problem, through a series of papers [23–25] In the 1st paper [23], we have shown that the way of gauge-invariant decomposition of nucleon spin is not necessarily unique, and proposed another gauge-invariant decomposition. The characteristic features of this decomposition is as follows. First, the quark part of this decomposition is common with the Ji decomposition, including both of spin and OAM parts. Second, the quark and gluon-spin parts are common with the Chen decomposition. A crucial difference with the Chen decomposition appears in the orbital parts. The sum of the quark and gluon OAMs in both decompositions is just the same, but each term is different. The difference of the gluon OAM in the two decompositions, which is equal to the difference of the quark OAM in the two decompositions with an extra minus sign, is given in the following form:

$$\boldsymbol{L}^g - \boldsymbol{L}'^g = -(\boldsymbol{L}^q - \boldsymbol{L}'^q) = \int \rho^a(\boldsymbol{x} \times \boldsymbol{A}^a_{\rm phys}) d^3x$$
$$= \int \psi^\dagger \boldsymbol{x} \times \boldsymbol{A}_{\rm phys} \psi \, d^3x, \qquad (1)$$

and we call it the *potential angular momentum* by the following reason. (In the above equation, $\boldsymbol{L}^q$ and $\boldsymbol{L}^g$ stand for the quark and gluon OAMs in our decomposition, while $\boldsymbol{L}'^q$ and $\boldsymbol{L}'^g$ the quark and gluon OAMs in the decomposition of Chen *et al.*) That is, the QED correspondent of this term is nothing but the angular momentum carried by the electromagnetic field or potential, which appears in the famous Feynman paradox of classical electrodynamics [52]. An arbitrariness of the decomposition arises, because this potential angular momentum term is *solely* gauge-invariant. This means that one has a freedom to shift this potential OAM term from the gluon OAM part to the quark OAM part in our decomposition, which in fact leads to the quark OAM in the Chen decomposition in such a way that

$$\boldsymbol{L}^q + \text{potential angular momentum}$$

$$= \int \psi^\dagger \boldsymbol{x} \times (\boldsymbol{p} - g\boldsymbol{A})\psi d^3x + g \int \psi^\dagger \boldsymbol{x} \times \boldsymbol{A}_{\text{phys}} \psi d^3x$$

$$= \int \psi^\dagger \boldsymbol{x} \times (\boldsymbol{p} - g\boldsymbol{A}_{\text{pure}})\psi d^3x = \boldsymbol{L}'^q. \tag{2}$$

Next, in the 2nd paper [24], we found that we can make a covariant extension of gauge-invariant decomposition of the nucleon spin. Covariant generalization of the decomposition has several advantages:

(i) First, it is useful to find relations to deep inelastic scattering (DIS) observables.
(ii) Second, it is vital to prove frame independence of the decomposition.
(iii) Third, it generalizes and unifies the previously-known nucleon spin decompositions.

Basically, we find two physically different decompositions. The decomposition (I) contains the well-known Ji decomposition [13], although it also allows gauge-invariant decomposition of gluon total angular momentum into its spin and OAM parts. The decomposition (II) contains in it three known decomposition, i.e. those of Bashinsky-Jaffe [15], of Chen *et al.* [18,19], and of Jaffe-Manohar [12], as we shall discuss below. The basis of our treatment is a decomposition of the full gauge field into its physical and pure-gauge parts, similar to Chen *et al.* [18,19]. Different from their treatment, however, we impose only the following quite general conditions:

$$F^{\mu\nu}_{\text{pure}} \equiv \partial^\mu A^\nu_{\text{pure}} - \partial^\nu A^\mu_{\text{pure}} - ig[A^\mu_{\text{pure}}, A^\nu_{\text{pure}}] = 0, \tag{3}$$

and

$$A^\mu_{\text{phys}}(x) \to U(x) A^\mu_{\text{phys}}(x) U^\dagger(x), \tag{4}$$

$$A^\mu_{\text{pure}}(x) \to U(x)\left(A^\mu_{\text{pure}}(x) + \frac{i}{g}\partial^\mu\right)U^\dagger(x). \tag{5}$$

The first is the pure-gauge condition for A^μ_{pure}, while the second are the gauge transformation properties for these two components. (These transformation properties indicates that the physics is basically contained in the physical part A^μ_{phys}, while the pure-gauge part A^μ_{pure} carries unphysical gauge degrees of freedom.) Actually, these conditions are not enough to fix gauge uniquely. However, the point of our theoretical scheme is that we can *postpone* a complete gauge fixing until a later stage, while accomplishing a gauge-invariant decomposition of $M^{\mu\nu\lambda}$ based on the above conditions alone. Still, we find the way of gauge-invariant decomposition is not unique and are left with two possibilities.

We start with the decomposition (II) given in the form:

$$M^{\mu\nu\lambda} = M'^{\mu\nu\lambda}_{q\text{-spin}} + M'^{\mu\nu\lambda}_{q\text{-OAM}} + M'^{\mu\nu\lambda}_{G\text{-spin}} + M'^{\mu\nu\lambda}_{G\text{-OAM}} + M'^{\mu\nu\lambda}_{\text{boost}}, \tag{6}$$

with

$$M'^{\mu\nu\lambda}_{q\text{-spin}} = M^{\mu\nu\lambda}_{q\text{-spin}}, \tag{7}$$

$$M'^{\mu\nu\lambda}_{q\text{-OAM}} = \bar{\psi}\gamma^\mu(x^\nu iD^\lambda_{\text{pure}} - x^\lambda iD^\nu_{\text{pure}})\psi, \tag{8}$$

$$M'^{\mu\nu\lambda}_{G\text{-spin}} = M^{\mu\nu\lambda}_{G\text{-spin}}, \tag{9}$$

$$M'^{\mu\nu\lambda}_{G\text{-OAM}} = -2\,\text{Tr}[F^{\mu\alpha}(x^\nu D^\lambda_{\text{pure}} - x^\lambda D^\nu_{\text{pure}})A^{\text{phys}}_\alpha], \tag{10}$$

and

$$M'^{\mu\nu\lambda}_{\text{boost}} = -\tfrac{1}{2}\text{Tr}F^2(x^\nu g^{\mu\lambda} - x^\lambda g^{\mu\nu}). \tag{11}$$

At first sight, this decomposition looks like a covariant generalization of Chen *et al.*'s decomposition, in the sense that the quark OAM part contains *pure gauge-covariant derivative*. However, a crucial difference is that we have not yet fixed the gauge (and the Lorentz frame) explicitly. The point is that, as long as the general conditions (3)–(5) are satisfied, each term of the decomposition (II) is never mixed up under general color gauge transformation of QCD, which means that each term is separately gauge-invariant [24]. These conditions are general enough, so that they are expected to be satisfied by most gauges used in QCD. The fact that the Bashinsky-Jaffe decomposition is contained in our more general decomposition (II) was explicitly verified in [24]. It is also logically plausible that the Chen *et al.* decomposition is contained in our decomposition (II). This is because, although their decomposition is given in a noncovariant manner like the formulation of electrodynamics to be discussed in Sec. II, their decomposition of the gluon field $\boldsymbol{A}$ into $\boldsymbol{A}_{\text{phys}}$ and $\boldsymbol{A}_{\text{pure}}$ naturally satisfies our general conditions (3)–(5). In view of the fact that both frameworks of Bashinsky-Jaffe and of Chen *et al.* are contained in our more general gauge-invariant decomposition, we naturally expect that both give the same answer at least for the momentum sum rule of QCD as well as for the longitudinal spin decomposition of the nucleon, which can be formulated frame-independently.

In a recent paper, Ji, Xu, and Zhao threw doubt on this viewpoint [39]. According to them, the Bashinsky-Jaffe decomposition is one gauge-invariant extension (GIE) of gauge-variant Jaffe-Manohar decomposition based on the light-cone gauge, whereas the Chen *el al.*'s decomposition is another GIE based on the Coulomb gauge. (Concerning the idea of GIE, see also [40].) Their claim is that, since they are different GIEs, there is no reason to expect that they give the same physical predictions. However, it seems to us that their conclusion is heavily influenced by the following observation. That is, the explicit calculations of the evolution matrices for the momentum fractions of quarks and gluons by Chen *et al.* based on the generalized Coulomb gauge are advertised to give totally different answers from the standardly-believed ones, which the treatment in the light-cone gauge can reproduce as we shall see later. However, no one has checked the validity of their Coulomb-gauge calculation yet. What would one conclude, if this discrepancy simply arises from some technical mistakes in the Coulomb-gauge treatment of the problem? We shall come back to this question at the end of Sec. IV.

Next we turn to the decomposition (I) given in the form:

$$M^{\mu\nu\lambda} = M^{\mu\nu\lambda}_{q\text{-spin}} + M^{\mu\nu\lambda}_{q\text{-OAM}} + M^{\mu\nu\lambda}_{G\text{-spin}} + M^{\mu\nu\lambda}_{G\text{-OAM}} + M^{\mu\nu\lambda}_{\text{boost}}, \tag{12}$$

with

$$M^{\mu\nu\lambda}_{q\text{-spin}} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\bar{\psi}\gamma_\sigma\gamma_5\psi, \tag{13}$$

$$M^{\mu\nu\lambda}_{q\text{-OAM}} = \bar{\psi}\gamma^\mu(x^\nu iD^\lambda - x^\lambda iD^\nu)\psi, \tag{14}$$

$$M^{\mu\nu\lambda}_{G\text{-spin}} = 2\,\mathrm{Tr}[F^{\mu\lambda}A^\nu_{\text{phys}} - F^{\mu\nu}A^\lambda_{\text{phys}}], \tag{15}$$

$$\begin{aligned} M^{\mu\nu\lambda}_{G\text{-OAM}} = &-2\,\mathrm{Tr}[F^{\mu\alpha}(x^\nu D^\lambda_{\text{pure}} - x^\lambda D^\nu_{\text{pure}})A^{\text{phys}}_\alpha], \\ &+ 2\,\mathrm{Tr}[(D_\alpha F^{\alpha\mu})(x^\nu A^\lambda_{\text{phys}} - x^\lambda A^\nu_{\text{phys}})], \end{aligned} \tag{16}$$

$$M^{\mu\nu\lambda}_{\text{boost}} = M'^{\mu\nu\lambda}. \tag{17}$$

It differs from the decomposition (II) in the orbital parts. The quark OAM part contains *full covariant derivative* contrary to the decomposition (II). Correspondingly, the gluon OAM part is also different. It contains a covariant generalization of the potential angular momentum term.

It was sometimes criticized that there are so many decompositions of the nucleon spin. As already explained, we do not take this viewpoint. We claim that there are only two physically nonequivalent decompositions. (We shall develop an argument which gives a support to this viewpoint, in the next section, by utilizing a plainer example from electrodynamics.) One is an extension of the Ji decomposition, which also fulfills the decomposition of the gluon total angular momentum into the intrinsic spin and orbital part, while the other is a decomposition that contains in it three known decompositions as *gauge-fixed forms* of more general expression. (We however recall that there a criticism to this idea [39].) The orbital OAMs appearing in these two decompositions are, respectively, the dynamical OAMs and the *generalized canonical* OAMs. Since both decompositions are gauge-invariant, there arises a possibility that they both correspond to observables.

A clear relation with observables was first obtained for the decomposition (I) [24]. The keys are the following identities, which hold in our decomposition (I). For the quark part, it holds that

$$x^\nu T^{\mu\lambda}_q - x^\lambda T^{\mu\nu}_q = M^{\mu\nu\lambda}_{q\text{-spin}} + M^{\mu\nu\lambda}_{q\text{-OAM}} + \text{total divergence}, \tag{18}$$

while for the gluon part we have

$$\begin{aligned} x^\nu T^{\mu\lambda}_g - x^\lambda T^{\mu\nu}_q - \text{boost} = &M^{\mu\nu\lambda}_{g\text{-spin}} + M^{\mu\nu\lambda}_{g\text{-OAM}} \\ &+ \text{total divergence}. \end{aligned} \tag{19}$$

Here, $T^{\mu\nu}_q$ and $T^{\mu\nu}_g$, respectively, stand for the quark and gluon parts of QCD energy-momentum tensor in the Belinfante symmetrized form. By evaluating the nucleon forward matrix element of the above identities, we can prove the following important relations.

First, for the quark part, we get

$$\begin{aligned} L_q &\equiv \langle p\uparrow|M^{012}_{q\text{-OAM}}|p\uparrow\rangle \\ &= \frac{1}{2}\int_{-1}^{1} x[H^q(x,0,0) + E^q(x,0,0)]dx - \frac{1}{2}\int_{-1}^{1}\Delta q(x)dx, \end{aligned} \tag{20}$$

with

$$M^{012}_{q\text{-OAM}} = \bar{\psi}\Big(\boldsymbol{x}\times\frac{1}{i}\boldsymbol{D}\Big)^3\psi \neq \begin{cases} \bar{\psi}\Big(\boldsymbol{x}\times\frac{1}{i}\nabla\Big)^3\psi \\ \bar{\psi}\Big(\boldsymbol{x}\times\frac{1}{i}\boldsymbol{D}_{\text{pure}}\Big)^3\psi. \end{cases} \tag{21}$$

We find that the proton matrix element of our quark OAM operator coincides with the difference between the 2nd moment of generalized parton distribution (GPD) $H + E$ and the 1st moment of the longitudinally polarized distribution of quarks. What should be emphasized here is that full covariant derivative appears, not a simple derivative operator nor pure-gauge covariant derivative. In other words, the quark OAM extracted from the combined analysis of GPD and the polarized parton distribution function (PDF) is dynamical (or mechanical) OAM not canonical OAM. This conclusion is nothing different from Ji's finding [13].

Also for the gluon part, we find that the difference between the 2nd moment of gluon GPD $H + E$ and the 1st moment of polarized gluon distribution coincides with the proton matrix element of our gluon OAM operator given as follows:

$$\begin{aligned} L_g &\equiv \langle p\uparrow|M^{012}_{g\text{-OAM}}|p\uparrow\rangle \\ &= \frac{1}{2}\int_{-1}^{1} x[H^g(x,0,0) + E^g(x,0,0)]dx - \int_0^1 \Delta g(x)dx, \end{aligned} \tag{22}$$

with

$$\begin{aligned} M^{012}_{g\text{-OAM}} = &2\,\mathrm{Tr}[E^j(\boldsymbol{x}\times\boldsymbol{D}_{\text{pure}})^3 A^{\text{phys}}_j]\text{: canonical OAM} \\ &+ 2\,\mathrm{Tr}[\rho(\boldsymbol{x}\times\boldsymbol{A}_{\text{phys}})^3]\text{: potential OAM term.} \end{aligned} \tag{23}$$

Namely, the gluon OAM extracted from the combined analysis of GPD and polarized PDF contains a potential OAM term, in addition to canonical OAM. (Notice that this also clarifies the reason why the sum of the gluon spin and OAM in the Jaffe-Manohar decomposition does not coincide with the total gluon angular momentum in the Ji decompostion.) It would be legitimate to call the whole part the gluon dynamical or mechanical OAM.

Here, we want to make several important remarks on the above sum rules. First, our decomposition has a Lorentz-frame-independent meaning. This should be clear from the fact that the GPDs and PDFs appearing in our sum rules are manifestly Lorentz-invariant quantities. Recently, Goldman argued that the nucleon spin decomposition is frame-dependent [53]. This is generally true. In fact, Leader recently proposed a sum rule for the transverse angular momentum [54]. In this sum rule, P_0, the energy of the nucleon, appears. It is clear that this sum rule is manifestly frame-dependent. Note, however, that our main interest here is the simplest and most fundamental longitudinal spin decomposition of the nucleon. We emphasize once again that the longitudinally spin decomposition is definitely frame-independent. We think it a welcome feature, since, then the decomposition can be thought to reflect intrinsic properties of the nucleon, which are independent of the velocity which the nucleon is running with. Underlying reason why the longitudinal spin sum rule is Lorentz-frame-independent seems very simple. The OAM component along the longitudinal direction comes from the motion in the perpendicular plane to this axis, and such transverse motion is not affected by the Lorentz transformation along this axis.

Although our decomposition looks quite satisfactory in many respects, one subtle question remained. It is a role of quantum-loop effects. Is the longitudinal gluon polarization ΔG gauge-invariant even at quantum level? This is a fairly delicate question. In fact, despite the existence of several formal proof showing the gauge-invariance of ΔG [45–49], it was sometimes claimed that ΔG has its meaning only in the light-cone gauge and infinite-momentum frame [43,55]. More specifically, in an influential paper, Hoodbhoy, Ji, and Lu claim that ΔG evolves differently in the Feynman gauge and the LC gauge [55]. However, the gluon-spin operator used in their Feynman gauge calculation is given by

$$M^{+12}_{g\text{-spin}} = 2\,\mathrm{Tr}[F^{+1}A^2 - F^{+2}A^1], \tag{24}$$

which is not gauge-invariant, and is delicately different from our gauge-invariant gluon-spin operator given as

$$M^{+12}_{g\text{-spin}} = 2\,\mathrm{Tr}[F^{+1}A^2_{\mathrm{phys}} - F^{+2}A^1_{\mathrm{phys}}], \tag{25}$$

The problem is how to incorporate this difference into the Feynman rule for evaluating 1-loop anomalous dimension of the quark and gluon-spin operators. This problem was attacked and solved in the paper [25]. We find that the calculation in the Feynman gauge (as well as in any covariant gauge including the Landau gauge) reproduces the answer obtained in the LC gauge, which is also the answer obtained in the famous Altarelli-Parisi method [56]. (This conclusion for the evolution of ΔG however contradicts the one given in [34].) Our finding is important also from another context. So far, a direct check of the answer of Altarelli-Pasiri method for the evolution of ΔG within the operator-product-expansion (OPE) framework was limited to the LC gauge, because it was believed that there is no gauge-invariant definition of gluon spin in the OPE framework. This was the reason why the question of gauge-invariance of ΔG has been left in unclear status for a long time.

After establishing satisfactory natures of the decomposition (I), now we turn our attention to another decomposition (II) According to Chen *et al.*, the greatest advantage of the decomposition (II is that their quark OAM operator $\boldsymbol{L}'_q$ satisfies the standard commutation relation of angular momentum:

$$\boldsymbol{L}'_q \times \boldsymbol{L}'_q = i\boldsymbol{L}'_q, \tag{26}$$

due to the property $\nabla \times \boldsymbol{A}_{\mathrm{pure}} = 0$. This property was claimed to be essential for its physical interpretation as an OAM. However, this is not necessarily true as is clear from the papers [57,58], which treats a similar problem in QED. It was shown there that the spin and OAM operators of the photons do not satisfy the ordinary commutation relation of angular momentum ($SU(2)$ algebra) separately. This is not surprising at all. In fact, it is true that the total momentum as well as the total angular momentum operators of a composite system must satisfy the Poincare algebra, because, in quantum field theory, a physical state of a composite particle must be one of the irreducible representations of the Poincare group. However, it is not an absolute demand of the Poincare symmetry that the momentum and the angular momentum of each constitute of a composite particle satisfies the Poincare algebra separately. Then, the claimed superiority of the decomposition (II) over (I) is not actually present. Nevertheless, since the decomposition (II) is also gauge-invariant, there still remains a possibility that it can be related to observables.

Recently, Hatta made an important step toward this direction [29] based on his decomposition formula of the physical- and pure-gauge components of gluon fields proposed by himself in [28]. Starting from the gauge-invariant expression of the Wigner distribution also called the generalized transverse-momentum-dependent distributions (GTMDs), which depends not only the longitudinal and transverse momenta but also the momentum transfer of the target nucleon, he showed that the nucleon matrix element of the generalized canonical OAM can be related to a weighted integral of a certain GTMD. It is important to recognize that this quantity does not appear in the standard classification of transverse-momentum-dependent distributions (TMDs) by the following reason. To explain it, we first recall the definition of the most fundamental GTMD appearing in the classification given in [59]:

$$\begin{aligned} & W^{[\gamma^+]}(x, \xi, \boldsymbol{q}_T^2, \boldsymbol{q}_T \cdot \boldsymbol{\Delta}_T, \boldsymbol{\Delta}_T^2; \eta) \\ &= \frac{1}{2} \int \frac{dz^- d^2 z_T}{(2\pi)^2} e^{k \cdot z} \langle p', \lambda' | \bar{\psi}\Big(-\frac{z}{2}\Big) \gamma^+ \mathcal{W}\Big(-\frac{z}{2}, \frac{z}{2} \Big| n\Big) \psi\Big(\frac{z}{2}\Big) | p, \lambda \rangle_{z^+=0} \\ &= \frac{1}{2M} \bar{u}(p', \lambda') \Big[F_{1,1} + \frac{i\sigma^{i+} q_T^i}{P^+} F_{1,2} + \frac{i\sigma^{i+} \Delta_T^i}{P^+} F_{1,3} + \frac{i\sigma^{ij} q_T^i \Delta_T^j}{M^2} F_{1,4} \Big] u(p, \lambda). \end{aligned} \tag{27}$$

The GTMD defined by the 2nd line of the above equation generally contains 4 pieces of invariant functions $F_{1,i}(x, \xi, \boldsymbol{q}_T^2, \boldsymbol{\Delta}_T^2, \eta)$ with $i = 1, \ldots, 4$, which are functions of the Bjorken variable x, the skewedness parameter ξ, the transverse-momentum square $\boldsymbol{q}_T^2$, the transverse-momentum-transfer square $\boldsymbol{\Delta}_T^2$, and the parameter η characterizing the nature of the functions under time-reversal. In the forward limit, the first and the second pieces, respectively, reduce to the usual spin-independent TMD and the naively time-reversal odd Sivers function. On the other hand, the last two terms disappear in the forward limit, $\boldsymbol{\Delta}_\perp \to 0$. Nonetheless, within the framework of a quark model, which does not pay much attention to the gauge-invariance issue, Lorce and Pasquini showed [60] that a weighted integral of this 4th function is related to the nucleon matrix element of the canonical OAM given as

$$L_{\mathrm{can}} = -\int dx d^2 q_T \frac{\boldsymbol{q}_T^2}{M^2} F_{1,4}^q(x, 0, \boldsymbol{q}_T^2, 0, 0). \tag{28}$$

This is just the sum rule, to which Hatta gave a gauge-invariant meaning, i.e. the meaning within the framework of QCD as a color gauge theory. In this sense, Hatta's work opened up a possibility that the OAM appearing in the decomposition (II) may also be related to observables. Since the relation between the OAM appearing in the decomposition (I) and the observables is already known, this means that we may be able to isolate the correspondent of potential angular momentum term appearing in Feynman's paradox as a difference between the two OAMs as

$$L_{\mathrm{pot}} = L_{\mathrm{mech}} - L_{\text{“can”}}. \tag{29}$$

However, one must be careful about the presence of very delicate problem on the sum rules containing GTMDs and/or ordinary TMDs. (See, for example, the textbook [61], which discusses the delicacies of TMDs in full detail.) Once quantum-loop effects are taken into account, the very existence of TMDs satisfying gauge-invariance and factorization (universality or process independence) simultaneously is being questioned. Is process-independent extraction of $L_{\text{“can”}}$ really possible? One must say that it is still a challenging open question.

III. WHAT IS "POTENTIAL ANGULAR MOMENTUM"?

We have shown that the key quantity, which distinguishes the two physically different nucleon spin decompositions, is what we call the potential angular momentum term. To understand its physical meaning more thoroughly, and also to understand the reason why there exist two physically different decompositions with gauge-invariance, we find it very instructive to study easier QED case, especially an interacting system of charged particles and photons [62–64]. The total Hamiltonian of such system is given by

$$H = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{2} \int d^3 r [\boldsymbol{E}^2 + \boldsymbol{B}^2]. \tag{30}$$

Here the 1st and the 2nd terms of the right-hand side (rhs), respectively, stand for the mechanical kinetic energy of the charged particles and the total energy of the electromagnetic fields. As is well-known, the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ of the photon can be decomposed into longitudinal and transverse components as

$$\boldsymbol{A} = \boldsymbol{A}_\parallel + \boldsymbol{A}_\perp, \tag{31}$$

with the properties

$$\nabla \times \boldsymbol{A}_\parallel = 0, \qquad \nabla \cdot \boldsymbol{A}_\perp = 0. \tag{32}$$

We emphasize that this longitudinal-transverse decomposition is *unique*, once the Lorentz frame of reference is fixed. Under a general gauge transformation given as

$$A^0 \to A'^0 = A^0 - \frac{\partial}{\partial t} \Lambda(x), \tag{33}$$

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda(x), \tag{34}$$

the longitudinal component $\boldsymbol{A}_\parallel$ transforms as

$$\boldsymbol{A}_\parallel \to \boldsymbol{A}'_\parallel = \boldsymbol{A}_\parallel + \nabla \Lambda(x), \tag{35}$$

while the transverse component $\boldsymbol{A}_\perp$ is invariant, i.e.

$$\boldsymbol{A}_\perp \to \boldsymbol{A}'_\perp = \boldsymbol{A}_\perp, \tag{36}$$

indicating that $\boldsymbol{A}_\parallel$ carries unphysical gauge degrees of freedom.

To avoid a misunderstanding, we think it is important to clarify the fact that the decomposition (31) itself has nothing to do with Coulomb-gauge fixing. The Coulomb-gauge condition is to require that

$$\nabla \cdot \boldsymbol{A} = 0. \tag{37}$$

Since $\nabla \cdot \boldsymbol{A}_\perp = 0$ by definition, this is equivalent to requiring that

$$\nabla \cdot \boldsymbol{A}_{\parallel} = 0. \quad (38)$$

This is the Coulomb-gauge fixing condition, which works to eliminate unphysical gauge degrees of freedom $\boldsymbol{A}_{\parallel}$. In fact, once this condition is imposed, $\boldsymbol{A}_{\parallel}$ is divergence-free as well as irrotational, so that we can take

$$\boldsymbol{A}_{\parallel} = 0 \quad (39)$$

without loss of generality.

Naturally, the separation of the vector potential $\boldsymbol{A}$ of the photon into the transverse (physical) and longitudinal (pure-gauge) components is frame-dependent. However, it is also true that we can start this decomposition in an arbitrary Lorentz frame. The Coulomb-gauge condition $\nabla \cdot \boldsymbol{A} = 0$ is definitely Lorentz noncovariant, different from the so-called Lorenz gauge condition $\partial_\mu A^\mu = 0$. It is known that the 4-vector potential A^μ in the Lorenz gauge satisfies the Lorenz gauge condition $\partial_\mu A^\mu = 0$ even after Lorentz transformation to another frame. On the other hand, when the 3-vector potential $\boldsymbol{A}$ in a certain Lorentz frame is prepared to satisfy the Coulomb-gauge condition $\nabla \cdot \boldsymbol{A} = 0$, the Lorentz-transformed vector potential $\boldsymbol{A}'$ does not satisfy $\nabla' \cdot \boldsymbol{A}' = 0$. Here, we need a further gauge transformation in order to get $\boldsymbol{A}'$ satisfying $\nabla' \cdot \boldsymbol{A}' = 0$. However, this does not make any trouble because we could start the whole consideration in the transformed frame and could impose the condition $\nabla \cdot \boldsymbol{A} = 0$ in that frame. (An equivalent but formally more convenient framework for showing the covariance of the Coulomb-gauge treatment would be to require somewhat nonstandard Lorentz transformation property for the 4-vector potential of the gauge field as described in the textbook of Bjorken and Drell [65] as well as in the recent paper [42].) The fact is that observables (which must of course be gauge-invariant) are independent of the choice of gauge. Both of the Lorentz gauge and the Coulomb gauge give exactly the same answer for physical observables. The is just the core of Maxwell's electrodynamics as a Lorentz-invariant gauge theory.

To return to our main discussion, in parallel with the above decomposition of the vector potential $\boldsymbol{A}$, the electric field can also be decomposed into longitudinal and transverse components as

$$\boldsymbol{E} = \boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}, \quad (40)$$

with

$$\boldsymbol{E}_{\parallel} = -\nabla A^0 - \frac{\partial \boldsymbol{A}_{\parallel}}{\partial t}, \quad (41)$$

$$\boldsymbol{E}_{\perp} = -\frac{\partial \boldsymbol{A}_{\perp}}{\partial t}, \quad (42)$$

while the magnetic field is intrinsically transverse

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{A}_{\perp} = \boldsymbol{B}_{\perp}. \quad (43)$$

As a consequence, the photon part of the total energy can be decomposed into two pieces, i.e. the longitudinal part and the transverse part, as

$$H = \sum_{i-1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{2} \int d^3 r \boldsymbol{E}_{\parallel}^2 + \frac{1}{2} \int d^3 r [\boldsymbol{E}_{\perp}^2 + \boldsymbol{B}_{\perp}^2]. \quad (44)$$

Now, by using the Gauss law $\nabla \cdot \boldsymbol{E}_{\parallel} = \rho$, it can be shown that the longitudinal part is nothing but the Coulomb energy between the charged particles (aside from the self-energies), so that we can write as

$$H = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + V_{\mathrm{coul}} + H_{\mathrm{trans}}, \quad (45)$$

with

$$V_{\mathrm{Coul}} = \frac{1}{4\pi} \sum_{i,j=1 (i \neq j)}^{N} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}, \quad (46)$$

$$H_{\mathrm{trans}} = \frac{1}{2} \int d^3 r [\boldsymbol{E}_{\perp}^2 + \boldsymbol{B}_{\perp}^2]. \quad (47)$$

Next we consider a similar decomposition of the total momentum. The total momentum of the system is a sum of the mechanical momentum of charged particles and the momentum of photon fields as

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i + \int d^3 r \boldsymbol{E} \times \boldsymbol{B}. \quad (48)$$

The total momentum of the electromagnetic fields can be decomposed into longitudinal and transverse parts as

$$\int d^3 r \boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{P}_{\mathrm{long}} + \boldsymbol{P}_{\mathrm{trans}}, \quad (49)$$

with

$$\boldsymbol{P}_{\mathrm{long}} = \int d^3 r \boldsymbol{E}_{\parallel} \times \boldsymbol{B}_{\perp}, \quad (50)$$

$$\boldsymbol{P}_{\mathrm{trans}} = \int d^3 r \boldsymbol{E}_{\perp} \times \boldsymbol{B}_{\perp}, \quad (51)$$

which gives the decomposition

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i + \boldsymbol{P}_{\mathrm{long}} + \boldsymbol{P}_{\mathrm{trans}}. \quad (52)$$

Again, by using the Gauss law, it can be shown that $\boldsymbol{P}_{\mathrm{long}}$ is also expressed as

$$\boldsymbol{P}_{\mathrm{long}} = \sum_i q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i), \quad (53)$$

so that we can write

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i + \sum_i q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i) + \boldsymbol{P}_{\text{trans}}. \tag{54}$$

We point out that the quantity $q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i)$ appearing in this decomposition is nothing but the *potential momentum* according to the terminology of Konopinski [66]. In the present context, it represents the momentum that associates with the longitudinal (electric) field generated by the particle i. Which of particles or photons should it be attributed to? This is a fairly delicate question. It is of the same sort of question as which of the charged particles or photons should the Coulomb energy be attributed to. To attribute it to a charged particle is closer to the concept of "action at a distance theory," while to attribute it to an electromagnetic field is closer to the concept of "action through medium." If there is no difference between their physical predictions, the choice is a matter of convenience. Let us see what happens if we combine the potential momentum term with the mechanical energy of charged particles. To this end, we recall that, under the presence of electromagnetic potential, the canonical momentum $\boldsymbol{p}_i$ of the charged particle i is given by the equation

$$\boldsymbol{p}_i \equiv \frac{\partial L}{\partial \dot{\boldsymbol{r}}_i} = m_i \dot{\boldsymbol{r}}_i + q_i \boldsymbol{A}(\boldsymbol{r}_i), \tag{55}$$

where L is the Lagrangian corresponding to the Hamiltonian (30). Using it, the total momentum $\boldsymbol{P}$ can be expressed in the following form:

$$\boldsymbol{P} = \sum_i (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)) + \boldsymbol{P}_{\text{trans}}, \tag{56}$$

where use has been made of the relation $\boldsymbol{A}(\boldsymbol{r}_i) - \boldsymbol{A}_\perp(\boldsymbol{r}_i) = \boldsymbol{A}_\parallel(\boldsymbol{r}_i)$. The discussion so far is totally independent of the choice of gauge. To make the following discussion as transparent as possible, we shall work for a while in a particular gauge, i.e. the Coulomb gauge, and will come back to more general case later. As was already explained, in the Coulomb gauge, we can set $\boldsymbol{A}_\parallel = 0$ without loss of generality. The above expression for the total momentum $\boldsymbol{P}$ then reduces to a very simple form given as

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i + \boldsymbol{P}_{\text{trans}}. \tag{57}$$

One observes that the total momentum of the charged particles and the photons is given as a sum of the canonical momenta of charged particles and the transverse momentum of the electromagnetic fields.

Next, let us consider a similar decomposition of the total angular momentum. The total angular momentum of the system is a sum of the mechanical angular momentum of charged particles and the angular momentum of photon fields as

$$\boldsymbol{J} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \int d^3 r \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}). \tag{58}$$

Similarly as before, the total angular momentum of the electromagnetic fields can be decomposed into longitudinal and transverse parts as

$$\int d^3 r \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{J}_{\text{long}} + \boldsymbol{J}_{\text{trans}}, \tag{59}$$

with

$$\boldsymbol{J}_{\text{long}} = \int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\parallel \times \boldsymbol{B}_\perp), \tag{60}$$

$$\boldsymbol{J}_{\text{trans}} = \int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\perp \times \boldsymbol{B}_\perp), \tag{61}$$

which leads to the relation

$$\boldsymbol{J} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \boldsymbol{J}_{\text{long}} + \boldsymbol{J}_{\text{trans}}. \tag{62}$$

Again, by using the Gauss law, $\boldsymbol{J}_{\text{long}}$ can also be expressed as

$$\boldsymbol{J}_{\text{long}} = \sum_i q_i \boldsymbol{r}_i \times \boldsymbol{A}_\perp(\boldsymbol{r}_i), \tag{63}$$

so that we can write as

$$\boldsymbol{J} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \sum_i q_i \boldsymbol{r}_i \times \boldsymbol{A}_\perp(\boldsymbol{r}_i) + \boldsymbol{J}_{\text{trans}}. \tag{64}$$

We recall that the quantity $q_i \boldsymbol{r}_i \times \boldsymbol{A}_\perp(\boldsymbol{r}_i)$ appearing in the above decomposition just corresponds to what we call the potential angular momentum [23]. In the present context, it represents the angular momentum that associates with the longitudinal (electric) field generated by the charged particle i. Again, if one combines it with the mechanical angular momentum of the charged particle i, the total angular momentum $\boldsymbol{P}$ of the system is represented as

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)) + \boldsymbol{J}_{\text{trans}}, \tag{65}$$

in general gauges, and as

$$\boldsymbol{J} = \sum \boldsymbol{r}_i \times \boldsymbol{p}_i + \boldsymbol{J}_{\text{trans}}, \tag{66}$$

in the Coulomb gauge.

Summarizing the above manipulations, we find (in the Coulomb gauge) the following much simpler-looking expressions for the total momentum and the total angular momentum of the interacting system of charged particles and the photons:

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i + \boldsymbol{P}_{\text{trans}}, \tag{67}$$

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i + \boldsymbol{J}_{\text{trans}}. \tag{68}$$

At first sight, it appears to indicate physical superiority of the canonical momentum and the canonical angular momentum over the mechanical ones. However, such a conclusion is premature, as is clear from the following consideration of the energy of the system. As already pointed out, the total Hamiltonian of the system is given as a sum of three terms, i.e. the mechanical energies of the charged particles, the Coulomb energies between them, and the energy of transverse photons. An important observation here is that, different from the cases of total momentum and angular momentum, when the sum of the mechanical energy and the Coulomb energy (the energy associate with $\boldsymbol{E}_{\parallel}$) is expressed in terms of the canonical momentum, it does not reduce to a simple form, because

$$\frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + V_{\mathrm{Coul}} \neq \sum_i \frac{\boldsymbol{p}_i^2}{2m_i}. \tag{69}$$

Instead, we have

$$\begin{aligned} H &= \sum_i \frac{1}{2m_i} (\boldsymbol{p}_i - q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i))^2 + V_{\mathrm{Coul}} \\ &= \sum_i \frac{\boldsymbol{p}_i^2}{2m_i} + V_{\mathrm{Coul}} \\ &\quad + \sum_i \frac{q_i}{2m_i} [\boldsymbol{p}_i \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) \cdot \boldsymbol{p}_i] + \sum_i \frac{q_i^2}{2m_i} \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}_i). \end{aligned} \tag{70}$$

Crucially important to recognize here is the difference of the two quantities,

$$\sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i{}^2, \tag{71}$$

and

$$\sum_i \frac{\boldsymbol{p}_i^2}{2m_i}. \tag{72}$$

As already mentioned, the former quantity represents the mechanical kinetic energy of charged particles, i.e. the kinetic energy of particles which associate with their translational motion. Usually, the latter quantity is also interpreted as the kinetic energy of charged particles, which means that we are not distinguishing these two quantities very clearly. The reason is that we are too much accustomed with weakly coupled systems of charged particles and photons. To understand it, let us consider the problem of a hydrogen atom. Assuming, for simplicity, that the proton is infinitely heavy, it reduces to a problem of one electron and photons, described by the following Hamiltonian:

$$H = \tfrac{1}{2} m \dot{\boldsymbol{r}}^2 + V_{\mathrm{Coul}} + H_{\mathrm{trans}} = H_0 + H_{\mathrm{trans}} + H_{\mathrm{int}}, \tag{73}$$

with

$$H_0 = \frac{\boldsymbol{p}^2}{2m} + V_{\mathrm{Coul}}(r), \tag{74}$$

$$H_{\mathrm{trans}} = \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \hbar \omega_{\boldsymbol{k}} a^{\dagger}_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda}, \tag{75}$$

$$H_{\mathrm{int}} = \frac{e}{2m} [\boldsymbol{p} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}) + \boldsymbol{A}_{\perp} \cdot \boldsymbol{p}] + \frac{e^2}{2m} \boldsymbol{A}_{\perp}(\boldsymbol{r}) \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}). \tag{76}$$

Here, H_0 is taken as an unperturbed Hamiltonian of the Hydrogen atom with the Coulomb interaction between the electron and the proton, H_{trans} is the Hamiltonian of the transverse photons, and H_{int} describes the interactions between the electron and the transverse photons. A general form of eigenstates of the above Hamiltonian is expressed as a direct product of the eigenstates of H_0 and those of H_{trans} as $|\psi_n\rangle \otimes |\{n_{\boldsymbol{k},\lambda}\}\rangle$, where

$$H_0 |\psi_n\rangle = E_n |\psi_n\rangle, \tag{77}$$

while $|\{n_{\boldsymbol{k},\lambda}\}\rangle$ is an abbreviation of the following occupation-number representation of transverse photons:

$$|\{n_{\boldsymbol{k},\lambda}\}\rangle = \prod_{\alpha} |n_{\boldsymbol{k}_\alpha,\lambda_\alpha}\rangle. \tag{78}$$

It is important to recognize that, in the ordinary description of hydrogen atom, one does not include Fock components of transverse (real) photons. (The formation of the hydrogen atom is entirely due to the Coulomb attraction between the proton and the electron, and the transverse photons have little to do with it.) Consequently, both of the total momentum or the total angular momentum of the hydrogen atom are saturated by the electron alone, and the photons carry none of them. This also means that there is no practical difference between the mechanical momentum

$$\boldsymbol{P}_{\mathrm{mech}} = m\dot{\boldsymbol{r}} = \boldsymbol{p} - e\boldsymbol{A}_{\perp}, \tag{79}$$

and the canonical momentum

$$\boldsymbol{P}_{\mathrm{can}} = \boldsymbol{p}, \tag{80}$$

since the expectation value of $\boldsymbol{A}_{\perp}$ in such restricted Fock space is vanishing. Exactly the same can be said for the difference between the mechanical angular momentum

$$\boldsymbol{J}_{\text{mech}} = m\boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{A}_\perp), \tag{81}$$

and the canonical angular momentum

$$\boldsymbol{J}_{\text{can}} = \boldsymbol{r} \times \boldsymbol{p}. \tag{82}$$

The difference between the mechanical kinetic energy

$$\frac{1}{2} m\dot{\boldsymbol{r}}^2 = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}_\perp)^2 \tag{83}$$

and the kinetic energy

$$\frac{1}{2m}\boldsymbol{p}^2 \tag{84}$$

is also ineffective in the static properties of the hydrogen atom. The fact is that the difference between these two quantities is nothing but the interaction Hamiltonian, which is treated perturbatively, thereby describing the processes of emissions and absorptions as well as the scatterings of transverse photons by the hydrogen atom.

One must recognize that the situation is totally different in QCD. Here, the nucleon is a strongly-coupled gauge system of quarks and gluons. One certainly needs to include Fock components of transverse gluons. Otherwise, the concept like the gluon distributions in the nucleon would never be invoked. In such circumstances, the difference between the mechanical angular momentum and the canonical angular momentum as well as the difference between the mechanical momentum and the canonical momentum are generally nonzero and may have sizable magnitude.

So far, we were mainly working in the Coulomb gauge, in order to avoid unnecessary complexities for the above physical consideration. Now we consider the problem of gauge-invariance more seriously. As we shall see below, it provides us with a new and interesting insight into the decomposition problem of the total momentum as well as the total angular momentum of the interacting system of charged particles and photons. (Since the argument goes in entirely the same manner for both of the total momentum and the total angular momentum, we concentrate below on more interesting angular momentum case.) We have already shown that the total angular momentum $\boldsymbol{J}$ can be decomposed into the following form in an arbitrary gauge:

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)) + \boldsymbol{J}_{\text{trans}}. \tag{85}$$

It is a well-known fact that the transverse part $\boldsymbol{J}_{\text{trans}}$ of photons can further be decomposed into the orbital and spin parts as

$$\boldsymbol{J}_{\text{trans}} = \int d^3 r E^l_\perp (\boldsymbol{r} \times \nabla) A^l_\perp + \int d^3 r \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp. \tag{86}$$

We emphasize that this decomposition is gauge-invariant, because $\boldsymbol{A}_\perp$ is gauge-invariant. (Naturally, $\boldsymbol{E}_\perp$ is gauge-invariant.) Then, we are led to a decomposition as follows:

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)) + \int d^3 r E^l_\perp (\boldsymbol{r} \times \nabla) A^l_\perp + \int d^3 r \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp. \tag{87}$$

When going to quantum theory (in the coordinate representation), the canonical momentum is replaced by a differential operator as

$$\boldsymbol{p}_i - q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i) \Rightarrow \frac{1}{i}(\nabla_i - i q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)). \tag{88}$$

Notice that, with the identification $\boldsymbol{A}_\parallel = \boldsymbol{A}_{\text{pure}}$, the rhs is basically the pure-gauge derivative

$$\boldsymbol{D}_{i,\text{pure}} = \nabla_i - i q_i \boldsymbol{A}_{\text{pure}}(\boldsymbol{r}_i), \tag{89}$$

introduced by Chen *et al.* [18,19]. Using it, Eq. (87) can now be written as

$$\boldsymbol{J} = \boldsymbol{L}'_p + \boldsymbol{L}'_\gamma + \boldsymbol{S}'_\gamma, \tag{90}$$

with

$$\boldsymbol{L}'_p = \sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_{i,\text{pure}}, \tag{91}$$

$$\boldsymbol{L}'_\gamma = \int d^3 r E^l_\perp (\boldsymbol{r} \times \nabla) A^l_\perp, \tag{92}$$

$$\boldsymbol{S}'_\gamma = \int d^3 r \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp. \tag{93}$$

One may recognize now that this just corresponds to a gauge-invariant decomposition of Chen *et al.* in the case of QED except that we are handling here the charged particles without intrinsic spin [18,19]. In fact, the gauge-invariance of the 1st term can readily be verified, by using the gauge transformation property of the longitudinal component of $\boldsymbol{A}_\parallel$

$$\boldsymbol{A}_\parallel(\boldsymbol{r}_i) \to \boldsymbol{A}_\parallel(\boldsymbol{r}_i) + \nabla\Lambda(\boldsymbol{r}_i), \tag{94}$$

and the gauge transformation property of the quantum-mechanical wave function of the charged particles given as

$$\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \to \left(\prod_i^N e^{i q_i \Lambda(\boldsymbol{r}_i)}\right) \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N). \tag{95}$$

As is obvious from our previous studies [23,24], however, the above decomposition (90) is not a unique possibility of gauge-invariant decomposition of the total angular momentum. To confirm it, we go back to Eq. (64), which we now write as

$$\boldsymbol{J} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\parallel \times \boldsymbol{B}_\perp) + \int d^3 r E^l_\perp (\boldsymbol{r} \times \nabla) A^l_\perp + \int d^3 r \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp \tag{96}$$

Combining the piece $\int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\parallel \times \boldsymbol{B}_\perp)$, which was previously written as $\sum_i q_i \boldsymbol{r}_i \times \boldsymbol{A}_\perp(\boldsymbol{r}_i)$, with the orbital part of $\boldsymbol{J}_{\rm trans}$, we are led to another decomposition:

$$\boldsymbol{J} = \boldsymbol{L}_p + \boldsymbol{L}_\gamma + \boldsymbol{S}_\gamma, \tag{97}$$

where

$$\boldsymbol{L}_p = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i, \tag{98}$$

$$\boldsymbol{L}_\gamma = \int d^3 r E_\perp^k (\boldsymbol{r} \times \nabla) A_\perp^k + \int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\parallel \times \boldsymbol{B}_\perp), \tag{99}$$

$$\boldsymbol{S}_\gamma = \int d^3 r \boldsymbol{E}_\perp \times \boldsymbol{B}_\perp. \tag{100}$$

Note that, using the relationship $\boldsymbol{p}_i = m_i \dot{\boldsymbol{r}}_i + q_i \boldsymbol{A}(\boldsymbol{r}_i)$, $\boldsymbol{L}_p$ can also be written as

$$\boldsymbol{L}_p = \sum_i \boldsymbol{r}_i \times (\boldsymbol{p}_i - q_i \boldsymbol{A}(\boldsymbol{r}_i)) \tag{101}$$

$$\rightarrow \sum_i \boldsymbol{r}_i \times \frac{1}{i}(\nabla_i - i q_i \boldsymbol{A}(\boldsymbol{r}_i)) \equiv \sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_i. \tag{102}$$

Obviously, this decomposition is also gauge-invariant. This gauge-invariant decomposition falls into the category of decomposition (I), while the previous decomposition into that of decomposition (II) according to the classification in [24]. As is clear by now, the difference between the two decompositions arises from the treatment of the potential angular momentum term $\sum_i q_i \boldsymbol{r}_i \times \boldsymbol{A}_\perp(\boldsymbol{r}_i)$, which is solely gauge-invariant. In the decomposition (I), it is included in the orbital angular momentum part of photons, while in the decomposition (II), it is included in the orbital angular momentum part of charged particles. As a consequence, what appears in the decomposition (I) is the mechanical (or dynamical) angular momentum given as

$$\boldsymbol{L}_p^{\rm mech} \equiv \boldsymbol{L}_p = \sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_i \equiv \sum_i \boldsymbol{r}_i \times \frac{1}{i}(\nabla_i - i q_i \boldsymbol{A}(\boldsymbol{r}_i)), \tag{103}$$

containing full gauge-covariant derivative, while what appears in the decomposition (II) is a generalized canonical angular momentum (with gauge-invariance) given by

$$\boldsymbol{L}_p^{\text{“can”}} \equiv \boldsymbol{L}'_p = \sum_i \boldsymbol{r}_i \times \frac{1}{i} \boldsymbol{D}_{i,\rm pure}$$
$$\equiv \sum_i \boldsymbol{r}_i \times \frac{1}{i}(\nabla_i - i q_i \boldsymbol{A}_\parallel(\boldsymbol{r}_i)), \tag{104}$$

which reduces to the ordinary canonical momentum in the Coulomb gauge, in which $\boldsymbol{A}_\parallel(\boldsymbol{r}_i) = 0$. All of these are anticipated facts from the analysis in our previous papers [23,24]. Here, we can say more. It is a widespread belief that, among the two quantities, i.e. the canonical angular momentum and the dynamical (or mechanical) angular momentum, what is closer to simple physical image of orbital motion is the former because it appears that the latter contains an *extra interaction term* between the charged particles and the photons. (This prepossession is further amplified by a simpler commutation relation of $\boldsymbol{L}'_p$, which is not possessed by $\boldsymbol{L}_p$ [32,53].) We now realize that the truth is just opposite. In fact, we have shown that the canonical angular momentum is a sum of the mechanical angular momentum and the longitudinal part of the photon angular momentum as

$$\boldsymbol{L}'_p = \boldsymbol{L}_p + \sum_i \boldsymbol{r}_i \times q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i) \tag{105}$$

$$= \boldsymbol{L}_p + \int d^3 r \boldsymbol{r} \times (\boldsymbol{E}_\parallel \times \boldsymbol{B}_\perp), \tag{106}$$

where

$$\boldsymbol{L}_p = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i = \sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i. \tag{107}$$

As is clear from the above expression (107) of $\boldsymbol{L}_p$, it is the mechanical angular momentum $\boldsymbol{L}_p$ not the canonical angular momentum $\boldsymbol{L}'_p$ that has a natural physical interpretation as orbital motion of particles. It may really sound paradoxical, but what contains an extra interaction term is rather the canonical angular momentum not the mechanical angular momentum !

IV. RELATION BETWEEN THE TWO INEQUIVALENT DECOMPOSITIONS OF THE NUCLEON MOMENTUM

In one of the two papers [18,19], which brought about a big argument on the nucleon spin decomposition problem, Chen *et al.* suspect a common wisdom of DIS physics that the gluons carry about half of the nucleon momentum in the asymptotic limit. According to them, this large fraction is due to an unsuitable definition of the gluon momentum in an interacting theory. It was claimed that, if the quark and gluon momenta are defined in a gauge-invariant and consistent way, the asymptotic limit of the gluon momentum fraction would be only about one-fifth as compared with the standardly believed value of one-half. We shall inspect below the validity of this astounding conclusion.

Their argument starts with the statement that the conventional gluon momentum fraction is based on the following decomposition of the total momentum operator in QCD:

$$\boldsymbol{P}_{\rm total} = \int d^3 x \psi^\dagger \frac{1}{i} \boldsymbol{D} \psi + \int d^3 x \boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{P}_q + \boldsymbol{P}_G, \tag{108}$$

where $\boldsymbol{D} = \nabla - ig\boldsymbol{A}$ is the standard covariant derivative. The scale evolution of $\boldsymbol{P}_q$ and $\boldsymbol{P}_G$ is governed by the following anomalous dimension matrix at the leading order [67,68]:

$$\gamma^P \equiv \begin{pmatrix} \gamma_{qq}^{(2)} & \gamma_{qG}^{(2)} \\ \gamma_{Gq}^{(2)} & \gamma_{GG}^{(2)} \end{pmatrix} = \frac{\alpha_S}{8\pi} \begin{pmatrix} -\frac{8}{9} n_g & \frac{4}{3} n_f \\ \frac{8}{9} n_g & -\frac{4}{3} n_f \end{pmatrix}, \quad (109)$$

with n_g and n_f being the number of gluon fields and the number of active quark flavors. This leads to the well-known asymptotic limit for the gluon momentum fraction,

$$\boldsymbol{P}_G = \frac{2n_g}{2n_g + 3n_f} \boldsymbol{P}_{\text{total}}. \quad (110)$$

Their objection to this common knowledge is based on another gauge-invariant decomposition proposed by themselves:

$$\boldsymbol{P}_{\text{total}} = \boldsymbol{P}'_q + \boldsymbol{P}'_G, \quad (111)$$

where

$$\boldsymbol{P}'_q = \int d^3x \psi^\dagger \frac{1}{i} \boldsymbol{D}_{\text{pure}} \psi, \quad (112)$$

$$\boldsymbol{P}'_G = \int d^3x E^k \mathcal{D}_{\text{pure}} A^k_{\text{phys}}, \quad (113)$$

with

$$D^\mu_{\text{pure}} \equiv \partial^\mu - igA^\mu_{\text{pure}}, \quad (114)$$

$$\mathcal{D}^\mu_{\text{pure}} \equiv \partial^\mu - ig[A^\mu_{\text{pure}}, \cdot]. \quad (115)$$

Although the detail of the calculation was not shown, they concluded that this decomposition leads to the following anomalous dimension matrix [19]:

$$\gamma^{P'} = \frac{\alpha_S}{8\pi} \begin{pmatrix} -\frac{2}{9} n_g & \frac{4}{3} n_f \\ \frac{2}{9} n_g & -\frac{4}{3} n_f \end{pmatrix}, \quad (116)$$

thereby predicting a totally different asymptotic limit for the gluon momentum fraction,

$$\boldsymbol{P}_G = \frac{n_g}{n_g + 6n_f} \boldsymbol{P}_{\text{total}}. \quad (117)$$

For the typical case of $n_f = 5$, this gives $\boldsymbol{P}'_G \simeq (1/5)\boldsymbol{P}_{\text{total}}$, as compared with the prediction of the standard scenario $\boldsymbol{P}_G \simeq (1/2)\boldsymbol{P}_{\text{total}}$.

Apparently, to discuss the momentum sum rule of QCD and its evolution, it is more convenient to handle the problem in a covariant way. Along the same line as explained in our previous papers [23,24], which established the fact that there exist two physically inequivalent decompositions of the QCD angular momentum tensor, we can show that there are two different decompositions of the QCD energy-momentum tensor, both of which are gauge-invariant. The decomposition (I) contains in it the standard decomposition given in the paper [12]:

$$T^{\mu\nu} = T^{\mu\nu}_q + T^{\mu\nu}_G, \quad (118)$$

with

$$T^{\mu\nu}_q = \tfrac{1}{2}\bar{\psi}(\gamma^\mu iD^\nu + \gamma^\nu iD^\mu)\psi, \quad (119)$$

$$T^{\mu\nu}_G = -\mathrm{Tr}[F^{\mu\alpha}D^\nu A_\alpha + F^{\nu\alpha}D^\mu A_\alpha] + \tfrac{1}{2}g^{\mu\nu}\,\mathrm{Tr}[F^2]. \quad (120)$$

Since the second term of $T^{\mu\nu}_G$ contributes only to the boost and does not contribute to the momentum sum rule of the nucleon, we shall drop it in the following argument. It can be shown that, up to a surface term, the gluon part can further be decomposed into two gauge-invariant pieces as

$$T^{\mu\nu}_G = -\mathrm{Tr}[F^{\mu\alpha}D^\nu_{\text{pure}}A_{\alpha,\text{phys}} + F^{\nu\alpha}D^\mu_{\text{pure}}A_{\alpha,\text{phys}}] - \mathrm{Tr}[D_\alpha F^{\mu\alpha}A^\nu_{\text{phys}} + D_\alpha F^{\nu\alpha}A^\mu_{\text{phys}}] + \text{surface term}. \quad (121)$$

Here, the 2nd term of the above equation is a covariant generalization of the potential momentum term as discussed in Sec. III. Under the imposed gauge transformation property of the physical and pure-gauge components of the gluon fields given by

$$A^\mu_{\text{phys}}(x) \to U(x)A^\mu_{\text{phys}}(x)U^\dagger(x), \quad (122)$$

$$A^\mu_{\text{pure}}(x) \to U(x)\left[A^\mu_{\text{pure}}(x) + \frac{i}{g}\partial^\mu\right]U^\dagger(x), \quad (123)$$

supplemented with the pure-gauge condition for the pure-gauge part of A^μ,

$$F^{\mu\nu}_{\text{pure}} \equiv \partial^\mu A^\nu_{\text{pure}} - \partial^\nu A^\mu_{\text{pure}} + ig[A^\mu_{\text{pure}}, A^\nu_{\text{pure}}] = 0, \quad (124)$$

it is easy to show that each term of (121) is separately gauge-invariant. Equations (118)–(120) combined with (121) gives our gauge-invariant decomposition (I) of the QCD energy-momentum tensor. Since the potential momentum term is still contained in the gluon part in this decomposition, it is practically the same as the standard decomposition.

On the other hand, if one combines the potential angular momentum term with the quark part by making use of the QCD equation of motion $(D_\alpha F^{\mu\nu})^a = g\bar{\psi}\gamma^\mu T^a\psi$, one is led to another gauge-invariant decomposition (II) of QCD energy-momentum tensor given as follows:

$$T^{\mu\nu} = T'^{\mu\nu}_q + T'^{\mu\nu}_G, \quad (125)$$

where

$$T'^{\mu\nu}_q = \tfrac{1}{2}\bar{\psi}(\gamma^\mu iD^\nu_{\text{pure}} + \gamma^\nu iD^\mu_{\text{pure}})\psi, \quad (126)$$

$$T'^{\mu\nu}_G = -\mathrm{Tr}[F^{\mu\alpha}D^\nu_{\text{pure}}A_{\alpha,\text{phys}} + F^{\nu\alpha}D^\mu_{\text{pure}}A_{\alpha,\text{phys}}]. \quad (127)$$

This decomposition is thought of as a covariant generalization of the decomposition of Chen *et al.*

The question is now whether these two decompositions of QCD energy-momentum tensor lead to different predictions for the quark and gluon momentum fractions and their evolution. As emphasized in [24], a remarkable feature of our gauge-invariant decompositions (I) and (II) is that we have not yet fixed gauge explicitly. This means that we can choose any gauge as long as the choice is consistent with the above-mentioned general conditions (122)–(124). Particularly useful is the fact that we can take the light-cone gauge as well [24], which is the most convenient gauge for discussing DIS observables.

We can then follow the argument given by Jaffe [69]. The simplest way of obtaining the momentum sum rule is to evaluate the nucleon matrix element of the $(++)$-component of the energy-momentum tensor. The momentum sum rule then follows from the normalization condition:

$$\frac{\langle P|T^{++}|P\rangle}{2(P^+)^2} = 1. \tag{128}$$

As emphasized by Jaffe, T^{++} simplifies dramatically in $A^+ = 0$ gauge, because of the simplification of D^+ and $F^{+\alpha}$,

$$D^+ = \partial^+ - igA^+ \rightarrow \partial^+, \tag{129}$$

$$F^{+\alpha} = \partial^+ A^\alpha - \partial^\alpha A^+ + g[A^+, A^\alpha] \rightarrow \partial^+ A^\alpha. \tag{130}$$

As a consequence, T^{++} reduces to a marvelously simple form as

$$T^{++} = T_q^{++} + T_G^{++} \rightarrow \psi_+^\dagger i\partial^+ \psi_+ + 2\,\mathrm{Tr}(\partial^+ \boldsymbol{A}_\perp)^2, \tag{131}$$

where ψ_+ is the standard $(+)$-component of ψ defined by $\psi_+ \equiv P_+ \psi$ and $P_+ = (1/2)\gamma^-\gamma^+$ with $\gamma^\pm = (1/\sqrt{2})(\gamma^0 \pm \gamma^3)$. The two terms here give the contributions of quarks and gluons, respectively, to P^+. Each term can be related to the 2nd moment of the positive definite parton momentum distribution:

$$\psi_+^\dagger i\partial^+ \psi_+ \rightarrow \int dx x q(x), \qquad (\partial^+ \boldsymbol{A}_\perp^a)^2 \rightarrow \int dx x g(x). \tag{132}$$

The normarization condition (128) then gives the well-known momentum sum rule of QCD,

$$1 = \int dx [q(x, Q^2) + g(x, Q^2)]. \tag{133}$$

This is a familiar story about the standard decomposition of the QCD energy-momentum tensor.

A question is what would change if one adopts the decomposition (II), which is thought to contain in it the decomposition of Chen *et al.*? To answer this question, we first recall the following relation between the quark part of T^{++} in the two decompositions:

$$T_q^{++} - T_q^{\prime ++} = g\bar{\psi}\gamma^+ A_{\mathrm{phys}}^+ \psi. \tag{134}$$

We emphasize that the difference is nothing but a special component of generalized potential momentum tensor. Remember now the fact that, different from Chen *et al.*'s treatment, we have a freedom to choose even the light-cone gauge. Since $A^+ = A^+_{\mathrm{phys}} = A^+_{\mathrm{pure}} = 0$ in this gauge, the difference between T^{++} and $T^{\prime ++}$ simply vanishes. We must therefore conclude that the two decompositions (I) and (II) give exactly the same answer, as far as the longitudinal momentum sum rule is concerned. This fact has been verified in a particular gauge, i.e. in the light-cone gauge. Note, however, that both of our decompositions (I) and (II) are manifestly gauge-invariant. It is therefore a logical consequence of gauge-invariance that the statement must hold in arbitrary gauges. (Naturally, it is of vital importance to confirm the validity of this statement through explicit calculations in other gauges than the light-cone gauge.)

Still, one might worry about the claim by Chen *et al.* that the two decompositions of the nucleon momentum lead to totally different evolution equations for the momentum fractions of quarks and gluons in the nucleon [19]. Let us next try to clarify this point. Before discussing the evolution equation corresponding to the decomposition (II), we think it is useful to recall some basic knowledge on the evolution matrix for the quark and gluon momentum fractions corresponding to the standard decomposition (I). Though somewhat trivial to remark, since the quark and the gluon parts of this standard decomposition is separately gauge-invariant, the evolution matrix should be independent of gauge choice. First we concentrate on the quark part of T^{++}, which consists of two parts in general gauge as

$$T^{++} = V_A + V_B, \tag{135}$$

with

$$V_A = \bar{\psi}\gamma^+ i\partial^+ \psi, \tag{136}$$

$$V_B = g\bar{\psi}\gamma^+ A^+ \psi. \tag{137}$$

The momentum space vertices corresponding to these operators are expressed by the following formulas supplemented with the diagram shown in Fig. 1:

$$V_A = \delta_{bc}\gamma^+ p^+, \tag{138}$$

$$V_B = g(T^a)_{bc}\gamma^+ g^{+\nu}. \tag{139}$$

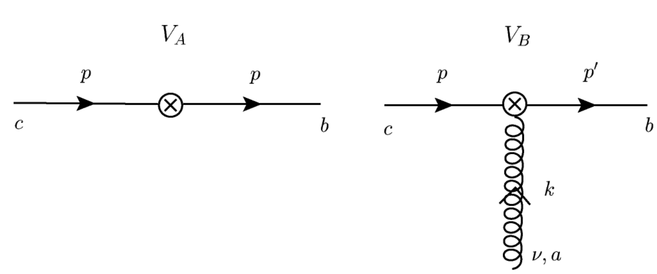

FIG. 1. Momentum space vertices for the quark part of T^{++}.

Note that $V_B \neq 0$ in general gauges, although $V_B = 0$ in the light-cone gauge.

Shown in Fig. 2 are the Feynman diagrams, which contribute to the anomalous dimension $\gamma_{qq}^{(2)}$ in general covariant gauge. The answer in the Feynman gauge is well-known. It is given by

$$\gamma_{qq}^{(2)} = \frac{\alpha_S}{2\pi}\left\{\frac{1}{6}C_F - C_F - \frac{1}{2}C_F\right\} = \frac{\alpha_S}{2\pi}\left(-\frac{4}{3}C_F\right), \tag{140}$$

with $C_F = 4/3$. Here, the three terms in the middle of the above equation, respectively, correspond to the contributions from the graphs (a), (b) and (c) of Fig. 2. On the other hand, in the light-cone gauge, there is no contribution from the graph (b), and the answer is given as

$$\gamma_{qq}^{(2)} = \frac{\alpha_S}{2\pi}\left\{-\frac{17}{6}C_F + 0 + \frac{3}{2}C_F\right\} = \frac{\alpha_S}{2\pi}\left(-\frac{4}{3}C_F\right). \tag{141}$$

Although individual term contributes differently, the final answer is just the same as that of Feynman gauge.

Now that we have convinced that the anomalous dimension $\gamma_{qq}^{(2)}$ corresponding to the standard decomposition of the energy-momentum tensor is independent of the choice of gauge, our next task is to obtain the anomalous dimension $\gamma_{qq}^{\prime(2)}$ corresponding to another gauge-invariant decomposition (II) of the QCD energy-momentum tensor. For this purpose, we recall again the fact that the quark parts of T^{++} in the two decomposition are connected through the relation (134). As pointed out before, the rhs of (134) vanishes in the light-cone gauge. This already indicates that the anomalous dimensions corresponding to the two decompositions are the same, i.e. $\gamma_{qq}^{(2)} = \gamma_{qq}^{\prime(2)}$. Let us verify this statement more explicitly by showing that the vertex $V_C \equiv g\bar{\psi}\gamma^+ A^+_{\rm phys}\psi$ does not contribute to the corresponding anomalous dimension even in other gauges than the light-cone gauge. A key factor here is the fact that the gluon field contained in the vertex V_C is its physical part $A^+_{\rm phys}$. By taking care of this fact, we recall somewhat nonstandard Feynman rule proposed in [25]. According to this rule, the momentum representation of the vertex V_C is given as

$$V_C = g(T^a)_{bc}\gamma^+ g^{+\nu} P_T^\nu, \tag{142}$$

which is delicately different from the vertex V_B in that it contains a kind of projection operator P_T^ν. This projection operator P_T^ν with the Lorentz index ν reminds us of the fact that we must use the modified gluon propagator

$$\tilde{D}^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\varepsilon}T^{\mu\nu}, \tag{143}$$

with

$$T^{\mu\nu} = \sum_{\lambda=1}^{2} \varepsilon^\mu(k,\lambda)\varepsilon^{\nu*}(k,\lambda), \tag{144}$$

whenever it is obtained with the contraction with the vertex V_C containing the Lorentz index ν. The Feynman diagram, which may potentially contribute to the anomalous dimension in question, is given by the same graph as the graph (b) of Fig. 2 except that the vertex V_B is replaced by V_C. An explicit calculation given in Appendix A shows that the contribution of this diagram vanishes. [Note that this conversely means that the contribution to $\gamma_{qq}^{(2)}$ from the graph (b) in the Feynman gauge comes totally from the vertex $g\bar{\psi}\gamma^+ A^+_{\rm pure}\psi$.]

Although slightly more trivial, we have also checked in Appendix A that the potential momentum term does not contribute to the anomalous dimension $\gamma_{qG}^{(2)}$. (The relevant diagram appearing in this proof is illustrated in Fig. 3.) In this way, we now confirm that

$$\gamma_{qq}^{(2)} = \gamma_{qq}^{\prime(2)}, \qquad \gamma_{qG}^{(2)} = \gamma_{qG}^{\prime(2)}. \tag{145}$$

As is well-known, because of the conservation of total momentum, the 2×2 evolution matrix of the quark and gluon momenta (in whatever decomposition) has only two independent elements such that

$$\gamma_{Gq}^{(2)} = -\gamma_{qq}^{(2)}, \qquad \gamma_{GG}^{(2)} = -\gamma_{qG}^{(2)}. \tag{146}$$

We therefore conclude that the anomalous dimension matrix corresponding to the two decompositions (I) and

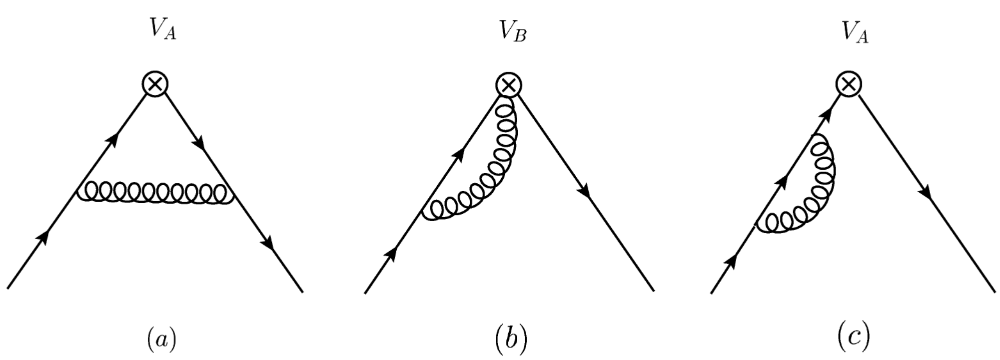

FIG. 2. One-loop diagrams contributing to the anomalous dimension of the quark part of the energy-momentum tensor. The diagram (c) corresponds to the quark field-strength renormalization. Graphs that are not symmetric with respect to the vertical lines through the operator vertex have to be counted twice.

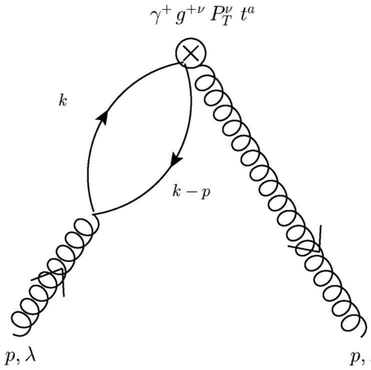

FIG. 3. The Feynman graph, which may potentially contribute to the anomalous dimension $\gamma_{qG}^{(2)}$.

(II) of the QCD energy-momentum tensor are exactly the same, i.e.

$$\begin{pmatrix} \gamma_{qq}^{(2)} & \gamma_{qG}^{(2)} \\ \gamma_{Gq}^{(2)} & \gamma_{GG}^{(2)} \end{pmatrix} = \begin{pmatrix} \gamma_{qq}^{\prime(2)} & \gamma_{qG}^{\prime(2)} \\ \gamma_{Gq}^{\prime(2)} & \gamma_{GG}^{\prime(2)} \end{pmatrix}. \tag{147}$$

This contradicts the conclusion of Chen *et al.* given in [19]. According to our analysis above, the gluons *do* carry about one-half of the total nucleon momentum in the asymptotic limit.

The readers might suspect that the conclusion above contradicts our previous statement that the Chen *et al.*'s decomposition is contained in our more general decomposition (II), so that the physical predictions should be the same. Note, however, that no one has yet checked the validity of their calculation based on the Coulomb gauge. Possible reasons of discrepancy might therefore be the following. One possibility is that they have made a mistake in their Coulomb-gauge calculation of the anomalous dimension matrix. In fact, the treatment of the gluon propagator in the Coulomb gauge is known to be a fairly delicate issue because of the so-called energy-divergence in loop integrations [70]. Another possibility is that what they have calculated does not precisely correspond to the evolution matrix of the longitudinal momentum fractions of quarks and gluons appearing in deep inelastic scattering physics.

V. RELATION BETWEEN THE TWO INEQUIVALENT DECOMPOSITIONS OF THE NUCLEON SPIN

In the previous section, we have shown that there certainly exist two generally inequivalent decompositions of the nucleon total momentum, analogous to the QED problem. Nonetheless, as long as the longitudinal momentum sum rules of the nucleon is concerned, the two decompositions turn out to give completely the same answer for the quark and gluon momentum fractions including their scale evolution.

Now, we turn to a more interesting problem of nucleon spin decomposition. We first recall the fact that there exist two different decompositions also for the nucleon spin, both of which are gauge-invariant. The QCD angular momentum tensor in the decomposition (I) is given by Eq. (12), while that in the decomposition (II) is given by Eq. (6). The theoretical basis for obtaining the nucleon spin sum rule is given by the equation [71]

$$\langle Ps|W^\mu s_\mu|Ps\rangle/\langle Ps|Ps\rangle = \tfrac{1}{2}, \tag{148}$$

where s_μ is the covariant spin vector of the nucleon, while

$$W^\mu = -\epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} P_\gamma/(2\sqrt{P^2}), \tag{149}$$

with

$$J^{\alpha\beta} = \int d^3x M^{0\alpha\beta}, \tag{150}$$

is the so-called Pauli-Lubansky vector [72]. Assuming that the nucleon is moving in the z direction with momentum P^μ and helicity $+1/2$, it holds that

$$J^{12}|P+\rangle = \tfrac{1}{2}|P+\rangle. \tag{151}$$

Thus we are led to the relation

$$\tfrac{1}{2} = \langle P+|J^{12}|P+\rangle/\langle P+|P+\rangle, \tag{152}$$

which provides us with a basis for obtaining longitudinal spin sum rule of the nucleon. Depending on the two decompositions of $M^{\mu\nu\lambda}$, this gives the following sum rules. The decomposition (I) gives

$$\tfrac{1}{2} = (\tfrac{1}{2}\Delta\Sigma + L_q) + (\Delta G + L_G) = J_q + J_G, \tag{153}$$

where

$$\Delta\Sigma = \langle P+|\int d^3x \psi^\dagger \gamma^0\gamma^3\gamma_5\psi|P+\rangle, \tag{154}$$

$$L_q = \langle P+|\int d^3x \psi^\dagger (x^1 D^2 - x^2 D^1)\psi|P+\rangle, \tag{155}$$

$$\Delta G = \langle P+|\int d^3x (E^1 A^2_{\rm phys} - E^2 A^1_{\rm phys})|P+\rangle, \tag{156}$$

$$\begin{aligned} L_G = &\langle P+|\int d^3x 2\,{\rm Tr}\{E^k(x^2\mathcal{D}^1_{\rm pure} - x^1\mathcal{D}^2_{\rm pure})A^k_{\rm phys}\}|P+\rangle \\ &+ \langle P+|\int d^3x 2\,{\rm Tr}\{(\boldsymbol{D}\cdot\boldsymbol{E})(x^2 A^2_{\rm phys} - x^2 A^1_{\rm phys})\}|P+\rangle, \end{aligned} \tag{157}$$

where we have neglected the normalization of the state, for simplicity.

On the other hand, the decomposition (II) leads to

$$\tfrac{1}{2} = (\tfrac{1}{2}\Delta\Sigma' + L'_q) + (\Delta G' + L'_G) = J'_q + J'_G, \tag{158}$$

where

$$\Delta\Sigma' = \Delta\Sigma, \quad (159)$$

$$L'_q = \langle P + | \int d^3x \psi^\dagger (x^1 D^2_{\rm pure} - x^2 D^1_{\rm pure}) \psi | P+\rangle, \quad (160)$$

$$\Delta G' = \Delta G, \quad (161)$$

$$L'_G = \langle P + | \int d^3x 2 \mathrm{Tr}\{E^k (x^2 \mathcal{D}^1_{\rm pure} - x^1 \mathcal{D}^2_{\rm pure}) A^k_{\rm phys}\} | P+\rangle. \quad (162)$$

The difference between the two decompositions resides in the orbital parts. Note that L_q and L'_q, respectively, correspond to the nucleon matrix elements of mechanical and generalized canonical OAM operators. What characterizes the difference of these two quantities is the forward matrix element of the potential angular momentum given by

$$\begin{aligned} L_q - L'_q &= -(L_G - L'_G) \\ &= \langle P + | \int d^3x g \psi^\dagger (x^1 A^2_{\rm phys} - x^2 A^1_{\rm phys}) \psi | P+\rangle. \end{aligned} \quad (163)$$

It is important to recognize the fact that $A^1_{\rm phys}$ and $A^2_{\rm phys}$ are physical (transverse) components of gluons, which cannot be transformed away even in the light-cone gauge. (This is totally different from the case of the nucleon momentum decomposition, where the transverse components do not appear in the difference between T^{++}_q and T'^{++}_q.) Since the quantum state vector of the nucleon as a strongly-coupled gauge system of quarks and gluons definitely contains Fock components of transverse gluons, we conclude that the difference between L_q and L'_q is generally nonzero. An explicit calculation by Burkardt and BC based on simple models appears to confirm it [73].

Nonetheless, one mysterious observation still remains to be clarified. The problem concerns the scale dependence of quark and gluon OAMs. Accepting that there are two different OAMs of both of quarks and gluons, i.e. (L_q, L_G) and (L'_q, L'_G), one might naturally expect different evolution equations for these two kinds of OAMs. Somewhat embarrassingly, the past studies indicate that the evolution equations of L_q and L_G are nothing different from those of L'_q and L'_G [43–50]. This can be confirmed as follows. First, the scale dependence of $\Delta\Sigma$ and ΔG at the leading order is widely-known [56,74,75], and given as

$$\frac{d}{dt}\begin{pmatrix} \Delta\Sigma \\ \Delta G \end{pmatrix} = \frac{\alpha_S(t)}{2\pi} \begin{pmatrix} 0 & 0 \\ \frac{3}{2}C_F & \frac{\beta_0}{2} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta G \end{pmatrix}, \quad (164)$$

where $t = \ln Q^2/\Lambda^2_{\rm QCD}$, $C_F = 4/3$, and $\beta_0 = 11 - (2/3)n_f$. On the other hand, the leading-log evolution equation of quark and gluon OAMs L'_q and L'_G was first derived by Ji, Tang, and Hoodbhoy [43]. It is given by

$$\frac{d}{dt}\begin{pmatrix} L'_q \\ L'_G \end{pmatrix} = \frac{\alpha_S(t)}{2\pi} \begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} L'_q \\ L'_G \end{pmatrix} + \frac{\alpha_S(t)}{2\pi} \begin{pmatrix} -\frac{2}{3}C_F & \frac{n_f}{3} \\ -\frac{5}{6}C_F & -\frac{11}{2} \end{pmatrix} \begin{pmatrix} \Delta\Sigma \\ \Delta G \end{pmatrix}. \quad (165)$$

To be more precise, their derivation is based on a gauge-noninvariant definition of L'_q and L'_G appearing in the Jaffe-Manohar decomposition. Luckily, their calculation was done in the light-cone gauge. This ended up with the result that the derived evolution equation coincides with the answer obtained from the gauge-invariant definition of L'_q and L'_G appearing in our decompositin (II). (Remember the similar situation which we encounter in the study of evolution equation of ΔG [25]. The point is that the Jaffe-Manohar decomposition is now taken as a gauge-fixed form of our more general decomposition with manifest gauge-invariance.)

Using the above evolution equations for $(\Delta\Sigma, \Delta G)$ and (L'_q, L'_G), one can easily write down the evolution equation of the quark and gluon total angular momentum in the decomposition (II), which are defined by $J'_q \equiv L'_q + \frac{1}{2}\Delta\Sigma$ and $J'_G \equiv L'_G + \Delta G$. One finds that

$$\frac{d}{dt}\begin{pmatrix} J'_q \\ J'_G \end{pmatrix} = \frac{\alpha_S(t)}{2\pi} \begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} J'_q \\ J'_G \end{pmatrix}. \quad (166)$$

As noticed by several authors [44,50], the evolution matrix appearing here is just the same as that of the momentum fractions of quarks and gluons. On the other hand, Ji showed that the scale evolution of the total angular momenta of quarks and gluons appearing in the decomposition (I) is controlled by the same evolution matrix as that of the quark and gluon momentum fractions as

$$\frac{d}{dt}\begin{pmatrix} J_q \\ J_G \end{pmatrix} = \frac{\alpha_S(t)}{2\pi} \begin{pmatrix} -\frac{4}{3}C_F & \frac{n_f}{3} \\ \frac{4}{3}C_F & -\frac{n_f}{3} \end{pmatrix} \begin{pmatrix} J_q \\ J_G \end{pmatrix}. \quad (167)$$

The reason is that the quark and gluon angular momenta J_q and J_G in the decomposition (I) are defined by the QCD angular momentum tensor $M^{\alpha\mu\nu}$, which is related to the energy-momentum tensor $T^{\mu\nu}$ through the relation

$$M^{\alpha\mu\nu} = T^{\alpha\nu}x^\mu - T^{\alpha\mu}x^\nu, \quad (168)$$

with

$$T^{\mu\nu} = T^{\mu\nu}_q + T^{\mu\nu}_G. \quad (169)$$

According to Ji, forming spatial moment of $T^{\mu\nu}_q$ and $T^{\mu\nu}_G$ does not change the short-distance singularity of the operators. It then follows that (J_q, J_G) and $(\langle x\rangle_q, \langle x\rangle_G)$ obey the same evolution equation. At any rate, one now realizes that (J_q, J_G) and (J'_q, J'_G) obey the same evolution equation at least at the one-loop order. Since $\Delta\Sigma$ and ΔG are common in the two decompositions, this also means that (L_q, L_G) and (L'_q, L'_G) obey the same evolution equation.

How can we understand this somewhat puzzling observation ? The answer is basically given in the paper by Ji [13]. He claims that the above observation can be understood, since the interaction-dependent term, $g \int d^3x \psi^\dagger \boldsymbol{x} \times \boldsymbol{A} \psi$, which characterizes the difference between the dynamical and canonical angular momenta of quarks, shall not affect the leading-log evolution in the light-cone gauge. Unfortunately, an explicit proof is not given there. Furthermore, the statement holds only in the light-cone gauge, because it is based on gauge-noninvariant expression $g \int d^3x \psi^\dagger \boldsymbol{x} \times \boldsymbol{A} \psi$ of the interaction-dependent part. To refine Ji's statement and also to make the role of gauge-invariance more manifest, we recall the fact that the difference of L_q and L'_q is given by the nucleon matrix element of potential angular momentum [see (163)], which is a manifestly gauge-invariant quantity. It is therefore possible to extend the validity of Ji's statement by showing that the (gauge-invariant) potential angular momentum term does not contribute to the evolution matrix also in other gauges than the light-cone gauge. The proof is given in Appendix B. (The relevant Feynman diagram is shown in Fig. 4.) This clarifies the reason why (L_q, L_G) and (L'_q, L'_G), appearing in the two generally different decompositions of the nucleon spin, obey the same evolution equation.

To avoid a misunderstanding, we want to reemphasize the following fact. In the case of longitudinal momentum sum rule discussed in the previous section, we showed that the two decompositions of the QCD energy-momentum tensor gives the same evolution equation for the momentum fractions of quarks and gluons. In this case, the numerical values of the quark and gluon momentum fractions in the two decompositions are also the same at an arbitrary energy scale, because the transverse components of the gluon fields never contribute to the longitudinal momentum sum rule, as can be seen in the expression (134). This is not the case for the longitudinal spin sum rule of the nucleon, however. Although the quark and gluon OAMs appearing in the two decompositions (I) and (II) are shown to obey the same evolution equation, there is no reason that their numerical values at an arbitrary energy scale also coincide. In fact, they are generally different, because the transverse (real) gluon fields do contribute to the difference between the two definitions of quark and gluon OAMs in the nucleon. [Remember the relation (163).] As emphasized in [76], a clear recognition of this fact is especially important if one tries to compare the predictions of low energy effective models on the nucleon spin contents with those of lattice QCD [76–83].

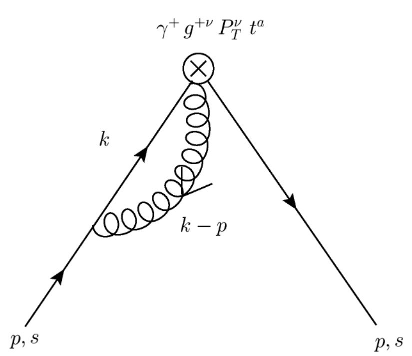

FIG. 4. The Feynman graph, which may potentially contribute to the evolution matrix for the quark orbital angular momentum.

VI. SUMMARY AND CONCLUSION

In summary, we first briefly review the current status of the nucleon spin decomposition problem with particular emphasis upon the fact that there exists two physically inequivalent gauge-invariant decompositions (I) and (II) of the nucleon spin. The difference between these two decompositions resides in the orbital parts of quarks and gluons, while intrinsic spin parts of quarks and gluons are just common. The OAMs of quarks and gluons appearing in the decomposition (I) are the gauge-invariant dynamical (or mechanical) OAMs, while the OAMs appearing in the decomposition (II) are the (generalized) canonical OAMs with gauge-invariance. The key ingredient, which characterizes the difference between these two OAMs is what we call the potential angular momentum. We clarify the physical meaning of this quantity by using an analogous but much simpler example from electrodynamics, i.e. a system of charged particles and photons. It was shown that the potential angular momentum represents angular momentum associated with the longitudinal component of the electric field generated by the charged particles. Remember the fact that the longitudinal component of the electric field is also the origin of the Coulomb interactions between the charged particles, although the generation of potential angular momentum needs the magnetic field as well. Related to the fact that the longitudinal component of the electric field does not show up in the absence of the charged particle sources, there arises the ambiguity as to which of charged particles or the photons the potential angular momentum should be attributed to. (One encounters the same arbitrariness if one attempts to attribute the Coulomb energy to either of the charged particles or the photons.) If we attribute the potential angular momentum to the charged particle property, we have an angular momentum decomposition, in which the angular momentum of the charged particles is given by the (generalized) canonical OAM. On the other hand, if we attribute it to the property of the photons, the orbital part of the charged particle is given by the mechanical (or dynamical) OAM. Although the choice is a matter of taste, it is important to recognize the fact that what is closer to the physical image of orbital (rotational) motion of charged particles is the mechanical OAM rather than the canonical OAM, in sharp contradiction to a widespread belief or prepossession. One confirms that the terminology mechanical OAM has a legitimate reason for it. This understanding may be of important physical significance, because, for example, one must recognize clearly which of dynamical or mechanical OAMs is a relevant quantity when one tries to explain the single-spin asymmetry of semi-inclusive hadron productions based on the orbital angular momenta of nucleon constituents.

Also addressed in the paper are several other issues left in the decomposition problem of nucleon spin and momentum. After verifying the fact that there exist two gauge-invariant decomposition of the QCD energy-momentum tensor into the quark and gluon contributions, which are generally nonequivalent, we have verified that the two decompositions give exactly the same answer as long as the longitudinal momentum sum rule of the nucleon is concerned. It was further proved that the two decompositions give the same answer also for the evolution equation for the momentum fractions of quarks and gluons, which contradicts Chen *et al*'s claim that the gluons carry much smaller momentum fraction in the asymptotic limit as compared with the standardly-believed value of about one-half.

We have also compared the evolution equations of OAMs of quarks and gluons appearing in the two decompositions of the nucleon spin. We confirmed the fact that these two types of OAMs obey exactly the same evolution equations as indicated by the preceding studies. We showed that the reason of this somewhat mysterious observation can be trace back to the fact that the potential angular momentum, which gives the difference between the two types of OAMs, does not contribute to the evolution matrix of the quark and gluon OAMs. We therefore believe that the present investigation has deepened our understanding about the relation between the two different decompositions of the nucleon spin..

ACKNOWLEDGMENTS

I would like to thank Elliot Leader and Cedric Lórce for continual and many enlightening discussions throughout the study. This work is supported in part by a Grant-in-Aid for Scientific Research for Ministry of Education, Culture, Sports, Science and Technology, Japan (No. C-21540268).

APPENDIX A: PROOF THAT THE POTENTIAL MOMENTUM TERM DOES NOT CONTRIBUTE TO THE ANOMALOUS DIMENSION MATRIX OF QUARK AND GLUON MOMENTUM FRACTIONS

The contribution of the potential momentum term to $\gamma_{qq}^{(2)}$ can be obtained by evaluating the matrix element

$$T_{qq} = \langle ps| \int d^3x g \bar{\psi}(x)\gamma^+ A^+_{\rm phys}(x)\psi(x)|ps\rangle, \qquad (A1)$$

where $|ps\rangle$ is one quark state with momentum p and spin s. The corresponding 1-loop diagram is given by the diagram (b) of Fig. 1 except that the vertex V_B is replaced by V_C. Taking care of the Feynman rule explained in the text, we obtain

$$T_{qq} = \int \frac{d^4k}{(2\pi)^4}\bar{u}(ps)g\gamma^+ g^{+\nu}t^a \frac{i\not{k}}{k^2+i\varepsilon}(-ig\gamma^\mu t^b)u(ps) \times \frac{-i\delta_{ab}}{(k-p)^2+i\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu(k-p,\lambda)\varepsilon^*_\nu(k-p,\lambda). \qquad (A2)$$

By using

$$T_{\mu\nu} \equiv \sum_{\lambda=1}^{2}\varepsilon_\mu(k,\lambda)\varepsilon^*_\nu(k,\lambda) = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}, \qquad (A3)$$

with n_μ being a lightlike vector with $n^2 = 0$ [25], we can write

$$T_{qq} = -g^2 C_F \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i\varepsilon)[(k-p)^2+i\varepsilon]} \times \bar{u}(ps)\gamma^+ g^{+\nu}\not{k}\gamma^\mu u(ps)\left[g_{\mu\nu} - \frac{(k-p)_\mu n_\nu + (k-p)_\nu n_\mu}{(k-p)\cdot n}\right]. \qquad (A4)$$

Averaging over the spins, we obtain

$$\bar{\sum}_{\rm spins}\bar{u}(ps)\gamma^+ g^{+\nu}\not{k}\gamma^\mu u(ps) g_{\mu\nu} = 4k^+p^+, \qquad (A5)$$

$$\bar{\sum}_{\rm spins}\bar{u}(ps)\gamma^+ g^{+\nu}\not{k}\gamma^\mu u(ps)\frac{(k-p)_\mu n_\nu + (k-p)_\nu n_\mu}{(k-p)\cdot n} = 4k^+p^+. \qquad (A6)$$

We thus find that the contributions from the two parts of the (modified) gluon propagator precisely cancel each other, which proves our statement that the potential momentum term does not contribute to $\gamma_{qq}^{(2)}$.

Next, we consider the following matrix element

$$T_{qG} = \langle p\lambda| \int d^3x g \bar{\psi}\gamma^+ A^+_{\rm phys}\psi|p\lambda\rangle, \qquad (A7)$$

where $|p\lambda\rangle$ is one gluon state with momentum p and polarization λ. The 1-loop diagram, that might potentially contribute to this matrix element, is shown in Fig. 3. This gives

$$T_{qG} = \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i\varepsilon)[(k-p)^2+i\varepsilon]} \times {\rm Tr}[\gamma^+ t_a \not{k}\gamma^\nu t_a(\not{k}-\not{p})]\varepsilon_+(p,\lambda)\varepsilon^*_\nu(p,\lambda). \qquad (A8)$$

Since the real gluon state has only transverse polarizations, we have

$$\varepsilon_+(p,\lambda) = 0. \tag{A9}$$

This ensures that the potential momentum term does not contribute to the anomalous dimension $\gamma_{qG}^{(2)}$.

APPENDIX B: PROOF THAT THE POTENTIAL ANGULAR MOMENTUM TERM DOES NOT CONTRIBUTE TO THE EVOLUTION MATRIX FOR ORBITAL ANGULAR MOMENTUM

We are interested here in the 1-loop contribution to the matrix element

$$\langle p+|\hat{L}_{\rm pot}|p+\rangle, \tag{B1}$$

with

$$\hat{L}_{\rm pot} = \int d^3x g\bar{\psi}(x)\gamma^+(x^1A^2_{\rm phys}(x) - x^2A^1_{\rm phys}(x))\psi(x). \tag{B2}$$

To avoid singular nature of the matrix element resulting from the explicit factor of x^μ, it is customary to first consider the off-forward matrix element and to take the forward limit afterwards. For the off-forward matrix element in a quark or gluon state, we have

$$\begin{aligned}\langle p'^+|\hat{L}_{\rm pot}|p+\rangle &= \int d^3x\{x^1\langle p'+|g\bar{\psi}(x)\gamma^+A^2_{\rm phys}(x)\psi(x)|p+\rangle \\ &\quad - x^2\langle p'+|g\bar{\psi}(x)\gamma^+A^1_{\rm phys}(x)\psi(x)|p+\rangle\} \\ &= (2\pi)^3\Big[-i\frac{\partial}{\partial p'_1}\delta^3(p'-p)\langle p'+|g\bar{\psi}(0)\gamma^+A^2_{\rm phys}(0)\psi(0)|p+\rangle \\ &\quad + i\frac{\partial}{\partial p'_2}\delta^3(p'-p)\langle p'+|g\bar{\psi}(0)\gamma^+A^1_{\rm phys}(0)\psi(0)|p+\rangle\Big].\end{aligned} \tag{B3}$$

When convoluted with a test function [43], this gives two terms. One is

$$\lim_{p'\to p}\Big[i\frac{\partial}{\partial p'_1}\langle p'+|g\bar{\psi}(0)\gamma^+A^2_{\rm phys}(0)|p+\rangle - i\frac{\partial}{\partial p'_2}\langle p'+|g\bar{\psi}(0)\gamma^+A^1_{\rm phys}(0)|p+\rangle\Big], \tag{B4}$$

which represents the generation of orbital angular momentum from quark and gluon helicities in the splitting processes. The other is

$$-\frac{1}{p^i}\langle p+|g\bar{\psi}(0)\gamma^+A^i_{\rm phys}(0)\psi(0)|p+\rangle, \quad (i\text{: not summed}), \tag{B5}$$

which represents the self-generation of orbital angular momentum in the splitting processes.

We first consider the former contribution, which has the structure

$$\lim_{p'\to p}\Big(i\frac{\partial}{\partial p'_1}\tilde{T}^2 - i\frac{\partial}{\partial p'_2}\tilde{T}^1\Big), \tag{B6}$$

with

$$\tilde{T}^i = \langle p'+|g\bar{\psi}(0)\gamma^+A^i_{\rm phys}(0)\psi(0)|p+\rangle. \tag{B7}$$

The 1-loop Feynman diagram contributing this matrix element is similar to that shown in Fig. 4. This gives

$$\begin{aligned}\tilde{T}^i =& \frac{1}{2p^+}\int\frac{d^4k}{(2\pi)^4}\bar{u}(p'+)g\gamma^+g^{i\mu}t^a\frac{i\not{k}}{k^2+i\varepsilon}(-g\gamma^\nu t^b)u(p+) \\ &\times\frac{-i\delta_{ab}}{(k-p)^2+i\varepsilon}\sum_{\lambda=1}^{2}\varepsilon_\mu(k-p,\lambda)\varepsilon^*_\nu(k-p,\lambda).\end{aligned} \tag{B8}$$

From this, we obtain

$$\begin{aligned}i\frac{\partial}{\partial p'+}\tilde{T}^2 =& \frac{g^2C_F}{2p^+}\int\frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial p'_1}\bar{u}(p'_1)\gamma^+\not{k}\gamma^\nu u(p+) \\ &\times\frac{1}{(k^2+i\varepsilon)[(k-p)^2+i\varepsilon]}\Big[\delta^i_\nu - \frac{(k-p)^in_\nu}{(k-p)\cdot n}\Big],\end{aligned} \tag{B9}$$

where we have used the fact that $n^i = 0$ for $i = 1, 2$. Using the explicit form of the light-cone spinors

$$u(p+) = \frac{1}{\sqrt{2}\sqrt{\sqrt{2}p^+}}\begin{pmatrix}\sqrt{2}p^+\\0\\\sqrt{2}p^+\\0\end{pmatrix},\quad u(p'+) = \frac{1}{\sqrt{2}\sqrt{\sqrt{2}p^+}}\begin{pmatrix}\sqrt{2}p^+\\p'_1+ip'_2\\\sqrt{2}p^+\\p'_1+ip'_2\end{pmatrix}, \tag{B10}$$

it can be shown that

$$\lim_{p'\to p}\frac{\partial}{\partial p'_1}\bar{u}(p')\gamma^+\not{k}\gamma^\nu u(p)\delta^i_\nu = 0, \tag{B11}$$

$$\lim_{p'\to p}\frac{\partial}{\partial p'_1}\bar{u}(p')\gamma^+\not{k}\gamma^\nu u(p)(k-p)^i n_\nu = 0. \qquad \text{(B12)}$$

Then, we find that

$$\lim_{p'\to p} i\frac{\partial}{\partial p'_1}\tilde{T}^2 = 0. \qquad \text{(B13)}$$

Similarly

$$\lim_{p'\to p} i\frac{\partial}{\partial p'_2}\tilde{T}^1 = 0, \qquad \text{(B14)}$$

In this way, we find that

$$\lim_{p'\to p}\left(i\frac{\partial}{\partial p'_1}\tilde{T}^2 - i\frac{\partial}{\partial p'_2}\tilde{T}^1\right) = 0. \qquad \text{(B15)}$$

Next we consider the term corresponding to self-generation of the orbital angular momentum in the splitting processes, which takes of the form:

$$-\frac{1}{p^i}\langle p+|g\bar{\psi}(0)\gamma^+A^i_{\text{phys}}(0)\psi(0)|p+\rangle = -\frac{1}{p^i}T^i. \qquad \text{(B16)}$$

We find that

$$\begin{aligned} T^i &= \frac{1}{2p^+}\int\frac{d^4k}{(2\pi)^4}\bar{u}(p+)g\gamma^+g^{i\mu}t^a\frac{i\not{k}}{k^2+i\varepsilon}(-g\gamma^\nu t^a)u(p+) \\ &\quad\times\frac{-i\delta_{ab}}{(k-p)^2+i\varepsilon}\left[g_{\mu\nu}-\frac{(k-p)_\mu n_\nu + n_\mu(k-p)_\nu}{(k-p)\cdot n}\right] \\ &= -\frac{ig^2C_F}{2p^+}\int\frac{d^4k}{(2\pi)^4}\bar{u}(p+)\gamma^+\not{k}\gamma^\nu u(p+) \\ &\quad\times\frac{1}{(k^2+i\varepsilon)[(k-p)^2+i\varepsilon]}\left[\delta^i_\nu - \frac{(k-p)^i n_\nu}{(k-p)\cdot n}\right]. \end{aligned} \qquad \text{(B17)}$$

Using the relation

$$\begin{aligned} &u(p+)\gamma^+\not{k}\gamma^\nu u(p+)\left[\delta^i_\nu - \frac{(k-p)^i n_\nu}{(k-p)\cdot n}\right] \\ &= 2(k^+p^i - p^+k^i)\frac{k^+ + p^+}{k^+ - p^+}, \end{aligned} \qquad \text{(B18)}$$

we have

$$\begin{aligned} T^i = &-\frac{2ig^2C_F}{2p^+}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i\varepsilon)[(k-p)^2+i\varepsilon]} \\ &\times(k^+p^i - p^+k^i)\frac{k^+ + p^+}{k^+ - p^+}. \end{aligned} \qquad \text{(B19)}$$

Carrying out k^- integration, shifting the variable $\boldsymbol{k}_\perp$ tp $\boldsymbol{k}'_\perp = \boldsymbol{k}_\perp - x\boldsymbol{p}_\perp$ and trading k^+ for xp^+, we obtain

$$T^i = \frac{g^2C_F}{2(2\pi)^3}\int\frac{d^2\boldsymbol{k}'_\perp}{\boldsymbol{k}'^2_\perp}\int_0^1(x-x)\frac{x+1}{x-1} = 0. \qquad \text{(B20)}$$

This means that the contributions from the two parts of the modified gluon propagator in (B17), i.e. the $g_{\mu\nu}$, and the other part, cancel each other out. We therefore confirm the fact that the potential angular momentum term, which distinguishes the two decompositions of the nucleon spin, does not contribute to the evolution matrix for the quark OAM. We emphasize that our proof here is not bound to the choice of gauge.

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