PHYSICAL REVIEW D **85,** 114038 (2012)

Anomaly-induced charges in baryons

Minoru Eto,[1,*] Koji Hashimoto,[2,†] Hideaki Iida,[2,‡] Takaaki Ishii,[3,§] and Yu Maezawa[2,∥]

[1]*Department of Physics, Yamagata University, Yamagata 990-8560, Japan*
[2]*Mathematical Physics Laboratory, RIKEN Nishina Center, Saitama 351-0198, Japan*
[3]*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, United Kingdom*

(Received 4 September 2011; published 21 June 2012)

We study the Skyrme model of baryons with quantum chiral anomaly of QCD in magnetic backgrounds, and suggest a possible induction of a novel structure of electric charge inside the baryons. Due to the anomaly-induced gauged Wess-Zumino term $\sim(\pi_0 + \text{multipion})\vec{E}\cdot\vec{B}$, the Skyrmions giving a local pion condensation $\langle(\pi_0 + \text{multipion})\rangle \neq 0$ would produce a local charge source, in the background magnetic field $\vec{B} \neq 0$. Since the appearance of the total additional electric charge on the baryon looks unrealistic and surprising, we discuss the validity of our detailed evaluation of the anomaly effects.

DOI: 10.1103/PhysRevD.85.114038 PACS numbers: 12.39.Dc, 11.30.Rd

I. INTRODUCTION

The chiral anomaly is one of the central concepts in QCD, and it manifests the nature of quantum field theories in an explicit way in our hadronic world. As the chiral anomaly is essentially coupled to electromagnetic sector, since the electromagnetism is a part of the chiral symmetry, the introduction of nontrivial electromagnetic backgrounds should add a good flavor of physics onto the chiral anomaly. In this paper, we report an interesting new effect induced by the chiral anomaly, for baryons in a background magnetic field.

We analyze baryons in a constant magnetic background in detail, and suggest that the baryon would acquire an additional electric charge due to the chiral anomaly. Since the result is counterintuitive and seems to be against charge conservation, we discuss in detail how things work in a concrete model of baryons. The mechanism of the generation of the charge is quite simple. It is well-known that Wess-Zumino-Witten (WZW) term [1,2] actually captures the chiral anomaly in terms of the hadronic degrees of freedom. In particular, this term serves as a manifestation of the famous $\pi_0 \to 2\gamma$ decay. Now, any baryon carries a cloud of pions around it, and so it is a source of the pions. Once we replace one of the two γ's in the Wess-Zumino-Witten term by the background magnetic field, we immediately see that the baryon can be a source of the electromagnetism (another γ), i.e.. the baryon can have an additional charge structure due to the chiral anomaly and the pion cloud. The schematic picture of this mechanism is illustrated in Fig. 1.

In this paper, we explicitly demonstrate this mechanism in detail, with a help of a concrete model of the pion-cloud picture of the baryons, the Skyrme model [3]. In the Skyrme model, baryons are given as a solitonic object made of a local pion condensate $\langle\pi(x)\rangle \neq 0$. Plugging the Skyrme solution to the Wess-Zumino-Witten term, it can be shown that the magnetic-field background can induce a novel charge structure inside the baryon (Skyrmion).

In particular, we give an argument that the total charge can also be generated, and as a result the Gell-Mann-Nishijima formula for baryon charges can be corrected under the magnetic field due to the anomaly

$$Q_e = e\left(I_3 + \frac{N_B}{2}\right) + \frac{Q_{\text{anm}}}{2}. \tag{1}$$

Here in the modified formula, Q_e is the electric charge of the baryon, I_3 is the third component of isospin, N_B is the baryon number, and the new term Q_{anm} is the charge generated by the anomaly and the background magnetic field.

The generation of the electric charge is counterintuitive, so one may be suspicious of the result. In this paper, we analyze the validity of the calculation in detail. One should also notice that, for example, in the renowned Witten effect [4,5], monopoles are accompanied with electric charges, in the presence of the θ term. We may regard our WZW term as an analogue of the θ term for the Witten effect. In addition, the chiral magnetic effect [6–9] in heavy ion collisions shares the same property too. So, it is fair to say that the generation of the electric charge is not a unique feature of our investigation, but is a common feature among parity-violating effects.

Quantum anomaly is literally quantum-mechanical, and thus is a tiny effect. However, when the coupled magnetic field is strong, this effect may be enhanced. So our physical motivation for this work is primarily oriented to the situation in which strong magnetic field is present with a finite density of baryonic matter. For this, one can come up with two important physical cases: one is a neutron star, in which neutrons are very dense and with a strong magnetic field, and the other is a heavy ion collision, in which nuclei

*eto@sci.kj.yamagata-u.ac.jp
†koji@riken.jp
‡hiida@riken.jp
§T.Ishii@damtp.cam.ac.uk
∥maezawa@ribf.riken.jp

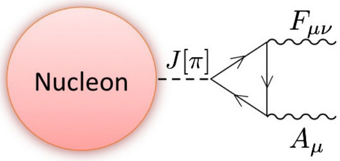

FIG. 1 (color online). A schematic figure for electric charge generation of a nucleon. In electromagnetic backgrounds, i.e., $F_{\mu\nu} \neq 0$, the chiral anomaly generates an additional coupling to the gauge fields A_μ.

are smashed and a strong electromagnetic field is expected to be created instantly. In this paper, we do not go into these concrete cases. We concentrate on providing a basis for that, and, in particular, evaluate in detail the anomaly WZW term with the quantized Skyrmions, under a constant magnetic field.

Since we have not explored the effects of the backreaction of the magnetic field to the Skyrmions, the result of our induced charge may not be conclusive. In this paper, we present some arguments that the backreaction may not be significant, but we shall wait for a further examinations in the Skyrme model to have a definite conclusion on the induced charge. In addition, we do not follow the time-dependent process of turning on the magnetic field gradually, so we cannot address the issue of the origin of the generated charges. We leave these as a future work.

The organization of this paper is as follows. In Sec. II, we provide a review of the Skyrme model and the WZW term, with a brief introduction to the Skyrmion solution. In Sec. III, we shall explicitly evaluate the anomaly term for the Skyrmions (baryons) in the magnetic field, and observe that there seems to appear the additional charge. We quantize the Skyrmion and evaluate the anomaly-induced electric current for an arbitrary baryon state. In Sec. IV, we evaluate the multipole moments of the anomaly-induced electric current and find a quadrupole, with a pion-mass dependence. In Sec. V, we discuss possible other effects due to the background magnetic field on the baryon. In Sec. VI, we evaluate classically the anomaly-induced charge for higher-charge (=multiple) Skyrmions. The final section is for our nonexhaustible conclusion and discussions. Appendix A is a study of the generated charge in a nonconstant magnetic field. In Appendix B, we show that the induced charge is due to a multipion effect (i.e., a pion cloud), and we compare our result with a point-particle description of baryons. The letter version of this paper is [10].

II. THE SKYRMIONS AND THE ANOMALY-INDUCED CHARGES

As briefly described in the introduction, it is indeed almost straightforward to calculate the effect of the anomaly term for baryons in the presence of the magnetic-field background, once we adopt a concrete model of the pion cloud. Here, we first review the Skyrme model, which realizes baryons as a condensation of the pions, and also review the gauged WZW term, which manifests the chiral anomaly in QCD.

A. The model

1. The Skyrme model

The chiral symmetry $SU(N)_L \times SU(N)_R$ acts on left-handed and right-handed quarks as

$$q_L \to U_L q_L, \qquad q_R \to U_R q_R, \quad \text{with} \quad U_{L,R} \in SU(N)_{L,R}. \tag{2}$$

When the chiral condensate $\bar{q}_R q_L$ develops a nonzero vacuum expectation value by some nonperturbative effects

$$\langle \bar{q}_R q_L \rangle = -v^3 \mathbf{1}_N, \quad \text{with} \quad v = \mathcal{O}(\Lambda_{\rm QCD}), \tag{3}$$

the axial part of the chiral symmetry is spontaneously broken as

$$SU(N)_L \times SU(N)_R \to SU(N)_{L+R}. \tag{4}$$

This gives rise to Nambu-Goldstone (NG) bosons, namely, the pions, which take value in the coset space $\frac{SU(N)_L \times SU(N)_R}{SU(N)_{L+R}}$,

$$U(x) = \exp\left(\frac{4i\pi^a(x)}{F_\pi} T^a\right), \qquad (a = 1, 2, \ldots, N^2 - 1). \tag{5}$$

Here $F_\pi = 108$ [MeV] is the pion decay constant and T^a is a generator of $SU(N)$ and we use the following standard normalization

$$\mathrm{Tr}[T^a T^b] = \tfrac{1}{2}\delta^{ab}. \tag{6}$$

The chiral symmetry acts on the NG modes as

$$U \to U_L U U_R^\dagger. \tag{7}$$

For later convenience, let us define left- and right-invariant Maurer-Cartan one-forms by

$$L_\mu \equiv U^\dagger \partial_\mu U, \qquad R_\mu \equiv \partial_\mu U U^\dagger. \tag{8}$$

These take their values in the algebra of $SU(N)_R$ and $SU(N)_L$, respectively. The chiral symmetry acts on them as

$$L_\mu \to U_R L_\mu U_R^\dagger, \qquad R_\mu \to U_L R_\mu U_L^\dagger. \tag{9}$$

We can think of U as an effective low-energy field. Its effective Lagrangian of the leading order to $\mathcal{O}(\partial^2)$ can be uniquely determined as

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{F_\pi^2}{16}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger + M_\pi^2(U + U^\dagger - 2)] \\ &= \frac{F_\pi^2}{16}\mathrm{Tr}[-R_\mu R^\mu + M_\pi^2(U + U^\dagger - 2)]. \end{aligned} \tag{10}$$

Here M_π stands for the pion mass $M_\pi = 137$[MeV] and our metric is $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. By expanding

L_μ and R_μ with respect to $1/F_\pi$, one gets

$$L_\mu = 4i\frac{\partial_\mu \pi^a}{F_\pi}T^a + 8i\epsilon^{abc}\frac{\pi^a\partial_\mu\pi^b}{F_\pi^2}T^c + \ldots, \quad (11)$$

$$R_\mu = 4i\frac{\partial_\mu \pi^a}{F_\pi}T^a - 8i\epsilon^{abc}\frac{\pi^a\partial_\mu\pi^b}{F_\pi^2}T^c + \ldots. \quad (12)$$

Plugging this into $\mathcal{L}^{(2)}$, one obtain a standard kinetic term of the pions and corrections

$$\mathcal{L}^{(2)} = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a - \frac{M_\pi^2}{2}\pi^a\pi^a - \frac{2}{3F_\pi^2}(\pi^a\pi^a\partial_\mu\pi^b\partial^\mu\pi^b - \pi^a\pi^b\partial_\mu\pi^a\partial^\mu\pi^b) + \frac{2M_\pi^2}{3F_\pi^2}(\pi^a\pi^a)^2 + \cdots. \quad (13)$$

We are interested in a topological soliton made by the pions in this work. The topological winding number is given by

$$\pi_3(SU(N)) = N_B \in \mathbb{Z}. \quad (14)$$

As will be shown, N_B is identified with the baryon number via the WZW term. However, it is easy to see, from a simple scaling argument, that no topological solitons can survive from collapsing in the theory with $\mathcal{L}^{(2)}$. So one needs higher-derivative corrections to $\mathcal{L}^{(2)}$. Therefore, we take a term of order $\mathcal{O}(\partial^4)$, which is so-called the Skyrme term

$$\mathcal{L}^{(4)} = \frac{1}{32e_s^2}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]), \quad (15)$$

with e_s being a dimensionless coupling constant. We will choose the parameter $e_s = 4.84$ by following Ref. [11]. Now we are ready to write down the Skyrme model with the right-invariant one-form as

$$\mathcal{L} = \frac{F_\pi^2}{16}\mathrm{Tr}[-R_\mu R^\mu + M_\pi^2(U + U^\dagger - 2)] + \frac{1}{32e_s^2}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]). \quad (16)$$

A Noether current of $SU(N)_L$ can be obtained by performing a local and infinitesimal $SU(N)_L$ rotation

$$\delta R_\mu = i\partial_\mu\phi_L. \quad (17)$$

Variation of the Skyrme Lagrangian is given by

$$\delta\mathcal{L} = \mathrm{Tr}\left\{\frac{i}{8}\left(-F_\pi^2 R_\mu + \frac{1}{e_s^2}[R_\nu, [R^\mu, R^\nu]]\right)\partial_\mu\phi_L\right\}. \quad (18)$$

Then the conserved current is given by

$$j_L^\mu = \frac{i}{8}\left(F_\pi^2 R^\mu - \frac{1}{e_s^2}[R^\nu, [R_\mu, R_\nu]]\right). \quad (19)$$

Similarly, the $SU(N)_R$ current takes the form

$$j_R^\mu = \frac{i}{8}\left(-F_\pi^2 L^\mu + \frac{1}{e_s^2}[L^\nu, [L_\mu, L_\nu]]\right). \quad (20)$$

These currents are related by

$$j_L^\mu = -U j_R^\mu U^\dagger, \quad (21)$$

where we have used

$$UL^\mu U^\dagger = U(U^\dagger\partial^\mu U)U^\dagger = R^\mu. \quad (22)$$

The equation of motion of the Skyrme model is identical to the current conservation law if the pion mass is zero

$$\partial_\mu j_L^\mu = 0, \quad \text{or} \quad \partial_\mu j_R^\mu = 0. \quad (23)$$

When the pion mass is nonzero, the equation of motion becomes

$$\partial_\mu j_L^\mu = -\frac{iF_\pi^2 m_\pi^2}{16}\mathrm{Tr}[U - U^\dagger]. \quad (24)$$

The vector and axial conserved currents are defined by

$$j_V^\mu = \frac{j_L^\mu + j_R^\mu}{2}, \qquad j_A^\mu = \frac{j_L^\mu - j_R^\mu}{2}. \quad (25)$$

The vector $SU(N)_{L+R}$ is nothing but the isospin, so we write its conserved charge as

$$I^a = \int d^3x j_V^{0a} = \int d^3x\,\mathrm{Tr}[(j_L^0 + j_R^0)T^a]. \quad (26)$$

2. Electromagnetic interaction

Let us next take the electromagnetic interaction into account. For simplicity, hereafter, we concentrate on the minimal case with two flavors $N = 2$. Since the electric charges of u and d quarks are $2/3$ and $-1/3$ respectively, the NG modes are rotated under the electromagnetic $U(1)$ as

$$U \to e^{-ieQ}Ue^{ieQ} = e^{-ieT^3}Ue^{ieT^3}, \quad Q = \frac{1}{6}\mathbf{1} + T^3, \quad (27)$$

where $T^3 = \tau^3/2$ with the Pauli matrix τ^a. Thus the electromagnetic $U(1)_{\rm em}$ is a subgroup of $SU(2)_{L+R}$.

Interactions of the NG modes and the electromagnetic fields are introduced by gauging the $U(1)_{\rm em} \subset SU(2)_{L+R}$ and replacing the partial derivative ∂_μ by a covariant derivative

$$\mathcal{D}_\mu U = \partial_\mu U + ieA_\mu[T^3, U]. \quad (28)$$

The left- and right-invariant one-forms are then replaced as

$$R_\mu \to \tilde{R}_\mu \equiv \mathcal{D}_\mu U U^\dagger, \qquad L_\mu \to \tilde{L}_\mu \equiv U^\dagger\mathcal{D}_\mu U. \quad (29)$$

Then the total Lagrangian can be read as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{F_\pi^2}{16}\mathrm{Tr}[-\tilde{R}_\mu\tilde{R}^\mu + (U + U^\dagger - 2)] + \frac{1}{32e_s^2}\mathrm{Tr}([\tilde{R}_\mu, \tilde{R}_\nu][\tilde{R}^\mu, \tilde{R}^\nu]). \quad (30)$$

The classical equation of motion is derived by variational method as before. One can easily obtain

$$\mathcal{D}_\mu \tilde{j}_L^\mu = -\frac{iF_\pi^2 M_\pi^2}{16}\mathrm{Tr}[U - U^\dagger], \quad (31)$$

$$\tilde{j}_L^{\,\mu} \equiv \frac{i}{8}\left(F_\pi^2 \tilde{R}^\mu - \frac{1}{e_s^2}[\tilde{R}_\nu, [\tilde{R}^\mu, \tilde{R}^\nu]]\right). \quad (32)$$

One can also express the E.O.M. in terms of the right-invariant one-form $\tilde{L}_\mu$ by just replacing $\tilde{R}_\mu$ with $\tilde{L}_\mu$. Note that since the electromagnetic charge Q breaks the chiral symmetry $SU(2)_L \times SU(2)_R \to U(1)_{\mathrm{em}}$, $\tilde{j}_L^\mu$ and $\tilde{j}_R^\mu$ are not conserved currents. The Maxwell equation is given by

$$\partial_\mu F^{\nu\mu} = e\tilde{j}_V^{3\nu}, \qquad \tilde{j}_V^{3\nu} = \mathrm{Tr}[T_3(\tilde{j}_L^\nu + \tilde{j}_R^\nu)]. \quad (33)$$

Thus the electromagnetic charge is given by

$$Q_e = e\int d^3x \tilde{j}_V^{3\nu=0} = eI^3. \quad (34)$$

3. WZW term and Chiral anomaly

In order to describe the baryons in the Skyrme model whose fundamental degrees of freedom are mesons, we have to consider not only elemental particles but also topological excitations. Since the NG modes do not carry the $U(1)_B$ charges, we need to add an extra terms to the above Lagrangian. It is the so-called Wess-Zumino-Witten term. With an electromagnetic field A_μ, the WZW term in the $N = 2$ flavor model is given by [12]

$$S_{\mathrm{WZW}}[A_\mu] = \int d^4x \frac{e}{2} j_B^\mu A_\mu, \quad (35)$$

with the baryonic current

$$j_B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[R_\nu R_\rho R_\sigma] - \frac{e}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(A_\rho P_\sigma). \quad (36)$$

Here we use $\epsilon^{0123} = -1$ and define

$$P_\sigma \equiv \frac{i}{2}\mathrm{Tr}[\tau^3(L_\sigma + R_\sigma)]. \quad (37)$$

This baryon current is clearly conserved due to the antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. On the other hand, j_B^μ appears to depend on gauge choices, at a glance. But this is not the case. One can rewrite the baryonic current as

$$j_B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[\tilde{R}_\nu\tilde{R}_\rho\tilde{R}_\sigma] - \frac{ie}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}\mathrm{Tr}[\tau^3(\tilde{L}_\sigma + \tilde{R}_\sigma)]. \quad (38)$$

This is manifestly gauge-invariant.

The first term in the current j_B^μ gives a topological number associated with $\pi_3(SU(2)_{L-R})$. Indeed, the integration of it over the space gives the topological winding number

$$N_B = \int d^3x \frac{1}{24\pi^2}\epsilon^{ijk}\mathrm{Tr}[R_iR_jR_k]. \quad (39)$$

The second term in the current j_{anm}^μ is a manifestation of the chiral anomaly. Actually, plugging Eqs. (11) and (12) into Eq. (35), one finds the famous $\pi^0 \to 2\gamma$ term by the anomaly

$$-\frac{N_c e^2}{48\pi^2 F_\pi}\epsilon^{\mu\nu\rho\sigma}A_\mu F_{\nu\rho}\partial_\sigma\pi^0 + \mathcal{O}(F_\pi^{-3}) = -\frac{N_c e^2}{12\pi^2 F_\pi}\pi^0\vec{E}\cdot\vec{B} + \mathcal{O}(F_\pi^{-3}), \quad (40)$$

where the equality holds up to a total derivative.

Let us next obtain the electric charge coupled to a photon fluctuation a_μ under a background electromagnetic field $\bar{A}_\mu$. To this end, we expand the gauge field as

$$A_\mu = \bar{A}_\mu + a_\mu. \quad (41)$$

Then the WZW action linear in a_μ gives

$$\begin{aligned} S_{\mathrm{WZW}}[a_\mu] &= \int d^4x\left(\frac{e}{2}j_B^\mu(\bar{A})a_\mu - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\bar{A}_\mu\partial_\nu(a_\rho P_\sigma)\right) \\ &= \int d^4x\epsilon^{\mu\nu\rho\sigma}\left[\left(\frac{e}{48\pi^2}\mathrm{Tr}[R_\nu R_\rho R_\sigma] - \frac{e^2}{16\pi^2}(\partial_\nu\bar{A}_\rho)P_\sigma\right)a_\mu - \frac{e^2}{16\pi^2}\bar{A}_\mu(\partial_\nu a_\rho)P_\sigma\right]. \end{aligned} \quad (42)$$

From this, we can read the electromagnetic current $j_{\mathrm{em,WZW}}^\mu = \delta S_{\mathrm{WZW}}/\delta a_\mu$ as

$$j_{\mathrm{em,WZW}}^\mu = j_{\mathrm{anm}}^\mu + \frac{\epsilon^{\mu\nu\rho\sigma}e}{48\pi^2}[\mathrm{Tr}[R_\nu R_\rho R_\sigma] + 3e\partial_\sigma(\bar{A}_\rho P_\nu)], \quad (43)$$

$$j_{\mathrm{anm}}^\mu \equiv -\epsilon^{\mu\nu\rho\sigma}\frac{e^2}{16\pi^2}(\partial_\nu\bar{A}_\rho)P_\sigma. \quad (44)$$

The current $j_{\mathrm{em,WZW}}^\mu$ is gauge-invariant [12]. The total electromagnetic current is

$$j_{\mathrm{em}}^\mu = e\tilde{j}_V^{3\mu} + j_{\mathrm{em,WZW}}^\mu. \quad (45)$$

This is gauge-invariant, and always conserved on the mass shell, from the gauge invariance.

The total electric charge is given by the spatial integral [13] of the zeroth component of this current (45)

$$Q_e = eI_3 + \frac{e}{2}N_B + \frac{e^2}{16\pi^2}\int d^3x B_i P_i. \tag{46}$$

Here B_i stands for a background magnetic field (we have used $\epsilon^{0ijk} = -\epsilon^{ijk}$), and I_3 is the isospin charge defined in (34) [14]. This is the expression of the gauge-invariant and conserved electric charge, which includes an extra term to the well-known Gell-Mann-Nishijima formula. The last term is a contribution from the chiral anomaly.

Note that, as we will see below shortly, the surface term in Eq. (43) does not contribute to the net electric charge if the pion mass is nonzero. So, we focus on the new last term of Eq. (46) coming from Eq. (44), in the following of this paper.

B. The Skyrmion: A nucleon as a topological soliton

Let us find a solution of the classical equations of motion derived previously,

$$\mathcal{D}_\mu \tilde{j}_L^\mu = 0, \qquad \partial_\mu F^{\nu\mu} = e\tilde{j}_V^\nu. \tag{47}$$

We solve these by dealing with the electromagnetic interaction as a perturbation. Then we expand the chiral field with respect to the electromagnetic coupling constant e as

$$\begin{aligned} U &= \exp(i\vec{\tau}\cdot(\vec{f}_0 + \vec{f}_1 + \cdots)),\\ A_\mu &= A_\mu^{(0)} + A_\mu^{(1)} + \cdots. \end{aligned} \tag{48}$$

At the zeroth order the chiral fields and electromagnetic fields are decoupled, so that the equations of motion are those of the Skyrme model without the electromagnetic interaction and the Maxwell equation without a source

$$\partial_\mu j_L^{(0)\mu} = -\frac{iF_\pi^2 M_\pi^2}{16}\mathrm{Tr}[U - U^\dagger], \qquad \partial_\mu F^{(0)\nu\mu} = 0, \tag{49}$$

with

$$j_L^{(0)\mu} = \frac{i}{8}\left(F_\pi^2 R^{(0)\mu} - \frac{1}{e_s^2}[R_\nu^{(0)}, [R^{(0)\mu}, R^{(0)\nu}]]\right). \tag{50}$$

The second equation in Eq. (49) is solved by considering a constant background magnetic field, say along the x_i-axis

$$\tfrac{1}{2}\epsilon^{ijk}F_{jk} = B^i \tag{51}$$

with B^i being a constant.

In order to solve the first equation in Eq. (49), it is useful to introduce a dimensionless coordinate

$$x_\mu \to \frac{1}{e_s F_\pi}x_\mu, \qquad \partial_\mu \to e_s F_\pi \partial_\mu. \tag{52}$$

In terms of this new coordinate, the Skyrme equation is written as

$$\partial_\mu(R^{(0)\mu} - [R_\nu^{(0)}, [R^{(0)\mu}, R^{(0)\nu}]]) = \frac{m_\pi^2}{2}\mathrm{Tr}[U^\dagger - U], \tag{53}$$

with a dimensionless mass in unit of $1/(e_s F_\pi)$

$$m_\pi \equiv \frac{M_\pi}{eF_\pi}. \tag{54}$$

Here and after, we will use this notation.

Let us make a standard hedgehog (radial) ansatz, for a static and topologically nontrivial solution with $N_B = 1$,

$$U_0(\vec{x}) = \exp(i\vec{f}_0\cdot\vec{\tau}) = \exp(if(r)\vec{\tau}\cdot\hat{\vec{x}}), \tag{55}$$

$$\hat{x}_i = \frac{x_i}{r}. \tag{56}$$

One can express U in a different fashion as

$$\begin{aligned} U_0 &= \mathbf{1}_2\cos|\vec{f}_0| + i\frac{\vec{f}_0}{|\vec{f}_0|}\cdot\vec{\tau}\sin|\vec{f}_0|\\ &= \mathbf{1}_2\cos f + i\hat{\vec{x}}\cdot\vec{\tau}\sin f. \end{aligned} \tag{57}$$

Then we obtain for a static configuration

$$R_i^{(0)} = if'\hat{x}_i\hat{\vec{x}}\cdot\vec{\tau} + \frac{i}{2r}(\tau_i - \hat{\vec{x}}\cdot\vec{\tau}\hat{x}_i)(\sin 2f + 2i\hat{\vec{x}}\cdot\vec{\tau}\sin^2 f). \tag{58}$$

Putting this into Eq. (53), we obtain the ordinary differential equation

$$\left(\frac{1}{4} + \frac{2\sin^2 f}{r^2}\right)f'' + \frac{1}{2r}f' + \frac{\sin 2f}{r^2}\left(f'^2 - \frac{1}{4} - \frac{\sin^2 f}{r^2}\right) = \frac{m_\pi^2}{4}\sin f. \tag{59}$$

The solution with a unit winding number corresponds to

$$\lim_{r\to\infty} f(r) = \pi, \qquad \lim_{r\to 0} f(r) = 0. \tag{60}$$

Numerical solutions with different m_π's are given in Fig. 2. We adopt the physical pion mass $m_\pi^{\mathrm{phys}} = 0.263$, which was determined from the mass splitting between nucleon and Δ [11]. We also show the profile functions for

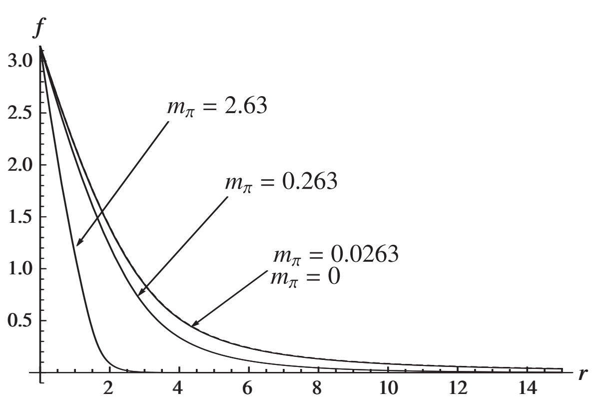

FIG. 2. Profile functions for the hedgehog solution.

$m_\pi = 2.63$ and 0.0263 in order to demonstrate a characteristic property of the profile function. We see that the larger (smaller) m_π gives the thinner (fatter) Skyrmion. The profile function with $m_\pi = 0.0263$ almost coincides to that with $m_\pi = 0$, see Fig. 2.

Asymptotic behavior of f can be found by solving the linearized equations of motion for large r

$$f'' + \frac{2}{r}f' - \frac{2f}{r^2} - m_\pi^2 f = 0. \tag{61}$$

This is solved by

$$f \simeq \left(\frac{C}{r^2} + \frac{A}{r}\right)e^{-m_\pi r}, \tag{62}$$

where C and $A(= Cm_\pi)$ are constant. For a massless pion, we find

$$f \simeq \frac{C}{r^2}, \tag{63}$$

with $C \simeq 8.638$. For a massive pion, the asymptotic form decays exponentially,

$$f \simeq \frac{A}{r}e^{-m_\pi r}, \tag{64}$$

where A is a constant that depends on m_π. For example, we find $A \simeq 2$ for $m_\pi = 0.263$.

III. ANOMALY-INDUCED CHARGE

We substitute the Skyrme solution to the electromagnetic current calculated from the WZW term, to evaluate the anomaly-induced electric charge of the baryons. We here use classical Skyrmions for an illustration first, then we move onto quantized Skyrmions, to obtain a formula for the anomaly-induced charge for baryon quantum states. The anomaly-induced charge is the last term of Eq. (46). Since we are considering a constant magnetic-field background, it is enough to see P_i defined in Eq. (37).

A. Classical evaluation

To evaluate P_i, let us first write down the left- and right-invariant one-forms as

$$R_i = i(\vec{\tau}\cdot\hat{\vec{x}})f'\hat{x}_i + \frac{i}{2}(\vec{\tau}\cdot\partial_i\hat{\vec{x}})\sin 2f + [(\partial_i\hat{\vec{x}}\cdot\hat{\vec{x}})\mathbf{1} + i(\partial_i\hat{\vec{x}}\times\hat{\vec{x}})\cdot\vec{\tau}]\sin^2 f, \tag{65}$$

$$L_i = i(\vec{\tau}\cdot\hat{\vec{x}})f'\hat{x}_i + \frac{i}{2}(\vec{\tau}\cdot\partial_i\hat{\vec{x}})\sin 2f + [(\partial_i\hat{\vec{x}}\cdot\hat{\vec{x}})\mathbf{1} - i(\partial_i\hat{\vec{x}}\times\hat{\vec{x}})\cdot\vec{\tau}]\sin^2 f. \tag{66}$$

Plugging these into Eq. (37), we have

$$P_i = -f'\hat{x}_i\hat{x}_3 - \frac{1}{2}(\partial_i\hat{x}_3)\sin 2f. \tag{67}$$

The topological charge density, P_1 and P_3 are shown in Fig. 3 at $m_\pi = 0$. The induced electric charge densities with nonzero m_π (see Fig. 4) are quite similar to those for the massless case in Fig. 3. A tiny difference comes from the similar but slightly different behaviors in profile functions $f(r)$, as shown in Fig. 2.

The spatial integrations of P_1 and P_3 become

$$\int d^3xP_1 = 0, \qquad \int d^3xP_3 = -\frac{4\pi}{3(e_sF_\pi)^2}c_0, \tag{68}$$

with

$$c_0 \equiv \int dr(r^2f' + r\sin 2f). \tag{69}$$

Note that the integration variable $r \to r/(e_sF_\pi)$ is the dimensionless coordinate, so that c_0 is a dimensionless number. This means that the net induced charge is zero

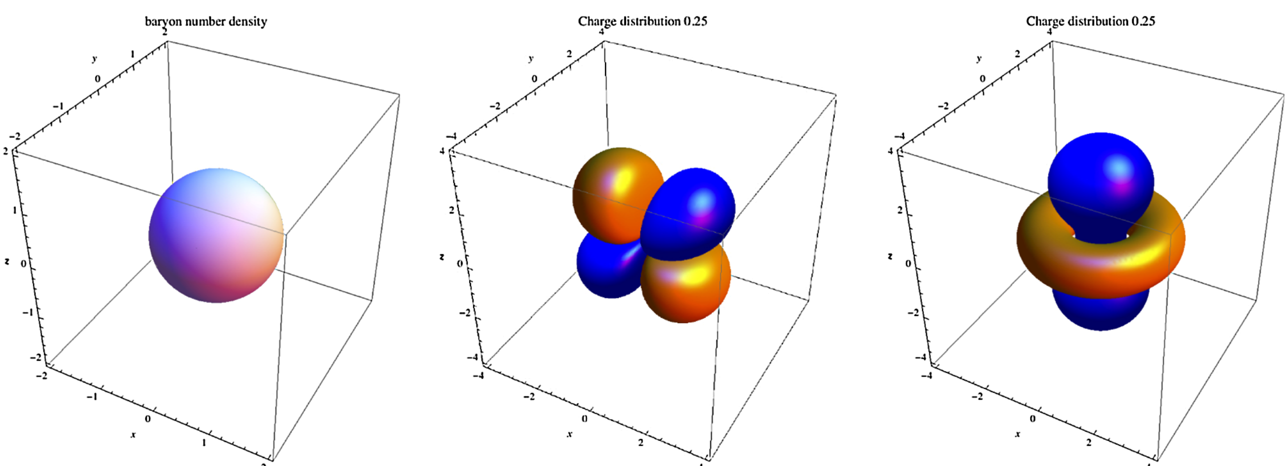

FIG. 3 (color online). $N_B = 1$ Skyrmion solution for $m_\pi = 0$. From left to right, contour plots of the baryon number density, $P_1 = \pm 0.2$ and $P_3 = \pm 0.2$, respectively.

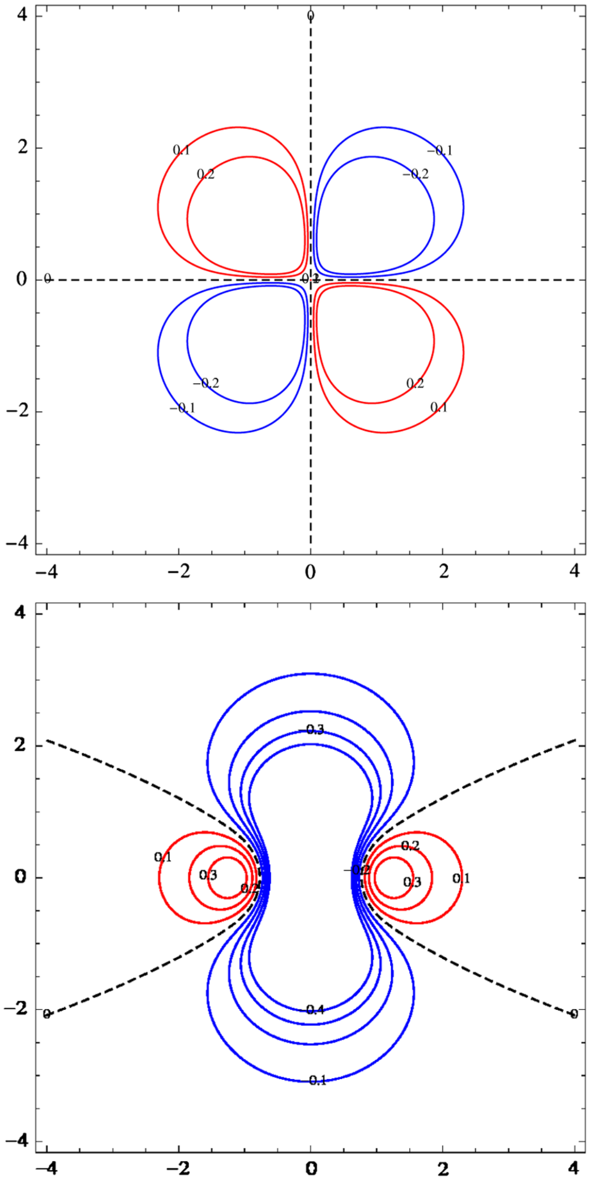

FIG. 4 (color online). The contour plots of the anomalous-charge densities of $B = 1$ Skyrmion for $\beta = 0.263$ on the cross section by the $y = 0$ plane. The top panel shows $-P_1/2$ with $B_1 \neq 0$ and the bottom panel shows $-P_3/2$ with $B_3 \neq 0$. The blue lines have positive values and the red ones have negative values. The black broken lines correspond to zero charge contours.

for $\vec{B} \propto (1, 0, 0)$, and (0,1,0), whereas it is nonzero for $\vec{B} \propto (0, 0, 1)$, where a nonzero correction appears to the Gell-Mann-Nishijima formula. The numerical coefficient c_0 can be rewritten as

$$c_0 = [r^2 f]_0^\infty + \int dr(-2rf + r\sin 2f). \tag{70}$$

The first term is a surface contribution that becomes nonzero only at $m_\pi = 0$ due to distinct behavior at large r shown in Eq. (63), and the second term expresses a pion-cloud effect discussed in Appendix B. Computational results of c_0 become

TABLE I. Numerical results of the coefficients c_0 and c_2. We neglect the surface contribution at $m_\pi = 0$ shown in Eq. (71) due to distinct behavior of the Skyrmion profile function $f(r)$ at $r \to \infty$, i.e., $f(r) \sim 1/r^2$, discussed in Sec. II B.

$m_\pi/m_\pi^{\rm phys}$	c_0	c_2
0	-1.41×10^1	$-\infty$
0.1	-1.37×10^1	-4.90×10^5
0.125	-1.37×10^1	-2.99×10^5
0.25	-1.34×10^1	-6.68×10^4
0.5	-1.23×10^1	-1.50×10^4
1	-1.02×10^1	-3.27×10^3
2	-7.32	-6.91×10^2
3	-5.67	-2.76×10^2
4	-4.62	-1.44×10^2
5	-3.88	-8.63×10^1
10	-2.20	-1.89×10^1
20	-1.15	-4.14
30	-7.97×10^{-1}	-1.74
50	-4.99×10^{-1}	-6.90×10^{-1}
100	-2.38×10^{-1}	-1.35×10^{-1}

$$c_0 = -14.1 + C \qquad (m_\pi = 0), \tag{71}$$

$$c_0 = -10.2 \qquad (m_\pi = 0.263), \tag{72}$$

where C is asymptotic factor shown in Eq. (63). Computational results for other values of the pion masses are also summarized in Table I.

Finally, we evaluate contributions from the surface term in Eq. (43). To this end, we need to compute

$$W = -\int d^3x \epsilon^{ijk} \partial_i (\bar{A}_j P_k). \tag{73}$$

Interestingly, plots of the integrand (density) are quite similar to those in Fig. 3 if we choose a gauge $\bar{A}_i = B_1(0, -z/2, y/2)$ or $\bar{A}_i = B_3(-y/2, x/2, 0)$. The surface term can be evaluated as

$$W = \int d\Omega_2 \left[\hat{x}_i \epsilon^{ijk} \bar{A}_j \left(f' \hat{x}_k \hat{x}_3 + \frac{(\partial_k \hat{x}_3) \sin 2f}{2} \right) \right]_{r\to\infty}.$$

As shown in Eq. (64), the profile function for $m_\pi \neq 0$ is exponentially small at large r, so that this integration vanishes.

Note that in the massless case the asymptotic behavior given in Eq. (62) leads to $W \neq 0$ as

$$W = \frac{32\pi i C}{3(e_s F_\pi)^2} B_3. \tag{74}$$

We see that this surface term with the constant C given in Eq. (62) cannot cancel the last term of Eq. (46). Anyway, since the physical pion mass is not zero, we consider the new last term in Eq. (46) as the anomaly-induced electric charge [15].

B. Evaluation with quantized Skyrmion

To evaluate the anomalous current and charge for each baryon state, the Skyrmion is quantized as a slowly rotating soliton. Quantizing the collective coordinates of soliton's moduli space G is achieved by the canonical quantization of a particle on a manifold G. In the case of two flavors, $G = SU(2) \simeq S^3$. We construct the angular momentum operators acting on baryon states and the harmonic functions on G corresponding to baryon wave function. Using these, we evaluate the expectation values of the anomalous currents.

We evaluate the expectation values $\langle j^\mu_{\rm anm}\rangle_B$ and the anomalous charge $Q^B_{\rm anm}$ for each baryon state B in the presence of a background magnetic field. We show that the spatial components of the currents vanish: $\langle j^i_{\rm anm}\rangle_B = 0$. We also obtain the anomalous charge $Q_{\rm anm}$, which is given by integrating $\langle j^0_{\rm anm}\rangle_B$.

1. Angular momentum operators and spherical harmonics on S^3

Let $g \in G$ be a group element of a group manifold G. Then there are operators $\mathcal{L}_a$ and $\mathcal{R}_a$ acting on g from left and right, respectively, and satisfying the commutation relations

$$[\mathcal{L}_a, \mathcal{L}_b] = if_{abc}\mathcal{L}_c, \qquad [\mathcal{R}_a, \mathcal{R}_b] = if_{abc}\mathcal{R}_c. \tag{75}$$

Here the roman indices correspond to those of the tangent space of G, and f_{abc} is the structure constant of the Lie algebra of G: $[T_a, T_b] = if_{abc}T_c$ and $\mathrm{Tr}(T_aT_b) = \delta_{ab}/2$. The actions of $\mathcal{L}_a$ and $\mathcal{R}_a$ on g are

$$\mathcal{L}_a g = -T_a g, \qquad \mathcal{L}_a g^{-1} = g^{-1}T_a, \qquad \mathcal{R}_a g = gT_a, \qquad \mathcal{R}_a g^{-1} = -T_a g^{-1}. \tag{76}$$

We restrict ourselves to the case $G = SU(2)$, and thus $f_{abc} = \epsilon_{abc}$. This is the case of the two-flavor Skyrmion.

The operators $\mathcal{L}_a$ and $\mathcal{R}_a$ are precisely the angular momentum operators with respect to the isometry $SU(2)_L \times SU(2)_R$ of S^3. We introduce the scalar spherical harmonics on S^3, $Y_{Jm\tilde{m}}$, where J is the same magnitude spins of both $SU(2)_L$ and $SU(2)_R$, and m and $\tilde{m}$ are the eigenvalues of their third components, respectively. The actions of the operators on the spherical harmonics are

$$\begin{aligned} \mathcal{L}^2 Y_{Jm\tilde{m}} &= \mathcal{R}^2 Y_{Jm\tilde{m}} = \sqrt{J(J+1)}Y_{Jm\tilde{m}},\\ \mathcal{L}_\pm Y_{Jm\tilde{m}} &= \sqrt{(J\mp m)(J\pm m+1)}Y_{J(m\pm 1)\tilde{m}},\\ \mathcal{R}_\pm Y_{Jm\tilde{m}} &= \sqrt{(J\mp \tilde{m})(J\pm \tilde{m}+1)}Y_{Jm(\tilde{m}\pm 1)},\\ \mathcal{L}_3 Y_{Jm\tilde{m}} &= mY_{Jm\tilde{m}}, \qquad \mathcal{R}_3 Y_{Jm\tilde{m}} = \tilde{m}Y_{Jm\tilde{m}}, \end{aligned} \tag{77}$$

where $\mathcal{L}_\pm = \mathcal{L}_1 \pm i\mathcal{L}_2$ and $\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2$.

It is convenient to introduce a D-function $D_{ab}(g)$ so as to see the relation of Eq. (76) and (77). It is defined by the adjoint action of g,

$$gT_ag^\dagger = T_bD_{ba}(g). \tag{78}$$

The action of $\mathcal{L}_a$ and $\mathcal{R}_a$ on $D_{ab}(g)$ becomes

$$\begin{aligned} \mathcal{L}_c D_{ab}(g) &= i\epsilon_{cad}D_{db}(g),\\ \mathcal{R}_c D_{ab}(g) &= -i\epsilon_{cdb}D_{ad}(g). \end{aligned} \tag{79}$$

With a little more algebra, it can be shown that appropriate linear combinations of $D_{ab}(g)$ precisely give the harmonic functions $Y_{Jm\tilde{m}}$.

Note that $\mathcal{L}_a$ and $\mathcal{R}_a$ are, respectively, the isospin and the spin operators for baryon states. See, for instance, Ref. [16] for a discussion in the three-flavor case. The adjoint action of g to the hedgehog Skyrmion gives $U(\mathbf{x}) = gU_0(\mathbf{x})g^\dagger = U_0(\mathbf{x}^{\rm rot})$. That is, $g\hat{\mathbf{x}}\cdot\boldsymbol{\tau}g^\dagger = \hat{\mathbf{x}}^{\rm rot}\cdot\boldsymbol{\tau}$. The transformation of the unit vector $\hat{\mathbf{x}}$ under the spatial rotation caused by $D(g)$ is

$$\hat{x}_a \longrightarrow \hat{x}^{\rm rot}_a = D_{ab}(g)\hat{x}_b. \tag{80}$$

It is natural to identify $SU(2)_R$ as the baryon spin, where $g \to gk_R$ with $k_R \in SU(2)_R$.

2. Absence of the spatial anomalous current

To evaluate $\langle j^i_{\rm anm}\rangle_B$ in the presence of a background magnetic field, we need to focus only on P_0, owing to the index structure of the WZW term. We first write down P_0 and then quantize it. Substituting a slowly rotating Skyrmion $U(\mathbf{x}, t) = g(t)U_0(\mathbf{x})g^\dagger(t)$ for P_0 (37), we obtain

$$P_0 = 2\sin(2f)\epsilon_{ac3}D_{ab}(g)\hat{x}_b\,\mathrm{Tr}[\tau_c\dot{g}g^\dagger]. \tag{81}$$

In the procedure of the canonical quantization, the time-derivative part is replaced with the angular momentum operator as follows [17]:

$$\begin{aligned} \mathcal{L}_a &= i\Lambda\,\mathrm{Tr}[\tau_a\dot{g}g^\dagger],\\ \Lambda &= \frac{8\pi}{3}\int dr r^2\sin^2 f\left[1 + 4\left(f'^2 + \frac{\sin^2 f}{r^2}\right)\right]. \end{aligned} \tag{82}$$

Hence P_0 can be written in terms of raising and lowering operators

$$\begin{aligned} P_0 = -\frac{1}{\sqrt{3}\Lambda}\sin(2f)[&\hat{x}_+(Y_{1--}\mathcal{L}_+ + Y_{1+-}\mathcal{L}_-)\\ &- \hat{x}_-(Y_{1-+}\mathcal{L}_+ + Y_{1++}\mathcal{L}_-)\\ &+ \hat{x}_3(Y_{1-0}\mathcal{L}_+ + Y_{1+0}\mathcal{L}_-)]_{\rm Weyl}, \end{aligned} \tag{83}$$

where the indices $\pm$ in $Y_{Jm\tilde{m}}$ mean ± 1, and $\hat{x}_\pm = \hat{x}_1 \pm i\hat{x}_2$. The Weyl ordering for the operators is understood.

Integrating a product of three spherical harmonics over S^3 gives

$$\int \frac{d\Omega_3}{2\pi^2}(Y_{J_1 m_1 \tilde{m}_1})^* Y_{J_2 m_2 \tilde{m}_2} Y_{J_3 m_3 \tilde{m}_3} = \sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}} C^{J_1 m_1}_{J_2 m_2 J_3 m_3} C^{J_1 \tilde{m}_1}_{J_2 \tilde{m}_2 J_3 \tilde{m}_3}, \tag{84}$$

where $C^{J_1 m_1}_{J_2 m_2 J_3 m_3}$ is a Clebsch-Gordan coefficient of $SU(2)$. The wave function of each baryon state is given by $Y_{Jm\tilde{m}}$. Our primary interest is in nucleons ($I = J = 1/2$) and Δ baryons ($I = J = 3/2$), but here we can keep J arbitrary. We use Eq. (84) to evaluate P_0 projected onto each baryon state.

In Eq. (83), we need to focus only on the last line

$$\mathcal{O}_W \equiv (Y_{1-0}\mathcal{L}_+ + Y_{1+0}\mathcal{L}_-)_{\rm Weyl} = Y_{1-0}\mathcal{L}_+ + Y_{1+0}\mathcal{L}_- + \sqrt{2}Y_{100}. \tag{85}$$

It is easily seen that each of the first and the second lines in Eq. (83) gives no contribution. Integrating Eq. (85) by using Eq. (84) along with baryon states of quantum numbers (J, I_3, S_3) for (iso)spin and the third components, we obtain

$$\langle \mathcal{O}_W\rangle_B = \int \frac{d\Omega_3}{2\pi^2}(Y_{JI_3-S_3})^* \mathcal{O}_W Y_{JI_3-S_3} = \sqrt{3}C^{J-S_3}_{10J-S_3}\big(\sqrt{(J-I_3)(J+I_3+1)}C^{JI_3}_{1-1J(I_3+1)} + \sqrt{(J+I_3)(J-I_3+1)}C^{JI_3}_{11J(I_3-1)} + \sqrt{2}C^{JI_3}_{10JI_3}\big), \tag{86}$$

where the minus sign appearing in front of S_3 is due to Eq. (79). This exactly vanishes once values of the Clebsch-Gordan coefficients are substituted [18]

$$C^{J\gamma}_{10J\gamma} = \frac{\gamma}{\sqrt{J(J+1)}}, \qquad C^{J\gamma}_{1\pm 1J(\gamma\mp 1)} = \mp\sqrt{\frac{(J\pm\gamma)(J\mp\gamma+1)}{2J(J+1)}}. \tag{87}$$

Thus we see $\langle j^i_{\rm anm}\rangle_B = 0$.

3. *Anomalous electric charge in baryons*

In $j^0_{\rm anm}$, rotation of the Skyrmion is encoded in $\hat{x}^{\rm rot}_3$. Hence, it is sufficient to evaluate $\langle \hat{x}^{\rm rot}_3\rangle_B$, which directly leads us to $\langle j^0_{\rm anm}\rangle_B$. Since this part does not contain derivatives in time, we simply integrate Eq. (80) by using Eq. (84). The result is

$$\langle \hat{x}^{\rm rot}_3\rangle_B = -\frac{I_3 S_3}{J(J+1)}\hat{x}_3. \tag{88}$$

Below, we will mainly focus on the case $J = 1/2$ for simplicity. However, thanks to Eq. (88), it is straightforward to consider higher-spin cases. For instance, this gives $\langle \hat{x}^{\rm rot}_3\rangle^N = -4I_3S_3\hat{x}_3/3$ for nucleons, and $\langle \hat{x}^{\rm rot}_3\rangle^\Delta = -4I_3S_3\hat{x}_3/15$ for Δ baryons.

Let us calculate the total electric charge from the anomalous effect. The matrix elements of P_μ are evaluated by applying Eq. (88),

$$\langle P_0\rangle^N_{I_3,S_3} = 0, \tag{89}$$

$$\langle P_a\rangle^{Na=1,2}_{I_3,S_3} = -\frac{16i}{3}I_3S_3\Big(f' - \frac{\sin(2f)}{2r}\Big)\hat{x}_a\hat{x}_3, \tag{90}$$

$$\langle P_3\rangle^N_{I_3,S_3} = -\frac{16i}{3}I_3S_3\Big[\Big(f' - \frac{\sin(2f)}{2r}\Big)\hat{x}^2_3 + \frac{\sin(2f)}{2r}\Big]. \tag{91}$$

The anomalous-charge density under a constant magnetic field $\mathbf{B}$ is indeed induced in nucleons

$$\langle j^0_{\rm anm}\rangle^N_{I_3,S_3} = \frac{ie^2N_c}{48\pi^2}B_i\langle P_i\rangle^N_{I_3,S_3}. \tag{92}$$

The integration of $\langle P_i\rangle^N_{I_3,S_3}$ over the whole space yields

$$\int d^3x\langle P_i\rangle^N_{I_3,S_3} = \begin{cases} 0 & (i = 1, 2), \\ -\frac{16\pi i}{9}(4I_3S_3)c_0 & (i = 3), \end{cases}$$

where c_0 is defined in Eq. (70). The numerical values of c_0 are shown in Table I for several pion masses. From Eq. (92) , we obtain the anomalous charge for nucleons

$$Q^N_{\rm anm} = \frac{4e^2N_c}{27\pi}I_3S_3\frac{c_0B_3}{(e_sF_\pi)^2}. \tag{93}$$

In this final expression we restored the rescaling factor e_sF_π by a dimensional counting. Equation (93) shows that an electric charge is actually induced by the anomalous effect *even for a neutron*.

As seen from Eq. (88), dividing the result in Eq. (93) by a factor of 5 gives the anomalous charge of Δ baryons.

The plot of the charge density of $\langle j^0_{\rm anm}\rangle$ for the quantized Skyrmion shows exactly the same as Fig. 3. For a magnetic field along x^1 direction, the charge density plot is symmetric, thus the total charge vanishes. However, obviously multipoles, in particular, a quadrupole, may show up. In the next section, we calculate multipoles in $\langle j^0_{\rm anm}\rangle$.

IV. MULTIPOLE MOMENTS OF ANOMALOUS CHARGES AND PION-MASS DEPENDENCE

When one regards charged baryons as pointlike particles, multipole moments are suitable physical quantities to describe an original charge distribution. In this section, we extend the calculations to the higher multipole moments due to the anomalous-charge distributions, and estimate pion-mass dependence of the anomalous charges and the multipole moments.

First, we can easily find that the dipole moment due to the anomalous charge vanishes

$$D_i \equiv \int d^3x x_i\langle j^0_{\rm anm}\rangle^N = 0 \qquad (i = 1, 2, 3). \tag{94}$$

On the other hand, the quadrupole moment

$$Q_{ij} \equiv \int d^3x(3x_ix_j - r^2\delta_{ij})\langle j^0_{\rm anm}\rangle^N, \tag{95}$$

is calculated for the nucleon as

$$Q_{ij} = e\frac{2N_c}{135\pi}(I_3S_3)\tilde{Q}_{ijk}\frac{eB_k}{(e_sF_\pi)^4}c_2, \tag{96}$$

$$\tilde{Q}_{ijk} \equiv \begin{pmatrix} -2\delta_{k3} & 0 & 3\delta_{k1} \\ 0 & -2\delta_{k3} & 3\delta_{k2} \\ 3\delta_{k1} & 3\delta_{k2} & 4\delta_{k3} \end{pmatrix}_{ij}, \tag{97}$$

where the numerical coefficient,

$$c_2 = \int dr[2r^4f' - r^3\sin(2f)], \tag{98}$$

is shown in Table I for several pion masses. This means that the leading multipole due to the anomalous contribution is the quadrupole moment. We note that the quadrupole is induced in response to all directions of the external magnetic fields, although the anomalous charge is induced only by B_3 (see Eq. (93)).

In order to extract the pion-mass dependence of the anomalous charge and the quadrupole moment, we calculate the Skyrmion profile function $f(r)$ for wide pion-mass range ($0.1 \le m_\pi/m_\pi^{\rm phys} \le 100$). Behavior of $f(r)$ for several pion masses is shown in Fig. 5, where the solid line is $f(r)$ at $m_\pi = m_\pi^{\rm phys}$. We find that the wave function shrinks with the pion mass increasing.

Since the pion-mass-dependence of the anomalous charge and the quadrupole moment appears in the numerical coefficients, c_0 and c_2, via r integration with $f(r; m_\pi)$, we focus on these coefficients. Figure 6 shows results of $-c_0$ (top) and $-c_2$ (bottom) as a function of $m_\pi/m_\pi^{\rm phys}$ in log-log scale. Numerical values of c_0 and c_2 are also summarized in Table I. We can see that c_0 becomes almost plateau at small pion mass ($m_\pi/m_\pi^{\rm phys} < 1$), whereas that decreases linearly at large pion mass ($m_\pi/m_\pi^{\rm phys} > 10$). We

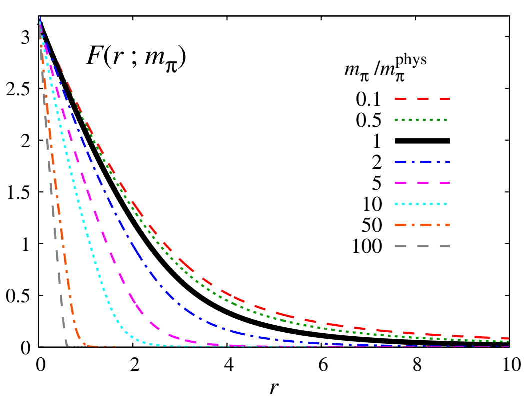

FIG. 5 (color online). Behavior of the Skyrmion profile functions, $f(r)$, for several pion masses.

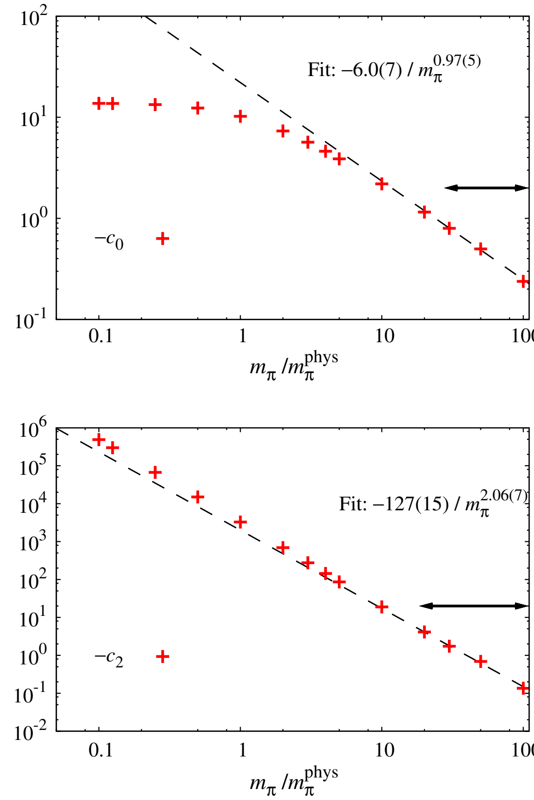

FIG. 6 (color online). Results of the numerical coefficients, $-c_0$ (top) and $-c_2$ (bottom), as a function of $m_\pi/m_\pi^{\rm phys}$ in log-log scale. The dotted lines mean fit results by a function A/m_π^n, with A and n being free parameters in the fit ranges shown by the arrows on the figures. Numerical results of the fit are also shown on the figures.

fit the results by a function, A/m_π^n, with A and n being free parameters, and obtain $c_0 \sim -6.0(7)/m_\pi^{0.97(5)}$ in a range of $30 \le m_\pi/m_\pi^{\rm phys} \le 100$, shown by the dashed line on Fig. 6 (top). This implies that the pion-mass dependence of the anomalous charge is $Q_{\rm anm} \propto 1/m_\pi$ at large pion mass.

On the other hand, c_2 behaves linearly for all pion-mass region. We also fit the results by A/m_π^n, and obtain $c_2 \sim -127(15)/m_\pi^{2.06(7)}$ in a range of $20 \le m_\pi/m_\pi^{\rm phys} \le 100$, shown by the dashed line on Fig. 6 (bottom). Although the fit is performed at a large pion-mass region, almost all results of c_2 are located around the dashed line. This implies that the quadrupole moment due to the anomaly behaves as $Q_{ij} \propto 1/m_\pi^2$.

Note that we have evaluated $j^\mu_{\rm anm}$, which is only a part of the total electromagnetic current. [19]

V. NO CONTRIBUTION FROM OTHER CORRECTIONS

In this section, we study other effects of the background magnetic field to the electric charge of the nucleon. Our aim is to show that the anomaly-induced electric charge is not canceled by the other electromagnetic effects.

The total electric charge is written as a modified Gell-Mann–Nishijima formula,

$$Q_e = eI_3 + \frac{eN_B}{2} + \frac{Q_{\rm anm}}{2}. \quad (99)$$

The first term stems from the electromagnetic current in the original Skyrme model. The second term is due to the baryon number coupling to the electromagnetic potential. The last one is the anomaly-induced electric charge, which is nonzero only when we have the background electromagnetic field.

We are working with the perturbative expansion with respect to the background magnetic field eB. What we found for the anomaly-induced charge is

$$Q_{\rm anm} = \mathcal{O}(e^2 B). \quad (100)$$

In the total charge formula (99), the second term is due to the baryon charge, which is a topological charge for the Skyrme model, thus not corrected by the background magnetic field. On the other hand, the first term can be corrected in the presence of B. If a correction of the order $\mathcal{O}(eB)$ appears from the I_3 term in the charge formula, then it may possibly cancel our anomaly-induced charge. In the following, we shall present an argument showing that there is no such correction of $\mathcal{O}(eB)$ to the I_3 term.

First, let us examine if there is a correction to the electromagnetic current itself in the Skyrme model. One may naively think that, since the Skyrme solution itself is corrected by the background electromagnetic field, the current may also be corrected. However, this is not the case for the Skyrmion. The reason is that for the Skyrmion, the electromagnetic $U(1)$ is identical to a part of the isospin, and the action itself has the isospin structure from the first place. In fact, the relevant I_3 is indeed expressed by a part of the flavor $SU(2)$ rotation, and thus the current I_3 is universally expressed as

$$I_3 = \frac{i}{2}\left[a_0 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_0} - a_1 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_1}\right] \quad (101)$$

Here, there is no room for eB to show up, thus we can safely use this expression for I_3 in the electric charge formula (99).

Then, the issue is whether the expectation value of I_3 in the background magnetic field is corrected or not. The background magnetic field modifies the wave function of the Skyrmion, so in principle this corrected wave function may give a correction to $\langle I_3\rangle$, which is of importance for us. We shall show in the following that there is no such correction at $\mathcal{O}(eB)$.

To proceed, we need to know how the Skyrme moduli wave function is corrected. Because of the background magnetic field, there appears a potential in the moduli space, then two of the moduli parameters are lifted to become pseudomoduli parameters. The corrected quantum mechanics of the moduli and the pseudomoduli is written as

$$S = 2\lambda \sum_{i=0}^{3} [(\dot{a}_i)^2] - eB_3 V(\vec{a}), \quad (102)$$

where the potential of the quantum mechanics is of the form $V(\vec{a}) = ((a_1)^2 + (a_2)^2)\tilde{V}(\vec{a})$. The function $\tilde{V}(\vec{a})$ is a polynomial of a_i (with a finite order), with just numerical coefficients. The potential $V(\vec{a})$ breaks the $SU(2)$ symmetry of the system down to the diagonal $U(1)$.

We briefly explain how to derive the form (102) of the induced potential. In the Skyrme model, the electromagnetic interaction enters as

$$\hat{R}_\mu \equiv D_\mu U U^\dagger, \qquad D_\mu U \equiv \partial_\mu + ieA_\mu[q, U]. \quad (103)$$

For a background magnetic field B_3, we consider $A_1 = -B_3 x^2$, thus among $\hat{R}_\mu$ the electromagnetic contribution appears only in $\hat{R}_1$ as

$$\hat{R}_1 = \partial_1 U U^\dagger + \delta\hat{R}_1, \quad (104)$$

$$\delta\hat{R}_1 \equiv -ieB_3 x^2 [q, GU_0 G^\dagger] G U_0^\dagger G^\dagger. \quad (105)$$

Here we have already substituted the Skyrme solution. Assuming that G is dependent on t and plugging this $\hat{R}$ into the action, we obtain a correction to the Skyrme Lagrangian at $\mathcal{O}(eB)$ as

$$\delta L = \frac{F_\pi^2}{8} {\rm Tr}[R_1 \delta\hat{R}_1] + \frac{1}{8e_s^2} {\rm Tr}[R_\mu, R_1][R^\mu, \delta\hat{R}_1]. \quad (106)$$

Since we know that the Skyrme solution has the particular dimension dependence $x \to x/(e_s F_\pi)$, we obtain

$$\delta S = \int d^4x \delta L = \int dt \frac{1}{e_s^3 F_\pi} eB_3 V(\vec{a}), \quad (107)$$

where $V(\vec{a})$ is a polynomial in a_i, with only dimensionless numerical coefficients. This is nothing but the potential in Eq. (102). As the potential $V(\vec{a})$ should vanish when G corresponds to the electromagnetic direction, i.e., the τ_3 direction, $V(\vec{a})$ is proportional to $(a_1)^2 + (a_2)^2$.

With the potential, the Skyrme wave function $\psi(\vec{a})$ is modified. We may apply a well-known perturbation technique for quantum mechanics, and obtain the corrected nucleon wave function as

$$\begin{aligned} |l = 1/2\rangle &= |l = 1/2\rangle_0 \\ &\quad + eB_3 \sum_{n=1}^{\infty} \frac{V_{l=1/2, l=n+1/2}}{E_{l=1/2} - E_{l=n+1/2}} |l = n + 1/2\rangle_0 \\ &\quad + \mathcal{O}((eB)^2). \end{aligned} \quad (108)$$

Here $V_{l=1/2,l=n+1/2}$ is the matrix element of the operator V appearing in the quantum mechanics (102), and the state with subscript 0 is the one without the perturbation. In the current case the states have degenerate energy, but the expression above is universal.

Now, using this corrected wave function, we evaluate the expectation value of I_3. Since we have

$$ {}_0\langle l = n + 1/2 | I_3 | l = 1/2 \rangle_0 = 0 \tag{109} $$

for $n \geq 1$, we obtain

$$ \langle I_3 \rangle = \langle I_3 \rangle_0 + \mathcal{O}((eB)^2), \tag{110} $$

where $\langle I_3 \rangle_0$ is the third component of the isospin of the leading (uncorrected) order wave function. Therefore, the electromagnetic correction to the charge formula starts at $\mathcal{O}((eB)^2)$, which is at higher order compared to the anomaly-induced charge $Q_{\rm anm}$. This means that our anomaly-induced charge $Q_{\rm anm}$ is the leading-order correction of $\mathcal{O}(eB)$, and cannot be canceled by the other effect of the background magnetic field.

Here we have presented an argument that the total induced charge due to the anomaly is not canceled by other corrections due to the magnetic field. This argument may be reinforced and supplemented by an explicit computation of a backreaction to the Skyrme configuration itself, from the magnetic field. The calculation of the backreaction is quite complicated, so we leave it to our future work.

VI. HIGHER-CHARGE SKYRMIONS

In this section, we study classical higher-charge Skyrmions and the anomaly-induced charges. To this end, we will utilize the so-called rational map ansatz [20], which is a reasonable method giving a good approximation. In this section we use another notation based on a standard textbook [21]. A main difference from the previous sections is the dimensionless coordinate

$$ x_\mu \to \frac{2x_\mu}{e_s F_\pi}, \qquad \partial_\mu \to \frac{e_s F_\pi \partial_\mu}{2}. \tag{111} $$

Let us first give a brief review on the rational map ansatz. A solution of the Skyrmion $U(\mathbf{x})$ with $U(\mathbf{x}) \to \mathbf{1}$ as $|\mathbf{x}| \to \infty$ gives a map from $\mathbf{R}^3 + \{\infty\} \simeq S^3$ to $SU(2) \simeq S^3$. The map is characterized by the homotopy group $\pi_3(SU(2)) = \mathbf{Z}$. More explicitly, it can be expressed as

$$ U(\mathbf{x}) = \exp(i f_B(r) \vec{\tau} \cdot \vec{n}(\theta, \phi)), \tag{112} $$

where $\{f_B, \mathbf{n}\}$ is a coordinate of $SU(2)$ under a constraint $f_B \in [0, \pi]$ and $|\vec{n}| = 1$: Namely, we decompose $SU(2)$ into $I_{[0,\pi]} \times S^2$. The parameters (r, θ, ϕ) are standard spherical coordinates on the space $\mathbf{R}^3 \simeq \mathbf{R}_{\geq 0} \times S^2$. In order to get the map of degree $N_B \in \mathbf{Z}$, we assume that f_B is a one-to-one map from $R_{\geq 0} \to I_{[0,\pi]}$. Then $\vec{n}(\theta, \phi)$ should give a map $S^2 \to S^2$ of degree N_B. Let us introduce the stereographic projection, which is useful to find the generic map of degree N_B,

$$ z(\theta, \phi) = e^{i\phi} \tan\frac{\theta}{2}. \tag{113} $$

For example, the $N_B = 1$ hedgehog ansatz (the one-to-one map from S^2 to S^2) can be expressed by

$$ \vec{n}(z, \bar{z}) = \left(\frac{z + \bar{z}}{1 + |z|^2}, -i \frac{z - \bar{z}}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right). \tag{114} $$

This can be easily extended to a N_B-to-one map from S^2 to S^2 by replacing z with any rational maps $w(z) : \mathbf{C}$ to $\mathbf{C}$

$$ w(z) = \frac{P(z)}{Q(z)}. \tag{115} $$

Here $P(z)$ and $Q(z)$ are holomorphic functions in z and we set $N_B = \max\{\deg P, \deg Q\}$. Thus, we obtain the map from S^2 to S^2 of degree N_B

$$ \vec{n} = \left(\frac{w + \bar{w}}{1 + |w|^2}, -i \frac{w - \bar{w}}{1 + |w|^2}, \frac{1 - |w|^2}{1 + |w|^2} \right). \tag{116} $$

Plugging this into Eq. (112), we reach the map from S^3 to $SU(2)$ of degree N_B. This is called the rational map ansatz.

The baryon number can be expressed by

$$ N_B = -\int \frac{f_B'}{2\pi^2} \left(\frac{\sin f_B}{r} \frac{1 + |z|^2}{1 + |w|^2} \right)^2 \left| \frac{dw}{dz} \right|^2 r^2 dr d\Omega_z, \tag{117} $$

where $d\Omega_z$ is the usual area element on a 2-sphere

$$ d\Omega_z = \frac{2i dz d\bar{z}}{(1 + |z|^2)^2} = \sin\theta d\theta dr. \tag{118} $$

By making use of the following pullback

$$ \left(\frac{1 + |z|^2}{1 + |w|^2} \right)^2 \left| \frac{dw}{dz} \right|^2 d\Omega_z = d\Omega_w, \tag{119} $$

it is easy to change the integral area over S^2 to the target space of the rational map S^2 as

$$ \begin{aligned} \text{r.h.s of Eq. (117)} &= -\int \frac{f_B'}{2\pi} \sin^2 f_B dr d\Omega_w \\ &= -\frac{2N_B}{\pi} \int_\pi^0 \sin^2 f_B df_B = N_B, \end{aligned} \tag{120} $$

where we have used $\int d\Omega_w = 4\pi N_B$. The Skyrmion energy in the $F_\pi/(4e_s)$ energy unit and $2/(e_s F_\pi)$ length unit with the rational map ansatz is given by

$$ \begin{aligned} E = 4\pi \int_0^\infty dr \Big[& r^2 f_B'^2 + 2N_B (f_B'^2 + 1) \sin^2 f_B \\ & + I \frac{\sin^4 f_B}{r^2} + 8 m_\pi^2 (1 - \cos f_B) \Big], \end{aligned} \tag{121} $$

where we have introduced

$$ I \equiv \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |w|^2} \right)^4 \left| \frac{dw}{dz} \right|^4 d\Omega_z. \tag{122} $$

In order to find the best approximation, we need to seek an appropriate rational map $w(z)$. We should choose w in such a way that I is minimized. Though this is not easy task, by using a numerical method, the rational maps $w(z)$

for several N_B were found in Ref. [20]. For instance, the following rational maps for $N_B = 1, 2, \ldots, 8$ are known as

$$w_1 = z, \tag{123}$$

$$w_2 = z^2, \tag{124}$$

$$w_3 = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}, \tag{125}$$

$$w_4 = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}, \tag{126}$$

$$w_5 = \frac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1}, \tag{127}$$

$$w_6 = \frac{z^4 + ic}{z^2(icz^4 + 1)}, \tag{128}$$

$$w_7 = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1}, \tag{129}$$

$$w_8 = \frac{z^6 - d}{z^2(dz^6 + 1)}, \tag{130}$$

with $a = 3.07$, $b = 3.94$, $c = 0.16$ and $d = 0.14$.

The last task is to determine the profile function f_B. Because no analytic solutions have been known, we need to solve equations of motion numerically

$$\left(1 + \frac{2N_B}{r^2}\sin^2 f_B\right)f_B'' + \frac{2}{r}f_B' + \frac{N_B \sin 2f_B}{r^2}\left(f_B'^2 - 1 - \frac{I}{N_B}\frac{\sin^2 f_B}{r^2}\right) - 4m_\pi^2 \sin f_B = 0, \tag{131}$$

with the boundary condition $f_B(0) = \pi$ and $f_B(\infty) = 0$. We show several numerical solutions for $N_B = 2$ with different pion masses in Fig. 7.

Now we are ready to evaluate the anomaly-induced electric charge from Eqs. (37) and (46). As before, what we need is only P_i, which can be obtained from

$$R_i = i(\vec{\tau}\cdot\vec{n})f_B'\hat{x}_i + \frac{i}{2}(\vec{\tau}\cdot\partial_i\vec{n})\sin 2f_B + \{(\partial_i\vec{n}\cdot\vec{n})\mathbf{1} + i(\partial_i\vec{n}\times\vec{n})\cdot\vec{\tau}\}\sin^2 f_B, \tag{132}$$

$$L_i = i(\vec{\tau}\cdot\vec{n})f_B'\hat{x}_i + \frac{i}{2}(\vec{\tau}\cdot\partial_i\vec{n})\sin 2f_B + \{(\partial_i\vec{n}\cdot\vec{n})\mathbf{1} - i(\partial_i\vec{n}\times\vec{n})\cdot\vec{\tau}\}\sin^2 f_B. \tag{133}$$

By plugging these into Eq. (37), we get

$$P_i = -f_B' n_3 \hat{x}_i - \frac{1}{2}(\partial_i n_3)\sin 2f_B. \tag{134}$$

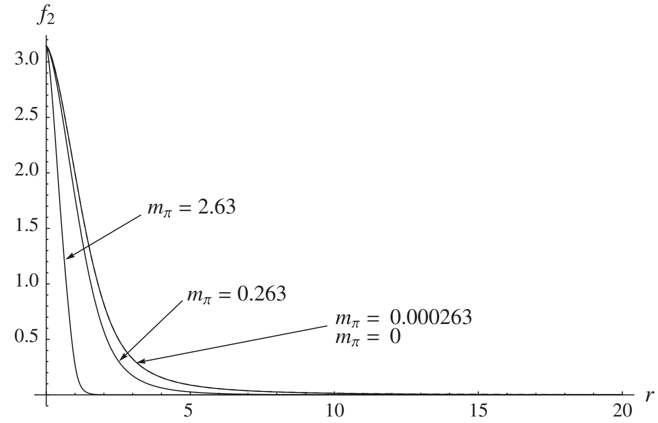

FIG. 7. The profile functions $f_{B=2}$ for $N_B = 2$.

Note that, as expected, replacement n_3 with $\hat{x}_3$ gives us a $N_B = 1$ hedgehog solution.

The induced charge densities for the $N_B = 2$ solution are shown in Fig. 8. As one can see, the $N_B = 1$ and $N_B = 2$ charge distributions are quite similar, even though the baryon charge distributions are totally different. However, one can find differences if paying attention to the detail structures. As can be seen in Figs. 4 and 9, the $N_B = 2$ densities are fatter than those of $N_B = 1$. Also the $N_B = 2$ configuration has an internal structure.

The anomaly-induced charges of the classical Skyrmions with $N_B = 1, 2, \ldots, 8$ under a constant background magnetic field $\vec{B} = (0, 0, B_3)$

$$Q_{\rm anm}^{\rm classical} = \frac{e^2}{16\pi^2(e_s F_\pi)^2}B_3\tilde{c}_0, \tag{135}$$

$$\tilde{c}_0 = 4\int d^3x P_3, \tag{136}$$

are summarized in the Table II. The prefactor 4 is needed because of the dimensionless coordinate $x_\mu \to 2x_\mu/e_s F_\pi$. Note that c_0 and $\tilde{c}_0$ for $N_B = 1$ are related by $\tilde{c}_0 = 4\pi c_0/3$. We find that the classical anomaly-induced charge is not proportional to the baryon charge N_B. It is intriguing that $N_B = 4$ and 7 Skyrmions have zero induced electric charge. From the values given in the Table II, we observe that higher-charge Skyrmions tend to cancel the total induced charge. A natural reason for this cancelation is as follows. Each Skyrmion has a classical orientation in spin and isospin space, and to form a bound state of the Skyrmions the orientations should be arranged to cancel each other. Our formula of the anomaly-induced charge depends on the signs of the quantum spin and isospin, so, accordingly the total anomaly-induced charge would tend to cancel each other. Although we have not performed quantization of the higher-charge Skyrmions, we expect that this cancelation should occur even at the quantized level.

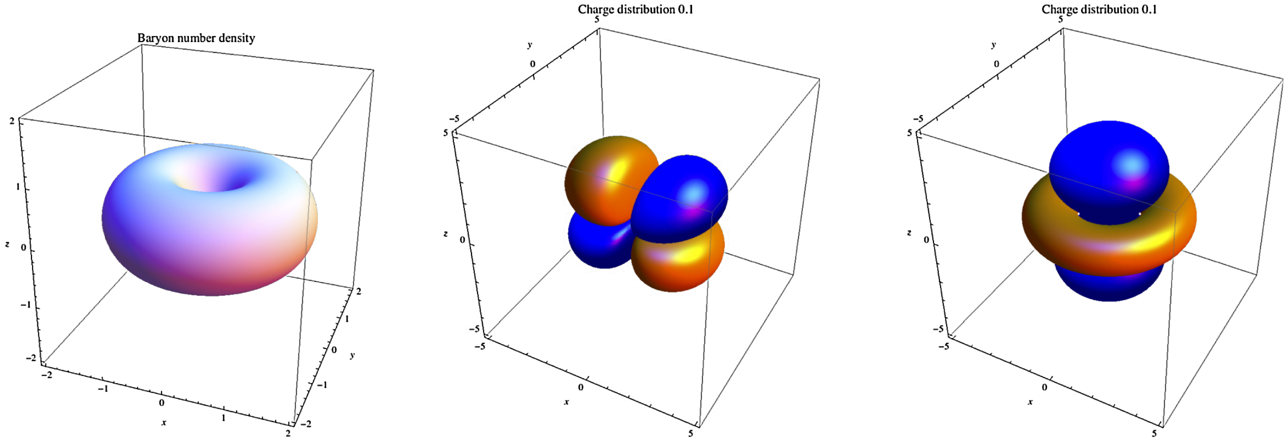

FIG. 8 (color online). $N_B = 2$ Skyrmion solution. From left to right, the baryon number density, P_1 and P_3, respectively.

Let us finally display the baryon number densities and anomaly-induced electric charges of the Skyrmions with $N_B = 3, 4, \cdots, 8$ and $N_B = 17$ [10], see Fig. 10. The anomaly-induced charge densities exhibit amusing shapes. Possible interpretation of the shape is an open question.

VII. CONCLUSION AND DISCUSSION

We have evaluated the gauged WZW term for quantized Skyrmions under a background magnetic field in the expansion of the electromagnetic coupling constant. We have found that there is an anomaly-induced charge structure due to the gauged WZW term. The detailed analysis of the total induced charge suggests that the pion cloud of the baryons can induce a net charge. The magnitude of the induced charge structure is $\mathcal{O}(e^2B)$, so it is quite small except for the case with a strong magnetic-field background. Although the generation of the charge is counterintuitive, our detailed calculations still suggest that the generated charge may not be canceled by backreactions. Although our discussion is still not conclusive, we like to present our detailed calculations in this paper.

We have calculated the anomaly-induced electric charge for any baryons, which appear as quantum excitations of Skyrmions (Sec. III). The induced charge is nonvanishing when the magnetic field is present along the axis of the quantization of the spin and the isospin of the baryon. In Sec. V, we argued that this induced charge may not be canceled by other possible electromagnetic corrections to the Skyrmion, although a complete verification may need an explicit calculation of the backreaction of the Skyrmion solution in the magnetic field. We further examined explicitly the anomaly-induced quadrupole moment (Sec. IV) and also the cases with multibaryons (Sec. VI).

It is nontrivial that an additional electric charge of baryons is generated in magnetic fields. So we do not claim explicitly that our result is conclusive. However, it is important that we provide a detailed calculation of the anomaly term and its possible effect, and expect that our calculations will serve as a further understanding of the Skyrme model in the background electromagnetic field. If our finding is true in nature, it may have an observable effect on physics related to neutron stars and heavy ion collisions [10].

Finally, we would like to discuss the possible origin of the anomaly-induced charge. One may wonder if the constant magnetic field may be too artificial and it might be a reason for the anomaly-induced charge. In Appendix A, we considered a magnetic field generated by a circular electric current, and we found that the calculated induced charge is again nonzero. It suggests that the induced charge is not an artifact of the everywhere-constant magnetic field.

Then what is the origin of the additional charge? A good indication comes from the peculiar property of baryons. As shown in Appendix B, we found that the total induced charge is due to the multipion effect in the nonlinear sigma model. As the Skyrmion profile extends to the spatial infinity, the charge distribution also has a tail that elongates to the spatial infinity. This would be the origin of the generation of the additional electric charge. Obviously, if quarks are completely confined, the total charge of any baryon should be quantized to be a half-integer. However, in reality, any baryon is surrounded by a pion cloud, which means that quark-antiquark pair can percolate out of the mean volume of the baryon. We can interpret our anomaly-induced charge as an effective charge carried by the pion cloud surrounding the baryon. To make sure our interpretation, it is important to calculate the complete effect of the magnetic field, i.e., the backreaction to the Skyrmion profile due to the magnetic field.

The anomaly-induced charge may appear to violate the charge conservation. In general, any electromagnetic current should be conserved on the mass shell when the total

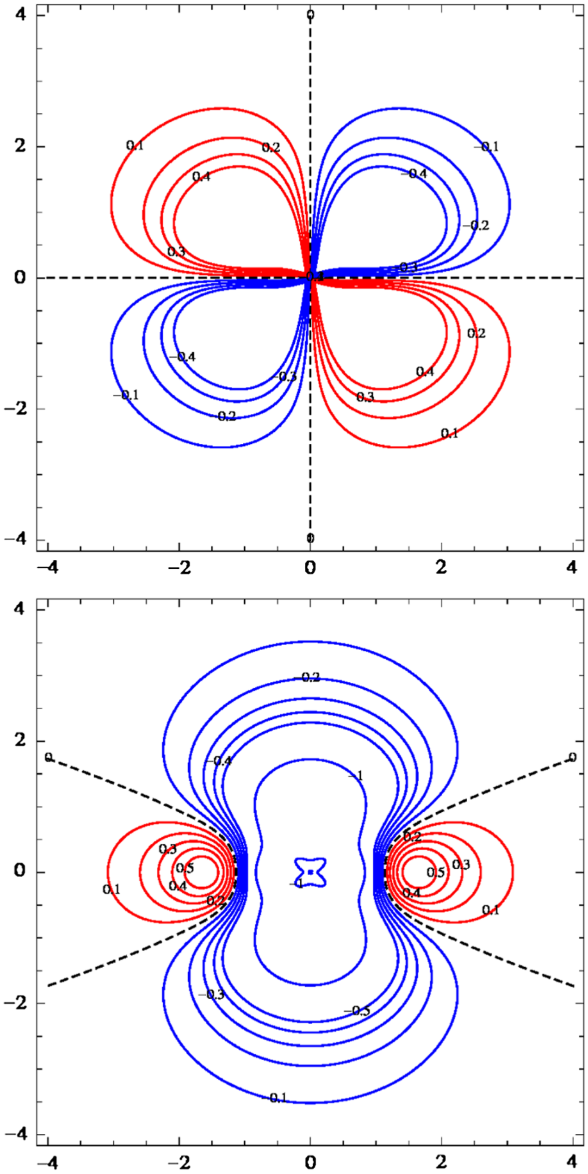

FIG. 9 (color online). The contour plots of the anomalous-charge densities of $B = 2$ Skyrmion on the cross section by the $y = 0$ plane. The top panel shows $-P_1/2$ with $B_1 \neq 0$ and the bottom panel shows $-P_3/2$ with $B_3 \neq 0$. The blue lines have positive values and the red ones have negative values. The black broken lines correspond to zero charge contours.

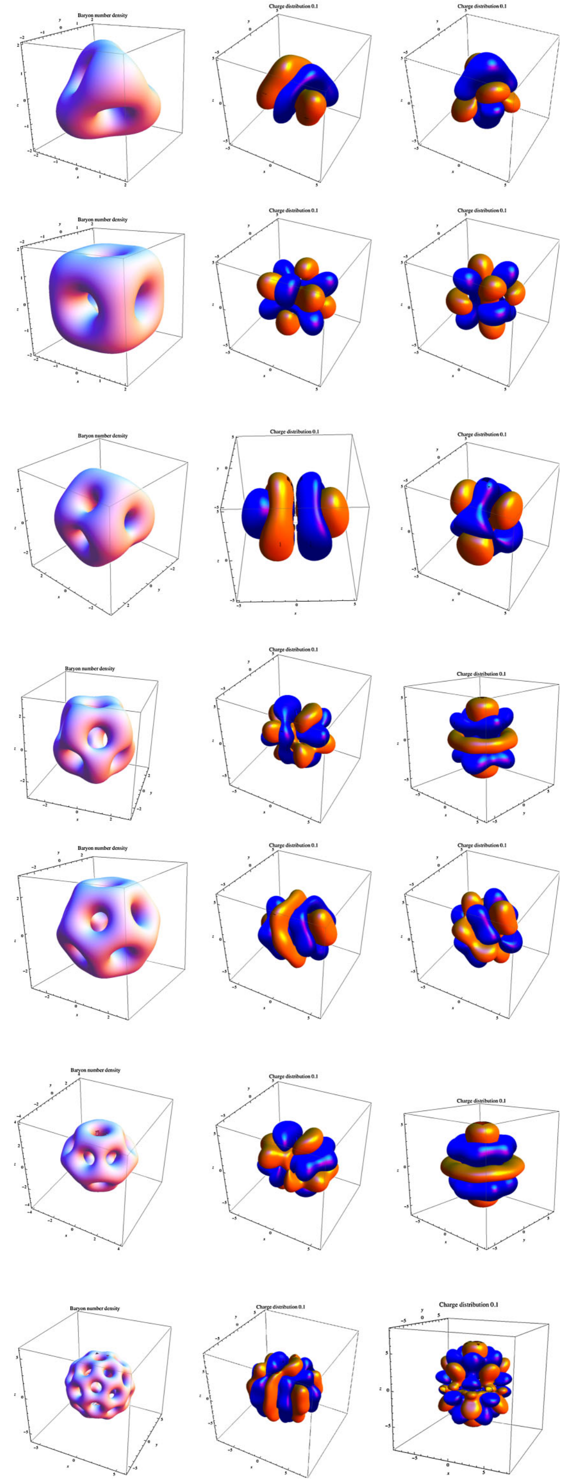

FIG. 10 (color online). Higher-charge Skyrmion solution. From left to right, the baryon number density, P_1 and P_3, respectively. $N_B = 3, 4, \cdots, 8$ and 17 are shown from top to bottom.

TABLE II. The anomaly-induced charge of $N_B = 1, 2, \cdots, 8$ Skyrmions under a constant magnetic-field background. The dimensionless pion mass is chosen to be $m_\pi = 0.263$.

N_B	1	2	3	4	5	6	7	8
$\tilde{c}_0$	−43.2	−105	−60.3	0.00	−13.3	28.7	0.00	−11.6

system is gauge-invariant, and this applies surely to our case. However, we considered in our paper only a static situation, so we have not considered the situation where one turns on the magnetic field gradually from zero to a nonzero value, in a time-dependent manner. To understand the origin of the additional charge concretely, one needs to calculate the backreaction and also the time-dependent magnetic fields. We leave it to our future work.

ACKNOWLEDGMENTS

The authors would like to thank Aleksey Cherman, Kenji Fukushima, Deog-Ki Hong, Nicholas Manton, Makoto Oka, Masashi Wakamatsu, Nodoka Yamanaka, and Koichi Yazaki for useful comments and discussions. The work of M. E. is supported in part by Grant-in Aid for Scientific Research (No. 23740226). K. H. is supported in part by the Japan Ministry of Education, Culture, Sports, Science and Technology. H. I. is supported by Grant-in-Aid for Scientific Research on Innovative Areas (No. 23105713). T. I. received support from JSPS.

APPENDIX A: ANOMALY-INDUCED CHARGE IN CIRCULAR ELECTRIC-CURRENT

In the above argument of the anomaly-induced charge, we have assumed a uniform external magnetic field. However the magnetic field should be always closed unless the magnetic monopole appears. In this appendix, we consider the anomaly-induced charge in the external magnetic field generated by a circular electric current, which is instructive for us, because the magnetic field is closed with finite circular radius, whereas that becomes uniform when a radius of the circular electric-current becomes infinity. Here we suppose that an electric field is not induced by the electric current. We will show that the anomalous charge is induced in the circular electric-current even with the finite radius.

Let us suppose the circular electric-current density with a radius a on xy-plane as

$$\mathbf{j}(\mathbf{r}) \equiv \frac{j_0 a}{2\pi}\delta(z)\delta(\sqrt{x^2+y^2}-a)(-\sin\zeta, \cos\zeta, 0), \tag{A1}$$

where we assume that magnitude of the electric-current is proportional to the radius a. The magnetic field generated by the electric-current density can be given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu}{4\pi}\,\mathrm{rot}\int d\tilde{\mathbf{r}}\frac{\mathbf{j}(\tilde{\mathbf{r}})}{|\tilde{\mathbf{r}}-\mathbf{r}|}, \tag{A2}$$

with μ being a magnetic permeability. For simplicity, we omit the factor $\mu/4\pi$ in the following. One can easily see that, in the large radius limit ($a \to \infty$), the magnetic field becomes

$$B_1 = B_2 = 0, \qquad B_3 = j_0. \tag{A3}$$

This is the same situation with the uniform external magnetic field to z-direction.

When the nucleon is located at the center of the circular electric current, the anomalous charge in the external magnetic field is given by an integration of $\langle j_{\mathrm{anm}}\rangle^N_{I_3,S_3}$, shown in Eq. (92), over the whole space,

$$Q_{\mathrm{anm}} = \frac{ie^2N_c}{48\pi^2}\int d^3x B_i\langle P_i\rangle^N_{I_3,S_3}. \tag{A4}$$

Notice that the magnetic field is also a function of the coordinate variables. Performing the integration over the whole angular-space, we can separate three components of the anomalous charge

$$\rho_{xy}(r) = \int d\Omega_2\hat{x}_1\hat{x}_3\tilde{B}_1 = \int d\Omega_2\hat{x}_2\hat{x}_3\tilde{B}_2,$$

$$\rho_{z,1}(r) = \int d\Omega_2\tilde{B}_3,\ \rho_{z,2}(r) = \int d\Omega_2\hat{x}_3^2\tilde{B}_3,$$

where $\tilde{B}_i \equiv B_i/j_0$. Then the anomalous charge can be rewritten as

$$Q_{\mathrm{anm}} = \frac{4eN_c}{27\pi}(I_3S_3)\frac{ej_0}{(e_sF_\pi)^2}(c_{xy}+c_{z,1}+c_{z,2}),$$

with the numerical coefficients

$$c_{xy} = \frac{3}{4\pi}\int_0^\infty dr[2r^2f' - r\sin(2f)]\rho_{xy}(r),$$

$$c_{z,1} = \frac{3}{8\pi}\int_0^\infty dr r\sin(2f)\rho_{z,1}(r),$$

$$c_{z,2} = \frac{3}{8\pi}\int_0^\infty dr[2r^2f' - r\sin(2f)]\rho_{z,2}(r).$$

Namely, we denote c_{xy} ($c_{z,1}$ and $c_{z,2}$) as component(s) of the anomalous charge induced by B_x and B_y (B_z) generated by the circular electric-current. With these definitions, one can also show that $c_{xy}+c_{z,1}+c_{z,2} = c_0$ at large radius limit ($a \to \infty$).

Figure 11 shows results of ρ_{xy}, $\rho_{z,1}$ and $\rho_{z,2}$ as a function of r in the case of $a = 1$. We find that ρ_{xy} shows small but finite value with a peak at $r = a$, which implies that the anomalous charge is induced by not only B_z but also B_x and B_y. $\rho_{z,1}$ shows intrinsic behavior: it becomes constant at $r < a$, whereas it vanishes at $r > a$. $\rho_{z,2}$ shows smooth behavior without any singularity at $r = a$.

Magnitude of the coefficients, $c_{xy}+c_{z,1}+c_{z,2}$, is shown in Fig. 12 as a function of the radius a for $m_\pi/m_\pi^{\mathrm{phys}} = 0.5$, 1.0 and 2.0. The arrows on the right

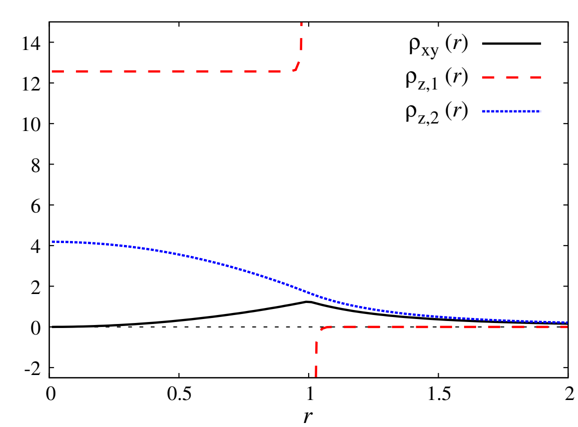

FIG. 11 (color online). Results of ρ_{xy}, $\rho_{z,1}$ and $\rho_{z,2}$ as a function of r in the case of $a = 1$.

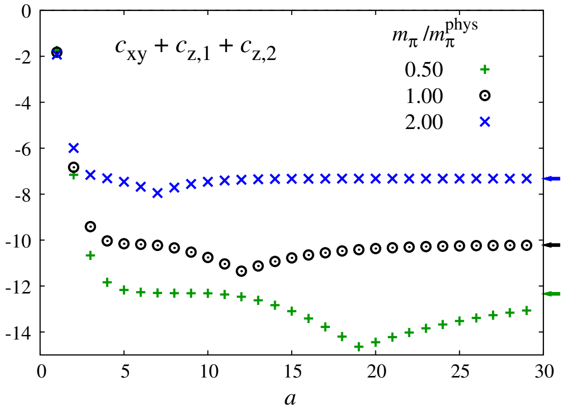

FIG. 12 (color online). Magnitude of $c_{xy} + c_{z,1} + c_{z,2}$ as a function of the radius a of the circular electric-current for $m_\pi/m_\pi^{\rm phys} = 0.5$, 1.0 and 2.0. The arrows on the right side denote magnitude of c_0.

side denote magnitude of c_0. The coefficients, $c_{xy} + c_{z,1} + c_{z,2}$, have finite value even with finite radius, which implies the anomaly-induced charge by the closed magnetic field, and converge to c_0 at large radius. It is also found that the coefficients shows minimum values at some radius, e.g., $a \sim 12$ for $m_\pi/m_\pi^{\rm phys} = 1.0$. This may be understood as follows: in the case of $a \gtrsim m_\pi^{-1}$, the Skyrmion feels a similar magnetic field to the uniform one to z-direction, which induces a similar anomalous charge, i.e., $c_{z,1} + c_{z,2} \sim c_0$. Furthermore, since the anomalous charge is also induced by B_x and B_y at finite radius discussed above, there is finite contribution, $|c_{xy}| > 0$. This extra contribution gives larger induced charge than that induced by the uniform magnetic field.

We also calculate the multipole moment due to the anomaly, and find that the results are similar to the case of the uniform magnetic field: the dipole moment vanishes, whereas the quadrupole moment Q_{ij} shows finite values only for diagonal parts ($i = j$).

APPENDIX B: MULTIPION EFFECT AND COMPARISON WITH POINT-PARTICLE PICTURE

Here we argue that the anomaly-induced electric charge is due to the pion cloud that exists around any baryon. The pion "cloud," which is the multipion effect, in the anomaly term is simply the terms with higher powers in the π field. The anomaly term in the gauged WZW term $S_{\rm WZW}$ can be expanded as

$$S_{\rm WZW} \sim \int d^4x A_0 B_3 P_3 \sim \int d^4x \,{\rm Tr}[\tau_3 U^\dagger \partial U] A_0 B_3 \sim \int d^4x [\partial \pi_0 + \pi\pi\partial\pi + \cdots] A_0 B_3. \quad \text{(B1)}$$

The first term is responsible for the famous $\pi_0 \to 2\gamma$ interaction, while the remaining terms are the pion cloud.

In the following, we shall see that, only with the first term, the anomaly-induced total charge $Q_{\rm anm}$ vanishes. So, our anomaly-induced total charge is due to the pion cloud.

For the Skyrme solution, we have $\pi_0 \sim f(r)\hat{x}_3$, so the total electric charge induced by the first term in Eq. (B1) is proportional to

$$\int d^3x \partial_3 \pi_0 = \int d^3x \partial_3 (f(r)\hat{x}_3) = 2\pi \int r^2 \sin\theta dr d\theta \Big[\Big(f' - \frac{f}{r}\Big)\cos^2\theta + \frac{f}{r}\Big] = \frac{4\pi}{3}\int_0^\infty dr (r^2 f' + 2rf) = \frac{4\pi}{3}[r^2 f]_{r=0}^{r=\infty}. \quad \text{(B2)}$$

The last expression vanishes for nonzero pion mass, because $f(r)$ decays exponentially at large r, and $f(0)$ is finite. So, the anomaly-induced total charge vanishes if one use only the single-pion term in the anomaly term (B1).

It was discussed in [22] that the anomaly-induced total charge of nucleon vanishes, by using a generic argument without using the specific Skyrme model. The argument [22] uses only the single-pion term, so our result is consistent with it.

Before going to the multipion term, we note that, in the chiral limit where the pion mass vanishes, the last expression is nonzero, since $f \sim r^{-2}$ at large r (see [23]). So, in the chiral limit, the contribution that comes from the single-pion term is nonzero. This is again consistent with the discussion in [22] where the pion momentum is neglected compared to the pion mass to show the vanishing total charge. Note that this discussion on the chiral limit is suggestive but not so firm since various observables in the Skyrme model diverges in the chiral limit.

Now, let us evaluate the multipion term in Eq. (B1). The representative 3-pion term is evaluated as

$$\int d^3x \pi\pi\partial\pi \sim \int_0^\infty r f(r)^3 dr, \quad \text{(B3)}$$

which is nonzero for any pion mass. Therefore, we conclude that our anomaly-induced total charge is due to the multipion effect.

The point-particle picture of [22] shows that the quadrupole moment is induced as a leading moment. So let us compare the conclusion of the Skyrmion with that of the point-particle picture.

The anomaly-induced quadrupole moment has been written in the point-particle picture as [22]

$$Q^{ij}_{\rm pp} = -\frac{N_c \alpha}{6\pi} \frac{g_A}{(f_\pi m_\pi)^2} N^\dagger \sigma^i \tau^3 N B^j, \quad \text{(B4)}$$

where $\alpha = e^2/4\pi$, and g_A and N are the axial coupling constant and the nucleon wave function, respectively. In the Skyrmion, the quadrupole moment due to the anomaly is given in Eq. (96), where the pion-mass dependence of the coefficient becomes

$$c_2 \simeq \frac{A}{(m_\pi/e_s F_\pi)^2}. \tag{B5}$$

Using the formula of $g_A = -\pi D/3e_s^2$ in the Skyrme model, where D is the numerical coefficients of r integration including the pion profile function [23], we can rewrite the quadrupole moment with familiar physical observables as

$$Q_{ij} = -\frac{8N_c\alpha}{45\pi}\frac{A}{D}I_3S_3\frac{g_A}{(F_\pi m_\pi)^2}\tilde{Q}_{ijk}B_k. \tag{B6}$$

We have checked that the numerical coefficient D does not show singular dependence on m_π. This implies that the pion-mass dependence of the quadrupole moment is qualitatively consistent between the point-particle picture and the Skyrme picture.

As a consequence of the comparison, we find no contradiction between the point-particle picture and the Skyrme picture. For further understanding of the anomaly-induced charge, calculations of the multipion effect in the point-particle picture are required.

[1] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).
[2] E. Witten, Nucl. Phys. **B223**, 422 (1983); Nucl. Phys. **B223**, 433 (1983).
[3] T. H. R. Skyrme, Proc. R. Soc. A **260**, 127 (1961).
[4] E. Witten, Phys. Lett. B **86**, 283 (1979).
[5] F. Wilczek, Phys. Rev. Lett. **58**, 1799 (1987).
[6] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, Nucl. Phys. A **803**, 227 (2008).
[7] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[8] M. Eto, K. Hashimoto, H. Iida, and A. Miwa, Phys. Rev. D **83**, 125033 (2011).
[9] S. A. Voloshin (STAR), Nucl. Phys. **A830**, 377c (2009).
[10] M. Eto, K. Hashimoto, H. Iida, T. Ishii, Y. Maezawa, arXiv:1103.5443.
[11] G. S. Adkins and C. R. Nappi, Nucl. Phys. **B233**, 109 (1984).
[12] Y. Brihaye, N. K. Pak, and P. Rossi, Phys. Lett. B **149**, 191 (1984).
[13] Note that, when we make the spatial integration, the last term in (43) drops off as it is a total derivative term. For massive pions, the pion profile of the Skyrmion always decays exponentially asymptotically, so the surface integral derived from the integration of this total derivative term always vanish.
[14] When we defined the isospin charge I_3 in (34), we have not included the gauged WZW term (35). However, it can be shown that the isospin charge coming from (35) vanishes, as follows. Let us act the isospin transformation in the pion field U in the gauged WZW term (35): $U \to U' \equiv GUG^\dagger$ where $G = \exp i\epsilon\tau_3$. To obtain the current, following the standard Noether's method, we regard the transformation parameter as a spacetime-dependent variable, $G = G(x)$. Then we find $L \to GU^\dagger G^\dagger(\partial G)UG^\dagger + GU^\dagger(\partial U)G^\dagger + G\partial G^\dagger$, and $R \to GU(\partial G^\dagger)GU^\dagger G^\dagger + G(\partial U)U^\dagger G^\dagger + (\partial G)G^\dagger$. Using $[G, \tau_3] = 0$, $(\partial G)G^\dagger + G\partial G^\dagger = \partial(GG^\dagger) = 0$, and $G^\dagger\partial G = -\partial G^\dagger G = i\tau_3\partial\epsilon(x)$, we find that $\mathrm{tr}[\tau_3(L+R)]$ is invariant under this local transformation. So, there is no additional contribution to the isospin charge from the gauged WZW term. The result is natural, since the gauged WZW term is basically $\pi_0 \to 2\gamma$ and this π_0 does not carry the isospin charge. We would like to thank the referee for pointing out the effect of the WZW term.
[15] As long as we think of the massless pion as the limit $m_\pi \to 0$, the surface term contribution can be always ignored.
[16] P. O. Mazur, M. A. Nowak, and M. Praszalowicz, Phys. Lett. B **147**, 137 (1984).
[17] I. Zahed and G. E. Brown, Phys. Rep. **142**, 1 (1986).
[18] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonsky, *Quantum Theory Of Angular Momentum* (World Scientific, Singapore, 1988).
[19] It is possible that the last term in (43) gives additional multipoles, although it is negligible for the total charge. However, the term itself is not gauge-invariant and once combined with the baryon number term (the second term in (43)) it becomes gauge-invariant. In this paper we evaluate only the gauge-invariant j^μ_{anm} for the quadrupole moment, and the other terms (which can be evaluated if a backreaction to the Skyrmion profile can be computed) are left for our future work.
[20] C. J. Houghton and N. S. Manton, and P. M. Sutcliffe, Nucl. Phys. **B510**, 507 (1998).
[21] N. S. Manton and P. M. Sutcliffe, *Topological Solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2004), p. 493.
[22] D. E. Kharzeev, H.-U. Yee, and I. Zahed, Phys. Rev. D **84**, 037503 (2011).
[23] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).