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Critical endpoint for deconfinement in matrix and other effective models

Kouji Kashiwa*
RIKEN/BNL, Brookhaven National Laboratory, Upton, New York 11973, USA

Robert D. Pisarski†
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973, USA
RIKEN/BNL, Brookhaven National Laboratory, Upton, New York 11973, USA

Vladimir V. Skokov‡
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973, USA
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We consider the position of the deconfining critical endpoint, where the first order transition for deconfinement is washed out by the presence of massive, dynamical quarks. We use an effective matrix model, employed previously to analyze the transition in the pure glue theory. *If* the parameters of the pure glue theory are unaffected by the presence of dynamical quarks, and if the quarks only contribute perturbatively, then for three colors and three degenerate quark flavors this quark mass is very heavy, $m^{\rm de} \sim 2.5$ GeV, while the critical temperature $T_{\rm de}$ barely changes, $\sim 1\%$ below that in the pure glue theory. The location of the deconfining critical endpoint is a sensitive test to differentiate between effective models. For example, models with a logarithmic potential for the Polyakov loop give much smaller values of the quark mass, $m^{\rm de} \sim 1$ GeV, and a large shift in $T_{\rm de} \sim 10\%$ lower than that in the pure glue theory.

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The nature of the phase transitions in QCD are of interest for a variety of reasons. In theory, all in-equilibrium thermodynamic quantities can be computed from first principles using numerical simulations on the lattice. In practice, and especially if one is interested in comparing to experiment, it is useful to have effective models. These allow one to compute quantities near thermal equilibrium, such as transport coefficients, which are much more difficult to extract from the lattice.

For a pure glue theory with three colors and no dynamical quarks, the deconfining phase transition is of first order. This follows from the global $Z(3)$ symmetry in the pure glue theory [1]. Adding dynamical quarks acts like a background $Z(3)$ field and so tends to weaken the first order transition. As the quark mass decreases, it is possible to reach a deconfining critical endpoint, at a mass $m^{\rm de}$, where the deconfining transition is of second order.

In this paper we consider the properties of the deconfining critical endpoint in effective models. This was done before in a zero parameter matrix model by Meisinger, Miller, and Ogilvie [2,3]. In this paper we perform the computation in one [4] and two [5] parameter matrix models. This is a useful exercise, since the solution of the zero parameter model does not agree with lattice data on the interaction measure of the pure $SU(3)$ glue theory. In contrast, the coefficients of the one and two parameter models are tuned to give increasingly good agreement with the lattice results [4,5]. For related matrix models, see [6–8]. We expect these models will give very similar results to ours.

For these matrix models, and for models based upon polynomials of the Polyakov loop [9–12], we find a heavy $m^{\rm de}$. For three colors and three degenerate quark flavors, $m^{\rm de}$ is about twice as heavy as the charm quark mass, $m^{\rm de} \sim 2.5$ GeV for the matrix model, and ~ 3.5 GeV for a polynomial loop model. This is in sharp contrast to models based on a potential motivated by the Vandermonde determinant, which involve the logarithm of the Polyakov loop [13–17]. They give a quark mass that is lighter than the charm quark, $m^{\rm de} \sim 1$ GeV.

Besides the value of $m^{\rm de}$, effective models can give detailed information about the properties of the deconfining critical endpoint. One obvious parameter is the temperature at which it occurs, $T_{\rm de}$. If T_d is the temperature of the deconfining phase transition in the pure glue theory, matrix and polynomial loop models find very small shifts in the temperature, $T_{\rm de} \sim 0.995 T_d$. The shift in logarithmic loop models is much larger, $T_{\rm de} \sim 0.9 T_d$.

We also present results for the interaction measure of the theory, which exhibits characteristic differences between the different models. For three degenerate flavors, in the matrix and polynomial loop models, the interaction measure exhibits two peaks, one near T_d and one near $3T_d$. This is not seen in logarithmic loop models.

There are lattice simulations [18,19] which address this problem. These give a result which is close to the value in a

*kashiwa@ribf.riken.jp
†pisarski@bnl.gov
‡vskokov@quark.phy.bnl.gov

logarithmic loop model, $m^{\rm de} \sim 1.4$ GeV [20–22]. They are not current, however. There are recent results from the WHOT Collaboration [23], but they do not present an estimate for $m^{\rm de}$ nor of T_d. Recently, the Wuppertal-Budapest Collaboration has published numerical results on the QCD equation of state with $2+1$ light flavors and a dynamical charm quark [24]. Thus they could also address the question of the deconfining critical endpoint with relative ease.

At the outset, we note that in our effective model, any dimensional parameter is a pure number times a common mass scale, which we chose at T_d. That is, our model determines the dimensionless ratios $m^{\rm de}/T_d$ and T_d/T_d. To illustrate the physics, when we quote quantities such as $m^{\rm de}$, we uniformly assume that $T_d = 270$ MeV but stress that this is an assumption.

I. EFFECTIVE MATRIX MODELS

A. Model for a pure glue theory

We wish to model the region near the deconfining transition temperature. In this region, which has been termed the "semi" quark gluon plasma, the expectation value of the Polyakov loop is less than unity. In a matrix model [2,4,5], to represent this we expand the timelike component of the vector potential about a constant value,

$$A_0^{ij} = \frac{2\pi T}{g} q_i \delta^{ij}. \tag{1}$$

For now we consider the general case of $SU(N)$, where indices in the fundamental representation run from $i, j = 1, 2, \ldots, N$. The matrix A_0 is an element of the $SU(N)$ Lie algebra, so that $\sum_i q_i = 0$.

In this background field, in the fundamental representation the Wilson line $\mathbf{L}$ and the Polyakov loop ℓ are

$$\mathbf{L} = {\rm e}^{2\pi i \mathbf{q}}; \qquad \ell = \frac{1}{N}\,{\rm tr}\mathbf{L}. \tag{2}$$

At one loop order, in this background field the potential for q is

$$\mathcal{V}^{\rm gl}_{\rm pert}(\mathbf{q}) = -\frac{(N^2-1)\pi^2}{45}T^4 + \frac{2\pi^2}{3}T^4 \sum_{i,j=1}^{N} V_2(q_i - q_j), \tag{3}$$

where

$$V_2(x) = x^2(1-|x|)^2, \qquad -1 \le x \le 1. \tag{4}$$

$V_2(x)$ is periodic in x, $x \to x+1$. This can be understood more generally. While the Wilson line is gauge variant, its eigenvalues, $\exp(2\pi i q_i)$, are gauge invariant.

In the pure glue theory, the potential is also invariant under $Z(N)$ transformations. A $Z(N)$ transformation is given, e.g., by shifting $q_i \to q_i + 1/N$ for $i = 1 \ldots (N-1)$, and $q_N \to -(N-1)/N$, under which $\mathbf{L} \to \exp(2\pi i/N)\mathbf{L}$. The fact that the potential is invariant under this transformation is elementary, since the differences $|q_i - q_j|$ either vanish or equal one, which by periodicity is equivalent to zero.

In the matrix model, to drive the transition to confinement, one adds a nonperturbative term [4]

$$\mathcal{V}^{\rm gl}_{\rm non}(\mathbf{q}) = -\frac{4\pi^2}{3}T^2T_d^2\Big(c_1 \sum_{i,j=1}^{N} V_1(q_i - q_j) + c_2 \sum_{i,j=1}^{N} V_2(q_i - q_j) + \frac{(N^2-1)}{60}c_3\Big), \tag{5}$$

where

$$V_1(x) = x(1-|x|), \qquad -1 \le x \le 1. \tag{6}$$

Again, the potential $V_1(x)$ is periodic in $x \to x+1$; it is also invariant under $Z(N)$ transformations.

B. Inclusion of dynamical quarks

In the background field of Eq. (1), at one loop order gluons generate the potential in Eq. (3). When dynamical quarks are added then, certainly the first thing to do is to compute the analogous contribution which they make to the q-dependent potential.

This has been computed in Ref. [2], but we include it here for completeness. For quarks of mass m, the one loop potential is given by

$$\begin{aligned}&{\rm lndet}(\gamma^\mu \partial_\mu + q\delta^{\mu 4} + im) \\ &\quad = 2\,{\rm lndet}[(\partial_0 + 2\pi T q)^2 - \vec{\partial}^2 + m^2].\end{aligned} \tag{7}$$

As fermions, in the Matsubara formalism, the frequencies are odd multiples of πT.

For simplicity we denote the background field simply by q. This is equivalent to an imaginary chemical potential for the quark. This makes it easy to do the integral. Instead of summing over the frequencies, we can just use the standard expression for the thermodynamic potential of a free fermion gas, replacing the chemical potential μ by $2\pi i q$. Thus we need to compute

$$-2T\int \frac{d^3p}{(2\pi)^3}[\ln(1 + e^{-E_{\rm p}/T - 2\pi i q}) + \ln(1 + e^{-E_{\rm p}/T + 2\pi i q})], \tag{8}$$

where $E_{\rm p} = \sqrt{\vec{p}^2 + m^2}$. We then expand each logarithm in a power series,

$$\ln[1 + e^{-E_{\rm p}/T - 2\pi i q}] = e^{-E_{\rm p}/T - 2\pi i q} - \tfrac{1}{2}e^{-2E_{\rm p}/T - 4\pi i q} + \cdots, \tag{9}$$

where the first term is the usual Boltzmann term.

This leaves the integral over momenta. The angular integral is trivial, leaving only the integral over p:

$$\int_0^\infty p^2 e^{-E_p/T} dp = -\frac{\partial}{\partial\beta}\int_0^\infty \frac{p^2}{E_p} e^{-\beta E_p} dp$$
$$= -\frac{\partial}{\partial\beta}\int_1^\infty m^2\sqrt{y^2-1}\, e^{-\beta m y} dy$$
$$= -\frac{\partial}{\partial\beta}\left[\frac{m}{\beta}K_1(\beta m)\right] = m^2 T K_2\left(\frac{m}{T}\right), \tag{10}$$

where K_ν is the modified Bessel function of the second kind and $\beta = 1/T$.

The modified Bessel function of the first kind is given by

$$I_{-\nu}(x) - I_\nu(x) = \frac{\Gamma(\frac{1}{2}-\nu)\sin(2\nu\pi)}{\pi^{3/2}}\left(\frac{x}{2}\right)^\nu \times \int_1^\infty (y^2-1)^{\nu-(1/2)} e^{-xy} dy, \tag{11}$$

where $\Gamma(x)$ is the usual Gamma function. The Bessel function of the second kind is

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}. \tag{12}$$

Putting all of the factors together, and summing over different colors, for a single flavor of massive quark its contribution to the potential is given by

$$\mathcal{V}^{\rm qk}_{\rm pert}(\mathbf{q}) = \frac{2m^2T^2}{\pi^2}\sum_{n=1}^\infty \frac{(-1)^n}{n^2} K_2\left(\frac{nm}{T}\right)\sum_{i=1}^N \cos(2\pi n q_i). \tag{13}$$

The dependence upon the q's can be rewritten more generally as

$$\sum_{i=1}^N \cos(2\pi n q_i) = \frac{{\rm tr}\mathbf{L}^n + {\rm tr}(\mathbf{L}^\dagger)^n}{2}. \tag{14}$$

C. Predictions for matrix model

We now concentrate on the results for three colors. For the pure glue theory, by a $Z(3)$ rotation we can assume that the Polyakov loop is real. With dynamical quarks, this remains true if all of the quark masses are real and the quark chemical potential vanishes. We then take a path in q space as

$$\mathbf{q} = \tfrac{1}{3}(r-1, 0, 1-r). \tag{15}$$

We have chosen to introduce r because the confined vacuum is then given by $r = 0$; the perturbative vacuum is $r = 1$.

For three colors, assuming that the Polyakov loop is real, uniquely specifies the q_i. This is not true for four or more colors [5]. We also have computed the position of the deconfining critical endpoint for this model in the limit of an infinite number of colors [25] and find, somewhat surprisingly, that the parameters for the deconfining critical endpoint are similar to that for three colors.

In writing the potential, it is convenient to introduce a dimensionless temperature, rescaled by T_d,

$$t = \frac{T}{T_d}. \tag{16}$$

The gluon potential is a sum of two terms,

$$\mathcal{V}^{\rm gl}(r,t) = \mathcal{V}^{\rm gl}_{\rm pert}(r,t) + \mathcal{V}^{\rm gl}_{\rm non}(r,t)$$
$$= \frac{8\pi^2}{45}T_d^4 t^2(t^2-c_2)[\mathcal{W}_{\rm gl}(r,t) + \mathcal{W}^0_{\rm gl}(t)]. \tag{17}$$

The first term is a simple quartic potential in r,

$$\mathcal{W}_{\rm gl}(r,t) = -z(t)r^2 - \tfrac{10}{9}r^3 + \tfrac{5}{3}r^4, \tag{18}$$

where

$$z(t) = \frac{5}{3} - \frac{50}{27}\left(\frac{1-c_2}{t^2-c_2}\right). \tag{19}$$

Here we have fixed the constant c_1 by requiring that the transition occurs at T_d; this results in the relation

$$c_1 = \tfrac{50}{27}(1-c_2). \tag{20}$$

The second constant, c_3, is fixed by requiring that the pressure vanish at T_d, which gives the condition

$$c_3 = 1 + c_1 - \tfrac{10}{9}(1-c_2). \tag{21}$$

This leaves a term independent of r in the potential,

$$\mathcal{W}^0_{\rm gl}(t) = \frac{1}{9}\left(\frac{t^2-1}{t^2-c_2}\right). \tag{22}$$

The overall constant for $SU(N)$ is $1/N^2$, which is $1/9$ for $N = 3$. The basic justification of this matrix model is an expansion in large N, so this constant term manifestly represents a correction of $\sim 1/N^2$.

For three colors, the model of Ref. [2] takes $c_2 = 0$. The values of c_1 and c_3 are then fixed by Eqs. (20) and (21), so this model has no free parameters. When $c_2 = 0$, though, the peak in the interaction measure, $(e-3p)/T^4$ (e is the energy density and p the pressure), is much broader than found from lattice simulations [4,5].

We next consider the one parameter model of Ref. [4]. This finds a good fit to the interaction measure, for values

$$c_1 = 0.315\,37, \qquad c_2 = 0.8297, \qquad c_3 = 1.126\,15. \tag{23}$$

The one parameter model gives a good fit to the interaction measure overall but has problems very close to T_d; see, e.g., Fig. 3 of Ref. [5]. In particular, the value of the latent heat for the one parameter model is much smaller than found on the lattice. To avoid this discrepancy, we consider the two parameter model of Ref. [5]. In this

model, c_1 and c_2 are constants, as before, but now c_3 is a function of temperature,

$$c_3(t) = c_3(\infty) + \frac{c_3(1) - c_3(\infty)}{t^2}. \tag{24}$$

Because $T^2 c_3$ enters into the potential, this is equivalent to a MIT "bag" constant,

$$B = \frac{8\pi^2}{45}[c_3(1) - c_3(\infty)]T_d^4. \tag{25}$$

Fixing the transition to occur at T_d relates c_1 and c_2 as in Eq. (20). Requiring that the pressure vanish in the confined phase fixes $c_3(1)$ to have the value of Eq. (21). That leaves two free parameters, c_2 and $c_3(\infty)$. An optimal fit was found to be given by

$$\begin{aligned} c_1 &= 0.830, & c_2 &= 0.5517, \\ c_3(1) &= 1.332, & c_3(\infty) &= 0.95. \end{aligned} \tag{26}$$

The corresponding MIT bag constant is $(244\ \mathrm{MeV})^4$.

The r-dependent terms in the potential are unchanged. The only change in the potential is the term independent of r, which becomes

$$\mathcal{W}^0_{\mathrm{gl}}(0, t) = \frac{1}{9} + \left(\frac{-c_1 - c_2 + c_3(t)}{t^2 - c_2}\right). \tag{27}$$

When $c_3(t)$ is a constant, this reduces to the previous form in Eq. (22).

The addition of dynamical quarks behaves as expected. The pure glue theory is invariant under $Z(3)$ transformations, so that the confining vacuum, where $r = 0$, is always a stationary point of the potential.

With quarks, the $Z(3)$ symmetry is lost. This is clear, as the terms in the quark potential, Eq. (13), involve

$$\mathrm{tr}\,\mathbf{L}^n = 1 + 2\cos\left(\frac{2\pi n(1 - r)}{3}\right). \tag{28}$$

While these quantities all vanish in the confined vacuum, what matters is the equation of motion for r, and it is easy to see that any derivative with respect to r is nonzero when $r = 0$. Thus the presence of dynamical quarks removes the $Z(3)$ symmetry of the pure glue theory and acts like a background $Z(3)$ field.

Including the quark term, a nonzero expectation value is generated for any Polyakov loop, $\mathrm{tr}\mathbf{L}^n$, at any nonzero temperature. There is a problem with the $1/N^2$ term in the gluon potential, Eq. (22), though. While small relative to the other gluon terms for $T > T_d$, with quarks, there is an expectation for the Polyakov loop(s), and so a pressure, below T_d. The term in Eq. (22) tends to give negative pressures below T_d, which is manifestly unphysical.

Such a negative pressure is not always present. For example, in models like this we could consider the limit in which both the number of colors, N, and the number of quark flavors, N_f, are large. Then for temperatures below T_d in the pure glue theory, the pressure is large, $\sim N^2$ or NN_f, which are both of the same order.

To avoid this problem, for the one parameter model we simply modify the gluon potential as

$$\mathcal{W}^0_{\mathrm{gl}}(t) \to \mathcal{W}^0_{\mathrm{gl}}(r, t) = \frac{1}{9}\left(\frac{t^2 - 1}{t^2 - c_2}\right)r^2. \tag{29}$$

That is, to suppress the contribution of $\mathcal{W}^0_{\mathrm{gl}}$, we promote it to a function of r, that is constructed to vanish when $r = 0$. We admit that this is an *ad hoc* procedure. In principle, one could envisage matching onto a hadron resonance model in the low temperature phase.

This modification changes the properties of the deconfining transition in the pure gauge theory, but these are very small. In particular, the transition temperature T_d does not shift, since like $\mathcal{W}^0_{\mathrm{gl}}(t)$, $\mathcal{W}^0_{\mathrm{gl}}(r, t)$ vanishes at T_d, $t = 1$. We have also checked that the change in the parameters for the deconfining critical endpoint is small. The main modification is that with this improvement, the pressure is always non-negative, even below T_d.

Similarly, for the two parameter model, by hand we change the potential as

$$\mathcal{W}^0_{\mathrm{gl}}(0, t) \to \mathcal{W}^0_{\mathrm{gl}}(r, t) = \left[\frac{1}{9} + \left(\frac{-c_1 - c_2 + c_3(t)}{t^2 - c_2}\right)\right]r^2. \tag{30}$$

Besides multiplying by r^2, we also investigated multiplying by ℓ^2; the results obtained were very similar. This is because in the end we are looking for a regime where the pressure is small, anyway.

To compute, we make the most minimal assumptions possible. Notably, we *assume* that the temperature which enters into the pure glue potential, T_d, remains unchanged by the addition of quarks. Ultimately, the validity of this assumption will be tested by the comparison of our model to the results of lattice simulations.

We also assume that the *only* nonperturbative terms are those of the pure glue theory. This actually is forced upon us by the $Z(N)$ structure of the theory. While quarks break the $Z(N)$ symmetry, numerical lattice simulations with three colors find that, at least for up to three light flavors, the breaking of $Z(3)$ is relatively mild.

This implies that any terms in the nonperturbative potential should have the symmetry of the pure glue theory and *not* that of the theory with dynamical quarks. We have considered the modification of the theory with terms such as $\sim T^2 T_d^2\,\mathrm{tr}\mathbf{L}^n$, characteristic of quarks. These terms break the $Z(3)$ symmetry and drastically change the expectation value of the loop, even above T_d. We have checked that if such terms are present, their numerical coefficients must be *very* small.

Last, we take the parameter c_2 to be that of the pure glue theory. This is really a rather mild assumption; the important point is that we do not allow the mass scale T_d to itself

change as we add dynamical quarks. We also take the values of the other constants in the nonperturbative potential, c_1 and c_3, Eq. (23), from the pure glue theory [5].

In short, we assume that confinement is driven by the transition in the pure glue theory. Upon adding quarks, the modification to the theory is entirely through their perturbative contribution to the q-dependent potential.

Notably, this means that we assume that the transition temperature the pure glue theory, T_d, remains unchanged after the addition of quarks. Of course adding quarks will shift the physical transition in the theory, which is why the temperature for the deconfining critical endpoint, $T_{\rm de}$, will differ from T_d.

While these assumptions are rather strong, they do allow us to make unique predictions for the theory with dynamical quarks from the pure glue model.

II. OTHER MODELS

Two types of the Polyakov loop effective potential have been widely used. The first is a potential which is a polynomial in the Polyakov loop [9–12]:

$$\frac{\mathcal{V}_{\rm poly}}{T^4} = -\frac{b_2(T)}{2}\ell^*\ell - \frac{b_3}{6}[\ell^3 + (\ell^*)^3] + \frac{b_4}{4}(\ell^*\ell)^2. \tag{31}$$

The mass term is convoluted:

$$b_2(T) = a_0 + a_1\frac{T_d}{T} + a_2\left(\frac{T_d}{T}\right)^2 + a_3\left(\frac{T_d}{T}\right)^3. \tag{32}$$

The coefficients are determined by fitting the equation of state and the expectation value of the Polyakov loop to lattice data of pure gauge theory [26,27] in Ref. [12]. These coefficients are given in Table I.

The second is motivated by the strong coupling expansion. Following Fukushima, one uses a logarithmic potential [13–17]:

$$\frac{\mathcal{V}_{\log}}{T^4} = -\frac{a(T)}{2}\ell^*\ell + b(T)\ln\{1 - 6\ell^*\ell + 4[\ell^3 + (\ell^*)^3] - 3(\ell^*\ell)^2\}. \tag{33}$$

Again, the coefficients are involved functions of temperature:

$$a(T) = a_0 + a_1\frac{T_d}{T} + a_2\left(\frac{T_d}{T}\right)^2, \qquad b(T) = b_3\left(\frac{T_d}{T}\right)^3. \tag{34}$$

The above parameterization for the temperature dependence was introduced by Rößner, Ratti, and Weise [14],

TABLE I. Parameters in the polynomial potential (31).

T_d [MeV]	a_0	a_1	a_2	a_3	b_3	b_4
270	6.75	−1.95	2.625	−7.44	0.75	7.5

TABLE II. Parameters in the logarithmic potential (33).

T_d [MeV]	a_0	a_1	a_2	b_3
270	3.51	−2.47	15.22	−1.75

with the constants determined by fitting lattice data to the pure $SU(3)$ theory. These coefficients are given in Table II.

For a single flavor and three colors, the quark contribution to the thermodynamic potential for the Polyakov loop models is

$$\mathcal{V}^{\rm qk}_{\rm pert}(\ell,\ell^*) = -2T\int\frac{d^3p}{(2\pi)^3}\{\ln[1 + 3(\ell + \ell^* e^{-\beta E_p}) \times e^{-\beta E_p} + e^{-3\beta E_p}] + \text{c.c.}\}. \tag{35}$$

We assume that there is no chemical potential for the quarks, so that $\ell^* = \ell$. In this case, Eq. (35) can be rewritten similarly to Eq. (13):

$$\mathcal{V}^{\rm qk}_{\rm pert}(\ell) = \frac{2m^2T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}K_2\left(\frac{nm}{T}\right) \times\left[1 + 2T_n\left(\frac{3\ell - 1}{2}\right)\right], \tag{36}$$

where $T_n(x)$ is the nth Chebyshev polynomial, $T_n(x) = \cos(n\arccos(x))$.

III. RESULTS

Most of the results can be understood by considering Eq. (13) in the nonrelativistic limit, when $m \gg T$. Then the contribution of a single flavor of quark to the potential is

$$\mathcal{V}^{\rm qk}_{\rm pert}(\mathbf{q}) \approx -\frac{\sqrt{2}}{\pi^{3/2}}T^{5/2}m^{3/2}\mathrm{e}^{-m/T}\,\mathrm{Re\,tr}\mathbf{L}. \tag{37}$$

We find that for a matrix model, and those with a polynomial potential, this is a good approximation.

If true, we can immediately make an interesting prediction. When the quark mass is heavy, the change in the critical temperature is small, and $T_{\rm de} \approx T_d$. For the one and two parameter matrix models [4,5], this holds to $\sim$1%, Eqs. (39) and (40).

Assuming that $T_{\rm de} \approx T_d$, consider a theory with N_f flavors of quarks, each with mass m. The mass for the deconfining critical endpoint is a function of N_f, $m^{\rm de}(N_f)$. *If* the nonrelativistic approximation holds, we trivially obtain the relation

$$\log(N_f) + \frac{3}{2}\log\left(\frac{m^{\rm de}(N_f)}{T_d}\right) - \frac{m^{\rm de}(N_f)}{T_d} \approx \text{constant}. \tag{38}$$

Once we know $m^{\rm de}$ for one value of N_f, this relation gives us its value for any N_f.

In general, $m^{\rm de}$ and $T_{\rm de}$ must be computed numerically. Even working with the exact result for the quark potential in Eq. (13), though, given the simplicity of the potentials

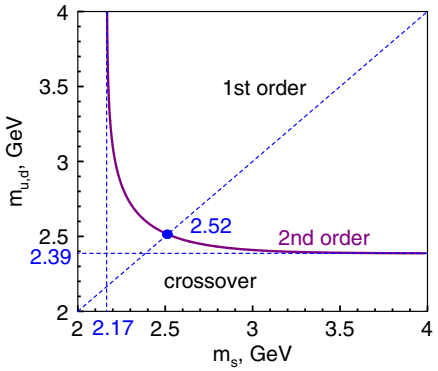

FIG. 1 (color online). The phase diagram for the deconfining phase transition in the matrix model without a bag constant, assuming $T_d \sim 270$ GeV.

the numerical effort is minimal. As always we assume that $T_d = 270$ GeV.

In Fig. 1 we present the results for the one parameter matrix model, without a bag constant. The critical temperature changes very little from the pure gauge theory,

$$T_{\rm de}^{\rm 1para} = 0.995 T_d. \tag{39}$$

The deconfining critical endpoint occurs for a *very* heavy mass: for one flavor, $m^{\rm de} = 2167$ MeV. The values for other flavors obey Eq. (38); as N_f increases, so does $m^{\rm de}$.

Going to a two parameter matrix model with a bag constant, we find that again the shift in the critical temperature is very small,

$$T_{\rm de}^{\rm 2para} = 0.990 T_d. \tag{40}$$

The quark mass $m^{\rm de}$ moves down by about 10% from the one parameter model, with $m^{\rm de} = 1836$ MeV for one flavor. This is the direction expected: in the one parameter model [4] the latent heat is too small. Going to a two parameter model, with a bag constant, increases the latent heat to agree with the lattice data [5]. If the latent heat is larger, it should take a larger background field, or a smaller quark mass, to wash out the first order deconfining phase transition.

For the zero parameter matrix model, for one flavor we find a slightly larger shift in the critical temperature, $T_{\rm de} = 0.973 T_d$, and a smaller quark mass, $m^{\rm de} = 1536$ MeV. These values agree with those of Ref. [2]. This quark mass is ~17% lighter than for the two parameter model, but remember that this model does not describe the interaction measure of the pure glue theory.

We compare the results for different models in Fig. 2. For the logarithmic Polyakov loop model, the masses are light, $m^{\rm de} \sim 1$ GeV. The temperature for the deconfining critical endpoint is significantly less than for the pure glue theory,

$$T_{\rm de}^{\rm Log\ PLM} = 0.90 T_d. \tag{41}$$

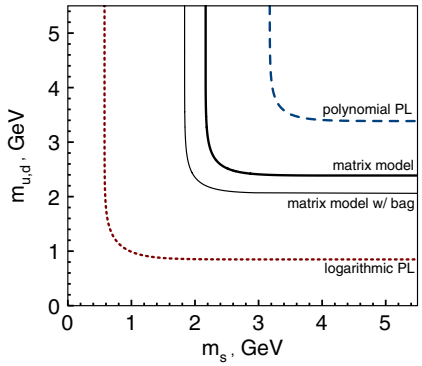

FIG. 2 (color online). The phase diagram of the deconfining phase transition for the matrix model without/with bag constant (bold/thin solid line) and the Polyakov loop models with the polynomial (dashed line) and logarithmic (dotted line) potentials. The lines correspond to the second order deconfining phase transition.

A polynomial Polyakov loop model gives a very large mass, $m^{\rm de} \sim 3.5$ GeV. The critical temperature is very close to the pure glue theory, $T_{\rm de} \sim 0.996 T_d$.

We comment that Ref. [11] used a polynomial Polyakov loop model but finds a result rather different from ours. Crucially, we are in accord on the value of the background field, h, at which the deconfining critical end point occurs. We differ in how to relate this background field to the quark mass. We assume that in a Polyakov loop model, the quark contribution is related following the one loop quark determinant, Sec. I B, which gives $h \sim m_{\rm qk}^{3/2} \exp(-m_{\rm qk}/T)$ when $m_{\rm qk} \gg T$, Eq. (36). Reference [11] uses a relation motivated by light quarks and takes $h \sim \exp(-m_\pi)$; doing so then gives $m_\pi^{\rm de} \sim 1.8$ GeV.

It is also interesting to compute the interaction measure at the deconfining critical endpoint. In our model, it is a sum of the interaction measure for the glue part of the theory plus the interaction measure for quarks. For a

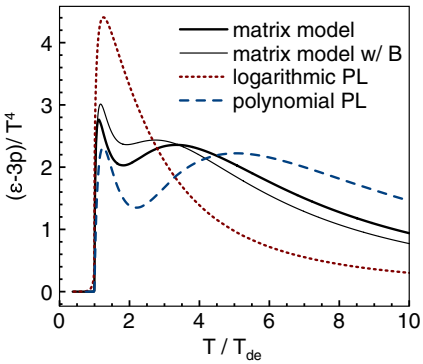

FIG. 3 (color online). The conformal anomaly $(\varepsilon - 3p)/T^4$ as a function of the temperature T, normalized by $T_{\rm de}$ for the matrix model without/with bag constant (bold/thin solid line) and the Polyakov loop models with the polynomial (dashed line) and logarithmic (dotted line) potentials. The calculations are performed for three flavors with $m = m^{\rm de}$.

massive quark, it is easy to see that the interaction measure has a peak at $\sim$0.4 m. For the logarithmic Polyakov loop potential, because the quark mass is relatively light, $m^{\rm de} \sim$ 1 GeV, the contribution of three degenerate flavors of quarks to the interaction measure just broadens and enhances the peak in the pure glue theory, Fig. 3.

In contrast, for either matrix model, or for a polynomial Polyakov loop potential, the quark mass is heavier than 2 GeV. This means that the interaction measure has a characteristic form: there is the usual peak from gluons, near $T_{\rm de}$, *plus* a second peak from quarks, at a heavier temperature.

This two peak structure in the interaction measure is special to three degenerate flavors of quarks. For one or two flavors, there is a peak in the interaction measure from quarks, but it does not stick out over the contribution from gluons.

IV. CONCLUSIONS

The existence and computation of the deconfining critical endpoint is a well known problem. In this paper we have shown that its properties can be used to differentiate between different effective models.

If the critical quark mass is very heavy, as in matrix and polynomial Polyakov loop models, then effects of lattice discretization for such heavy quarks will be severe. It may be useful instead to compute the background field at which the deconfining critical endpoint occurs, e.g., by adding a term proportional to $h(\mathrm{tr}\mathbf{L} + \mathrm{tr}\mathbf{L}^\dagger)/2$ to the Yang-Mills action, where h is the external field. This is easily computed to one loop order in the Boltzmann approximation. This result receives corrections perturbatively, proportional to $\sim g^2$, etc. Nevertheless, computing with a background field is elementary in numerical simulations on the lattice and should indicate if $m^{\rm de}$ is relatively light, $\sim$1 GeV, or heavy, $\sim$2 GeV.

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