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Neutrino velocity and neutrino oscillations

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We study distances of propagation and the group velocities of muon neutrinos in the presence of mixing and oscillations assuming that Lorentz invariance holds. Oscillations lead to distortion of the ν_μ wave packet, which, in turn, changes the group velocity and the distance ν_μ travels. We find that the change of the distance, $d_{\rm osc}$, is proportional to the length of the wave packet, σ_x, and the oscillation phase, ϕ_p, acquired by neutrinos in the π- and K- meson decay tunnel where the neutrino wave packet is formed: $d_{\rm osc} \propto \sigma\phi_p$. Although the distance $d_{\rm osc}$ may effectively correspond to the superluminal motion, the effect is too tiny ($\sim 10^{-5}$ cm) to be reconciled with the OPERA result. We analyze various possibilities to increase $d_{\rm osc}$ and discuss experimental setups in which $d_{\rm osc}$ (corresponding to the superluminal motion) can reach an observable value $\sim$1 m.

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I. INTRODUCTION

Neutrinos are the lightest known massive particles and therefore their velocity at accessible energies can be the closest one to the velocity of light. The velocity of neutrinos (and therefore the difference of the velocities of neutrinos and photons) can be affected by various factors: gravitational field, modifications of metric, local space-time environment, matter through which neutrinos propagate, local effects induced by the presence of new fields in space, possible interactions with "dark" components in the Universe, etc. Last but not least, the velocity can be affected by Lorentz violation. Therefore, measurements of neutrino velocity may reveal new phenomena related to physics beyond the Standard Model. They provide important probes of properties of space and time as well as dark sectors of the Universe.

At high energies, neutrino velocity has been measured in several accelerator experiments [1–4]. At low energies, observations of neutrino burst from supernova (SN) 1987a [5] place severe constraints on neutrino velocity. Many of the factors mentioned above, which affect neutrino velocity, have been discussed in detail [6] in connection to the controversial OPERA result [3].

To address properly the issue of possible superluminal velocity of particles, one must be aware that it can arise even within the conventional framework based on special theory of relativity. In fact, superluminal propagation of light is a well-known subject in optics [7]. The effect is related to a distortion of the photon pulse during propagation in media, so that suppression of the trailing edge of the pulse leads to an increase of group velocity. The superluminal motion does not contradict causality and no information can be transmitted with a velocity larger than the velocity of light. Indeed, the effect has been observed in several experiments (e.g., [8–11]).

A similar mechanism for neutrinos has been discussed in a series of papers [12–16] in the context of analyzing the OPERA result. In [12–15] the superluminal motion of the muon neutrinos has been considered in the presence of flavor mixing, in which a muon neutrino is described by coherent combination of the mass eigenstates. Distortion of the ν_μ wave packet is produced by interplay of two effects: (i) the coordinate independent ν_μ oscillations in time and (ii) a relative shift of the wave packets of mass eigenstates due to the difference of their group velocities.[1] In this proposal, an additional contribution to the distance of ν_μ propagation, d, and to the group velocity of muon neutrinos is proportional to the difference of the group velocities of the neutrino mass eigenstates $\Delta v \approx \Delta m^2/2E^2$, which is extremely small. Since $d \sim 1/P_{\mu\mu}$, the effect can be enhanced if the survival probability of a muon neutrino, $P_{\mu\mu}$, is very small. The latter, in turn, requires mixing to be very close to the maximal and the oscillation phase to be very close to π. Apparently an additional distance of ν_μ propagation is restricted by the length of the neutrino wave packet.

In this paper, we give a comprehensive treatment of the group velocities of neutrinos in vacuum and in matter within the framework of Lorentz invariance. As in the earlier works [12–16], we consider the superluminal motion of muon neutrinos in the presence of flavor mixing, in which a muon neutrino is described by coherent superposition of the neutrino mass eigenstates. We show that the dominant effect, which most affects the velocity of muon neutrinos, is a distortion of the ν_μ wave packet by the neutrino oscillations within the size of ν_μ wave packets in

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[1]We will give a detailed explanation of the effect in the Appendix.

the coordinate space. The distortion can be significant because neutrino wave packets are large, as will be discussed in Sec. III. On the other hand, the effect of relative shift of the wave packets of mass eigenstates due to their velocity difference is negligible compared to the dominant one.

The paper is organized as follows. In Sec. II we present general formulas for the distance traveled by oscillating neutrinos during a given interval of time. In Sec. III we construct a wave packet of the muon neutrino produced in the pion decay and describe its properties. In Sec. IV we give explicit expressions for the distances traveled by muon neutrinos. We consider the limit of the small size of a decay tunnel when the wave packet has an approximate boxlike (rectangular) shape. Effect of relative shift of the wave packets of mass eigenstates on the distance is computed. In Sec. V we estimate the distances and velocities for existing experiments and propose experimental setups in which the effective superluminal motion might be observed. In Sec. VI we consider group velocities of neutrinos in matter. We conclude in Sec. VII. In the Appendix, a simple explanation of the shift effect proposed before is given.

II. OSCILLATION PROBABILITY AND DISTANCE OF ν_μ PROPAGATION

For simplicity we will consider the two-neutrino mixing

$$\nu_\mu = c\nu_1 + s\nu_2, \qquad \nu_\tau = c\nu_2 - s\nu_1, \tag{1}$$

where $c \equiv \cos\theta$, $s \equiv \sin\theta$, and θ is the mixing angle; ν_i ($i = 1, 2$) are the mass eigenstates that correspond to eigenvalues m_i. We will show in Sec. VI that this is a good approximation for the three-neutrino mixing case. Recall that, in experiments such as OPERA, MINOS, T2K, etc., the neutrinos oscillate in matter, and therefore in the presence of nonzero 1–3 mixing one should use the mixing angles and mass splitting in matter. However, apart from the resonance regions $E \sim 0.1$ GeV and $E = (4\text{–}8)$ GeV, the two-neutrino description with vacuum values of the mixing angles and $\Delta m^2 \equiv m_2^2 - m_1^2$ gives a very good approximation.

Evolution of the muon neutrino state after it exits the region of formation of wave packets (i.e., a decay tunnel) is described as

$$|\nu_\mu(t)\rangle = cf_1(x - v_1 t)e^{-iE_1 t + ip_1 x}|\nu_1\rangle + sf_2(x - v_2 t)e^{-iE_2 t + ip_2 x}|\nu_2\rangle, \tag{2}$$

where f_i are the shape factors, E_i and p_i are the average energies and momenta of the wave packets, respectively, and v_i are the group velocities of the mass eigenstates. The shape factors are normalized as

$$\int dx|f_i(x - v_i t)|^2 = 1. \tag{3}$$

If a muon neutrino is detected at time t, the wave function of ν_μ (the amplitude of probability to find ν_μ) is given by

$$\psi_{\nu_\mu}(x, t) = \langle\nu_\mu|\nu_\mu(t)\rangle = c^2 f_1(x - v_1 t)e^{-iE_1 t + ip_1 x} + s^2 f_2(x - v_2 t)e^{-iE_2 t + ip_2 x}. \tag{4}$$

Here, we assumed that the shape factor of a detected muon neutrino is constant in the detection area and does not depend on time. Alternatively, one can consider f_i as the effective shape factor that already includes the process of detection.

The ν_μ survival probability at t is obtained as

$$P_{\mu\mu}(t) \equiv \int dx|\psi_{\nu_\mu}(x, t)|^2. \tag{5}$$

In what follows, we will compute the averaged coordinate $\langle x(t)\rangle$ of the muon neutrino at time t defined as

$$\langle x(t)\rangle \equiv \frac{\int dx x|\psi_{\nu_\mu}(x, t)|^2}{\int dx|\psi_{\nu_\mu}(x, t)|^2}, \tag{6}$$

where the denominator is nothing but $P_{\mu\mu}(t)$.

In a very good approximation, we can take at the initial time

$$f_1(x) = f_2(x) \tag{7}$$

so that the difference between f_1 and f_2 at an arbitrary time arises solely due to the difference of group velocities of ν_1 and ν_2. The difference $\Delta v \equiv v_1 - v_2$ produces a relative shift and eventually separation of the wave packets of mass eigenstates in the process of propagation. The separation leads to loss of coherence between the mass eigenstates. Numerically,

$$\Delta v \equiv v_1 - v_2 \simeq \frac{\Delta m^2}{2E^2} = 5 \times 10^{-22}\left(\frac{\Delta m^2}{10^{-3}\ \text{eV}^2}\right)\left(\frac{E}{1\ \text{GeV}}\right)^{-2}.$$

Using the expression (4) and the normalization condition (3) we find

$$P_{\mu\mu}(t) = c^4 + s^4 + 2c^2 s^2 I, \tag{8}$$

where

$$I \equiv \int dx f_1(x - v_1 t)f_2(x - v_2 t)\cos\Phi(x, t), \tag{9}$$

and the relative (oscillation) phase of mass eigenstates, $\Phi(x, t)$, is given by

$$\Phi(x, t) \equiv \Delta E t - \Delta p x, \qquad \Delta E \equiv E_1 - E_2, \qquad \Delta p \equiv p_1 - p_2.$$

The phase $\Phi(x, t)$ can be split into two pieces as

$$\Phi(x, t) \equiv \phi + \chi, \tag{10}$$

where

$$\phi \equiv \Delta E t - \Delta p v t, \qquad v \equiv \frac{v_1 + v_2}{2} \tag{11}$$

is the standard oscillation phase that depends on time only and

$$\chi \equiv -\Delta p(x - vt). \tag{12}$$

The phase χ depends also on the coordinate x and, as we will see, describes the change of the total oscillation phase within the wave packets.

Let us introduce the average coordinates of the mass eigenstates as

$$\langle x_i \rangle \equiv \int dx x[f_i(x - v_i t)]^2 = v_i t + \int dz z[f_i(z)]^2. \tag{13}$$

If $f_i(z)$ is symmetric with respect to the central point of packet, z_0: $f_i(z_0 - h) = f_i(z_0 + h)$, we obtain (performing shift of integration $z \to z - z_0$) $\langle x_i \rangle = v_i t + z_0$.

Using $\langle x_i \rangle$ we define the averaged coordinate of the two mass eigenstates,

$$\bar{x} \equiv \frac{1}{2}(\langle x_1 \rangle + \langle x_2 \rangle) = \frac{1}{2}(v_1 + v_2)t + \int dz z[f_i(z)]^2, \tag{14}$$

and the half-difference of the coordinates

$$\Delta x \equiv \frac{1}{2}(\langle x_1 \rangle - \langle x_2 \rangle) = \frac{1}{2}\Delta v t.$$

Then, the numerator of $\langle x(t) \rangle$ in (6) can be written as

$$\int dx x|\psi_{\nu_\mu}(x, t)|^2 = (c^4 + s^4)\bar{x} + \frac{\Delta v t}{2} + 2s^2c^2 J, \tag{15}$$

where

$$J \equiv \int dx x f_1(x - v_1 t) f_2(x - v_2 t) \cos\Phi(x, t) \tag{16}$$

is the average coordinate of the overlap of the two mass eigenstates.

Using definitions (6), (8), and (15), we can present $\langle x(t) \rangle$ in the form

$$\langle x(t) \rangle = \bar{x} + \frac{\Delta v t \cos 2\theta}{2P_{\mu\mu}} + \frac{\sin^2 2\theta}{2P_{\mu\mu}}(J - \bar{x}I). \tag{17}$$

The first term in (17) is the distance traveled by massive neutrinos, and the two others describe the additional distances due to oscillations and relative shift of the wave packets.

The distance of ν_μ propagation during a time interval from t_0 to t equals

$$d(t) \equiv \langle x(t) \rangle - \langle x(t_0) \rangle. \tag{18}$$

As t_0 we will take the moment in time when the neutrino wave packets are completely formed. For neutrinos from pion decay, $t_0 = l_p/v_\pi$ is the time when the pion reaches the end of the decay tunnel. According to (17), the distance $d(t)$ can be presented as

$$d(t, t_0) = d_{\text{light}} + d_{\text{mass}} + d_{\text{osc-}a} + d_{\text{shift-}s}, \tag{19}$$

where

$$d_{\text{light}} \equiv c(t - t_0)$$

is the distance traveled by the light, and

$$d_{\text{mass}} = -\frac{\bar{m}^2}{2E^2}(t - t_0) = -\frac{m_1^2 + m_2^2}{4E^2}(t - t_0)$$

(where E is the neutrino energy) is the contribution to the distance due to nonzero neutrino mass;

$$d_{\text{osc-}a} = -\frac{\sin^2 2\theta}{2}\left[\frac{J - \bar{x}I}{P_{\mu\mu}}\bigg|_t - \frac{J - \bar{x}I}{P_{\mu\mu}}\bigg|_{t_0}\right]$$

is, as we will see, the contribution to the distance from oscillation distortion of the muon neutrino wave packet and from shift of the packets in the case of asymmetric shape factors; and

$$d_{\text{shift-}s} = \frac{\Delta v \cos 2\theta}{2}\left[\frac{t}{P_{\mu\mu}(t)} - \frac{t_0}{P_{\mu\mu}(t_0)}\right] \tag{20}$$

is the contribution from the relative shift of the wave packets and coordinate-independent oscillations in the case of symmetric shape factors. Notice that the terms in the brackets evaluated at $t = t_0$ are negligible, since usually $t \gg t_0$ and $P_{\mu\mu}(t_0) \sim 1$. However, it may not be always the case for $d_{\text{osc-}a}$ because $(J - \bar{x}I)$ is a nonlinear function of t.

As we will see, the distance traveled by muon neutrinos is an oscillatory function of time. Correspondingly, the velocity changes with time. Therefore, we can speak about the average velocity for a given time interval $(t - t_0)$ defined as $v(t) = d(t, t_0)/(t - t_0)$.

III. NEUTRINO WAVE PACKETS FROM PION DECAY

For definiteness we will consider neutrinos from pion decay. Actually, these neutrinos dominate in the neutrino fluxes of the MINOS, T2K, and OPERA experiments. The contribution from K- mesons can be considered similarly. We describe here the wave packets of muon neutrinos [the wave function ψ_{ν_μ} in Eq. (4)]. The strict derivation of the expressions for the wave packets is given in [17], and here we explain properties of the shape factors and phases using simple physics arguments.

Let us consider first the shape factors. Pions are produced in the strong interaction of protons with a solid state target. Protons have wave packets of very small size and can be considered pointlike. The target nuclei are localized within the atomic scale. Therefore we can take in our computations that pions are produced at a fixed space-time point $x = t = 0$. The wave packets of pions are very short in the configuration space, so that pions also can be considered pointlike with space-time trajectories of motion $x = v_\pi t$. Here v_π is the group velocity of pion. In experiments under consideration, pions are ultrarelativistic with Lorentz factor $\gamma_\pi \equiv E_\pi/m_\pi \gg 1$, where E_π and m_π are the pion energy and the mass, respectively.

Pions decay in a decay tunnel of size l_p, which includes the length of the horn area and the decay pipe. Essentially l_p is the distance from the target to the end point of the decay pipe where surviving pions are absorbed. We neglect the effects of interactions of pions with particles in the decay pipe. In the decay tunnel, the quantum state is a superposition of an undecayed pion state and a state of muon and muon-neutrino. The neutrino waves emitted from different points of the pion trajectory are coherent, thus forming a single wave packet.

The decay length of a pion equals

$$l_{\rm decay} = v_\pi \tau_\pi = v_\pi \gamma_\pi \tau_\pi^0,$$

where τ_π and τ_π^0 are the lifetime of the pion in the laboratory frame and the lifetime in the pion rest frame, respectively. $\tau_\pi^0 = \Gamma_0^{-1}$ with Γ_0 being the pion decay rate at rest.

If $l_p \leq l_{\rm decay}$ (high energies, long-lived parent particles, short decay tunnel), the wave packets of mass eigenstate ν_i have the size

$$\sigma_i = l_p\left(\frac{v_i}{v_\pi} - 1\right) \approx \frac{l_p}{2\gamma_\pi^2}, \tag{21}$$

when a neutrino is emitted in the forward direction with respect to the pion velocity. The wave packet size is proportional to l_p; the factor in brackets represents shrinking of the packet due to pion motion [18]. In the second equality in (21), we used $m_i \ll m_\pi$. Numerically[2]

$$\sigma \simeq 50\ {\rm cm}\left(\frac{\gamma_\pi}{10}\right)^{-2}\left(\frac{l_p}{100\ {\rm m}}\right) = 17.7\ {\rm cm}\left(\frac{E}{1\ {\rm GeV}}\right)^{-2}\left(\frac{l_p}{100\ {\rm m}}\right). \tag{22}$$

If $l_{\rm decay} \ll l_p$ (low energies, short-lived parent particles, long decay tunnel), the size of the wave packet is determined by the decay length:

$$\sigma_i = l_{\rm decay}\left(\frac{v_i}{v_\pi} - 1\right) = \gamma_\pi \tau_\pi^0 (v_i - v_\pi). \tag{23}$$

For relativistic pions ($\gamma_\pi \gg 1$), $\sigma \approx \frac{1}{2}\tau_\pi^0 \gamma_\pi^{-1}$ instead of (21). In the nonrelativistic limit, $v_\pi \to 0$, the size becomes the largest: $\sigma_i = \tau_\pi^0 v_i \simeq 7.8$ m.

[2]For two-body pion decay the neutrino energy E is determined by a Lorentz factor of the pion, γ_π, or vice versa: $\gamma_\pi \simeq 16.8(\frac{E}{1\ {\rm GeV}})$ to a good approximation for $E \geq 500$ MeV and forward-going neutrinos.

Thus, in general,

$$\sigma_i = l_{\rm form}\left(\frac{v_i}{v_\pi} - 1\right) \approx \frac{l_{\rm form}}{2\gamma_\pi^2}, \tag{24}$$

where $l_{\rm form}$ is the region of formation of the neutrino wave packet:

$$l_{\rm form} \sim \min\{l_p, l_{\rm decay}\}.$$

In what follows we will neglect differences between the sizes of wave packets of the mass eigenstates, taking $\sigma_1 \approx \sigma_2$. It is an excellent approximation because the difference is of the order $\sim l_p \Delta v \sim 10^{-16}$ cm for $l_p = 1$ km, $E = 1$ GeV, and $\Delta m^2 = 2.5 \times 10^{-3}\ {\rm eV}^2$.

The neutrino shape factor from pion decay can be written as [17]

$$f(x) = f_0 e^{(\Gamma/2(v - v_\pi))(x-\sigma)} \Pi(x, [0, \sigma]), \tag{25}$$

where $\Pi(x, [0, \sigma]) = 1$ in the interval $0 \leq x \leq \sigma$ and Π vanishes outside this interval. The box function reflects a finite-sized neutrino production region with sharp edges related to the definite points of pion production and absorption at the end of the decay tunnel. The exponential factor follows from the amplitude of the pion decay with the rate enhanced by the factor $1/(v - v_\pi)$, which corresponds to shrinking of the neutrino wave packet emitted in a forward direction [see Eq. (24)]. This is nothing but the Doppler effect.

The normalization condition (3) gives

$$f_0^2 = \frac{y}{\sigma}\frac{1}{1 - e^{-y}}, \tag{26}$$

where

$$y \equiv \frac{\Gamma \sigma}{v - v_\pi} = \frac{l_{\rm form}}{l_{\rm decay}}. \tag{27}$$

If $l_{\rm decay} > l_p$, we have $l_{\rm form} = l_p$, so that y is the size of the decay tunnel in units of the decay length. Numerically,

$$y = \frac{l_p}{l_{\rm decay}} \approx \frac{l_p \Gamma_0}{\gamma_\pi} \simeq 1.28 \times \left(\frac{\gamma_\pi}{10}\right)^{-1}\left(\frac{l_p}{100\ {\rm m}}\right). \tag{28}$$

If $l_{\rm decay} < l_p$ and if we neglect effects of the exponentially suppressed tails, then $l_{\rm form} \simeq l_{\rm decay}$, so that $y \simeq 1$.

In the approximation $v_i \approx v$, the difference of momenta of the neutrino mass eigenstates equals [17]

$$\Delta p \equiv p_1 - p_2 \approx -\frac{\Delta m^2}{2E(v - v_\pi)} \approx -\frac{\Delta m^2}{2E} 2\gamma_\pi^2. \tag{29}$$

The factor in the denominator of (29) is the same as the one that describes shrinking of the wave packet size in comparison with the size of the wave packet formation region. The shrinking is accompanied by an increase of frequencies by the same factor. This is again the Doppler effect related to neutrino emission from a moving pion. Consequently, the oscillation effect within the packet is enhanced: it is determined by the oscillations within the wave packet formation region, rather than within the size of the packet itself in the laboratory frame.

IV. DISTANCES OF ν_μ PROPAGATION

We will use the formulas derived in Sec. II and the wave function of the muon neutrino constructed in Sec. III to find $P_{\mu\mu}$ and $\langle x(t)\rangle_{\rm osc}$. We first compute the effect of oscillations, which gives the dominant effect, neglecting a relative shift of the ν_1 and ν_2 wave packets due to a difference of group velocities. Then, we will estimate the effect of the shift, neglecting oscillations along the wave packets.

A. Oscillation effect

Let us set $v_1 = v_2 = v$ ($\Delta v = 0$) and compute I and J given in (9) and (16). We use the shape factors (25) and the phase Φ with ϕ defined in (11) and χ given according to (12) and (29) by

$$\chi = \frac{\Delta m^2}{2E}\frac{x - vt}{v - v_\pi}.$$

We obtain the probability

$$P_{\mu\mu}(t) = c^4 + s^4 + 2s^2c^2(\cos\phi I_c - \sin\phi I_s), \qquad (30)$$

where

$$\begin{aligned} I_c &\equiv \int dz f^2(z)\cos|\Delta p|z \\ &= \frac{y}{1-e^{-y}}\frac{1}{y^2+\phi_p^2}(-ye^{-y} + y\cos\phi_p + \phi_p\sin\phi_p), \\ I_s &\equiv \int dz f^2(z)\sin|\Delta p|z \\ &= \frac{y}{1-e^{-y}}\frac{1}{y^2+\phi_p^2}(\phi_p e^{-y} + y\sin\phi_p - \phi_p\cos\phi_p). \end{aligned} \qquad (31)$$

Here

$$\phi_p \equiv \frac{\Delta m^2}{2E}\frac{\sigma}{(v - v_\pi)} \approx \frac{\Delta m^2}{2E} l_{\rm form} = 2\pi\frac{l_{\rm form}}{l_\nu} \qquad (32)$$

is the phase change within the wave packet. Here $l_\nu = 4\pi E/\Delta m^2$ is the oscillation length. The second and third equalities in Eq. (32) are valid for an ultrarelativistic pion: $v_\pi \approx 1$. Thus, ϕ_p is given by the phase acquired within the distance of formation of the neutrino wave packet, $l_{\rm form}$. For $l_p < l_{\rm decay}$ we have

$$\begin{aligned} \phi_p &= \frac{\Delta m^2}{2E} l_p \\ &\approx 2.5\times 10^{-4}\left(\frac{\Delta m^2}{10^{-3}\ {\rm eV}^2}\right)\left(\frac{E}{1\ {\rm GeV}}\right)^{-1}\left(\frac{l_p}{100\ {\rm m}}\right). \end{aligned} \qquad (33)$$

For $l_p > l_{\rm decay}$ inserting σ from (23) into (32), we obtain $\phi_p = 2\pi\gamma_\pi\tau_\pi^0 v/l_\nu$.

We now calculate $\langle x(t)\rangle$ (17) for $\Delta v = 0$. The average distance traveled by mass eigenstates (14) with the wave packets (25) equals

$$\bar{x} = vt + \sigma\left(\frac{1}{1-e^{-y}} - \frac{1}{y}\right). \qquad (34)$$

A straightforward computation of the contribution from the oscillation effect to the distance of ν_μ propagation gives

$$\langle x\rangle_{\rm osc} = -\frac{\sin^2 2\theta}{2P_{\mu\mu}}\frac{\sigma}{y^2+\phi_p^2}(\kappa_s\sin\phi - \kappa_c\cos\phi). \qquad (35)$$

Here, the coefficients κ_s and κ_c in front of $\sin\phi$ and $\cos\phi$ equal

$$\kappa_s = \frac{y}{1-e^{-y}}\left[\frac{-2y\phi_p e^{-y} + q\sin\phi_p - n\cos\phi_p}{y^2+\phi_p^2} - \left(\frac{1}{1-e^{-y}} - \frac{1}{y}\right)(\phi_p e^{-y} + y\sin\phi_p - \phi_p\cos\phi_p)\right]$$

and

$$\kappa_c = \frac{y}{1-e^{-y}}\left[\frac{(y^2-\phi_p^2)e^{-y} + q\cos\phi_p + n\sin\phi_p}{y^2+\phi_p^2} - \left(\frac{1}{1-e^{-y}} - \frac{1}{y}\right)(-ye^{-y} + y\cos\phi_p + \phi_p\sin\phi_p)\right], \qquad (36)\ (37)$$

where

$$q \equiv y^2(y-1) + \phi_p^2(y+1),$$
$$n \equiv \phi_p(y^2 - 2y + \phi_p^2).$$

In realistic setups, $y \sim 1$ and $\phi_p \ll y$. For $\phi_p \to 0$ we obtain $\langle x\rangle_{\rm osc} = 0$, because there is no room in which an oscillation effect develops if $l_p \ll l_\nu$. In the limit of small ϕ_p the distance equals

$$\langle x\rangle_{\rm osc} = -\frac{\sin^2 2\theta}{2P_{\mu\mu}}\frac{\sigma\phi_p}{y^2}\left[\eta_s\sin\phi - \eta_c\frac{\phi_p}{y}\cos\phi\right], \qquad (38)$$

where

$$\eta_s = 1 - \left(\frac{y}{1-e^{-y}}\right)^2 e^{-y},$$

$$\eta_c = 2 - \left(\frac{y}{1-e^{-y}}\right)\left(1 - y + \frac{y^2}{2}\right) - \left(\frac{y}{1-e^{-y}}\right)^2\left(1 - \frac{y}{2}\right).$$

At $y = 1$, $\eta_s/y^2 \approx 1/12.61$, and it depends very weakly on y. It exactly equals to $1/12$ in the case of a boxlike packet that corresponds to $y \to 0$ (see Sect. IV B).

The effect of oscillations on the group velocity of the muon neutrino is illustrated in Fig. 1. In the case of

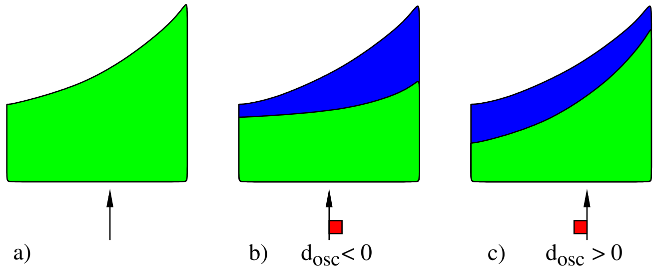

FIG. 1 (color online). The wave packets of the muon neutrino from pion decay without oscillations (a) and with oscillations at two different times (b), (c). The light-shadowed (green) parts of the shape factors show the ν_μ fraction, whereas the dark-shadowed (blue) parts correspond to the ν_τ fraction that appears due to oscillations. The arrows indicate positions of "centers of mass" of the ν_μ parts. The small (red) boxes show shifts of the centers due to oscillations with respect to the center in the no-oscillation case. Panel (b) corresponds to the baselines $0 < L < l_{osc}/2$, when the front edge of the wave packet is suppressed. Panel (c) is for $l_{osc}/2 < L < l_{osc}$, when the trailing edge is suppressed.

Fig. 1(b), the oscillations suppress the front edge of the ν_μ packet more strongly than the trailing edge. As a result, the "center of mass" of the packet shifts backward and therefore the distance of propagation is reduced. In the case of Fig. 1(c), realized in another moment of time, the trailing edge is suppressed more strongly; the center of mass shifts forward and the distance of propagation is increased.

B. Limit of small production region, $y \ll 1$

Expression for $\langle x \rangle$ obtained in Sec. IVA simplifies substantially in the limit $y \to 0$. It corresponds to short decay tunnels ($l_p \ll l_{dec}$) or to high-energy parts of the neutrino spectra in real experiments. In this limit $\Gamma \to 0$ and therefore

$$f(x) = \frac{1}{\sqrt{\sigma}} \Pi(x, [0, \sigma]), \tag{39}$$

and we will refer to this as the "boxlike" packet.

To obtain $P_{\mu\mu}$ and $\langle x \rangle_{osc}$ for the wave packet (39), we take the limit $y \to 0$ in (30) and (35) while keeping ϕ_p finite. The ν_μ survival probability becomes

$$P_{\mu\mu}(t) = c^4 + s^4 + \sin^2 2\theta \left[\frac{\sin\frac{\phi_p}{2}}{\phi_p} \right] \cos\left(\phi + \frac{\phi_p}{2}\right). \tag{40}$$

In the limit $y \to 0$ the second term in (34) tends to $\sigma/2$ and, consequently, $\bar{x} = vt + \frac{\sigma}{2}$.

Finally, $\langle x \rangle_{osc}$ takes the form

$$\langle x(t) \rangle_{osc} = -\frac{\sin^2 2\theta}{2P_{\mu\mu}} \frac{\sigma}{\phi_p^2} \left(2 \sin\frac{\phi_p}{2} - \phi_p \cos\frac{\phi_p}{2} \right) \times \sin\left(\phi + \frac{\phi_p}{2}\right). \tag{41}$$

If ϕ_p is small (for existing setups $\phi_p \sim 10^{-3}$), $\langle x \rangle_{osc}$ varies with distance traveled as $\sim \sin\phi$. For short baselines, when $\phi < \pi/2$, the additional distance $\langle x(t) \rangle_{osc} < 0$; i.e., oscillations suppress velocity (see Fig. 1). In addition, for small ϕ the distance is suppressed by ϕ. At around $\pi/2$, $3\pi/2$, etc., the oscillation effect is maximal for $\langle x(t) \rangle_{osc}$. Furthermore, at $\phi \sim \pi/2$ ($3\pi/2$) the oscillation probability is a decreasing (increasing) function of x. Therefore, the effective shape of the ν_μ wave packet deforms in such a way that the effective velocity is smaller (larger) than the normal velocity of massive particles at $\phi \sim \pi/2$ ($3\pi/2$).

In the limit $\phi_p \to 0$, the expressions in (40) and (41) give

$$P_{\mu\mu}(t) \approx c^4 + s^4 + \frac{1}{2} \sin^2 2\theta \left[\cos\left(\phi + \frac{\phi_p}{2}\right) - \frac{\phi_p^2}{6} \cos\phi \right], \tag{42}$$

$$\langle x(t) \rangle_{osc} \approx -\frac{\sin^2 2\theta}{24 P_{\mu\mu}} \sigma \phi_p \sin\left(\phi + \frac{\phi_p}{2}\right), \tag{43}$$

where we have used the fact that the factor inside the parentheses in front of the sine factor in (41) is approximately equal to $\frac{1}{12}\phi_p^3$. We have not expanded the cosine and sine factors in (42) or (43) since ϕ or its deviations from $n\pi$ could become smaller than ϕ_p at certain times.

According to (43), the contribution to the distance is proportional to [3]

$$\sigma\phi_p = 1.25 \times 10^{-2}\ \text{cm} \left(\frac{\Delta m^2}{10^{-3}\ \text{eV}^2} \right) \left(\frac{E}{1\ \text{GeV}} \right)^{-1} \times \left(\frac{\gamma_\pi}{10} \right)^{-2} \left(\frac{l_p}{100\ \text{m}} \right)^2. \tag{44}$$

Thus, in the limit $\Delta v \to 0$ the total distance traveled by neutrinos equals

$$d(t, t_0) = v(t - t_0) + [\langle x(t) \rangle_{osc} - \langle x(t_0) \rangle_{osc}],$$

where the first term is the distance traveled by a massive neutrino. The second term takes the largest positive value $\approx 2 \times 10^{-3}$ cm, at around $\phi \sim \frac{3\pi}{2}$ and for the typical values of the parameters in the parentheses of Eq. (44). This term dominates over the first one.

C. Estimation of effect of shift of the wave packets

For realistic experimental setups $\Delta v t \ll \sigma < l_{decay}$. The difference of group velocities of the mass eigenstates produces a relative shift of their wave packets and therefore an additional distortion of the ν_μ wave packet. Because of

[3]Though γ_π and E are related with each other in pion decay, we represent both factors independently for possible use of the formula for neutrinos from other sources such as muon decay, etc.

the shift, three different spatial parts of the ν_μ wave packet appear (we assume that $v_1 > v_2$):

(1) The front edge part: $(v_2 t + \sigma) \le x \le (v_1 t + \sigma)$. In this part only the ν_1 packet is present; therefore $\psi_{\nu_\mu} \approx c^2 f(\sigma) = c^2 f_0$ and $|\psi_{\nu_\mu}|^2 = c^4 f_0^2$. The length of this part equals $\Delta v t$ and we can safely neglect the change of $f(x)$ within this interval.

(2) The overlapping part: $(v_1 t) \le x \le (v_2 t + \sigma)$. Here both wave packets are nonzero and interfere between each other:

$$|\psi_{\nu_\mu}|^2 \approx f(x)^2 \left| c^2 + s^2 \cos\Phi(x) \times \left(1 + \frac{\Delta v t \Gamma}{2(v - v_\pi)}\right) \right|^2. \quad (45)$$

(3) The trailing edge part: $(v_2 t) \le x \le (v_1 t)$. Here only the ν_2 wave packet is present and $\psi_{\nu_\mu} \approx s^2 f(0)$, so that $|\psi_{\nu_\mu}|^2 \approx s^4 f_0^2 e^{-(\Gamma\sigma/v - v_\pi)}$.

Apparently, oscillations do not affect contributions from the front and trailing edge parts. These parts produce additional asymmetry of the ν_μ wave packet that is proportional to

$$f_0^2(c^4 - s^4 e^{-(\Gamma\sigma/v - v_\pi)})\Delta v t = (c^4 - s^4 e^{-(\Gamma\sigma/v - v_\pi)}) \times \left(\frac{y}{1 - e^{-y}}\right)\frac{\Delta v t}{\sigma}.$$

Consequently, the contribution to the distance of ν_μ propagation $d_{\text{shift-}a} \sim \Delta v t$. As we will show later, the asymmetry produced by the overlapping region (which depends on oscillations) is even smaller than $\frac{\Delta v t}{\sigma}$. Therefore in the first approximation we can neglect dependence of the oscillation effect inside the wave packet size on distance and take the phase Φ to be constant.

Let us introduce the dimensionless parameter

$$\epsilon \equiv \frac{\Delta v t \Gamma}{2 v_\pi} = \frac{\Delta v t}{2}\frac{1}{l_{\text{decay}}} \quad (46)$$

which is the half-shift of the wave packets in units of the decay length. Its value can be estimated as

$$\epsilon \approx \frac{\Delta v t}{2}\frac{\Gamma_0}{v_\pi \gamma_\pi} \approx 3.20 \times 10^{-18}\left(\frac{\Delta m^2}{10^{-3}\ \text{eV}^2}\right)\left(\frac{E}{1\ \text{GeV}}\right)^{-2}\left(\frac{L}{10^3\ \text{km}}\right)\left(\frac{\gamma_\pi}{10}\right)^{-1}, \quad (47)$$

where $L = ct$ stands for baseline distance.

When the oscillation effect in the coordinate space within the wave packet size is neglected, we have $J = \cos\phi \int dx x f_1(x - v_1 t) f_2(x - v_2 t)$ and $I = \cos\phi \times \int dx f_1(x - v_1 t) f_2(x - v_2 t)$. Then straightforward computations give

$$I = \cos\phi \frac{e^{-\epsilon} - e^{\epsilon - y}}{1 - e^{-y}},$$
$$J = \cos\phi \frac{1}{1 - e^{-y}}\left[(e^{-\epsilon} - e^{\epsilon - y})\left(vt - \frac{\sigma}{y}\right) - \frac{\Delta v t}{2}(e^{-\epsilon} + e^{\epsilon - y}) + \sigma e^{-\epsilon}\right]. \quad (48)$$

For the average distance traveled by the mass eigenstates we have $\bar{x}$ given in (34) with $v = 0.5(v_1 + v_2)$.

Using (48) we obtain correction to the distance of ν_μ propagation:

$$\langle x(t)\rangle_{\text{shift-}a} = \cos\phi \frac{\sin^2 2\theta}{2P_{\mu\mu}}\left(\frac{\Delta v t}{2}\right)F(y, \epsilon), \quad (49)$$

where

$$F(y, \epsilon) \equiv \frac{1}{(1 - e^{-y})^2}\left[y e^{-y}\frac{e^{-\epsilon} - e^{\epsilon}}{\epsilon} - (1 - e^{-y})(e^{-\epsilon} + e^{\epsilon - y})\right], \quad (50)$$

and the probability $P_{\mu\mu}$ is given in (8) with I from (48).

As can be seen from (47), ϵ is negligibly small for typical values of experimental parameters. In the limit of small ϵ we have

$$F(y) = \frac{1}{(1 - e^{-y})^2}[2y e^{-y} - 1 + e^{-2y}]. \quad (51)$$

Also in most of the experimental settings $y \sim 1$, and consequently $F(y, \epsilon) \sim \mathcal{O}(1)$. In particular, $F(1) = 0.322$. Then according to (49) $|d(t)| \approx \langle x(t)\rangle_{\text{shift-}a} \sim \Delta v t$, assuming $\cos\phi \sim \mathcal{O}(1)$ and $P_{\mu\mu} \sim \mathcal{O}(1)$. Notice that the $\langle x(t_0)\rangle$ term in $d(t)$ can be ignored for baselines $L > 100$ km, since $l_p \ll L$. Numerically,

$$\Delta v t \approx \frac{\Delta m^2}{2E^2} L = 0.5 \times 10^{-13}\left(\frac{\Delta m^2}{10^{-3}\ \text{eV}^2}\right)\left(\frac{E}{1\ \text{GeV}}\right)^{-2}\left(\frac{L}{10^3\ \text{km}}\right)\ \text{cm}. \quad (52)$$

Therefore, contribution to $d(t)$ due to a velocity difference of mass eigenstates is extremely small.

For small y we obtain $F \approx -y/3$. In the limit of $y \to 0$, which corresponds to long lifetimes compared to the travel time inside a decay tunnel, the function F and therefore the distance vanish: $\langle x(t)\rangle_{\text{shift}} = 0$. This limit corresponds to symmetric (boxlike) wave packets. Therefore, the contribution to the distance (49) described here originates from asymmetric profile of the shape factors.

If $y \to \infty$, we obtain $F \to -1$. Let us underline that $F < 0$, and therefore the correction is negative (reduces the effective velocity). It can be positive if formally $\Gamma < 0$, that is, when the amplitude of the shape factor is a decreasing function of x or, equivalently, increases with time.

TABLE I. The values of $y = l_{\rm form}/l_{\rm decay}$, sine of the oscillation phase, ϕ, the wave packet length, σ, as well as the contributions to the distance of ν_μ propagation from the mass terms, $d_{\rm mass}$, from the relative shift of the wave packets, $d_{{\rm shift}-s}$, and from oscillations, $d_{\rm osc}$. We use $\Delta m^2 = 2.5 \times 10^{-3}$ eV^2, and $\sin^2 2\theta = 0.97$.

Experiment	E (GeV)	y	$\sin\phi$	σ (cm)	$-d_{\rm mass}$ (cm)	$d_{\text{shift-}s}$ (cm)	$d_{\rm osc}$ (cm)
OPERA	17	0.491	0.270	0.671	1.58×10^{-16}	2.80×10^{-17}	-1.0×10^{-5}
OPERA	1	1	-0.99	23.3	4.56×10^{-14}	1.65×10^{-14}	$+1.6 \times 10^{-3}$
MINOS	3	1	1.00	7.7	5.10×10^{-15}	1.70×10^{-15}	-4.6×10^{-4}
MINOS	1	1	-0.995	23.3	4.59×10^{-14}	1.62×10^{-14}	$+1.54 \times 10^{-3}$
T2K	0.6	1	0.052	38.8	5.12×10^{-14}	2.94×10^{-13}	-2.2×10^{-3}
T2K	0.4	1	-0.997	58	1.14×10^{-13}	4.25×10^{-14}	$+3.9 \times 10^{-3}$

Let us now estimate corrections to the distance due to the overlapping part by taking into account the coordinate-dependent oscillation effect. The correction due to $\Delta v t$ in (45) does not depend on x, and therefore can be "absorbed" in redefinition of the mixing parameters:

$$|\psi_{\nu_\mu}|^2 \approx (1 + rs^2)^2 f(x)^2 |c_1^2 + s_1^2 \cos\Phi(x)|^2,$$

where

$$c_1^2 = \frac{c^2}{1 + rs^2}, \qquad s_1^2 = \frac{s^2(1+r)}{1 + rs^2},$$

and

$$r \equiv \frac{\Delta v t \Gamma}{2(v - v_\pi)} \sim \frac{\Delta v t}{\sigma}.$$

According to (49) the resulting additional distance equals

$$\langle x \rangle_{\text{shift-}a} \propto \frac{(1 + rs^2)^2 s_1^2 c_1^2}{P_{\mu\mu}} = (1 + r) s^2 c^2.$$

Therefore the correction to distance of propagation due to the shift is

$$\Delta\langle x \rangle_{\text{shift-}a} = r \langle x \rangle^0_{\text{shift-}a} \sim \Delta v t \frac{\langle x \rangle^0_{\text{shift-}a}}{\sigma} \ll \Delta v t,$$

where $\langle x \rangle^0_{\text{shift-}a}$ is the shift for constant survival probability.

V. NEUTRINO VELOCITY IN VARIOUS EXPERIMENTAL SETTINGS

A. Distances for existing experimental setups

Here we present numerical estimates of the correction terms in $d(t)$, ignoring the length of pion decay tunnels compared to the baseline distance, so that $d(t) \simeq \langle x(t) \rangle$. According to (19) the corrections to normal distance traveled by massive neutrinos consists of three different terms. It may be instructive to compare behavior of these terms in the limit of small phases $\phi \ll 1$ and $\phi_p \ll 1$ when all the contributions become linear in the time interval:

$$\begin{aligned} d_{\rm osc} &= -\frac{\sin^2 2\theta}{2P_{\mu\mu}(t)} \frac{\sigma \phi_p}{12} \frac{\Delta m^2}{2E}(t - t_0) \\ &= -\frac{\sin^2 2\theta}{2P_{\mu\mu}(t)} \frac{\sigma E \phi_p}{12} \Delta v (t - t_0), \\ d_{\text{shift-}s} &= \frac{\cos 2\theta}{2P_{\mu\mu}(t)} \Delta v (t - t_0), \\ d_{\text{shift-}a} &= \frac{\sin^2 2\theta}{2P_{\mu\mu}(t)} F(y, \epsilon) \Delta v (t - t_0). \end{aligned} \tag{53}$$

It follows immediately from these equations that $d_{\rm osc} \gg d_{\rm mass} \gtrsim d_{\rm shift}$. The contribution $d_{\rm osc}$ is strongly enhanced by the factor σE in comparison to the other contributions.

All the terms in (19) quickly decrease with energy:

$$d_{\rm osc} \propto \frac{1}{E^4}, \qquad d_{\rm mass} \propto \frac{1}{E^2},$$
$$d_{\text{shift-}s} \propto \frac{1}{E^2}, \qquad d_{\text{shift-}a} \propto \frac{1}{E^5}.$$

The contribution $d_{\rm shift}$ vanishes for maximal mixing. Other contributions depend on θ rather weakly or are independent of θ.

In Table I we present numerical estimates of the correction terms $d(t)_i \approx \langle x \rangle_i$ for the existing experiments. We take $\Delta m^2 = 2.5 \times 10^{-3}$ eV^2, and $\sin^2 2\theta = 0.97$, which corresponds to the largest departure from the maximal mixing allowed by the Super-Kamiokande atmospheric neutrino data at 90% CL [19].[4] We use Eqs. (35) and (38) to compute $d_{\rm osc}$. To find $d_{\rm mass}$ we have assumed that $m_2^2 + m_1^2 \approx \Delta m^2 = 2.5 \times 10^{-3}$ eV^2, which corresponds to the strong mass hierarchy. $d_{\text{shift-}s}$ has been computed according to Eq. (20). Notice that $d_{\text{shift-}s} = |d_{\rm mass}| \cos 2\theta / P_{\mu\mu}$. As follows from (49) for the exponential (asymmetric) wave packet

$$d_{\text{shift-}a} = d_{\text{shift-}s} \frac{F(y) \sin^2 2\theta \cos\phi}{2 \cos 2\theta},$$

and therefore $d_{\text{shift-}a} \lesssim d_{\text{shift-}s}$ unless $\cos 2\theta$ is very small.

[4] This is to avoid the maximal value of θ that leads to vanishing $d_{\rm shift}$.

In Table I we also show the size of the neutrino wave packet σ that gives the absolute upper bound on additional contribution to the distance of propagation, the parameter y, and the sine of oscillation phase. The phase ϕ_p acquired by neutrinos in the wave packet formation region equals $\phi_p = 4 \times 10^{-4}$ for $E = 17$ GeV in OPERA experiments and $\phi_p \sim 8.2 \times 10^{-4}$ in all other setups with $y \sim 1$. The contributions have been computed for the average energy and for a representative value of E at low energies in the spectrum in each experiment. We used the following baselines and decay tunnel lengths: OPERA: $L = 730$ km and $l_p = 1095$ m; MINOS: $L = 735$ km and $l_p = 715$ m; T2K: $L = 295$ km and $l_p = 118$ m.

For all the cases we find $d/\sigma \lesssim 10^{-3}$. The superluminal motion is realized when $d_{\rm osc} > |d_{\rm mass}|$ and this condition can be satisfied for all existing setups. However, as follows from Table I, the oscillation effects cannot explain the OPERA result in [3]. Indeed, we have shown that

(i) the additional distance of the $\nu_\mu-$ propagation is too small: it is restricted by size of the neutrino wave packet. Since the signal in OPERA is not suppressed, $P_{\mu\mu} \sim 1$, the effect cannot be related to oscillations into sterile neutrinos. Therefore, for the known Δm^2, the distance is further suppressed by the phase ϕ_p acquired by neutrinos in the production region.

(ii) the distance has strong dependence on neutrino energy E; d increases as E decreases.

In MINOS at certain energies, increase of propagation distance can be realized due to smallness of $P_{\mu\mu}$. In the case of $l_{\rm decay} \ll l_p$ ($y \gg 1$), the effect can be enhanced if one includes in consideration the exponentially suppressed tails of the wave packets utilizing l_p as the wave packet formation region instead of $l_{\rm decay}$. (Of course, this alternative requires an extremely intense beam of neutrinos.) From Eq. (38) we obtain in the $y \to \infty$ limit

$$\langle x\rangle_{\rm osc} \to -2\pi \frac{\sin^2 2\theta}{2P_{\mu\mu}} \frac{l_{\rm decay}^2}{l_\nu}\left(\frac{v - v_\pi}{v_\pi}\right)(\sin\phi + \phi_p \cos\phi) \tag{54}$$

So, in comparison to the case $y = 1$, for $y \gg 1$ the distance increases by a factor $\eta_s(\infty)/\eta_s(1) = 12.61$. Therefore in all the cases in Table I, where $y = 1$ the distance would be at most 12.61 times larger. In this way we obtain $d_{\rm osc} \sim 0.02$ cm for OPERA (1 GeV), MINOS (1 GeV), and T2K (0.4 GeV) instead of the numbers in Table I. Even with such an enhancement we have $d_{\rm osc}/\sigma < 10^{-3}$, and observation of these distances seems practically impossible.

B. Enhancement of contributions

Let us consider a possibility to increase the additional distance of ν_μ propagation and to observe effective superluminal motion of ν_μ. With nanotechnology the accuracy of time measurement would allow probing changes of distances of propagation as small as 1 m. Notice that one can evaluate the change of time of ν_μ propagation for a given baseline L dividing the corrections to the distance of propagation by v, e.g., $\Delta t_{\rm osc} = -d_{\rm osc}/v \approx -d_{\rm osc}/c$.

In this discussion we will use the results for boxlike shape factor that are simple and transparent, and at the same time are valid up to $\mathcal{O}(1)$ coefficients also for the exponential shape factors. The result for the boxlike wave packet (41) can be rewritten in the following form

$$\langle x\rangle_{\rm osc} = -\sigma g(\phi_p)\frac{\sin^2 2\theta}{2P_{\mu\mu}} \sin\left(\phi + \frac{\phi_p}{2}\right), \tag{55}$$

where

$$g(\phi_p) \equiv \frac{2}{\phi_p^2}\left(\sin\frac{\phi_p}{2} - \frac{\phi_p}{2}\cos\frac{\phi_p}{2}\right), \tag{56}$$

and $P_{\mu\mu} \equiv P_{\mu\mu}(\phi + \frac{\phi_p}{2})$.

According to (55) there are several ways to increase $d_{\rm osc} \approx \langle x\rangle_{\rm osc}$:

(1) Increase $\sigma = \sigma(\tau, l_{\rm decay}, v_\pi, \theta^0_{\nu\pi})$. For this, one should lower the neutrino energy, use parent particles with long lifetimes, increase the size of the decay region, and/or use off-axis neutrino beams. According to [18]

$$\sigma_x = \frac{\tau^0_\pi}{\gamma_\pi(1 + v_\pi \cos\theta^0_{\nu\pi})}, \tag{57}$$

where $\theta^0_{\nu\pi}$ is the angle between a neutrino momentum in the pion rest frame and a momentum of a pion in the laboratory frame. For $\theta^0_{\nu\pi} = 0$ it reproduces our previous results. With an increase of $\theta^0_{\nu\pi}$ the size of the wave packet increases. Simultaneously, the energy of neutrino decreases in such a way that $\sigma_x E$ is invariant [18]. For $\theta^0_{\nu\pi} = \pi/2$, for example, $\sigma_x = 2\sigma^0_x$ and energy becomes half. Thus, for off-axis experiments the additional distance of propagation can be larger. For the backward-emitted neutrinos, $\theta^0_{\nu\pi} = \pi$, the size of the wave packet becomes the largest one: $\sigma_x = 2\gamma_\pi v \tau^0_\pi = 2v\tau_\pi$. Then $\sigma_x = 2l_p$, if $l_{\rm decay} > l_p$. The energy becomes $4\gamma_\pi^2$ times smaller, thus for $E_\pi = 1.4$ GeV it will be about 2 MeV.

(2) Increase of the phase $\phi_p = l_p(l_p, l_{\rm decay}, l_\nu)$ acquired over the neutrino production region. This is possible with increase of Δm^2, decrease of the neutrino energy, etc.

(3) Selection of certain values of the oscillation phase, $\phi(L, l_\nu)$, by the selecting particular values of the baseline and/or neutrino energy. By varying the oscillation phase ϕ we find that maximal value of the distance is given by

$$\langle x(t)\rangle_{\rm osc}^{\rm max} = -S\sigma g(\phi_p) \times \frac{\sin^2 2\theta}{2\sqrt{(c^4+s^4)^2 - (\sin^2 2\theta \frac{\sin\phi_p/2}{\phi_p})^2}}, \quad (58)$$

where $S \equiv {\rm sign}[\sin(\phi + \phi_p/2)]$ and it is achieved at ϕ determined from the equation

$$\cos\left(\phi + \frac{\phi_p}{2}\right) = -\frac{\sin^2 2\theta}{c^4+s^4}\left(\frac{\sin\frac{\phi_p}{2}}{\phi_p}\right). \quad (59)$$

(4) Select baseline and/or neutrino energy so that $P_{\mu\mu}$ is small.

Here we consider two possibilities to increase $d_{\rm osc} \approx \langle x(t)\rangle_{\rm osc}$: (i) decrease of the survival probability $P_{\mu\mu}$, which implies a strong suppression signal, and (ii) increase of σ, $g(\phi_p)$, and $\sin\phi$ without substantial decrease of $P_{\mu\mu}$ and therefore the signal. Let us explore them in order.

(1) Small $P_{\mu\mu}$ can be achieved by selecting certain values of $\phi(L, E)$. We consider the case of small ϕ_p in which

$$g(\phi_p) = \frac{\phi_p}{12}.$$

Notice that according to (40) the minimum of $P_{\mu\mu}$ is obtained at $\phi = \pi - \phi_p/2$ and equals

$$P_{\mu\mu}^{\rm min} = \cos^2 2\theta + \sin^2 2\theta \frac{\phi_p^2}{48}.$$

However, for this value of ϕ the distance $d_{\rm osc}$ vanishes to the second order in ϕ_p. On the other hand, for $\phi = \pi$ we obtain

$$P_{\mu\mu}^{\rm min} = \cos^2 2\theta + \sin^2 2\theta \frac{\phi_p^2}{12},$$

and the distance equals

$$d_{\rm osc} = \frac{\sin^2 2\theta}{\cos^2 2\theta + \sin^2 2\theta \frac{\phi_p^2}{12}} \frac{\sigma\phi_p^2}{48}. \quad (60)$$

In the case of maximal mixing this equation gives $d_{\rm osc} = \sigma/4$.

In the limit $\phi_p \to 0$ we obtain from (58)

$$d_{\rm osc}^{\rm max} = \frac{\sigma\phi_p}{24} \frac{\sin^2 2\theta}{\sqrt{\cos^2 2\theta + \frac{\phi_p^2}{48}\sin^4 2\theta}}, \quad (61)$$

which corresponds to

$$\cos\phi \approx -\frac{2s^2c^2}{c^4+s^4}.$$

For maximal mixing we find from (61)

$$d_{\rm osc}^{\rm max} = \frac{\sigma}{2\sqrt{3}},$$

which can be considered as the maximal possible additional distance of propagation due to oscillations. This, however, corresponds to the very small survival probability, $P_{\mu\mu} = \frac{\phi_p^2}{24}\sin^2 2\theta$.

For large ϕ_p and $\phi = \pi$ we obtain

$$d_{\rm osc} = \frac{\sigma}{2} \frac{\sin^2 2\theta}{\cos^2 2\theta + 0.5\sin^2 2\theta(1 - \frac{\sin\phi_p}{\phi_p})} \times \left[\frac{1-\cos\phi_p}{\phi_p^2} - \frac{\sin\phi_p}{2\phi_p}\right],$$

and therefore $d_{\rm osc} \lesssim \sigma/2$.

(2) Let us maximize other factors in (55) for $P_{\mu\mu} = \mathcal{O}(1)$. The function $g(\phi_p)$ (56) reaches maximum

$$g(\phi_p) = \frac{1}{4}\sin\frac{\phi_p}{2} \approx 0.2$$

at ϕ_p determined from the condition

$$\tan(\phi_p/2) = \frac{\phi_p/2}{1 - \frac{(\phi_p/2)^2}{2}}.$$

The smallest value of the angle that satisfies this equation is $\phi_p \approx \pi + \epsilon$, where $\epsilon > 0$. (Other values give local extrema.) With further increase of ϕ_p (above π) the function g decreases as $1/\phi_p$, and for small ϕ_p it has a linear dependence: $g \propto \phi_p$.

The oscillatory factor in (55) gives maximal distance of ν_μ propagation when $\phi + \phi_p/2 = 3\pi/2$ so that $\sin(\phi + \phi_p/2) = -1$. This, in turn, can be achieved by selecting the baseline and/or neutrino energy. For $\phi_p \approx \pi$, which corresponds to the maximal value of g, one needs to have $\phi = \pi + 2\pi k$ (k is an integer). For these values of phase ϕ the probability is equal to the average probability: $P_{\mu\mu} = c^4 + s^4$. Consequently, in the case of maximal mixing we obtain

$$\langle x(t)\rangle_{\rm osc} = \sigma g(\pi) \approx 0.2\sigma. \quad (62)$$

For fixed value of ϕ_p the contribution $\langle x(t)\rangle_{\rm osc}$ can be slightly larger if $\phi + \phi_p/2$ deviates from $3\pi/2$. For $\phi_p = \pi$ and maximal mixing the equation (58) gives $\langle x(t)\rangle_{\rm osc} = 0.26\sigma$. At the same time the probability becomes smaller than before: $P_{\mu\mu} \approx 0.3$. In essence, what we have here is an enhancement of the additional distance due to decrease of probability.

Thus, the only way to obtain significant additional distance of ν_μ propagation, which amounts to a significant fraction of the length of the wave packet size $\langle x(t)\rangle_{\rm osc}/\sigma \sim \mathcal{O}(1)$ without strong suppression of signal, is to increase the phase acquired by neutrinos over the production region, so that $\phi_p = \mathcal{O}(1)$.

Let us consider possible setups that can realize $\phi_p = \mathcal{O}(1)$ and increase σ. Apparently the region of formation of neutrino wave packets is restricted by the length of the decay tunnel. At most $l_p = (1\text{–}2)$ km and therefore according to (33) we have two possibilities:

(1) For known $\Delta m^2 = 2.5 \times 10^{-3}$ eV^2 one can take low energies: $E =$ a few MeV. For muon neutrinos this can be arranged with slowly moving muons that have two orders of magnitude longer lifetime. The detection should then be via the neutral current interactions. Alternatively, one can detect backward-moving neutrinos from the pion decay.
For electron neutrinos one can use not only muon decay but also nuclei decay (beta beams). ν_e detection via the charged current interactions is of course easier. In this case, however, the effect is suppressed by the small 1–3 mixing: $\sin^2 2\theta_{13} < 0.1$. It allows one to achieve $\langle x(t)\rangle_{\rm osc}/\sigma \sim (0.01 - 0.1)$. At low energies the size of the wave packet can be large: $\sigma \sim (0.1\text{–}1) l_p$ and, therefore, $\langle x(t)\rangle_{\rm osc} \sim (1\text{–}10)$ m. However, realization of these setups will require us to overcome a significant loss of signal due to selection of low-energy parts of the muon decay spectrum, or the backward-going neutrinos and the small cross sections, etc.

(2) If sterile neutrinos exist with large mass splittings, $\Delta m^2 \sim 1$ eV^2, the phase ϕ_p can be large at higher energies: $E \sim 1$ GeV. Here again the effect will be suppressed by small allowed mixing: $\sin^2 2\theta_{14} < 0.1$. As a result, $\langle x(t)\rangle_{\rm osc}/\sigma \sim (0.01\text{–}0.02)$ can be achieved.
For low energy pions the size of the neutrino wave packet and the neutrino formation region are determined by the decay length, and therefore

$$\phi_p = \frac{\Delta m^2}{2E} \gamma_\pi \tau_\pi^0 v_\pi = \frac{\Delta m^2}{2m_\pi} \frac{E_\pi}{E} \tau_\pi^0. \qquad (63)$$

The phase depends weakly on energy and turns out to be of order unity for $\Delta m^2 > 10$ eV^2. The length of the wave packet equals $\sigma = \tau_\pi^0/2\gamma_\pi$ and for $E_\pi = 0.5$ GeV we obtain $\sigma \sim 1$ m. This gives $\langle x(t)\rangle_{\rm osc} \sim (1\text{–}2)$ cm.
Larger distances can be achieved for neutrinos from the muon decay. In the GeV energy range $l_{\rm decay} > l_p$. So, according to (22) $\sigma \sim (10\text{–}20)$ m, and $\langle x(t)\rangle_{\rm osc} \sim (10\text{–}20)$ cm.
To summarize, we have shown that the large additional distance of ν_μ propagation, $d_{\rm osc} \sim (0.1\text{–}10)$ m, can be obtained in rather nonstandard experimental setups with low-energy accelerators of muons and long decay tunnels, with use of the electron neutrinos or in the presence of large Δm^2.

VI. OSCILLATION IN MATTER AND NEUTRINO VELOCITIES

In long-baseline experiments, neutrinos propagate in matter. In this connection let us discuss influence of the matter effect on neutrino velocity [20]. Notice that in the absence of mixing the dispersion relation in matter reads

$$E = \sqrt{p^2 + m^2} + V,$$

where V is the matter potential that can be written at low energies as $V_0 \approx \beta\sqrt{2} G_F n$ with G_F and n being the Fermi constant and the electron number density in matter, respectively. The constant β depends on the neutrino flavor. Since at low energies the potential V_0 does not depend on the momentum of a neutrino, $dV_0/dp = 0$, the group velocity remains unchanged in matter:

$$\frac{dE}{dp} = \frac{p}{E}.$$

V depends on energy due to the W boson propagator. For the elastic scattering in a forward direction $q^2 = 0$, and therefore the energy dependence of V appears when the W exchange occurs in the s channel. In usual media this is possible for $\bar\nu_e$ only when $\bar\nu_e$ annihilates with electrons. In this case

$$V = V_0 \frac{m_W^2}{m_W^2 - s} \approx V_0 \frac{m_W^2}{m_W^2 - 2m_e p - m_e^2},$$

and hence

$$\frac{dV}{dp} = V_0 \frac{2m_e m_W^2}{(m_W^2 - s)^2}.$$

If $s \ll m_W^2$ we obtain the energy-independent contribution to velocity:

$$\Delta v \approx \frac{2m_e V_0}{m_W^2} = 1.8 \times 10^{-29} \left(\frac{\rho}{3\ \text{g/cm}^3}\right)\left(\frac{Y_e}{0.5}\right), \qquad (64)$$

where ρ is the matter density and Y_e is the electron fraction. This conclusion does not depend on whether V is independent of x or not. The contribution to the neutrino velocity in (64) is too small to affect any of our discussions, and therefore it can be ignored.

Since neutrinos are mixed, the propagating degrees of freedom are neutrino eigenstates in matter. The energy of these states (eigenstates of the Hamiltonian) are given by

$$E_{1,2}^m = p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_1 + V_2}{2} \pm \frac{1}{2}\sqrt{\left(V - 2\frac{\Delta m^2}{4p}\cos 2\theta\right)^2 + 4\left(\frac{\Delta m^2}{4p}\right)^2 \sin^2 2\theta}.$$

Differentiating by p we obtain

$$v_{1,2}^m \approx 1 - \frac{m_1^2 + m_2^2}{4p^2} \mp \frac{\Delta m^2}{4p^2} \frac{1 - \cos 2\theta \frac{2Vp}{\Delta m^2}}{\sqrt{(\cos 2\theta - \frac{2Vp}{\Delta m^2})^2 + \sin^2 2\theta}}. \tag{65}$$

In the limit of zero potential or small energies, it reproduces the usual result. At the resonance point, where the denominator is minimal, we find

$$v_{1,2}^m \approx 1 - \frac{m_1^2 + m_2^2}{4p^2} \mp \frac{\Delta m^2}{4p^2} \sin 2\theta. \tag{66}$$

At very high energies or large matter potential the velocities equal

$$v_{1,2}^m \approx 1 - \frac{m_1^2 + m_2^2}{4p^2} \pm \frac{\Delta m^2}{4p^2} \cos 2\theta. \tag{67}$$

Thus, as follows from (65) the correction to the velocity due to mixing in matter alone *cannot* lead to superluminal motion.

For the $\nu_\mu - \nu_\tau$ mixings in the limit of zero 1–3 mixing (neglecting loop corrections), the difference of potentials is zero, $V = 0$, and the situation is reduced to the vacuum case described in Secs. II, III, and IV. With nonzero θ_{13} in the three-neutrino case, the eigenvalues acquire additional dependence on momentum related to the matter potentials. Furthermore, the mixing angle and energy splitting in matter should be taken into account. However, at $E > 6$ GeV (i.e., above resonance energy in Earth), this dependence is weak, and it can be neglected in the first approximation.

It is possible to extend these statements to the case of full three-flavor neutrino mixing. The energy eigenvalues in matter can be written as $E_i^m = p + \frac{\lambda_i}{2p}$ $(i = 1, 2, 3)$, where λ_i are given by [21]

$$\begin{aligned} \lambda_1 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}[u + \sqrt{3(1-u^2)}], \\ \lambda_2 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}[u - \sqrt{3(1-u^2)}], \\ \lambda_3 &= \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2 - 3t}, \end{aligned} \tag{68}$$

with

$$\begin{aligned} s &= \Delta_{21} + \Delta_{31} + a, \\ t &= \Delta_{21}\Delta_{31} + a[\Delta_{21}(1 - s_{12}^2 c_{13}^2) + \Delta_{31}(1 - s_{13}^2)], \\ u &= \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c_{12}^2 c_{13}^2}{2(s^2 - 3t)^{3/2}}\right)\right]. \end{aligned} \tag{69}$$

In (69) we have used the notations $\Delta_{ij} \equiv m_i^2 - m_j^2$ and $a \equiv Vp$. Notice that p-dependence of λ_i is only through a. Therefore,

$$\begin{aligned} v_i - 1 &= -\frac{\lambda_i}{2p^2} + \frac{1}{2p}\frac{d\lambda_i}{da}\frac{da}{dp} = -\frac{\lambda_i}{2p^2} + \frac{1}{2p^2}\frac{d\lambda_i}{da}a \\ &= -\frac{\lambda_i}{2p^2}\left[1 - \frac{d(\log\lambda_i)}{d(\log a)}\right]. \end{aligned} \tag{70}$$

According to (68), λ_i is a monotonically increasing function of a, but with a growth rate slower than a. (This feature can be seen in the plot for λ_i as a function of a, given, e.g., in [22,23].) Therefore, $\frac{d(\log\lambda_i)}{d(\log a)} < 1$. Asymptotically, the largest eigenvalue λ_3 behaves as a, approaching the equality. Thus, $v_i - 1 < 0$, which excludes the possibility of a neutrino's superluminal velocity due to matter effect.

VII. CONCLUSION

Several factors alter the shape of the wave packet of a muon neutrino and, consequently, influence the distance of ν_μ propagation for a given time and, hence, the velocity of a neutrino: (i) the relative shift of the wave packets of the mass eigenstates, (ii) oscillations, (iii) absorption, and (iv) production. In this paper, we focused on the first two factors that are mutually correlated: both are due to the mass-squared difference, and therefore the separation of the wave packets is always accompanied by oscillations and vice versa.

(1) We have computed the distances of ν_μ propagation in the presence of mixing and oscillations. The oscillations lead to distortion of the shape factor of the ν_μ wave packet. This, in turn, changes the effective distance traveled by neutrinos and therefore the group velocity. This is essentially related to the oscillation effect within the neutrino wave packet. The oscillatory pattern is squeezed and therefore the effect is enhanced in the same way as the size of the neutrino wave packet shrinks in comparison to the pion decay length or size of the decay tunnel.
We find that the distance of the ν_μ propagation is proportional to the length of the wave packet σ_x and the oscillation phase ϕ_p acquired by neutrinos along the decay path of the parent particles (pions, K mesons) where a neutrino wave packet is formed $d_{\rm osc} \propto \sigma\phi_p$. Furthermore, $d_{\rm osc}$ has an oscillatory behavior with distance determined by the oscillation length. For small distances, $L < l_\nu/2$, the oscillations reduce the group velocity. The distance $d_{\rm osc}$ becomes positive for baselines $L = l_\nu/2 - l_\nu$, etc. In this range of baselines motion can be effectively superluminal. The additional distance is restricted by the size of the neutrino wave packet: $d_{\rm osc} < \sigma_{x\nu}$. The additional distance strongly decreases with increase of energy: $d_{\rm osc} \sim 1/E^4$.

(2) Distortion of the $\nu_\mu-$ wave packet is also produced due to a relative shift of the wave packets of the mass eigenstates even in the case when the oscillation effect does not depend on a coordinate. This effect previously considered in the literature is proportional to $\Delta vt = \Delta m^2/2E^2 t$, and therefore negligible in comparison with the oscillation effect.

(3) For the OPERA setup with $\Delta m^2 = 2.5 \times 10^{-3}$ eV^2 and $E \sim 17$ GeV, we obtain $d_{\rm osc} \approx -10^{-5}$ cm and other contributions are vanishingly small. Therefore, the OPERA result [3], which corresponds to the distance $\sim +20$ m, cannot be explained. For average energy $E = 17$ GeV the oscillations reduce the distance propagated by neutrinos and consequently the group velocity. Furthermore, the distance $d_{\rm osc}$ rapidly decreases with increase of the neutrino energy.

(4) We estimated the additional distances of ν_μ propagation for different experimental setups. In particular, we find $d_{\rm osc} = (0.01\text{–}0.04)$ cm for MINOS and T2K. Change of the time of ν_μ propagation (for a given L) can be obtained as $\Delta t \approx -d_{\rm osc}/c$.

(5) Larger additional distance can be obtained for neutrinos from decays of particles with longer lifetimes: muons and nuclei (beta beams). The $d_{\rm osc}$ becomes larger with a decrease in neutrino energy and for large Δm^2, if such exists. It can be as large as several meters. However, this requires rather nonstandard experimental setups and extremely intense neutrino fluxes.

(6) Measurements of additional distances of the flavor neutrino propagation opens a way to determine sizes of neutrino wave packets and test certain quantum mechanical features of neutrino production. In any case, the oscillation effect should be taken into account in analysis of future measurements of the neutrino velocities.

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APPENDIX: MAXIMAL EFFECT OF SHIFT OF THE EIGENSTATES

Here we clarify the effect of shift of the wave packets discussed in the literature. Following [12,14,15], we take $\Delta p = 0$, so that the oscillation phase is the same along the wave packet (i.e., by themselves, oscillations do not produce distortion of the ν_μ packet). For simplicity we take the boxlike shape factors for the mass eigenstates.

Let us consider the evolution of a neutrino state produced as ν_μ. According to (1) in the ν_μ state the ν_1 and ν_2 packets have amplitudes c and s, respectively. (Here we omit normalization factors that are irrelevant for the final result.) The amplitude of probability to find ν_μ in the ν_1 part of the state is c^2 and in the ν_2 part: s^2. The measurement result of ν_μ in the fully evolved neutrino state is then determined by interference of the ν_μ parts of ν_1 and ν_2. The interference pattern, in turn, is determined by the relative (oscillation) phase of the two mass eigenstates, Φ, and is described by $\cos\Phi$. The phase does not depend on x being the same over the whole wave packet.

At time t after production, the centers of the ν_1 and ν_2 packets have coordinates $x = v_1 t + \sigma/2$ and $x = v_2 t + \sigma/2$ correspondingly. The average value of the two coordinates is $\bar{x} = \bar{v}t = (v_1 + v_2)t/2 + \sigma/2$. The relative shift of the packets equals $\Delta vt = \Delta m^2 t/2E^2$. Correspondingly, the front edges of ν_1 and ν_2 are at $v_1 t + \sigma$ and $v_2 t + \sigma$.

Consider the shape of the ν_μ wave packet and $|\psi_{\nu_\mu}|^2$. As we marked in Sec. IV C, due to the relative shift of the wave packets of mass eigenstates there are three different spatial parts of the ν_μ wave packet ($v_1 > v_2$):

(1) In the front edge part, $(v_2 t + \sigma) \le x \le (v_1 t + \sigma)$, only ν_1 packet is present, so that $\psi_{\nu_\mu} = c^2$ and $|\psi_{\nu_\mu}|^2 = c^4$. The size of this part is given by the shift Δvt.

(2) In the overlapping part, $(v_1 t) \le x \le (v_2 t + \sigma)$, both wave packets are nonzero and $|\psi_{\nu_\mu}|^2 = |c^2 + s^2\cos\Phi|^2$.

(3) In the trailing edge part, $(v_2 t) \le x \le (v_1 t)$, only ν_2 wave packet is present and $\psi_{\nu_\mu} = s^2$, so that $|\psi_{\nu_\mu}|^2 = s^4$.

In the first approximation the distance of ν_μ propagation is determined by the position of the "center of mass" of the wave packet squared. In the case of maximal mixing whole, the picture is completely symmetric with respect to $\bar{x}$ and therefore $\langle x \rangle = \bar{x}$. If mixing deviates from maximal one, the shift of the packets leads to asymmetric distortion of the ν_μ wave packet. This, in turn, leads to a shift of the "center of mass" from $\bar{x}$:

$$\langle x \rangle = \bar{x} + \delta_x.$$

For nonmaximal mixing (with $c > s$) the forward edge is higher by $c^4 - s^4 = \cos 2\theta$ than the trailing edge. This difference should be compensated by the shift of the center of mass by amount δ_x in the overlapping region. This compensation leads to the condition

$$\Delta vt \cos 2\theta = 2\delta_x |c^2 + s^2 \cos\Phi|^2.$$

Therefore

$$\delta_x = \frac{\cos 2\theta \Delta vt}{2|c^2 + s^2\cos\Phi|^2}.$$

The maximal shift would correspond to $\cos\Phi = -1$. It implies the destructive interference in the overlapping region, when $\Phi = \pi$, so that $|\psi_{\nu_\mu}|^2 = |c^2 - s^2|^2 = \cos^2 2\theta$, and therefore

$$\delta_x = \Delta v t \frac{\cos 2\theta}{2\cos^2 2\theta} = \Delta v t \frac{1}{2\cos 2\theta}, \qquad \text{(A1)}$$

which reproduces results in [12,14,15]. Thus, a superluminal motion here is a result of interplay of the coordinate-independent oscillations and the relative shift of wave packets due to different group velocities. Apparently the shift δ_x is restricted by the size of the wave packet. In this case it is clear that v has no physical meaning. It is the velocity of the "center of mass": some effective point in the flat overlapping part of the shape factor. Neither a single body nor real structure in the shape factor (edges of different regions) is moving with $v > c$.

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