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Matrix theory compactifications on twisted tori

Athanasios Chatzistavrakidis[1,*] and Larisa Jonke[1,2,†]

[1]*Bethe Center for Theoretical Physics and Physikalisches Institut, University of Bonn, Nussallee 12, D-53115 Bonn, Germany*
[2]*Theoretical Physics Division, Rudjer Bošković Institute, Bijenička 54, 10000 Zagreb, Croatia*

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We study compactifications of Matrix theory on twisted tori and noncommutative versions of them. As a first step, we review the construction of multidimensional twisted tori realized as nilmanifolds based on certain nilpotent Lie algebras. Subsequently, matrix compactifications on tori are revisited, and the previously known results are supplemented with a background of a noncommutative torus with nonconstant noncommutativity and an underlying nonassociative structure on its phase space. Next, we turn our attention to three- and six-dimensional twisted tori, and we describe consistent backgrounds of Matrix theory on them by stating and solving the conditions which describe the corresponding compactification. Both commutative and noncommutative solutions are found in all cases. Finally, we comment on the correspondence among the obtained solutions and flux compactifications of 11-dimensional supergravity, as well as on relations among themselves, such as Seiberg-Witten maps and T-duality.

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I. INTRODUCTION

An attractive way to gain access to the nonperturbative regime of superstring theories and M theory passes through certain matrix models. The most prominent instances include the matrix model of Banks-Fischler-Shenker-Susskind (BFSS) [1], also known as Matrix theory, and the type IIB matrix model of Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) [2]. They were suggested as nonperturbative definitions of M theory and type IIB superstring theory, respectively. As such, they provide frameworks where brane dynamics can be studied and nonperturbative duality symmetries can be tested.

An interesting program in the study of matrix models, already initiated in Refs. [1,3], relates to their toroidal compactification. This is defined by a restriction of the action functional of the model under certain conditions incorporating the geometry of the compactification space. In Ref. [4], Connes, Douglas, and Schwarz perform a detailed study of matrix compactifications on multidimensional tori and unveil striking relations to noncommutative geometry. They argue that noncommutative deformations of tori are tantamount to turning on fluxes in supergravity compactifications. Such a correspondence was supported by subsequent work on the subject [5–8]. Matrix compactifications on spaces other than tori were considered in Refs. [9,10].

In this paper, we revisit matrix compactifications on tori and perform a study of compactifications on multidmensional twisted tori. The latter are smooth manifolds corresponding to nontrivial fiber bundles with a toroidal fiber over a toroidal base. They may be described equivalently as nilmanifolds, namely, coset spaces obtained by quotiening an appropriate discrete group out of a nilpotent Lie group.[1] Such a description, including illuminating examples, is provided in Sec. II. In order to set the stage for the compactification of Matrix theory, we review in Sec. III the basics of the BFSS and IKKT matrix models, as well as their toroidal compactification. Apart from the previously obtained results, we describe a solution of the associated conditions which corresponds to a noncommutative deformation of the torus with nonconstant noncommutativity. This solution carries an underlying nonassociative structure on the corresponding phase space.

The approach of matrix compactifications on tori is used, with the necessary modifications, in order to study compactifications on twisted tori. An analysis of the case of the twisted 3-torus is performed, where a set of solutions to the corresponding conditions is identified for commutative and noncommutative twisted 3-tori. The solution already found in Ref. [8] is recovered too. A similar analysis is carried out for a particular six-dimensional twisted torus. The resulting solutions are presented in a form which may be directly generalized to any other higher-dimensional nilmanifold.

The solutions of Matrix theory on tori and twisted tori are related in certain ways among themselves as well as with flux compactifications of string/M theory in the supergravity approximation. Adopting the Connes-Douglas-Schwarz correspondence between noncommutative deformations and supergravity fluxes, we suggest that the toroidal background with nonconstant noncommutativity may be associated with a constant 4-form flux in 11-dimensional supergravity. Likewise, noncommutative twisted tori may be related to compactifications where

*than@th.physik.uni-bonn.de
†larisa@irb.hr

[1]The relevance of nonsemisimple Lie groups and their algebras for matrix models was pointed out in Ref. [11] and found an interesting application in Ref. [12].

both Neveu-Schwarz (NS) fluxes and geometric fluxes are present [13–15]. Moreover, in Sec. IV, we present certain transformations relating the noncommutative solutions to the commutative ones and inducing the Seiberg-Witten (SW) maps [16] between the corresponding gauge theories. Finally, we comment on the T-duality between toroidal backgrounds with nonconstant B field and NS-fluxless twisted tori. Sec. V contains our conclusions, and the Appendix contains the geometric data of a certain class of interesting five- and six-dimensional nilmanifolds.

II. TWISTED TORI AS NILMANIFOLDS

A. General considerations

In order to generate masses in four dimensions induced by a higher-dimensional supergravity, Scherk and Schwarz introduced a general consistent compactification scheme in their pioneering paper [17]. Specific realizations of this scheme are provided by the so-called twisted tori, and the low-energy effective action resulting from a dimensional reduction of the heterotic string on them was studied in detail in Ref. [13]. Later, it was shown that higher-dimensional twisted tori also provide consistent backgrounds of Type II string theories [18,19] and M theory [14,15].

Twisted tori may be described essentially in two complementary ways. First of all, they arise as T-duals of square tori endowed with a constant NS 3-form flux[2] [18]. Here we shall concentrate on the second description, which is directly related to the study of nilmanifolds.

Nilmanifolds are smooth manifolds constructed as quotients of nilpotent Lie groups by discrete subgroups of them [20]. Thus a nilmanifold $\mathcal{M}$ may be described as a coset space A/Γ of a nilpotent group A and a discrete group $\Gamma \subset A$. The nilpotent Lie groups of dimension up to six and their corresponding Lie algebras $\mathcal{A}$ are fully classified (see, e.g., Ref. [21]). Here we shall follow the notation appearing in the tables of Ref. [21], denoting a nilpotent Lie algebra as $\mathcal{A}_{d,i}$, where d is the dimension of the algebra (the number of its generators) and i is just an enumerating index according to the aforementioned tables. Moreover, when there is some parameter on which the algebra depends, it will appear as superscript, e.g. $\mathcal{A}^{\alpha}_{d,i}$, if there is one parameter α. Let us also note that the generators of an algebra will be denoted as X_a, $a = 1, \ldots, d$.

Since we are willing to use nilmanifolds for compactification, it is self-evident that they had better be compact manifolds. It turns out that a necessary condition for compactness is that the group A is unimodular, i.e., its structure constants satisfy ${f^a}_{ab} = 0$ (this was already discussed in Ref. [17]). This condition is in general not sufficient, but for nilpotent groups it is enough to require that their structure constants are rational [20]. Hereforth, we rely on the above assumptions.

[2]Clearly this corresponds to a case of nonconstant B field.

A fact which facilitates the coset construction of nilmanifolds is that the group elements $g \in A$ may always be expressed as upper triangular matrices. Then the algorithm for the construction of a nilmanifold consists of the following steps:

(1) Express the basis elements X_a of the nilpotent Lie algebra $\mathcal{A}_{d,i}$ as upper triangular $d \times d$ matrices.
(2) Choose a representative general group element $g \in A_{d,i}$. A convenient choice is: $g = \prod_{a=1}^{d} \exp(x^a X_a)$, $x^a \in \mathbb{R}$.
(3) Consider the general discrete subgroup element $\gamma \in \Gamma$ as the restriction of $g \in A_{d,i}$ for integer coefficients: $\gamma = \prod_{a=1}^{d} \exp(\gamma^a X_a)$, $\gamma^a \in \mathbb{Z}$.
(4) The subgroup Γ acts on $A_{d,i}$ by matrix multiplication. Thus as a final step, one can construct the left coset $\mathcal{M} = A_{d,i}/\Gamma$.

Having constructed the nilmanifold as above, it is then easy to study its geometry. Most important, it is straightforward to compute the Lie algebra 1-form

$$e = g^{-1} dg, \tag{2.1}$$

which is Lie algebra valued, and therefore it might be decomposed as $e = e^a X_a$. The quantities e^a correspond to the usual vielbein basis, and they may be expressed in terms of the coordinate basis 1-forms dx^a as

$$e^a = U(x)^a_b dx^b, \tag{2.2}$$

for some x-dependent twist matrix $U(x)$. The vielbeins satisfy the Maurer-Cartan equations

$$de^a = -\frac{1}{2} f^a_{bc} e^b \wedge e^c, \tag{2.3}$$

where f^a_{bc} are constant coefficients, which are identified with the structure constants of the Lie algebra $\mathcal{A}_{d,i}$. In the context of flux compactifications, they are also referred to as geometric fluxes.

Finally, in accord with the above, it is very useful to read off the coordinate identifications which are made in the process of the compactification of the nilpotent group. For the square torus, e.g., in three dimensions with coordinates x^1, x^2, x^3, and unit radii, these identifications are very simple and they read as

$$(x^1, x^2, x^3) \sim (x^1 + 1, x^2, x^3) \sim (x^1, x^2 + 1, x^3) \sim (x^1, x^2, x^3 + 1). \tag{2.4}$$

The corresponding identifications for nilmanifolds are slightly less simple, but they are easily obtained from the twist matrix $U(x)$, as it will become evident in the following. For example, in the case of a three-dimensional nilmanifold with coordinates x^a, $a = 1, 2, 3$ and unit radii, we shall show that they are

$$(x^1, x^2, x^3) \sim (x^1 + 1, x^2, x^3) \sim (x^1, x^2, x^3 + 1) \sim (x^1 + x^3, x^2 + 1, x^3). \quad (2.5)$$

A very illuminating consequence is that the latter identifications allow the interpretation of the three-dimensional nilmanifold as twisted fibration of a 2-torus over a circle, namely, a 2-torus in the x^1, x^3 directions whose geometry varies as it traverses the base circle in the x^2 direction. This is the reason why we may call it a twisted torus. In fact, the above observation holds in higher dimensions as well, thus providing a correspondence between nilmanifolds and twisted fibrations of toroidal fibers over toroidal bases.

B. A three-dimensional example

Let us now provide a couple of examples of the previously described procedure. We start with the simplest possible case, based on the unique three-dimensional nilpotent Lie algebra, which is the algebra of the Weyl group. We use the notation $\mathcal{A}_{3,1}$, in accord with Ref. [21]. The only nontrivial commutation relation of this algebra is

$$[X_2, X_3] = X_1. \quad (2.6)$$

Following the steps which were described in the previous subsection, we first write down a basis for the algebra in terms of 3×3 upper triangular matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.7)$$

Then, any element of the corresponding group $A_{3,1}$ may be parametrized as

$$g = \begin{pmatrix} 1 & x^2 & x^1 \\ 0 & 1 & x^3 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.8)$$

This is clearly a noncompact group. According to the previous discussion, in order to produce a compact manifold out of it, a compact discrete subgroup Γ has to be considered. Such a subgroup is given by those elements $g \in A_{3,1}$ which have integer values of x^a. Then the quotient $A_{3,1}/\Gamma$ is indeed a compact nilmanifold.

The Lie algebra invariant 1-form e is given by

$$e = \begin{pmatrix} 0 & dx^2 & dx^1 - x^2 dx^3 \\ 0 & 0 & dx^3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.9)$$

Clearly, its components are

$$e^1 = dx^1 - x^2 dx^3, \qquad e^2 = dx^2, \qquad e^3 = dx^3, \quad (2.10)$$

which evidently satisfy the Maurer-Cartan equations, since $de^2 = de^3 = 0$ and $de^1 = -e^2 \wedge e^3$. The twist matrix has the form

$$U = \begin{pmatrix} 1 & 0 & -x^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.11)$$

and therefore the required identifications are

$$(x^1, x^2, x^3) \sim (x^1 + 1, x^2, x^3) \sim (x^1, x^2, x^3 + 1) \sim (x^1 + x^3, x^2 + 1, x^3). \quad (2.12)$$

From now on, we shall refer to Eq. (2.12) (and similar ones) as twisted identifications.

Moreover, it is straightforward to determine the vector fields $\tilde{e}_a$, which are dual to the 1-forms of Eq. (2.10). They are

$$\tilde{e}_1 = \partial_1, \qquad \tilde{e}_2 = \partial_2, \qquad \tilde{e}_3 = \partial_3 + x^2 \partial_1. \quad (2.13)$$

These do not deserve to be collectively called Killing vector fields since only $\tilde{e}_1$ generates an isometry while the other two do not [18]. However, it should be noted that these vector fields do not commute but instead they satisfy

$$[\tilde{e}_2, \tilde{e}_3] = \tilde{e}_1. \quad (2.14)$$

This observation will be important in Sec. III.

Another simple observation is that rescaling the central element X_1 of the algebra by an integer, i.e., $X_1 \longrightarrow \frac{1}{N} X_1$, $N \in \mathbb{Z}$, leads to the commutation relation

$$[X_2, X_3] = N X_1.$$

Then the effect on the above geometric data is that one has to replace x^2 in Eqs. (2.9), (2.10), and (2.11) with $N x^2$, while the last identification in Eq. (2.12) becomes $(x^1 + N x^3, x^2 + 1, x^3)$. This is related to the presence of quantized flux, as we will discuss.

Let us briefly recall that the above twisted 3-torus also can be obtained by T-dualizing a square torus with N units of NS 3-form flux, proportional to its volume form, turned on. Indeed, let us choose a gauge where the B field is

$$B_{31} = N x^2, \quad (2.15)$$

and the metric is the standard metric on the 3-torus, $ds^2 = \delta_{ab} dx^a dx^b$. Then one can use the Buscher rules [22] to perform a T-duality along the x^1 direction. For completeness, let us be reminded of these rules for the metric and the B field when a T-duality is performed along the direction i:

$$G_{ii} \xrightarrow{T_i} \frac{1}{G_{ii}}, \quad G_{ai} \xrightarrow{T_i} \frac{B_{ai}}{G_{ii}}, \quad G_{ab} \xrightarrow{T_i} G_{ab} - \frac{G_{ai}G_{bi} - B_{ai}B_{bi}}{G_{ii}},$$
$$B_{ai} \xrightarrow{T_i} \frac{G_{ai}}{G_{ii}}, \quad B_{ab} \xrightarrow{T_i} B_{ab} - \frac{B_{ai}G_{bi} - G_{ai}B_{bi}}{G_{ii}}, \quad (2.16)$$

where T_i denotes the T-duality action. In the present case, the result is that in the T-dual frame the B field vanishes and the dual metric corresponds exactly to that of the twisted 3-torus, i.e., it is given by $ds^2 = \delta_{ab}e^a e^b$, with the 1-forms as in Eq. (2.10). Therefore, a twisted torus background is T-dual to a square torus background with nonconstant B field. We shall return to this point again, after we will have studied matrix compactifications.

C. A six-dimensional example

Let us now move on to a less simple six-dimensional example. In six dimensions, there are several nilpotent Lie algebras and therefore several cases of nilmanifolds. In fact, excluding algebras which are algebraic sums of lower-dimensional ones, there are 22 nilpotent Lie algebras[3] up to isomorphism [21].

In the present subsection, we consider the algebra $\mathcal{A}_{6,5}^{\alpha}$, where the superscript α denotes that there is an additional parameter in this case.[4] This algebra has the following commutation relations:

$$[X_1, X_3] = X_5, \qquad [X_1, X_4] = X_6, \qquad [X_2, X_3] = \alpha X_6, \qquad [X_2, X_4] = X_5. \tag{2.17}$$

A basis is given by the following 6×6 upper triangular matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & \alpha \\ & & & 0 & 1 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

The general group element is found to be

$$g = \begin{pmatrix} 1 & 0 & x^2 & x^1 & x^5 & x^6 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & x^4 & \alpha x^3 \\ & & & 1 & x^3 & x^4 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}, \tag{2.18}$$

while the 1-form e can be computed and it has the following form:

$$e = \begin{pmatrix} 0 & 0 & dx^2 & dx^1 & dx^5 - x^2dx^4 - x^1dx^3 & dx^6 - \alpha x^2dx^3 - x^1dx^4 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & dx^4 & \alpha dx^3 \\ & & & 0 & dx^3 & dx^4 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \tag{2.19}$$

with components

$$e^i = dx^i, \qquad i = 1, \ldots, 4, \qquad e^5 = dx^5 - x^2dx^4 - x^1dx^3, \qquad e^6 = dx^6 - \alpha x^2dx^3 - x^1dx^4. \tag{2.20}$$

Then the twist matrix is

[4] A different case was examined in Ref. [11].

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -x^1 & -x^2 & 1 & 0 \\ 0 & 0 & -\alpha x^2 & -x^1 & 0 & 1 \end{pmatrix}, \tag{2.21}$$

and the twisted identifications are found to be

$$\begin{aligned} &(x^1, x^2, x^3, x^4, x^5, x^6) \\ &\quad \sim (x^1, x^2, x^3 + c, x^4, x^5, x^6) \\ &\quad \sim (x^1, x^2, x^3, x^4 + d, x^5, x^6) \\ &\quad \sim (x^1, x^2, x^3, x^4, x^5 + e, x^6) \\ &\quad \sim (x^1, x^2, x^3, x^4, x^5, x^6 + f) \\ &\quad \sim (x^1 + a, x^2, x^3, x^4, x^5 + ax^3, x^6 + ax^4) \\ &\quad \sim (x^1, x^2 + b, x^3, x^4, x^5 + bx^4, x^6 + \alpha bx^3), \\ &\quad a, b, c, d, e, f \in \mathbb{Z}. \end{aligned} \tag{2.22}$$

Under Eq. (2.22) we obtain the desired twisted compactification.

The vector fields which are dual to the 1-forms of Eq. (2.20) in the present case read as

$$\begin{aligned} \tilde{e}_i = \partial_i, \quad i = 1, 2, 5, 6, \quad \tilde{e}_3 = \partial_3 + x^1\partial_5 + \alpha x^2 \partial_6, \\ \tilde{e}_4 = \partial_4 + x^2\partial_5 + x^1\partial_6. \end{aligned} \tag{2.23}$$

Of them, only $\tilde{e}_5$ and $\tilde{e}_6$ are Killing vector fields, i.e., they generate an isometry. The full set of vector fields of Eq. (2.23) satisfies the algebra of Eq. (2.17).

As before, the central elements X_5 and X_6 may be rescaled by integer numbers as $X_5 \to \frac{1}{M} X_5$ and $X_6 \to \frac{1}{N} X_6$, leading to the modified commutation relations

$$\begin{aligned} [X_1, X_3] = MX_5, \qquad [X_1, X_4] = NX_6, \\ [X_2, X_3] = \alpha N X_6, \qquad [X_2, X_4] = MX_5. \end{aligned} \tag{2.24}$$

Then in Eq. (2.20), e^5 and e^6 change to

$$\begin{aligned} e^5 = dx^5 - Mx^2 dx^4 - Mx^1 dx^3, \\ e^6 = dx^6 - \alpha N x^2 dx^3 - Nx^1 dx^4, \end{aligned} \tag{2.25}$$

which modify the twist matrix and the twist identifications accordingly.

The above background can also be obtained by T-dualizing a square 6-torus with appropriate quantized 3-form fluxes. In particular, consider a 6-torus with the standard square metric, endowed with fluxes generated by the nonconstant B field with values

$$B_{53} = Mx^1, \quad B_{54} = Mx^2, \quad B_{63} = \alpha N x^2, \quad B_{64} = Nx^1. \tag{2.26}$$

The corresponding fluxes are

$$H_{153} = H_{254} = M \text{ and } H_{263} = \alpha N, \quad H_{164} = N. \tag{2.27}$$

Then one can use the Buscher rules of Eq. (2.16) to show that in performing two consequtive T-dualities along the directions x^5 and x^6, the six-dimensional nilmanifold previously described is obtained, i.e., the B field vanishes and the metric is given by $ds^2 = \delta_{ab} e^a e^b$ with 1-forms as in Eqs. (2.20) and (2.25) respectively.

The same procedure may be followed for any other nilmanifold in any dimension. In the Appendix we collect the resulting twist matrices and twisted identifications for a class of nilmanifolds in five and six dimensions.

III. MATRIX THEORY COMPACTIFICATIONS

A. The BFSS and IKKT matrix models

Let us start by briefly describing the two basic string-inspired matrix models (MMs), widely known as BFSS [1] and IKKT [2]. The BFSS MM, also referred to as Matrix theory, was suggested as a nonperturbative definition of M theory. Its action, determining the dynamics of N D0 branes in uncompactified space-time, is given by the following functional:

$$\begin{aligned} \mathcal{S}_{\rm BFSS} = \frac{1}{2g} \int dt \mathrm{Tr}\Big(\dot{\mathcal{X}}_a \dot{\mathcal{X}}_a - \frac{1}{2}[\mathcal{X}_a, \mathcal{X}_b]^2 \\ + 2\psi^T \dot{\psi} - 2\psi^T \Gamma^a [\psi, \mathcal{X}_a] \Big), \end{aligned} \tag{3.1}$$

where $\mathcal{X}_a(t)$, $a = 1, \ldots, 9$ are nine time-dependent $N \times N$ Hermitian matrices, ψ are their fermionic superpartners, and Γ^a furnish a representation of $SO(9)$. In the following, we shall be concerned mainly with the bosonic part of the above action.

The equations of motion resulting from the variation of the action Eq. (3.1) with respect to $\mathcal{X}_a$, setting $\psi = 0$, are

$$\ddot{\mathcal{X}}_a + [\mathcal{X}_b, [\mathcal{X}^b, \mathcal{X}_a]] = 0, \tag{3.2}$$

where indices are raised and lowered with δ_{ab} and therefore it does not make any difference whether they are upper or lower. For static configurations, it is clear that the first term in Eq. (3.2) may be dropped.

On the other hand, the IKKT MM was suggested as a nonperturbative definition of Type IIB superstring theory and may be regarded as the D-instanton analog of the previous model. It is described by the following action:

$$S = \frac{1}{2g} \mathrm{Tr}\Big(-\frac{1}{2}[\mathcal{X}_a, \mathcal{X}_b]^2 - \bar{\psi}\Gamma^a[\mathcal{X}_a, \psi] \Big), \tag{3.3}$$

where now the number of matrices $\mathcal{X}_a$ is ten, namely, the index a takes the values $0, \ldots, 9$ and ψ is a spinor of $SO(10)$ in the Euclidean model where the indices are raised and lowered with the metric δ_{ab}.

The corresponding equations of motion are

$$[\mathcal{X}_b, [\mathcal{X}^b, \mathcal{X}_a]] = 0, \tag{3.4}$$

which are formally the same as the time-independent equations of the BFSS model.[5]

B. Compactification on tori

In the following, we shall consider compactifications of the previous MMs, focusing on the BFSS model. Let us start by reviewing the cases of compactification on multidimensional tori and noncommutative versions of them [4]. Moreover, we supplement this discussion by providing the matrix analog of a square 3-torus with NS 3-form flux in the same approach. This will provide the guidelines for the investigation of compactifications on twisted tori and their noncommutative versions, which follows in the next subsections.

A matrix compactification on a d-dimensional torus is defined by a restriction of the matrix action under certain periodicity conditions incorporating the cycles of the torus. The simplest example involves compactification on a circle, but it is more illuminating for our purposes to start with the case of a 3-torus.

For a T^3 extending, say, in the directions $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, the compactification involves three invertible unitary matrices U_i obeying

$$\begin{aligned} &\mathcal{X}_1 + R_1 = U_1\mathcal{X}_1U_1^{-1}, \qquad \mathcal{X}_2 + R_2 = U_2\mathcal{X}_2U_2^{-1},\\ &\mathcal{X}_3 + R_3 = U_3\mathcal{X}_3U_3^{-1}, \qquad \mathcal{X}_a = U_i\mathcal{X}_aU_i^{-1},\\ &\qquad a\neq i, \qquad a = 1,\dots,9, \qquad i = 1,2,3, \end{aligned} \tag{3.5}$$

where R_i are complex constants.

Compactification on T^3. A simple solution of the conditions of Eq. (3.5) is given by

$$\begin{aligned} &\mathcal{X}_i = iR_i\mathcal{D}_i, \qquad \mathcal{X}_m = \mathcal{A}_m(U_i), \qquad m = 4,\dots,9,\\ &U_1 = e^{ix^1}, \qquad U_2 = e^{ix^2}, \qquad U_3 = e^{ix^3}, \end{aligned} \tag{3.6}$$

where x^i are coordinates on T^3 and $\mathcal{D}_i$ are covariant derivatives

$$\mathcal{D}_i = \partial_i - i\mathcal{A}_i(U_j), \tag{3.7}$$

with $\partial_i \equiv \partial/\partial x^i$. This is in fact the unique solution, up to gauge equivalence, for the standard (i.e. commutative) 3-torus. Note that in the present case it holds that

$$[U_i, U_j] = 0, \tag{3.8}$$

which is enough to guarantee that the U-dependence of the gauge potentials $\mathcal{A}$ ensures that the conditions of Eq. (3.5) are indeed satisfied.

Furthermore, substituting the above form of the solution in the bosonic sector of the BFSS action functional Eq. (3.1), one gets

$$\begin{aligned} \mathcal{S} = \frac{1}{2g}\int dt\int d^3x\,\mathrm{Tr}\Big(&-R_i^2\dot{\mathcal{D}}_i\dot{\mathcal{D}}_i + \dot{\mathcal{A}}_m\dot{\mathcal{A}}_m\\ &+\frac{1}{2}R_iR_j[\mathcal{D}_i,\mathcal{D}_j]^2 - iR_i[\mathcal{D}_i,\mathcal{A}_m]^2 - \frac{1}{2}[\mathcal{A}_m,\mathcal{A}_n]^2\Big), \end{aligned} \tag{3.9}$$

corresponding to the bosonic part of the action of a (1 + 3)-dimensional supersymmetric Yang-Mills (SYM) theory.

The above are readily generalized in d dimensions. In that case, one has to determine d unitary matrices U_i, $i = 1,\dots,d$, such that

$$\begin{aligned} &\mathcal{X}_i + R_i = U_i\mathcal{X}_iU_i^{-1}, \qquad \mathcal{X}_a = U_i\mathcal{X}_aU_i^{-1},\\ &\qquad a\neq i, \qquad a = 1,\dots,9, \qquad i = 1,\dots,d, \end{aligned} \tag{3.10}$$

with d complex constants R_i. The solution is readily given by the direct generalization of the previous one, namely,

$$\begin{aligned} &\mathcal{X}_i = iR_i\mathcal{D}_i, \qquad \mathcal{X}_m = \mathcal{A}_m(U_i),\\ &m = d+1,\dots,9, \qquad U_i = e^{ix^i}, \end{aligned} \tag{3.11}$$

where x^i are the coordinates of the torus T^d, while again the Us are commuting, i.e., Eq. (3.8) remains true. The corresponding $(1 + d)$-dimensional SYM action is the direct generalization of Eq. (3.9).

Compactification on noncommutative T^3_θ. The conditions of Eqs. (3.5) and (3.10), apart from the solution on a standard 3-torus and d-torus, respectively, allow in fact for more general configurations [4]. Focusing on the general d-dimensional case, this may be easily seen by considering the operator

$$Q_{(ij)} = U_iU_jU_i^{-1}U_j^{-1}, \qquad i\neq j. \tag{3.12}$$

It is readily verified that, rather generally, this operator satisfies

$$[Q_{(ij)}, \mathcal{X}_a] = 0, \qquad \forall\ a = 1,\dots,9. \tag{3.13}$$

Let us mention that the latter condition carries less information than Eq. (3.10), since its derivation assumes an underlying associative structure. This will turn out to be important in certain instances to follow. Keeping this remark in mind, from Eq. (3.13) it follows that $Q_{(ij)}$ is a scalar operator, which implies that

$$U_iU_j = \lambda_{ij}U_jU_i, \tag{3.14}$$

with complex constants $\lambda_{ij} = e^{2\pi i\theta^{ij}}$. The special case of $\theta^{ij} = 0$, or equivalently $\lambda_{ij} = 1$, corresponds to commuting Us, and thus it implements the previously discussed case of the standard d-torus T^d.

However, in general the θ^{ij} are not vanishing, in which case the Us are not commuting operators. It is straightforward to verify [6] that a solution of the conditions Eq. (3.10) in the present case is obtained for

[5]Note, however, that in the IKKT model, Euclidean time is also treated as a matrix variable.

$$\mathcal{X}_i = iR_i\hat{\mathcal{D}}_i, \qquad \mathcal{X}_m = \mathcal{A}_m(\hat{U}_i), \\ m = d+1,\ldots,9, \qquad U_i = e^{i\hat{x}^i}. \tag{3.15}$$

The following important remarks are in order. First of all, we have introduced hatted quantities in Eq. (3.15) because the gauge potentials cannot depend anymore just on Us as in the commutative case. In order for the conditions in Eq. (3.10) to be satisfied, the dependence has to be on the set of operators $\hat{U}_i$, which commute with all U_i, i.e.,

$$[\hat{U}_i, U_j] = 0. \tag{3.16}$$

Moreover, a direct implication of Eq. (3.14) is that the $\hat{x}^i$ do not commute as well, but instead they satisfy the relation

$$[\hat{x}^i, \hat{x}^j] = -2\pi i\theta^{ij}. \tag{3.17}$$

Thus they may be interpreted as the "coordinates" of a noncommutative d-torus T^d_θ (more precisely, they should be called coordinate operators, since they are not coordinates in the classical sense). Likewise, the covariant derivative $\hat{\mathcal{D}}_i$ is now defined as

$$\hat{\mathcal{D}}_i = \hat{\partial}_i - i\mathcal{A}_i(\hat{U}_j). \tag{3.18}$$

Let us note that the operators $\hat{\partial}_i$ commute among themselves, and therefore the vacuum solution for $\mathcal{X}_i$ satisfies the equations of motion in Eq. (3.2), since $[\mathcal{X}_i, \mathcal{X}_j] = 0$. This statement remains valid for all the solutions that we describe here and in the following.

The general procedure one may follow in order to determine the set of operators $\hat{U}_i$ is described in Ref. [4]. This includes cases where twisted gauge bundles are considered, which is beyond the scope of the present paper.[6] In the present case, the $\hat{U}$s have the simple form

$$\hat{U}_i = e^{i\hat{x}^i - 2\pi\theta^{ij}\hat{\partial}_j}, \tag{3.19}$$

and they satisfy the requirement of Eq. (3.16) as well as

$$\hat{U}_i\hat{U}_j = e^{2\pi i\hat{\theta}^{ij}}\hat{U}_j\hat{U}_i, \qquad \hat{\theta}^{ij} = -\theta^{ij}. \tag{3.20}$$

Therefore, the compactification on a noncommutative torus leads to a SYM theory where the gauge fields and scalars $\mathcal{A}_a$, $a = 1,\ldots,9$ are fields on a dual noncommutative torus with parameter $\hat{\theta}$. Clearly, the same holds for fermions when they are included in the action. The form of this SYM action is similar to Eq. (3.9) but with all commutators exchanged with $\star$-commutators, e.g.,

$$[\mathcal{A}_m, \mathcal{A}_n] \to [\mathcal{A}_m, \mathcal{A}_n]_\star \equiv \mathcal{A}_m \star \mathcal{A}_n - \mathcal{A}_n \star \mathcal{A}_m. \tag{3.21}$$

The $\star$ product is a deformation of the ordinary product for functions, and it encodes the noncommutative algebraic information of the theory. In order to represent the (noncommutative) algebra on the space of commuting coordinates, one needs to map the basis in the algebra[7] to the basis of monomials of commuting coordinates. This map provides a representation of the elements of the abstract algebra in terms of functions of commuting coordinates, and these functions are then multiplied with the $\star$ product. Furthermore, one needs to map derivatives from the abstract algebra to the space of commuting coordinates.

In the present case of constant noncommutativity $\hat{\theta}^{ij}$, the $\star$ product is given by the well-known expression

$$f \star g = e^{(i/2)(\partial/\partial x^i)2\pi\hat{\theta}^{ij}(\partial/\partial y^j)} f(x)g(y)|_{y\to x}, \tag{3.22}$$

corresponding to the Moyal-Weyl product, while the derivatives in the algebra are mapped to the usual derivatives on the space of commuting coordinates.

In the spirit of noncommutative geometry [24,25], the U_i are the generators of an algebra $\mathcal{A}_\theta$ serving as the defining algebraic structure of the noncommutative torus. In analogy, $\hat{U}_i$ comprise the algebra $\mathcal{A}_{\hat{\theta}}$ of the dual noncommutative torus, where the gauge theory resides. Moreover, $\hat{\mathcal{D}}$ serves as a linear connection defined on a projective module over $\mathcal{A}_{\hat{\theta}}$, the analog of a connection on a vector bundle in ordinary smooth geometry. We shall not go any further on these matters. Detailed expositions may be found, for example, in Refs. [4,24].

Compactification on noncommutative T^3_x. Apart from the solutions that we already reviewed, here we suggest that there are alternative ways to solve the conditions of Eq. (3.5) of the compactification on a 3-torus. These correspond to noncommutative tori with nonconstant noncommutativity, which we denote as T^3_x.

Let us consider again the set of matrices Eq. (3.15), specializing to the case of $d = 3$, although our discussion may be directly generalized for any d. Moreover, let us require that the only nonvanishing commutation relation of $\hat{x}^i$ is

$$[\hat{x}^1, \hat{x}^3] = iN\hat{x}^2, \tag{3.23}$$

for an arbitrary scale N, while keeping the Heisenberg commutation relations $[\hat{x}^i, \hat{\partial}_j] = -\delta^i_j$. It is directly verified that the conditions Eq. (3.5) are satisfied, and therefore the above solution serves as a consistent background for the compactification of the matrix model. However, it is clear that Eq. (3.13) is not satisfied. This is due to the following subtlety. In the present case the associator, $[\hat{\partial}_2, \hat{x}^1, \hat{x}^3] \equiv [\hat{\partial}_2, [\hat{x}^1, \hat{x}^3]] + \text{(cyclic permutations)}$, does not vanish. Indeed, we find

$$[\hat{\partial}_2, \hat{x}^1, \hat{x}^3] = iN. \tag{3.24}$$

[6] In the richer case of twisted gauge bundles, the vacuum solution for $\mathcal{X}_i$ still satisfies the equations of motion in Eq. (3.2), although in a less trivial way since $[\mathcal{X}_i, \mathcal{X}_j] \neq 0$.

[7] An algebra needs to fulfil the Poincaré-Birkoff-Witt property [23].

Because of this nonassociativity, the relation Eq. (3.13) does not directly follow from the conditions of Eq. (3.5) anymore. Therefore, although the latter are satisfied, as they should, the former is not. This, of course, does not spoil the consistency of the solution. Let it be stressed that the associator Eq. (3.24) is the only nonvanishing one, while all the rest of the Jacobi identities still hold, e.g., $[x_i, x_j, x_k] = 0$, $[\partial_i, \partial_j, \partial_k] = 0$, $[\partial_i, \partial_j, x_k] = 0$. This ensures the compatibility of the solution with the gauge invariance of the matrix model action. Indeed, the latter is based only on the Jacobi identities $[U_i,[X_j,U_i^{-1}]]+(\text{cyclic permutations})=0$, $[U_i,[X_j,X_k]]+(\text{cyclic permutations})=0$, which stem from the vanishing associators and therefore they are satisfied.

According to the above, it is reasonable to interpret the resulting compactification as one on a noncommutative torus T_x^3 with an additional nonassociative structure on the corresponding phase space. In the present case, the noncommutativity is nonconstant and the nonassociativity a constant one. In Sec. IV, we shall associate this solution to the background of a square 3-torus carrying an constant NS 3-form flux[8] which was described in Sec. II. It is worth mentioning that indications of nonassociativity in backgrounds carrying a nonvanishing 3-form flux were given recently in a different context [27–29].

For the solution at hand, it is straightforward to check that the Us satisfy the commutation relations

$$[U_1,U_2]=0,\quad [U_2,U_3]=0,\quad U_1U_3=e^{-iN\hat{x}^2}U_3U_1. \tag{3.25}$$

A set of operators $\hat{U}_i$ commuting with the Us is given by

$$\hat{U}_1 = e^{i\hat{x}^1+N\hat{x}^2\hat{\partial}_3},\qquad \hat{U}_2 = e^{i\hat{x}^2},\qquad \hat{U}_3 = e^{i\hat{x}^3-N\hat{x}^2\hat{\partial}_1}, \tag{3.26}$$

and they satisfy the commutation relations

$$[\hat{U}_1,\hat{U}_2]=0,\qquad [\hat{U}_2,\hat{U}_3]=0,\qquad \hat{U}_1\hat{U}_3 = e^{iN\hat{x}^2}\hat{U}_3\hat{U}_1, \tag{3.27}$$

which are dual to Eq. (3.25). These operators generate the algebra of the dual torus and they provide the dependence of the gauge potentials.[9]

The noncommutativity of the resulting theory may be again encoded in the appropriate $\star$ product, which does not have the Moyal-Weyl form due to the $\hat{x}$-dependence in Eq. (3.27). Indeed, the relevant $\star$ product is now given by

$$f\star g = e^{(i/2)Nx^2((\partial/\partial y^1)(\partial/\partial z^3)-(\partial/\partial y^3)(\partial/\partial z^1))}f(y)g(z)|_{y,z\to x}. \tag{3.28}$$

Note that this $\star$ product is associative.[10] However, the nonassociativity Eq. (3.24) is mapped to the usual phase space by mapping the derivatives in the algebra to the usual derivatives on the space of commuting coordinates:

$$[\partial_j, x^i] = \delta^i_j,\qquad [\partial_2,[x^1,x^3]_\star] = iN \neq 0. \tag{3.29}$$

It is worth mentioning that a class of nonassociative but commutative gauge theories was studied in Ref. [31]. It would be interesting to examine in detail how the nonassociativity encountered in the present context manifests itself within the gauge theory over the (dual) noncommutative torus Eq. (3.27).

It is reasonable to ask whether the background Eq. (3.15) may be further generalized along the same lines, i.e., with nonconstant noncommutativity. First of all, the conditions Eq. (3.5) impose the constraint that the commutation relations between coordinates and momenta remain unchanged, i.e., $[\hat{x}^i,\hat{\partial}_j] = -\delta^i_j$. Furthermore, in the present paper, we discuss trivial gauge bundles, and therefore we assume that $[\hat{\partial}_i,\hat{\partial}_j]=0$. In order to be able to provide a solid argument on possible extensions of Eq. (3.23) let us also require that the commutators between the $\hat{x}^i$ are linear in $\hat{x}^i$. A final constraint, imposed again by the conditions Eq. (3.5), is that there should exist a set of operators, constructed from $\hat{x}^i$ and $\hat{\partial}_i$, which commute with $\hat{x}^i$. The most general algebra respecting the above constraints and requirements is

$$\begin{aligned}&[\hat{x}^1,\hat{x}^2]=2\pi i\theta^{12},\\ &[\hat{x}^1,\hat{x}^3]=iN\hat{x}^2+2\pi i\theta^{13}-2\pi N\theta^{12}\hat{\partial}_1+2\pi N\theta^{23}\hat{\partial}_3,\\ &[\hat{x}^2,\hat{x}^3]=2\pi i\theta^{23}.\end{aligned} \tag{3.30}$$

The solution Eq. (3.23), which was described previously, is a special case of Eq. (3.30) with all θs vanishing, and it will be useful in Sec. IV.

These commutation relations lead to

$$U_1U_3=\lambda_{13}U_3U_1,\qquad \lambda_{13}=e^{-iN\hat{x}^2-2\pi i\theta^{12}+2\pi N\theta^{12}\hat{\partial}_1-2\pi N\theta^{23}\hat{\partial}_3}, \tag{3.31}$$

$$U_1U_2=\lambda_{12}U_2U_1,\qquad \lambda_{12}=e^{-2\pi i\theta^{12}}, \tag{3.32}$$

$$U_2U_3=\lambda_{23}U_3U_2,\qquad \lambda_{23}=e^{-2\pi i\theta^{23}}. \tag{3.33}$$

[8]A different approach on this matter may be found in Ref. [26], where instead the conditions Eq. (3.5) are modified.

[9]More exactly, the gauge potentials of Eq. (3.18) should be modified in the present case. In particular, $\mathcal{A}_2(\hat{U})$ should be replaced by $\hat{\mathcal{A}}_2 = \mathcal{A}_2(\hat{U}) + i\mathcal{A}_2(\hat{U})\partial_1 + i\mathcal{A}_2(\hat{U})\partial_3$ in order to ensure that the space of gauge potentials is well-defined under gauge transformations $\hat{U}$. This modification leaves the compactification conditions satisfied, up to gauge transformations. A more detailed account on the resulting gauge theory will be given elsewhere.

[10]Nonassociative $\star$ products were studied in Ref. [30], where they are also associated to nonvanishing H flux backgrounds.

One can construct the operators $\hat{U}_i$, which commute with Us as $\hat{U}_i = e^{i\hat{y}^i}$, where

$$\begin{aligned} \hat{y}^1 &= \hat{x}^1 - 2\pi i\theta^{12}\hat{\partial}_2 - 2\pi i\theta^{13}\hat{\partial}_3 - iN\hat{x}^2\hat{\partial}_3 \\ &\quad + 2\pi N\theta^{12}\hat{\partial}_1\hat{\partial}_3 - 2\pi N\theta^{23}(\hat{\partial}_3)^2, \\ \hat{y}^2 &= \hat{x}^2 - 2\pi i\theta^{23}\hat{\partial}_3 + 2\pi i\theta^{12}\hat{\partial}_1, \\ \hat{y}^3 &= \hat{x}^3 + 2\pi i\theta^{23}\hat{\partial}_2 + 2\pi i\theta^{13}\hat{\partial}_1 + iN\hat{x}^2\hat{\partial}_1 \\ &\quad + 2\pi N\theta^{23}\hat{\partial}_3\hat{\partial}_1 - 2\pi N\theta^{12}(\hat{\partial}_1)^2. \end{aligned} \tag{3.34}$$

These operators satisfy dual ($\lambda \to 1/\lambda$) relations to Eqs. (3.31), (3.32), and (3.33). Relaxing the requirements that were posed above, it is in principle possible to reach more general algebraic structures. However, this task is beyond the scope of this paper.

Finally, it is natural to ask whether nonassociativity could be avoided in the present approach. The answer is yes; indeed one can describe a solution where the $\hat{x}^i$ satisfy Eq. (3.23); moreover, it holds that

$$[\hat{\partial}_2, \hat{x}^1] = iN\hat{\partial}_3, \tag{3.35}$$

while the U_1 in Eq. (3.15) is modified to $U_1 = e^{i\hat{x}^1 + N\hat{x}^2\hat{\partial}_3}$. Then one obtains a consistent solution of the conditions Eq. (3.5), while ensuring that all the Jacobi identities for the algebra of $\hat{x}^i$ and $\hat{\partial}_i$ are satisfied. However, in that case the Us turn out to be commutative, i.e., they generate a commutative algebra of functions. Thus the gauge theory turns out to be a commutative one and the information from the nontrivial commutation relations of $\hat{x}^i$ is lost at the level of the action. This situation is not interesting for our purposes, and the corresponding solutions will not be discussed further.

C. Compactification on twisted 3-tori

Having discussed the compactification of the MMs on multidimensional tori and their noncommutative counterparts, the stage is all set to move on to the study of twisted tori. We shall follow the same lines as before, starting with the special case of a standard (i.e., commutative) twisted torus and subsequently treating the case of a noncommutative one. In the present subsection, we focus on the simplest case of the twisted 3-torus, while in the next one, higher-dimensional twisted tori are discussed.

Compactification on twisted $\tilde{T}^3$. The simplest example of a twisted torus arises for $d = 3$, as we described in Sec. II. In that case, a (twisted) compactification is achieved by imposing and solving an appropriately extended set of constraints, incorporating the twisted identifications Eq. (2.12). These twisted constraints involve three unitary matrices U_1, U_2, U_3, and they are

$$\begin{gathered} \mathcal{X}_1 + R_1 = U_1\mathcal{X}_1U_1^{-1}, \quad \mathcal{X}_2 + R_2 = U_2\mathcal{X}_2U_2^{-1}, \\ \mathcal{X}_3 + R_3 = U_3\mathcal{X}_3U_3^{-1}, \quad \mathcal{X}_1 + R_2\mathcal{X}_3 = U_2\mathcal{X}_1U_2^{-1}, \\ \mathcal{X}_a = U_i\mathcal{X}_aU_i^{-1}, \\ a \neq i, \quad a = 1,\ldots,9, \quad i = 1,2,3, \quad (a,i) \neq (1,2). \end{gathered} \tag{3.36}$$

The latter constraints generalize the ones for the square torus appearing in Eq. (3.5), thus incorporating the twist of the three-dimensional nilmanifold $\tilde{T}^3$. Indeed, as in the torus case the constraints Eq. (3.5) reflect the defining relations Eq. (2.4) of T^3, likewise the constraints Eq. (3.36) are tantamount to the defining relations Eq. (2.12) of the nilmanifold. Therefore, the restriction of the matrix action under Eq. (3.36) defines a compactification on the $\tilde{T}^3$ (see also Ref. [8]).

A solution of the above constraints is now given by

$$\begin{gathered} \mathcal{X}_i = iR_i\mathcal{D}_i, \quad \mathcal{X}_m = \mathcal{A}_m, \quad m = 4,\ldots,9, \\ U_1 = e^{ix^1}, \quad U_2 = e^{ix^2 - ((R_2R_3)/R_1)x^1\partial_3}, \quad U_3 = e^{ix^3}. \end{gathered} \tag{3.37}$$

Let us comment on the above solution. First of all, for this solution the coordinates x^i are ordinary commutative coordinates on the twisted 3-torus $\tilde{T}^3$. Unlike the coordinates, the matrices U_i do not commute, but instead they satisfy a single nontrivial commutation relation,

$$U_2U_3 = e^{-i((R_2R_3)/R_1)x^1}U_3U_2. \tag{3.38}$$

This relation is not associated to any noncommutative properties of the manifold. It just reflects in the present framework the non-Abelian nature of the algebra of vector fields along the three directions of the twisted torus [see Eq. (2.14)], and therefore it is totally expected. Regarding the gauge potentials, note that $\mathcal{A}_1$ in $\mathcal{X}_1$ has to be modified to $\hat{\mathcal{A}}_1 = \mathcal{A}_1 + i\frac{R_2R_3}{R_1}\mathcal{A}_3\partial_2$ in order to satisfy the fourth equation in Eq. (3.36). The same modification has to be done in the noncommutative case that follows. Furthermore, since the compactification manifold is commutative, the gauge potentials $\mathcal{A}_i$ and the scalar fields $\mathcal{A}_m$ depend directly on the commuting coordinates x^i.

Compactification on noncommutative twisted $\tilde{T}^3_x$. In analogy to the square tori, it is also possible to consider a solution of the conditions Eq. (3.36) where all the U_i have the same form, namely $U_i = e^{i\hat{x}^i}$, but with noncommuting coordinates $\hat{x}^i$. This was already noticed and elaborated on in Ref. [8]. Setting

$$\begin{aligned} [\hat{x}^2, \hat{x}^3] &= iR\hat{x}^1, \qquad [\hat{x}^1, \hat{x}^j] = 0, \\ [\hat{\partial}_1, \hat{x}^2] &= iR\hat{\partial}_3, \qquad [\hat{\partial}_i, \hat{x}^i] = 1, \end{aligned} \tag{3.39}$$

where $R = \frac{R_2R_3}{R_1}$, the conditions Eq. (3.36) are satisfied and Eq. (3.38) is reproduced. It is worth noting that although the present case resembles Eq. (3.23), it is in fact very different.

Here, due to Eq. (3.39), associativity is guaranteed and the compactification is on a noncommutative twisted torus.

As far as the dependence of the fields of the gauge theory is concerned, in the present noncommutative case, they depend on the set of operators $\hat{U}$s commuting with the Us. This set is given as follows:

$$\hat{U}_1 = e^{i\hat{x}^1}, \quad \hat{U}_2 = e^{i\hat{x}^2 + R\hat{x}^1\hat{\partial}_3}, \quad \hat{U}_3 = e^{i\hat{x}^3 - R\hat{x}^1\hat{\partial}_2}, \tag{3.40}$$

satisfying the relation

$$\hat{U}_2\hat{U}_3 = e^{iR\hat{x}^1}\hat{U}_3\hat{U}_2, \tag{3.41}$$

dual to Eq. (3.38). Once more, one can define a $\star$ product of the form

$$f \star g = e^{(i/2)Rx^1((\partial/\partial y^2)(\partial/\partial z^3)-(\partial/\partial y^3)(\partial/\partial z^2))} f(y)g(z)|_{y,z\to x}, \tag{3.42}$$

to represent the algebra relations Eq. (3.39) on the space of commuting coordinates. However, in the present case the derivatives in the algebra are not mapped in the usual derivatives $\hat{\partial}_i \rightarrow \partial_i^* \neq \partial_i$. One constructs ∂_i^* derivatives by comparing $(\hat{\partial}\hat{f})(\hat{x})$ and $(\partial_i^* \star f)(x)$ expanded in the appropriate basis. This results in a perturbative (in the noncommutative parameter) expression for ∂_i^*, which can be generalized to a closed expression in some cases. In the present example, we obtain $\partial_2^* = \partial_2$, $\partial_3^* = \partial_3$, $\partial_1^* = \partial_1 + \frac{iR}{2}\partial_2\partial_3 + \mathcal{O}(R^2)$.

Compactification on noncommutative twisted $\tilde{T}^3_{\theta,x}$. Let us finally go one step further and follow a general approach for the treatment of the conditions of Eq. (3.36), which will lead us to more general solutions than the ones we obtained previously for the twisted 3-torus. This may be made clear by considering once more the operator

$$Q \equiv Q_{(ij)} = U_i U_j U_i^{-1} U_j^{-1}.$$

It is straightforward to show that

$$[Q_{(13)}, \mathcal{X}_a] = [Q_{(12)}, \mathcal{X}_a] = 0, \quad \forall\ a, \tag{3.43}$$

however,

$$[Q_{(23)}, \mathcal{X}_1] = -R_2R_3 Q_{(23)} \neq 0. \tag{3.44}$$

Therefore, we encounter a mixed situation, where the two operators $Q_{(12)}$ and $Q_{(13)}$ commute with all the $\mathcal{X}_a$, but the remaining one does not. It follows that the former ones are scalar operators and thus

$$U_1U_3 = \lambda_{13}U_3U_1, \qquad \lambda_{13} = e^{-2\pi i\theta^{13}}, \tag{3.45}$$

$$U_1U_2 = \lambda_{12}U_2U_1, \qquad \lambda_{12} = e^{-2\pi i\theta^{12}}, \tag{3.46}$$

while this is not true for $Q_{(23)}$. The commutation relation between U_2 and U_3 does not have the same form and it is in fact expected to be $\hat{x}$-dependent as previously. Let us note that for $\lambda_{12} = \lambda_{13} = 1$, we recover either the case of the commutative twisted torus or the one on noncommutative twisted torus with purely nonconstant noncommutativity, both discussed previously. Therefore, any solution in the present case should reduce either to the solution Eq. (3.37) or to Eq. (3.39) in the limit of $\theta^{12} = \theta^{13} = 0$.

Without further ado, let us write down the solution of the conditions Eq. (3.36) for general θ^{12} and θ^{13},

$$\mathcal{X}_i = iR_i\hat{\mathcal{D}}_i, \quad \mathcal{X}_m = \mathcal{A}_m, \quad m = 4, \ldots, 9, \quad U_i = e^{i\hat{x}^i}, \tag{3.47}$$

where

$$\begin{aligned}
&[\hat{x}^1, \hat{x}^2] = 2\pi i\theta^{12}, \qquad [\hat{x}^1, \hat{x}^3] = 2\pi i\theta^{13}, \\
&[\hat{x}^2, \hat{x}^3] = iR\hat{x}^1 + 2\pi i\theta^{23}. \qquad [\hat{\partial}_1, \hat{x}^2] = iR\hat{\partial}_3,
\end{aligned} \tag{3.48}$$

with $R \equiv \frac{R_2R_3}{R_1}$ as before. In the present case, *all* the U_i do not commute among themselves. In particular, along with Eqs. (3.45) and (3.46) we obtain

$$U_2U_3 = e^{-2\pi i\theta^{23} - iR\hat{x}^1}U_3U_2. \tag{3.49}$$

For the solution Eqs. (3.47) and (3.48) we can find the set of $\hat{U}$s which give the connection on a trivial gauge bundle. They have the form $\hat{U}_i = e^{i\hat{y}^i}$, where

$$\begin{aligned}
\hat{y}^1 &= \hat{x}^1 - 2\pi i\theta^{13}\hat{\partial}_3 - 2\pi i\theta^{12}\hat{\partial}_2, \\
\hat{y}^2 &= \hat{x}^2 + 2\pi i\theta^{12}\hat{\partial}_1 - iR\hat{x}^1\hat{\partial}_3 - 2\pi i\theta^{23}\hat{\partial}_3 - \pi R\theta^{13}(\hat{\partial}_3)^2, \\
\hat{y}^3 &= \hat{x}^3 + 2\pi i\theta^{13}\hat{\partial}_1 + iR\hat{x}^1\hat{\partial}_2 + 2\pi i\theta^{23}\hat{\partial}_2 + 2\pi R\theta^{13}\hat{\partial}_2\hat{\partial}_3 \\
&\quad + \pi R\theta^{12}(\hat{\partial}_2)^2.
\end{aligned} \tag{3.50}$$

We observe that in the latter case, the commutation relations of the coordinates $\hat{x}^i$ involve both constant and nonconstant parts.

Summarizing, we were able to find consistent solutions of Eq. (3.36), corresponding to compactifications of Matrix theory on the ordinary twisted 3-torus or on different versions of noncommutative twisted 3-tori. Let us remind at this point that these solutions solve the equations of motion Eq. (3.2), since $[\mathcal{X}_i, \mathcal{X}_j] = 0$ in the vacuum. This remains true in the six-dimensional case which follows.

D. Compactification on twisted 6-tori

The approach of the previous subsection for the matrix compactification on twisted 3-tori may be directly generalized to twisted tori of any dimension, such as the ones described in Sec. II C and the Appendix. Higher-dimensional twisted tori comprise richer structures than the three-dimensional case, since they generically include more than one twist and therefore they can be associated to string/M theory compactifications with more fluxes. Indeed, in Sec. II C we saw that the twisted torus based on the six-dimensional nilpotent group $A^{\alpha}_{6,5}$ is T-dual to a square 6-torus with NS fluxes given in Eq. (2.27). In the present section, we study the compactification of Matrix theory on this example.

Let us therefore consider the case of the twisted 6-torus constructed by the algebra $\mathcal{A}^{\alpha}_{6,5}$. For simplicity we set the parameter $\alpha = 1$, since it does not affect the generality of the discussion. Guided by the identifications of Eq. (2.22), it is easy to write down the necessary constraints, which now involve six unitary matrices U_i, $i = 1, \ldots, 6$, and they take the form

$$\begin{aligned} &\mathcal{X}_i + R_i = U_i \mathcal{X}_i U_i^{-1}, \quad \mathcal{X}_5 + R_1 \mathcal{X}_3 = U_1 \mathcal{X}_5 U_1^{-1}, \\ &\mathcal{X}_5 + R_2 \mathcal{X}_4 = U_2 \mathcal{X}_5 U_2^{-1}, \quad \mathcal{X}_6 + R_1 \mathcal{X}_4 = U_1 \mathcal{X}_6 U_1^{-1}, \\ &\mathcal{X}_6 + R_2 \mathcal{X}_3 = U_2 \mathcal{X}_6 U_2^{-1}, \quad \mathcal{X}_a = U_i \mathcal{X}_a U_i^{-1}, \\ &\qquad a \neq i, \quad (a, i) \neq (5,1), (5,2), (6,1), (6,2). \end{aligned} \tag{3.51}$$

The solutions we consider again involve trivial gauge bundles, and therefore we choose the $\mathcal{X}_a$ to be

$$\mathcal{X}_i = iR_i \hat{\mathcal{D}}_i \text{ and } \mathcal{X}_m = \mathcal{A}_m, \quad m = 7, \ldots, 10, \tag{3.52}$$

where the gauge potentials in the hatted covariant derivative generically depend on some $\hat{U}_i$ which commute with all the U_i, as before. More precisely, $\mathcal{A}_i$ are modified according to the relation $\hat{\mathcal{A}}_i = \mathcal{A}_i + i\frac{R_j R_k}{R_i} f^{jk}{}_i \mathcal{A}_k \partial_j, j < k$.

Commutative twisted $\tilde{T}^6$. The commutative case is relatively easy to describe. It amounts to choosing the following set of unitary operators:

$$\begin{aligned} U_1 &= e^{ix^1 - ((R_1R_3)/R_5)x^5\partial_3 - ((R_1R_4)/R_6)x^6\partial_4}, \\ U_2 &= e^{ix^2 - ((R_2R_4)/R_5)x^5\partial_4 - ((R_2R_3)/R_6)x^6\partial_3}, \\ U_s &= e^{ix^s}, \qquad s = 3, 4, 5, 6. \end{aligned} \tag{3.53}$$

Here, x^i are ordinary commutative coordinates, and it is straightforward to show that the only nontrivial commutation relations for the U_i are

$$\begin{aligned} U_1 U_3 &= e^{-i((R_1R_3)/R_5)x^5} U_3 U_1, \\ U_1 U_4 &= e^{-i((R_1R_4)/R_6)x^6} U_4 U_1, \\ U_2 U_3 &= e^{-i((R_2R_3)/R_6)x^6} U_3 U_2, \\ U_2 U_4 &= e^{-i((R_2R_4)/R_5)x^5} U_4 U_2. \end{aligned} \tag{3.54}$$

As we also discussed in the three-dimensional case, for the commutative manifold these relations just reflect the non-Abelian nature of the algebra of vector fields on it, which were described in Eq. (2.23). Therefore, the compactification here is on an ordinary manifold, the gauge potentials and the scalar fields depend directly on the commutative coordinates x^i, and the resulting theory is a $(1 + 6)$-dimensional YM theory.

Noncommutative twisted $\tilde{T}^6_{\theta,x}$. After the brief exposition of the commutative case, let us now turn our attention to the general case. Considering the operators $Q_{(ij)}$, one obtains that the ones which do not commute with $\mathcal{X}_a$ are the ones with the index pairing $(i, j) = \{(1,3), (1,4), (2,3), (2,4)\} \equiv \mathcal{I}$, which satisfy

$$[Q_{(13)}, \mathcal{X}_5] = -R_1R_3 Q_{(13)}, \quad [Q_{(14)}, \mathcal{X}_6] = -R_1R_4 Q_{(14)}, \tag{3.55}$$

$$[Q_{(23)}, \mathcal{X}_6] = -R_2R_3 Q_{(23)}, \quad [Q_{(24)}, \mathcal{X}_5] = -R_2R_4 Q_{(24)}. \tag{3.56}$$

The rest of the commutators among $Q_{(ij)}$ and $\mathcal{X}_a$ vanish. This means that

$$U_i U_j = e^{2\pi i \theta^{ij}} U_j U_i, \qquad (i, j) \notin \mathcal{I}. \tag{3.57}$$

Then we may consider the solution Eq. (3.52) with $U_i = e^{i\hat{x}^i}$ and the following commutation relations:

$$\begin{aligned} &[\hat{x}^i, \hat{x}^j] = iR_{(ij)} f^{ij}{}_k \hat{x}^k + 2\pi i \theta^{ij}, \qquad [\hat{\partial}_i, \hat{\partial}_j] = 0, \\ &[\hat{\partial}_i, \hat{x}^j] = \delta_i^j + iR_{(jk)} f^{jk}{}_i \hat{\partial}_k, \qquad j < k. \end{aligned} \tag{3.58}$$

The $f_{ij}{}^k$ are the structure constants of the algebra $\mathcal{A}_{6,5}$,

$$f_{13}{}^5 = f_{14}{}^6 = f_{23}{}^6 = f_{24}{}^5 = 1, \tag{3.59}$$

which are antisymmetric in the lower indices. Moreover, the quantities $R_{(ij)}$ in the example at hand are

$$R_{(13)} = \frac{R_1R_3}{R_5}, \quad R_{(14)} = \frac{R_1R_4}{R_6}, \quad R_{(23)} = \frac{R_2R_3}{R_6}, \quad R_{(24)} = \frac{R_2R_4}{R_5}. \tag{3.60}$$

Clearly, the subscripts in parentheses are labels and not indices, and therefore they are not summed in Eq. (3.58). Furthermore, it holds that $R_{(ij)} = R_{(ji)}$.

It is straightforward to check that the above set of U_i and $\mathcal{X}_a$ furnishes a consistent solution of Eq. (3.51). The last relation in Eq. (3.58) is crucial for this consistency; moreover, it guarantees the associativity of the full algebra of coordinates and momenta. However, apart from the above, it is important to be able to construct the set of operators $\hat{U}_i$ which commute with U_i. This is only possible upon imposing some additional constraints. Indeed, these operators can be written as $\hat{U}_i = e^{i\hat{y}^i}$, where

$$\hat{y}^i = \hat{x}^i - 2\pi i \theta^{ij} \hat{\partial}_j - iR_{(ij)} f^{ij}{}_k \hat{x}^k \hat{\partial}_j + \pi R_{(ij)} f^{i\{j}{}_k \theta^{l\}k} \hat{\partial}_l \hat{\partial}_j, \tag{3.61}$$

where the notation $\eta^{\{j}\xi^{l\}}$ for the superscripts denotes symmetrization or antisymmetrization as follows:

$$\eta^{\{j}\xi^{l\}} = \begin{cases} \eta^j \xi^l - \eta^l \xi^j, & \text{for } i = 1, 2, \quad (j, l) \in \mathcal{I}, \\ \eta^j \xi^l + \eta^l \xi^j, & \text{for } i = 3, 4, \quad (j, l) \in \mathcal{I}, . \\ \eta^j \xi^l, & \text{for } (j, l) \notin \mathcal{I} \end{cases}$$

For $i = 5, 6$, the related term is absent due to Eq. (3.59). The $\hat{U}_i$ constructed in that way commute with the U_i under the following conditions:

TABLE I. Solutions of the BFSS model compactified on three- and six-dimensional tori and twisted tori. C stands for commutative, NC stands for noncommutative, and NA for nonassociative. Indices run from 1 to 3 for the three-dimensional cases and from 1 to 6 for the six-dimensional ones. The last column contains the associated supergravity fluxes for each compactification, and it is discussed in Sec. IV.

	$[\hat{x}^i, \hat{x}^j]$	$Q_{(ij)}$	Torus type	Twist	SuGra Flux
1	0	1	3d C	No	...
2	$-2\pi i\theta^{ij}$	$e^{2\pi i\theta^{ij}}$	3d NC	No	B_{ij}
3	$iN\epsilon_{ij2}\hat{x}^2$	$e^{-iN\epsilon_{ij2}\hat{x}^2}$	3d NC, NA	No	H_{123}
4	0	$e^{-i((R_2R_3)/R_1)\epsilon_{ij1}x^1}$	3d C	Yes	${f_{23}}^1$
5	$i\frac{R_2R_3}{R_1}\epsilon_{ij1}\hat{x}^1$	$e^{-i((R_2R_3)/R_1)\epsilon_{ij1}\hat{x}^1}$	3d NC	Yes	H_{123}, ${f_{23}}^1$
6	$i\frac{R_2R_3}{R_1}\epsilon_{ij1}\hat{x}^1 + 2\pi i\theta^{ij}$	$e^{-i((R_2R_3)/R_1)\epsilon_{ij1}\hat{x}^1-2\pi i\theta^{ij}}$	3d NC	Yes	B_{ij}, H_{123}, ${f_{23}}^1$
7	0	$e^{-iR_{(ij)}{f^{ij}}_k\hat{x}^k}$	6d C	Yes	${f_{ij}}^k$
8	$iR_{(ij)}{f^{ij}}_k\hat{x}^k + 2\pi i\theta^{ij}$	$e^{-iR_{(ij)}{f^{ij}}_k\hat{x}^k-2\pi i\theta^{ij}}$	6d NC	Yes	B_{ij}, H_{ijk}, ${f_{ij}}^k$

$$\theta^{56} = 0, \qquad \frac{\theta^{36}}{R_3R_6} = \frac{\theta^{45}}{R_4R_5}, \qquad \frac{\theta^{15}}{R_1R_5} = \frac{\theta^{26}}{R_2R_6},$$
$$\frac{\theta^{16}}{R_1R_6} = \frac{\theta^{25}}{R_2R_5}, \qquad \frac{\theta^{35}}{R_3R_5} = \frac{\theta^{46}}{R_4R_6}. \tag{3.62}$$

It is worth noting that an appropriate $\star$ product may be defined in the present case as well. It is of a mixed form and it is given by

$$f \star g = e^{(i/2)(R_{(ij)}{f^{ij}}_k x^k + 2\pi\theta^{ij})(\partial/\partial y^i)(\partial/\partial z^j)} f(y)g(z)|_{y,z\to x}. \tag{3.63}$$

Because of the last relation in Eq. (3.58) the derivatives in the algebra are not mapped to the usual derivatives, but instead $\hat{\partial}_i \to \partial_i^* = \partial_i + \frac{i}{2}R_{(jk)}{f^{jk}}_i\partial_j\partial_k + \mathcal{O}(R^2)$, $j < k$.

The above procedure may be directly generalized for the compactification of Matrix theory on any higher-dimensional nilmanifold, such as the five- and six-dimensional ones of the Appendix. Then, the general expressions Eqs. (3.58) and (3.61) will still hold for an appropriate set of index pairs $\mathcal{I}$ and the structure constants ${f_{ij}}^k$ of the corresponding nilpotent algebra.

IV. DUALITIES AND SEIBERG-WITTEN MAPS

Having studied several backgrounds corresponding to compactifications of the BFSS model in Sec. III, let us now discuss how they may be related among themselves as well as with known backgrounds of M theory/Type IIA string theory in the supergravity approximation. It is useful at this stage to summarize the solutions that we have described so far in order to facilitate their comparison. This is done in Table I.

Connes-Douglas-Schwarz conjecture. The starting point is that the BFSS model corresponds to a nonperturbative definition of M theory in the infinite momentum frame [1]. Therefore, one expects that known backgrounds of 11-dimensional supergravity (the field theory limit of M theory) should be reproduced in the matrix framework. Indeed, Connes, Douglas, and Schwarz suggested that that the deformation parameters θ^{ij} defining the noncommutative tori as in Sec. III B correspond to moduli of the 11-dimensional supergravity [4]. The latter contains a 3-form C_{IJK}, which is the gauge potential of the 4-form field strength of the theory, where I, J, K are 11-dimensional indices. Then the claim is that

$$\theta^{ij} \propto \int dx^i dx^j C_{ij-}, \tag{4.1}$$

where $-$ denotes the light cone direction x^- [4]. It is also useful to rephrase this statement in the language of the Type IIA theory, which is obtained by 11-dimensional supergravity upon compactification on a circle. In that process, the 3-form C gives rise to the NS 2-form field B of the Type IIA supergravity.[11] Therefore, in IIA language, Eq. (4.1) may be restated as

$$\theta^{ij} \propto \int dx^i dx^j B_{ij}. \tag{4.2}$$

In the following, we shall retain this auxilliary Type IIA language in our discussion.

According to the above, the deformation of a commutative torus to a noncommutative one in the matrix model corresponds to turning on background values for the B field in Type IIA string theory. Let us now use this statement as a guiding principle in order to unveil relations between the backgrounds we studied in Sec. III and known Type IIA backgrounds.[12] For this purpose we will use the following notation: When we want to refer to a solution of the compactified matrix model, such as the ones appearing in Table I, we write, e.g., "MM on T^3_θ" for the solution on the noncommutative 3-torus with constant noncommutativity θ^{ij}. Similarly, for the auxiliary Type IIA background, we

[11] We do not discuss here issues related to the Ramond-Ramond forms of the Type IIA theory. A related discussion may be found in Ref. [32].

[12] We use here the term "background" in a somewhat loose sense. Some of the situations we discuss are not fully consistent string backgrounds and need to be appropriately lifted [18]. However, here we are interested in relations between fluxes and deformations, and this discussion in beyond our scope.

write, e.g. "IIA on $\tilde{T}^3$" for a compactification on the twisted 3-torus.

In the above notation, the Connes-Douglas-Schwarz conjecture reads as

$$\text{MM on} T^3_\theta \overset{CDS}{\longleftrightarrow} \text{IIA on} T^3_B, \qquad (4.3)$$

corresponding to Eq. (4.2). The left-hand side (lhs) of the conjecture refers to the solution Eq. (3.15) described in Sec. III B, which is solution 2 in Table I. This correspondence was further justified in Refs. [5,7].

Along the same lines, we suggest the following correspondences for the solutions which appear in Table I:

(i) Solution 3 of Table I describes the compactification of the BFSS model on a torus with nonconstant noncommutativity. On the supergravity side, this should correspond to a background with nonconstant B field and therefore a constant 3-form flux H. Schematically,

$$\text{MM on} T^3_x \overset{CDS}{\longleftrightarrow} \text{IIA on} T^3_H, \qquad (4.4)$$

The right-hand side (rhs) of this correspondence was already referred to in Sec. II B, Eq. (2.15). In other words, we suggest that deforming the torus to a noncommutative one with nonconstant noncommutativity corresponds to turning on a constant 3-form flux through the torus on the supergravity side. In 11-dimensional language, this situation corresponds to a constant 4-form background.

(ii) Turning to the compactification of the BFSS model on twisted 3-tori, solution 5 of Table I should be associated with a nonconstant B field on a twisted 3-torus on the supergravity side,

$$\text{MM on} \tilde{T}^3_x \overset{CDS}{\longleftrightarrow} \text{IIA on} \tilde{T}^3_H. \qquad (4.5)$$

This was discussed in detail in Ref. [8], while supergravity backgrounds with both geometric fluxes and NS fluxes were studied, for example, in Refs. [13–15]. Moreover, in the present study, we described more general solutions associated to twisted 3-tori, given by Eqs. (3.47) and (3.48). On the supergravity side, these should be associated to twisted 3-tori with mixed (constant and nonconstant) B field.

(iii) Finally, as far as the compactification on a twisted 6-torus is concerned, the situation is very similar. In particular, for solution 8 in Table I and vanishing θ^{ij},

$$\text{MM on} \tilde{T}^6_x \longleftrightarrow \text{IIA on} \tilde{T}^6_H, \qquad (4.6)$$

where the NS fluxes H are given in Eq. (2.27).

The above correspondences are plausible in view of previous work on the subject, but in order for them to be fully demonstrated, one has to study in detail the resulting $(1 + d)$-dimensional theory. We plan to perform such an analysis in a forthcoming publication.

T-duality and Seiberg-Witten maps. Previously, we discussed relations of matrix model backgrounds with certain supergravity ones. However, here we suggest that there exist also relations among the matrix backgrounds themselves and among the gauge theories on them.

Looking at the solution of the constraints for the compactification on a noncommutative torus T^3_θ, Eqs. (3.15) and (3.17), we observe that there exists a mapping from the noncommutative torus to a commutative one:

$$f\colon \hat{x}^i \to x^i - \pi i \theta^{ij}\partial_j, \qquad f\colon \hat{\partial}_i \to \partial_i, \qquad (4.7)$$

where $\hat{x}^i$ and $\hat{\partial}_i$ are the coordinates and the corresponding derivatives on the noncommutative torus, while x^i and ∂_i are the usual commuting coordinates and derivatives. Under this mapping, the Us and $\hat{U}$s of the solution Eq. (3.15) go to:

$$f\colon U_i = e^{i\hat{x}^i} \to e^{ix^i + \pi\theta^{ij}\partial_j}, \quad f\colon \hat{U}_i = e^{i\hat{x}^i - 2\pi\theta^{ij}\hat{\partial}_j} \to e^{ix^i - \pi\theta^{ij}\partial_j}, \qquad (4.8)$$

preserving the algebras of Us and $\hat{U}$s. Thus the mapping Eq. (4.7) induces the Seiberg-Witten map, i.e., the transformation from noncommutative Yang-Mills fields Eq. (3.21) to ordinary Yang-Mills fields Eq. (3.6) over the same commutative torus. This construction of the SW map goes along the lines of the original construction introduced in the seminal paper, Ref. [16]. More abstractly, it is enough to assume that the noncommutative gauge transformation is induced by the ordinary one [23]. One uses the representation of the elements of the noncommutative algebra in terms of functions of commuting coordinates which are multiplied with the $\star$ product. Assuming that the noncommutative gauge transformation is induced by the ordinary one provides enough data to express the noncommutative fields and gauge parameter as functions of the commutative ones. As the analysis of the Yang-Mills theory resulting from the compactification is beyond the scope of this paper, we do not construct the SW map(s) explicitly.

A similar map can be constructed in the nonassociative case, i.e., for the compactification on T^3_x defined by the relations Eqs. (3.23), (3.24), and (3.25). There exists a mapping from the aforementioned solution of Eq. (3.5) into the solution of the same condition but on a commutative torus:

$$\begin{aligned} &f\colon \hat{x}^1 \to x^1 + \frac{iN}{2}x^2\partial_3, \qquad f\colon \hat{x}^2 \to x^2, \\ &f\colon \hat{x}^3 \to x^3 - \frac{iN}{2}x^2\partial_1, \qquad f\colon \hat{\partial}_i \to \partial_i, \end{aligned} \qquad (4.9)$$

where

$$[x^i, x^j] = 0, \qquad [\partial_i, \partial_j] = 0, \qquad [\partial_i, x^i] = 1,$$
$$[\partial_2, x^3] = \frac{iN}{2}\partial_1, \qquad [\partial_2, x^1] = -\frac{iN}{2}\partial_3. \tag{4.10}$$

Note that the last two relations imply

$$[\partial_2, x^1, x^3] = iN.$$

The operators U_i and $\hat{U}_i$ are mapped to

$$U_1 = e^{ix^1-(N/2)x^2\partial_3}, \qquad U_2 = e^{ix^2},$$
$$U_3 = e^{ix^3+(N/2)x^2\partial_1}, \qquad \hat{U}_1 = e^{ix^1+(N/2)x^2\partial_3}, \tag{4.11}$$
$$\hat{U}_2 = e^{ix^2}, \qquad \hat{U}_3 = e^{ix^3-(N/2)x^2\partial_1},$$

and satisfy the same algebras as before. This induces the SW map between gauge theories over noncommutative and commutative tori with the (same) nonassociativity in the phase space. It is easy, but not very illuminating, to construct a similar mapping for the more general solution described by relations Eqs. (3.30), (3.31), (3.32), (3.33), and (3.34).

As a final example of the relation between the gauge field theories over the compact spaces we discuss, we provide a map for a twisted compactification given by the relations Eqs. (3.47), (3.48), (3.49), and (3.50):

$$f: \hat{x}^i \to x^i + \pi i\theta^{ij}\partial_j + i\delta^{i3}Rx^1\partial_3, \quad f: \hat{\partial}_i \to \partial_i. \tag{4.12}$$

In the gauge sector this map induces the SW map between noncommutative and ordinary Yang-Mills theories over a (commutative) twisted torus.

Let us close this section with the following observation. It is well-known that there exists a T-duality among a square torus with nonconstant background B field, i.e., with constant H flux, and a twisted torus with vanishing B field. This T-duality was briefly reviewed in Sec. II B. Schematically, this means that

$$\text{IIA on } T^3_H \overset{T}{\longleftrightarrow} \text{IIB on } \tilde{T}^3, \tag{4.13}$$

where T-duality relates Type IIA and Type IIB string theory as usual. We observe that the lhs of the T-duality appears in Eq. (4.4), while the rhs is directly associated with the solution Eq. (3.37) on a commutative twisted 3-torus (solution 4 in Table I). This allows us to construct the following diagram:

$$\begin{array}{ccc} \text{IIA on } \mathrm{T}^3_H & \overset{T}{\longleftrightarrow} & \text{IIB on } \tilde{\mathrm{T}}^3 \\ \updownarrow & & \updownarrow \\ \text{MM on } \mathrm{T}^3_x & \longleftrightarrow & \text{MM on } \tilde{\mathrm{T}}^3. \end{array}$$

The vertical arrow on the lhs of this diagram is the correspondence Eq. (4.5), while the one on the rhs simply relates two situations without fluxes or deformations. Then the horizontal arrow between the two MM solutions provides a possible realization of T-duality at the level of Matrix theory, which deserves further investigation. A similar diagram holds for the six-dimensional case as well, where the T-duality is performed in two different directions as explained in Sec. II C.

V. CONCLUSIONS

In the present paper, we studied compactifications of Matrix theory on twisted tori and noncommutative versions of them. Our starting point was the construction of twisted tori realized as nilmanifolds based on nilpotent Lie algebras. Certain explicit examples were provided, and their T-duality to square tori endowed with constant NS 3-form flux was discussed. Next, the toroidal compactification of the BFSS matrix model was revisited. Apart from the previously obtained results [4], we described a solution of the compactification conditions which corresponds to a noncommutative deformation of the torus with nonconstant noncommutativity. This solution carries an underlying nonassociative structure on the corresponding phase space. Thenceforth we moved on to study compactifications on twisted tori. Analyzing the case of the twisted 3-torus, we identified a set of solutions to the corresponding conditions for commutative and noncommutative twisted 3-tori. A similar analysis was carried out for a particular six-dimensional twisted torus leading to solutions which were presented in a form allowing direct generalization to any other higher-dimensional nilmanifold.

In addition, we presented arguments relating known backgrounds of M theory/Type IIA string theory in the supergravity approximation to the solutions of the BFSS model corresponding to the compactifications we studied. In particular, along the lines of the Connes-Douglas-Schwarz correspondence [4], noncommutative deformations of tori and twisted tori were associated to turning on fluxes in 11-dimensional supergravity. Moreover, star products associated with the corresponding noncommutative algebras were constructed and relations connecting noncommutative and commutative backgrounds, inducing the Seiberg-Witten map between the corresponding gauge theories, were determined. Finally, we indicated a possible realization of T-duality between twisted and untwisted tori in Matrix theory. However, these issues should be addressed in more detail by analyzing the resulting gauge theories. In this process, the spectra of Bogomol'nyi-Prasad-Sommerfield states should be carefully studied, along the lines of Refs. [4,8,33]. We plan to report on this issue on a future publication.

An interesting future direction along the lines of the present paper would be to identify ways to describe analogs of nongeometric backgrounds in the framework of Matrix theory. Such backgrounds arise by performing a T-duality along the base of twisted tori instead of the fiber [18]. Relations between nongeometry, T-folds [34], and noncommutativity were already reported in

Refs. [27,29,35]. Moreover, in recent work [36], it was argued that a ten-dimensional perspective of nongeometric fluxes may be gained by describing backgrounds in terms of variables yielding the geometry globally well-defined. This path goes through generalized geometry and uses an antisymmetric bivector field as a sign of nongeometry. It would be interesting to investigate whether such a bivector can be traced in a noncommutative deformation of the compactification manifold in Matrix theory.

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APPENDIX A: GEOMETRIC DATA FOR HIGHER-DIMENSIONAL NILMANIFOLDS

In this Appendix, we collect the twist matrices and the identifications for a class of nilmanifolds. These data are useful in order to fully determine the geometry of each case and classify the associated geometric fluxes.

Before proceeding, let us explain the requirements which single out the cases which we shall present. According to the tables of Ref. [21], there exists a certain number of isomorphism classes of nilpotent Lie algebras in each dimension which are not algebraic sums of lower-dimensional ones. We focus our attention on such cases. They include one three-dimensional case, which was treated in the main text, one four-dimensional case, six five-dimensional, and 22 six-dimensional cases. Out of the latter, the $\mathcal{A}^{\alpha}_{6,5}$ was treated in the main text. Here we shall not present all the above cases. Instead, we find it reasonable to impose the restriction that the Lie algebra satisfies the equations of motion of the BFSS and IKKT matrix models, in the former case at least for time-independent backgrounds,

$$[X_b,[X^b,X_a]]=0 \Leftrightarrow f^{ab}_c f^{dc}_b X_d = 0. \tag{A1}$$

Such cases were studied from a different perspective in Ref. [11]. It turns out that the relevant algebras are the $\mathcal{A}_{5,1}$, $\mathcal{A}_{5,4}$, $\mathcal{A}_{6,3}$, $\mathcal{A}_{6,4}$, $\mathcal{A}^{-1}_{6,14}$, plus the already studied cases of $\mathcal{A}_{3,1}$ and $\mathcal{A}^{\alpha}_{6,5}$. Let us now proceed to their geometric data.

$\mathcal{A}_{5,1}$. Let us first note that this case was also studied in Ref. [18]. The commutation relations of the algebra are

$$[X_3,X_5]=X_1, \qquad [X_4,X_5]=X_2. \tag{A2}$$

Then we find the invariant 1-forms

$$\begin{aligned} &e^1 = dx^1 - x^3 dx^5, \quad e^2 = dx^2 - x^4 dx^5,\\ &e^i = dx^i, \quad i=3,4,5, \end{aligned} \tag{A3}$$

which determine the twist matrix $e^a = U(x)^a_b dx^b$,

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & -x^3\\ 0 & 1 & 0 & 0 & -x^4\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A4}$$

The identification conditions for the compactification are

$$\begin{aligned} (x^1,x^2,x^3,x^4,x^5) &\sim (x^1+a,x^2,x^3,x^4,x^5)\\ &\sim (x^1,x^2+a,x^3,x^4,x^5)\\ &\sim (x^1+ax^5,x^2,x^3+a,x^4,x^5)\\ &\sim (x^1,x^2+ax^5,x^3,x^4+a,x^5)\\ &\sim (x^1,x^2,x^3,x^4,x^5+a), \qquad a\in\mathbb{Z}. \end{aligned} \tag{A5}$$

$\mathcal{A}_{5,4}$. The commutation relations in the present case are

$$[X_2,X_4]=X_1, \qquad [X_3,X_5]=X_1. \tag{A6}$$

The corresponding 1-forms are found to be

$$\begin{aligned} &e^1 = dx^1 - x^2dx^4 - x^3dx^5, \qquad e^i = dx^i,\\ &i=2,3,4,5, \end{aligned} \tag{A7}$$

and the twist matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & -x^2 & -x^3\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A8}$$

The identification conditions for the compactification are

$$\begin{aligned} (x^1,x^2,x^3,x^4,x^5) &\sim (x^1+a,x^2,x^3,x^4,x^5)\\ &\sim (x^1+ax^4,x^2+a,x^3,x^4,x^5)\\ &\sim (x^1+ax^5,x^2,x^3+a,x^4,x^5)\\ &\sim (x^1,x^2,x^3,x^4+a,x^5)\\ &\sim (x^1,x^2,x^3,x^4,x^5+a), \quad a\in\mathbb{Z}. \end{aligned} \tag{A9}$$

$\mathcal{A}_{6,3}$. The commutation relations are given as

$$[X_1,X_2]=iX_6, \qquad [X_1,X_3]=iX_4, \qquad [X_2,X_3]=iX_5, \tag{A10}$$

leading to the 1-forms

$$\begin{aligned} &e^1 = dx^1, \qquad e^2 = dx^2, \qquad e^3 = dx^3, \\ &e^4 = dx^4 - x^1 dx^3, \qquad e^5 = dx^5 - x^2 dx^3, \\ &e^6 = dx^6 - x^1 dx^2. \end{aligned} \tag{A11}$$

Thus the twist matrix turns out to be

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -x^1 & 1 & 0 & 0 \\ 0 & 0 & -x^2 & 0 & 1 & 0 \\ 0 & -x^1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A12}$$

and the twisted identifications are

$$\begin{aligned} (x^1,x^2,x^3,x^4,x^5,x^6) &\sim (x^1,x^2,x^3+a,x^4,x^5,x^6) \\ &\sim (x^1,x^2,x^3,x^4+a,x^5,x^6) \\ &\sim (x^1,x^2,x^3,x^4,x^5+a,x^6) \\ &\sim (x^1,x^2,x^3,x^4,x^5,x^6+a) \\ &\sim (x^1+a,x^2,x^3,x^4+ax^3,x^5,x^6+ax^2) \\ &\sim (x^1,x^2+a,x^3,x^4,x^5+ax^3,x^6), \quad a\in\mathbb{Z}. \end{aligned} \tag{A13}$$

$\mathcal{A}_{6,4}$. The commutation relations in the present case are

$$[X_1,X_2]=X_5, \quad [X_1,X_3]=X_6, \quad [X_2,X_4]=X_6. \tag{A14}$$

Then we find

$$\begin{aligned} &e^5 = dx^5 - x^1 dx^2, \qquad e^6 = dx^6 - x^1 dx^3 - x^2 dx^4, \\ &e^i = dx^i, \qquad i = 1,2,3,4, \end{aligned} \tag{A15}$$

and the twist matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -x^1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -x^1 & -x^2 & 0 & 1 \end{pmatrix}, \tag{A16}$$

The identification conditions for the compactification are

$$\begin{aligned} (x^1,x^2,x^3,x^4,x^5,x^6) &\sim (x^1+a,x^2,x^3,x^4,x^5+ax^2,x^6+ax^3) \\ &\sim (x^1,x^2+a,x^3,x^4,x^5,x^6+ax^4) \\ &\sim (x^1,x^2,x^3+a,x^4,x^5,x^6) \\ &\sim (x^1,x^2,x^3,x^4+a,x^5,x^6) \\ &\sim (x^1,x^2,x^3,x^4,x^5+a,x^6) \\ &\sim (x^1,x^2,x^3,x^4,x^5,x^6+a), \quad a\in\mathbb{Z}. \end{aligned} \tag{A17}$$

$\mathcal{A}^{-1}_{6,14}$. The Lie algebra commutation relations are

$$\begin{aligned} &[X_1,X_3] = X_4, \qquad [X_1,X_4] = X_6, \\ &[X_2,X_3] = X_5, \qquad [X_2,X_5] = -X_6. \end{aligned} \tag{A18}$$

The invariant 1-forms are found to be

$$\begin{aligned} &e^i = dx^i, \quad i=1,2,3, \quad e^4 = dx^4 - x^1 dx^3, \\ &e^5 = dx^5 - x^2 dx^3, \\ &e^6 = dx^6 - x^1 dx^4 + ((x^1)^2 + (x^2)^2) dx^3 + (x^2 x^3 - x^5) dx^2, \end{aligned} \tag{A19}$$

while the following additional relations are obtained

$$\begin{aligned} &x^1 dx^1 + x^2 dx^2 = 0 \Leftrightarrow (x^1)^2 + (x^2)^2 = \text{const}, \\ &x^2 dx^3 + x^3 dx^2 = 0 \Leftrightarrow x^2 x^3 = \text{const} \end{aligned} \tag{A20}$$

The twist matrix is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -x^1 & 1 & 0 & 0 \\ 0 & 0 & -x^2 & 0 & 1 & 0 \\ 0 & x^2x^3 - x^5 & (x^1)^2 + (x^2)^2 & -x^1 & 0 & 1 \end{pmatrix}, \tag{A21}$$

and determining the identification conditions for the compactification in the present case turns out to be complicated due to the x^5 dependence of U.

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