Lifshitz scaling and hyperscaling violation in string theory

K. Narayan
Chennai Mathematical Institute, SIPCOT IT Park, Padur PO, Siruseri 603103, India
(Received 13 March 2012; published 18 May 2012)

We explore string and M-theory constructions of holographic theories with Lifshitz scaling exponent z and hyperscaling violation exponent θ, finding a range of z, θ values. Some of these arise as effective metrics from dimensional reduction of certain kinds of null deformations of AdS spacetimes appearing in the near-horizon geometries of extremal D3-, M2-, and M5-brane theories. The AdS_5 solution, in particular, gives rise to $\theta = 1$ in $d = 2$ (boundary) space dimensions. Other solutions arise as the IIA D2- and D4-brane solutions with appropriate null deformations, and we discuss the phase structure of these systems.

DOI: 10.1103/PhysRevD.85.106006 PACS numbers: 11.25.−w, 11.25.Tq

I. INTRODUCTION

Gauge/gravity duality [1] has enabled fascinating explorations of strongly coupled quantum field theories, with various investigations over the last few years exploring nonrelativistic and condensed matter systems [2]. These typically have reduced symmetries compared to anti–de Sitter space theories. An interesting class of theories exhibits Lifshitz scaling symmetry of the form $t \to \lambda^z t$, $x_i \to \lambda x_i$, $r \to \lambda r$, with z the dynamical exponent, and r is the radial coordinate in the gravity duals,

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2}. \qquad (1)$$

These Lifshitz spacetimes arise in effective gravity theories with a negative cosmological constant with Abelian gauge fields [3,4], and in various constructions in string theory [5–13] (see also earlier work [14–18]). A simple subclass [5,6] of such constructions involves the dimensional reduction of null deformations of AdS $\times$ X spacetimes that arise in familiar brane constructions; for instance the $AdS_5 \times X^5$ null deformation is of the form

$$ds^2 = \frac{1}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + g_{++}(dx^+)^2 + d\Omega_S^2, \qquad (2)$$

with $g_{++}(x^+)$ sourced by one or more fields. The long wavelength geometry upon dimensional reduction along the x^+ direction resembles a $z = 2$, $d = 3 + 1$ Lifshitz spacetime, dual to a $2 + 1$-dimensional field theory (see e.g. [19] for some recent progress on the field theory side).

Effective gravity theories with Abelian gauge fields as well as scalar fields [3,4,20–27] are in fact quite rich and have been shown to contain larger classes of solutions exhibiting interesting scaling properties. In particular, there exist (zero temperature) metrics with Lifshitz scaling and hyperscaling violation

$$ds^2 = r^{-2(1-(\theta/d))}(-r^{-2(z-1)}dt^2 + dx_i^2 + dr^2). \qquad (3)$$

These metrics, rewritten as $ds^2 = r^{2\theta/d}(-\frac{dt^2}{r^{2z}} + \frac{dx_i^2+dr^2}{r^2})$, can be seen to be conformal to Lifshitz spacetimes (1). Here d is the "boundary" spatial dimension (i.e. the dimension of the x_i) and θ the hyperscaling violation exponent. These spacetimes exhibit the scaling

$$t \to \lambda^z t, \qquad x_i \to \lambda x_i, \qquad r \to \lambda r, \qquad ds \to \lambda^{\theta/d} ds. \qquad (4)$$

Various interesting discussions in this context, including condensed matter perspectives, appear in [28,29]. Aspects of holography for these metrics have been discussed in [30]. In particular, a basic requirement for obtaining physically sensible dual field theories is the null energy condition $T_{\mu\nu}n^\mu n^\nu \geq 0$, $n_\mu n^\mu = 0$, which gives using the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$,

$$(d-\theta)(d(z-1)-\theta) \geq 0, \qquad (z-1)(d+z-\theta) \geq 0. \qquad (5)$$

It is interesting to look for configurations in string theory which in certain limits give rise to effective spacetime descriptions of the form (3). Indeed the authors of [30] already made the interesting observation that black Dp-brane supergravity solutions that arise naturally in string theory give rise to effective Lorentz invariant ($z = 1$) metrics with nontrivial hyperscaling violation.[1] Condensed matter motivations apart, it is useful to explore the space of possible spacetimes (3) and Lifshitz and hyperscaling violation exponents z, θ that arise from string/brane configurations. With this perspective, we study various classes of null deformations of AdS spaces in string and M theory here and argue that they give rise to effective metrics (3) with a range of nontrivial z, θ exponents. Some of these (Sec. II) comprise the dimensional

[1] Note that this has parallels with discussions in [23]. We have also been informed that the solutions in [22] [similar to (3)] have string constructions in [24], with broken scaling related to dimensional reduction.

reduction of null normalizable deformations of the form of AdS shock waves. In this class, the null normalizable deformation for AdS_5 (arising from the extremal limit of D3-brane stacks) gives rise to a solution with $d = 2$, $z = 3$, $\theta = 1$; this is thus in the family $\theta = d - 1$, which has been argued to correspond to a gravitational dual of a theory containing hidden Fermi surfaces, as discussed in [28,29]. Others arise from the type-IIA string description of null deformations of (extremal) M2- and M5-brane solutions in M theory (Sec. III). These latter supergravity solutions are best regarded as good descriptions in some regime of the full phase structure of these theories along the lines of [31].

Dimensional reduction.—In what follows, we will discuss the dimensional reduction of various higher dimensional spacetimes to obtain appropriate metrics of the form (3), so we state the basic expressions we use. Consider a (higher dimensional) metric $ds^2 = g^D_{\mu\nu}dx^\mu dx^\nu + h(x^\mu)d\sigma^2_{D_I}$ that we want to dimensionally reduce on the "internal" D_I-dimensional σ space to obtain an effective D-dimensional theory; here the warp factor for the internal space depends only on the D-dimensional spacetime coordinates x^μ. This has an effective action of the schematic form $S \sim \int d^Dx\sqrt{g^D}h^{D_I/2}(R + \ldots)$; to go to the effective Einstein frame, we perform a Weyl transformation $g^D_{E,\mu\nu} = e^{2\Omega}g^D_{\mu\nu}$, with $R_E = e^{-2\Omega}(R + \ldots)$. Thus we obtain a D-dimensional spacetime with Einstein metric

$$ds^2 = g^D_{\mu\nu}dx^\mu dx^\nu + h(x^\mu)d\sigma^2_{D_I} \to ds^2_E = h^{D_I/(D-2)}g_{\mu\nu}dx^\mu dx^\nu. \tag{6}$$

II. AdS_5 NULL NORMALIZABLE DEFORMATIONS AND HYPERSCALING VIOLATION

The gravity/fiveform sector of IIB string theory contains as a solution the spacetime

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^2(dx^+)^2 + R^2d\Omega_5^2,$$
$$R^4 \sim g^2_{YM}N\alpha'^2, \tag{7}$$

with no other sources, with Q a parameter of dimension (boundary) energy density, and $d\Omega_5^2$ being the metric on S^5 (or other Einstein space). Equivalently, the 5-dimensional part of the metric is a solution to $R_{MN} = -\frac{4}{R^2}g_{MN}$ arising in the effective 5-dimensional gravity system with negative cosmological constant. This is essentially a deformation of the familiar $AdS_5 \times S^5$ solution arising as the near-horizon geometry of N D3-branes stacks (in the extremal limit), with the boundary metric modification being $\delta g_{++} \sim \frac{1}{r^2}O(r^4)$. From the dual $\mathcal{N} = 4$ super Yang-Mills point of view, this is thus a normalizable deformation [32] and appears to be a nontrivial state of the gauge theory (by comparison, the solutions (2) comprise non-normalizable deformations). These solutions are of the form of shock waves in AdS and have been studied elsewhere e.g. [33,34] (see also [35]). This metric (7) has also appeared in [11], as a certain double-scaled "zero temperature" limit of a black 3-brane solution, and some properties of this solution have been discussed there.

Here we argue that upon dimensional reduction along a compactified x^+ direction, the resulting metric is conformal to $z = 3$ Lifshitz spacetimes in bulk $3 + 1$ dimensions, the conformal factor giving rise to hyperscaling violation. Indeed the 5-dimensional part of the metric (7) can be rewritten as (relabelling $x^- \equiv t$)

$$ds^2 = R^2\left(-\frac{dt^2}{Qr^6} + \frac{dx_i^2 + dr^2}{r^2} + Qr^2\left(dx^+ - \frac{dt}{Qr^4}\right)^2\right). \tag{8}$$

Then along the lines of [5,6], we regard the x^+ direction as compact[2] and dimensionally reduce on it, using (6). This gives the effective (bulk) $3 + 1$-dimensional Einstein metric

$$ds^2_E = (R^2Qr^2)^{1/(4-2)}R^2\left(-\frac{dt^2}{Qr^6} + \frac{dx_i^2 + dr^2}{r^2}\right) = \frac{R^3\sqrt{Q}}{r}\left(-\frac{dt^2}{Qr^4} + dx_i^2 + dr^2\right), \tag{9}$$

electric gauge field $A = -\frac{dt}{Qr^4}$, and scalar $e^\phi \sim r$. (Closely related solutions have also been discussed in appropriate dimensional reductions of certain limits of Schrodinger solutions [36].) We have retained the nontrivial scales R, Q to illustrate their higher dimensional origin. This dimensionally reduced metric has boundary spatial dimension $d = 2$ and is of the form (3) with

$$2\left(1 - \frac{\theta}{2}\right) = 1 \Rightarrow \theta = 1 = d - 1,$$
$$2(z - 1) = 4 \Rightarrow z = 3. \tag{10}$$

The family $\theta = d - 1$ has been argued to correspond to a gravitational dual description of a theory with hidden Fermi surfaces [28,29] (see also [30]). It would thus be interesting to obtain a deeper understanding of the present brane configuration.

We note that the higher dimensional metric (7) exhibits x^+ translations and the scaling symmetry

$$x_i \to \lambda x_i, \qquad r \to \lambda r,$$
$$x^- \to \lambda^3 x^-, \qquad x^+ \to \lambda^{-1}x^+, \tag{11}$$

while that of (9) are (4) with $d = 2$, $z = 3$, $\theta = 1$. Thus we see that the higher dimensional metric exhibits $z = 3$

[2]We note that $g_{++} > 0$ implies that constant x^- surfaces are spacelike while constant x^+ surfaces are null, somewhat similar to (2) discussed in [5,6]; thus x^- is the natural time coordinate.

Lifshitz scaling in the t, x_i, r subspace, while the hyperscaling violation arises from the x^+-dimensional reduction.

From the higher dimensional point of view, the spacetime (7) is asymptotically AdS_5: from the lower dimensional perspective, we have an asymptotically Schrodinger spacetime arising from the x^+ discrete light cone quantization of AdS_5 in light cone coordinates [37]. In this context, it is worth noting that there is in fact a slightly bigger class of solutions in a gravity-dilaton family with a nonzero g_{++} containing both normalizable and (dilaton Φ sourced) non-normalizable pieces,

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + \left[\frac{1}{4}R^2(\Phi')^2 + QR^2r^2\right](dx^+)^2 + R^2 d\Omega_5^2,$$
$$\Phi = \Phi(x^+). \qquad (12)$$

In the lower dimensional viewpoint, these interpolate between an asymptotic $z = 2$ 3 + 1-dimensional Lifshitz spacetime (2) [5] for small r and the $z = 3$, $\theta = 1$ hyperscaling violating metric (9) above for large r. It may be interesting to explore such interpolating solutions further; in this context, the metric (9) would appear as an effective IR metric with some UV completion.

A. Holographic stress tensor, scalar modes

The holographic stress tensor [38–42] for these AdS shock-wave-like spacetimes has been discussed in e.g. [33,34]. To quickly review, consider an asymptotically AdS solution to Einstein gravity with negative cosmological constant, with metric of the form (we set $R = 1$ for convenience here)

$$ds^2 = \frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu = \frac{dr^2}{r^2} + \frac{1}{r^2}(g^{(0)}_{\mu\nu} + r^2 g^{(2)}_{\mu\nu} + r^4 g^{(4)}_{\mu\nu} + \ldots)dx^\mu dx^\nu \quad (r \to 0), \qquad (13)$$

in the Fefferman-Graham expansion about the boundary $r = 0$. Then holographic renormalization methods [41,42] give rise to relations between the metric coefficients $g^{(0)}_{\mu\nu}, g^{(2)}_{\mu\nu}, g^{(4)}_{\mu\nu}, \ldots$, and physical observables such as the holographic stress tensor. In particular, for a flat boundary metric, we have

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu} \Rightarrow g^{(2)}_{\mu\nu} = 0, \qquad \langle T_{\mu\nu}\rangle = \frac{1}{4\pi G_5} g^{(4)}_{\mu\nu}. \qquad (14)$$

For the AdS_5 shock wave spacetime (7), this gives $T_{++} \sim$ const. This can be checked directly also [38] by defining the quasilocal stress tensor as $\tau_{\mu\nu} = \frac{2}{\sqrt{h}}\frac{\delta I}{\delta h^{\mu\nu}}$, where $h_{\mu\nu}$ is the induced boundary metric on the timelike near-boundary surface at $r =$ const, and $I = I_{\text{bulk}} + I_{\text{surf}} + I_{ct} = \frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{-g}(R + 12) - \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{h}\times(K + 3)$ is the total action including the surface term and counterterm engineered to remove near-boundary ($r \to 0$) divergences (with a flat boundary metric), and $K = h^{\mu\nu}K_{\mu\nu}$ is the trace of the extrinsic curvature $K_{\mu\nu}$. Then the quasilocal stress tensor and the gauge theory stress tensor expectation value are

$$\tau_{\mu\nu} = \frac{1}{8\pi G_5}(K_{\mu\nu} - Kh_{\mu\nu} - 3h_{\mu\nu}),$$
$$\langle T_{\mu\nu}\rangle = \lim_{r\to 0}\frac{1}{r^2}\tau_{\mu\nu}, \qquad (15)$$

where the overall $\frac{1}{r^2}$ factor arises from a regulated definition of the (induced) boundary metric. For (7), the only departures from the AdS_5 expressions are in $\{++\}$ components,[3] and we have

$$K_{\mu\nu} = -\frac{1}{r^2}\eta_{\mu\nu} + Qr^2\delta_{\mu,+}\delta_{\nu,+} \Rightarrow T_{++} = \frac{2Q}{8\pi G_5}, \qquad (16)$$

in agreement with the result above. Thus these shock wave spacetimes correspond to a wave on the boundary with nonzero constant energy momentum component T_{++}.

Now we consider a massless scalar field probe propagating in the 5-dimensional part of the spacetime (7): the action $S = \int d^5x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ for modes with no x^+ dependence ($\partial_+\phi = 0$) simplifies to

$$\int d^4x\frac{dx^+}{r^5}(-Qr^6(\partial_-\phi)^2 + r^2(\partial_i\phi)^2 + r^2(\partial_r\phi)^2), \qquad (17)$$

which is seen to map to that for a scalar in the background (9).

B. General AdS_D null normalizable deformations

Along the lines above, we have the (purely gravitational) AdS_D deformation,

$$ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^{D-3}(dx^+)^2, \qquad (18)$$

the x_i being d-dimensional (boundary) spatial coordinates, and $D = d + 3$, with Q a parameter of dimension energy density in $(D - 1)$ dimensions. This is a solution to $R_{MN} = -\frac{D-1}{R^2}g_{MN}$, i.e. to gravity with a negative cosmological constant, and has the interpretation of an AdS_D shock wave along the lines of the previous sections. In particular, this includes the null normalizable deformations of the M2-brane $AdS_4 \times X^7$ and M5-brane $AdS_7 \times X^4$ solutions in M theory, dimensionally reduced on the X^{11-D}

[3] The extrinsic curvature is $K_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$, where n_μ is the outward pointing unit normal to the surface $r =$ const. With $r = 0$ being the boundary here, we have $n = -\frac{dr}{r}$, giving $K_{\mu\nu} = \frac{r}{2}h_{\mu\nu,r}$.

space. Recalling that conformal dimensions satisfy $\Delta(\Delta - D + 1) = m^2R^2$ for AdS_D, we see that these are also normalizable deformations, the boundary metric being deformed as $\delta g_{++} \sim \frac{1}{r^2} O(r^{D-1})$. This metric (18) exhibits the scaling symmetry

$$x_i \to \lambda x_i, \qquad r \to \lambda r, \qquad x^- \to \lambda^{2+d/2} x^-, \\ x^+ \to \lambda^{-d/2} x^+. \tag{19}$$

Relabelling $x^- \equiv t$, the solution (18) can be rewritten as

$$ds^2 = R^2\left(-\frac{dt^2}{Qr^{D+1}} + \frac{dx_i^2 + dr^2}{r^2} + Qr^{D-3}\left(dx^+ - \frac{dt}{Qr^{D-1}}\right)^2\right), \tag{20}$$

and dimensionally reduced on the x^+ dimension using (6) to obtain

$$ds_E^2 = \frac{R^2(R^2Q)^{1/(D-3)}}{r}\left(-\frac{dt^2}{Qr^{D-1}} + dx_i^2 + dr^2\right). \tag{21}$$

The dimensionally reduced metric above has boundary spatial dimension $d = D - 3$ and is of the form (3) with

$$z = \frac{d}{2} + 2, \qquad \theta = \frac{d}{2}. \tag{22}$$

For the special case of $d = 2$, this θ value coincides with $\theta = d - 1$, as we have seen above.

It is worth discussing the general form of the solutions from the lower dimensional point of view (the numerical constants "#" below can be fixed); the D-dimensional action reduces as

$$\int d^D x\sqrt{-g^{(D)}}(R^{(D)} - 2\Lambda) = \int dx^+ d^{D-1}x\sqrt{-g^{(D-1)}}(R^{(D-1)} \\ - \#\Lambda e^{-2\phi/(D-3)} - \#(\partial\phi)^2 \\ - \# e^{2(D-2)\phi/(D-3)} F_{\mu\nu}^2), \tag{23}$$

where the scalar is $g_{DD} = e^{2\phi}$, the (purely electric) gauge field is $A = -\frac{dt}{r^{D-1}}$ and the $(D-1)$-dimensional metric undergoes a Weyl transformation as $g_{\mu\nu}^{(D-1)} = e^{2\phi/(D-3)} g_{\mu\nu}^{(D)}$. It is straightforward to check that the solution (21) is consistent with the equations of motion, with the scalar of the form $e^{2\phi} = r^{D-3}$. These are of the general form of the effective actions studied in [20–27].

III. PHASES OF AdS NULL DEFORMATIONS IN *M* THEORY

A. M2 branes with null deformations and D2 branes

Null deformations of $\mathrm{AdS}_4 \times X^7$ solutions obtained from near-horizon regions of (extremal) M2-brane stacks in M theory were discussed in [5,6] to obtain $z = 2$ Lifshitz spacetimes in bulk $2 + 1$ dimensions.[4] Here we have[5]

$$ds^2 = \frac{r^4}{R^4}(-2dx^+dx^- + dx_i^2) + \frac{1}{2}R^2(\phi')^2(dx^+)^2 \\ + R^2\frac{dr^2}{r^2} + R^2 d\Omega_7^2, \\ G_4 = \frac{6r^5}{R^6} dx^+ \wedge x^- \wedge dx \wedge dr + C d\phi(x^+) \wedge \Omega_3, \\ R^6 \sim N l_p^6, \tag{24}$$

with the scalar $\phi = \phi(x^+)$ (and $\phi' \equiv \frac{d\phi}{dx^+}$), $C \sim R^3$ being a normalization constant, and Ω_3 is a harmonic threeform on some Sasaki-Einstein 7-manifold X^7. With a trivial scalar $\phi = \mathrm{const}$, this is the $\mathrm{AdS}_4 \times X^7$ solution. The conditions $d\Omega_3 = 0$, $d \star \Omega_3 = 0$, $d(\star d\phi) = 0$ ensure that the Bianchi identity and the flux equation $d \star G_4 + \frac{1}{2} G_4 \wedge G_4 = 0$ are satisfied by the fourform flux. In particular, taking $X^7 = X^3 \times X^4$, and $\Omega_3 = \mathrm{vol}(X^3)$, these are automatically satisfied.

Now let us take the 11-dimensional circle to be in the X^4-space, and study the IIA description of this M2-brane AdS_4-null-deformed system after dimensional reduction on the 11th circle. Before we do this, let us recall the standard dimensional reduction of M2 branes to D2 branes (see e.g. [31]),

$$ds_{11}^2 = H^{-2/3} dx_\parallel^2 + H^{1/3}(dr^2 + r^2 d\Omega_7^2) \\ = e^{-2\Phi/3} ds_{10}^2 + e^{4\Phi/3}(dx_{11} + A_\mu dx^\mu)^2, \tag{25}$$

where ds_{10}^2, Φ, A_μ are the IIA string frame metric, dilaton and gauge field. With $H \sim \frac{R^6}{r^6}$, we have the M2 branes localized in the 8-dimensional transverse space. Taking the 11th dimension to be compact and small, we can take $H \sim \frac{N}{r^5}$ to then dimensionally reduce, as discussed in [31], and obtain the 10-dimensional D2-brane solution (r now being the radial coordinate in the seven noncompact transverse dimensions).

In the present case, since the null deformation along the x^+ direction is entirely along the brane world volume directions, we expect that it simply filters through the dimensional reduction on the 11th circle and appears along with $dx_\parallel^2$ in the reduced metric. To elaborate, the extra metric component g_{++} is unaffected by the harmonic function being smeared as $H \to \frac{N}{r^5}$ in the 10-dimensional solution; this extra g_{++} is the only modification induced by the null deformation to the standard dimensional reduction

[4] The 11-dimensional supergravity equations are $R_{MN} = \frac{1}{12} G_{MB_1B_2B_3} G_N^{B_1B_2B_3} - \frac{1}{144} g_{MN} G_{B_1B_2B_3B_4} G^{B_1B_2B_3B_4}$, and the flux equation $d \star G_4 + \frac{1}{2} G_4 \wedge G_4 = 0$, along with the Bianchi identity for G_4; see e.g. [43] for conventions.

[5] In this entire section, we find it convenient to define the radial coordinate r so that $r \to \infty$ is the boundary of the corresponding AdS space.

of the M2 branes to D2 branes, and gives here a D2-brane solution with null deformation. We then have the 10-dimensional IIA metric and dilaton

$$ds_{st}^2 = \frac{r^{5/2}}{R_2^{5/2}}\left(dx_{\|}^2 + \frac{R^6(\phi')^2}{r^4}(dx^+)^2\right) + \frac{R_2^{5/2}dr^2}{r^{5/2}} + \frac{R_2^{5/2}}{r^{1/2}}d\Omega_6^2,$$
$$e^\Phi = g_s\frac{R_2^{5/4}}{r^{5/4}}, \qquad R_2^5 \sim g_{YM}^2 N\alpha'^3, \qquad g_{YM}^2 = \frac{g_s}{\sqrt{\alpha'}},$$
$$R^6 \sim Nl_P^6 \sim g_s R_2^5\sqrt{\alpha'}, \tag{26}$$

with r now the radial coordinate in the seven noncompact transverse dimensions (and we have used the relation $l_P = g_s^{1/3}\sqrt{\alpha'}$ between the 11-dimensional Planck length, the string coupling, and the string length). We recall that the scalar ϕ here arises from the fourform flux; for $\phi = \text{const}$, this is the usual D2-brane supergravity solution [31,44], with $F^{(4)}_{+-ir} \sim \frac{r^4}{R_2^5}$. The solution (26) can be checked independently from the IIA supergravity equations of motion. Note first that the M-theory G_4-flux deformation in (24) has no components along the 11th circle and thus reduces in IIA to simply a deformation of $F_4 = dA_3$. This means that the effective action we need to study is simply of the form $S_{10} \sim \int d^{10}x\sqrt{-g}[e^{-2\Phi}(R + (\nabla\Phi)^2) - |F_4|^2]$, with the modifications arising only in the metric and F_4. Since the F_4 modification is lightlike with nonzero $F_{+i_1i_2i_3}$ alone, the equation of motion for F_4 is automatically satisfied. The equations of motion thus differ from those of the usual D2-branes solution only in $R_{++} \sim e^{2\Phi}(F_{+ABC}F_+^{ABC} - \# g_{++}F_4^2)$, which can be seen to be consistent. The resulting 10-dimensional spacetime is a consistent solution, independent of any compactification on the x^+ direction. The 10-dimensional Einstein metric here is

$$ds_E^2 = e^{-\Phi/2}ds_{st}^2 = \frac{r^{25/8}}{R_2^{25/8}}\left(dx_{\|}^2 + \frac{R^6(\phi')^2}{r^4}(dx^+)^2\right) + R_2^{15/8}\frac{dr^2}{r^{15/8}} + R_2^{15/8}r^{1/8}d\Omega_6^2. \tag{27}$$

Keeping the x^+ direction noncompact, we dimensionally reduce this metric on the S^6 using (6) (with dimensionless conformal factor $h = \frac{r^{1/8}}{R_2^{1/8}}$, so as to obtain an effective metric of the right physical dimension); this gives

$$ds_{E,4d}^2 = \frac{r^{7/2}}{R_2^{7/2}}(-2dx^+dx^- + dx_i^2) + \frac{R^6(\phi')^2}{R_2^{7/2}r^{1/2}}(dx^+)^2 + R_2^{3/2}\frac{dr^2}{r^{3/2}}. \tag{28}$$

Now for $\phi = \text{const}$, we see that this metric is of the form (3) with $z = 1$, $\theta = -\frac{1}{3}$, in agreement with [30]. For $\phi' \neq 0$, let us now consider compactifying the x^+ dimension to obtain, using (6), relabelling $x^- \equiv t$, and redefining $d\rho \sim r^{-5/2}dr$,

$$ds_{E,3d}^2 = c_1\rho^{-2}(-c_2\rho^{-8/3}dt^2 + dx^2 + d\rho^2), \tag{29}$$

with dimensionful constants c_1, c_2. Now $d = 1$ and this is of the form (3) with $z = \frac{7}{3}$, $\theta = 0$. This is simply a Lifshitz spacetime with no hyperscaling violation. We note that this dimensional reduction is not standard Kaluza-Klein reduction, but we expect that the long wavelength geometry (e.g. for zero modes on the x^+ circle) is of the above form, along the lines of [5].

It is worth mentioning that the 10-dimensional solution (26) approaches the standard D2-brane solution for large r, i.e. in the UV. Far in the UV, the supergravity solution breaks down and perturbative $2+1$-dimensional super Yang-Mills theory (with a null deformation) is a good description; it would be interesting to understand this deformation of the gauge theory better. We recall that in the IR, the dual field theory description is expected to be a discrete light cone quantization of an appropriate lightlike deformation of the M2-brane Chern-Simons theory [45].

As a 10-dimensional solution (27), we see that the size of the x^+ dimension (Einstein frame, with coordinate size L_+) and that of the 11th circle compare as $\frac{R_+}{R_{11}} = \frac{\sqrt{g_{++}}L_+}{e^{2\Phi/3}l_P} \sim r^{19/48}\frac{R^3\phi' L_+}{R_2^{115/48}l_P}$. Thus the x^+ circle is large relative to the 11th circle for r sufficiently large; in this intermediate regime, an x^+ compactification in the 10-dimensional D2-brane solution appears sensible.

B. M2 branes with null normalizable deformations

In this case, the G_4 flux is the same as for the usual M2-brane solution while the metric (18) with $d = 3$ can be recast as (after reinstating the X^7)

$$ds^2 = \frac{r^4}{R^4}(-2dx^+dx^- + dx_i^2) + \frac{QR^5}{r^2}(dx^+)^2 + R^2\frac{dr^2}{r^2} + R^2d\Omega_7^2,$$
$$G_4 = \frac{6r^5}{R^6}dx^+ \wedge x^- \wedge dx \wedge dr, \qquad R^6 \sim Nl_P^6. \tag{30}$$

On the IIA dimensional reduction as described previously, this gives the 10-dimensional string frame metric and dilaton for D2 branes with null normalizable deformation [with R_2 etc. defined in (26)]

$$ds_{st}^2 = \frac{r^{5/2}}{R_2^{5/2}}\left(dx_{\|}^2 + \frac{QR^9}{r^6}(dx^+)^2\right) + R_2^{5/2}\frac{dr^2}{r^{5/2}} + \frac{R_2^{5/2}}{r^{1/2}}d\Omega_6^2,$$
$$e^\Phi = g_s\frac{R_2^{5/4}}{r^{5/4}}. \tag{31}$$

This is consistent with the IIA supergravity equations of motion; the only new piece is $R_{++} \sim -e^{2\Phi}g_{++}F_4^2$, which can be seen to be consistent. Dimensionally reducing the

10-dimensional Einstein metric on the S^6 and compactifying the x^+ dimension, using (6), we obtain

$$ds^2 = c_1\rho^{-2/3}(-c_2\rho^{-4}dt^2 + dx^2 + d\rho^2), \qquad (32)$$

with dimensionful constants $c_1 = \frac{QR^9}{R_2^{16/3}}$, $c_2 = \frac{R_2^{10}}{QR^9}$. This effective metric is of the form (3) with $d = 1$, $z = 3$, $\theta = \frac{2}{3}$.

The 10-dimensional gravity solution breaks down in the far UV, where perturbative super Yang-Mills theory is a good description; the null normalizable deformation would appear to be a shock-wave-like state in the gauge theory.

In the 10-dimensional Einstein metric, the size of the x^+ dimension and the 11th circle compare as $\frac{R_+}{R_{11}} = \frac{\sqrt{g_{++}}L_+}{e^{2\Phi/3}l_P} \sim \frac{1}{r^{49/24}}\frac{QR^9}{R_2^{95/24}}\frac{L_+}{l_P}$. It thus appears that for r sufficiently small, there exists a regime of scales where an x^+ compactification in the 10-dimensional solution is sensible.

These solutions thus are of the form of null-deformed D2-brane systems, which flow from the x^+-dimensional reduction of a UV perturbative super Yang Mills regime through a IIA supergravity region to an 11-dimensional $\mathrm{AdS}_4 \times X^7$ null-deformed phase in the IR.

C. M5 branes with null deformation and D4 branes

We have the null deformation for the $\mathrm{AdS}_7 \times X^4$ solution ($i = 1\ldots4$) obtained from the near-horizon region of (extremal) M5-brane stacks in M theory,

$$\begin{aligned} ds^2 &= \frac{r}{R}[-2dx^+dx^- + dx_i^2] + R^2(\phi')^2(dx^+)^2 + R^2\frac{dr^2}{r^2} \\ &\quad + R^2 d\Omega_4^2, \\ G_4 &= C\mathrm{vol}(X^4) + C' d\phi(x^+) \wedge H_3, \qquad R^3 \sim Nl_P^3, \end{aligned} \qquad (33)$$

(C, C' being constants), i.e. H_3 is a harmonic form ($dH_3 = 0, d\star H_3 = 0$), and the 11-dimensional spacetime is of the form $\mathrm{AdS}_7 \times X_4$, with X_4 of the form $X_4 \equiv X_3 \times S^1$. In particular, we can take $H_3 = \mathrm{vol}(X_3)$ as the volume form on X_3. This thus reduces to the effective gravity-scalar system corresponding to an AdS_7-null deformation, with the equation $R_{MN} = -6g_{MN} + \frac{1}{2}\partial_M\phi\partial_N\phi$, M, $N = \mu, r$.

The M5-brane solution without any null deformation arises as $ds_{11}^2 = H^{-1/3}dx_\parallel^2 + H^{2/3}dx_\perp^2$ with $H \sim \frac{R^3}{r^3}$ in the near-horizon region. Using the second equation in (25) and dimensionally reducing the null-deformed solution (33) on the 11th circle with the M5s wrap, we obtain the 10-dimensional dilaton and string frame metric for D4 branes with null deformation

$$\begin{aligned} ds_{st}^2 &= \frac{r^{3/2}}{R_4^{3/2}}\left(dx_\parallel^2 + \frac{R^3(\phi')^2}{r}(dx^+)^2\right) \\ &\quad + R_4^{3/2}\frac{dr^2}{r^{3/2}} + R_4^{3/2}r^{1/2}d\Omega_4^2, \\ e^\Phi &= g_s\frac{r^{3/4}}{R_4^{3/4}}, \qquad R_4^3 \sim g_{YM}^2 N\alpha', \qquad g_{YM}^2 \sim g_s\sqrt{\alpha'}. \end{aligned} \qquad (34)$$

This can be seen independently from the IIA supergravity equations too. We first note that the M-theory G_4-deformation above has no components along the 11th circle. Therefore, as before in the case of D2 branes, this deformation reduces in IIA to purely a modification of $F_4 = dA_3$, with an effective 10-dimensional action $S_{10} \sim \int d^{10}x\sqrt{-g}[e^{-2\Phi}(R + (\nabla\Phi)^2) - |F_4|^2]$, the modifications arising only in the metric and F_4. Since the F_4 modification is lightlike with nonzero $F_{+i_1i_2i_3}$ alone, the equation of motion for F_4 is automatically satisfied. The equations of motion thus differ from those of the usual D4-branes solution only in $R_{++} \sim e^{2\Phi}(F_{+ABC}F_+^{ABC} - \#g_{++}F_4^2)$, which can be seen to be consistent. Dimensionally reducing the 10-dimensional Einstein metric on the S^4, using (6), we obtain a 6-dimensional metric that, for $\phi = \mathrm{const}$, is of the form (3) with $d = 4$, $z = 1$, $\theta = -1$, in agreement with [30]. Now with $\phi' \neq 0$, we compactify the x^+-direction obtaining the effective 5-dimensional metric

$$ds_E^2 = c_1\rho^{-8/3}(-c_2\rho^{-2}dt^2 + dx_i^2 + d\rho^2), \qquad (35)$$

with dimensionful constants c_1, c_2. This is of the form (3) with $d = 3$, $z = 2$, $\theta = -1$. This is again not standard Kaluza-Klein reduction, but we expect the long wavelength geometry to be of the above form, along the lines of [5]. We expect a range of scales for the regime of validity of the x^+ compactification of the 10-dimensional solution, as before.

D. M5 branes with null normalizable deformations

The $\mathrm{AdS}_7 \times X^4$ null normalizable solution (18) can be recast as

$$\begin{aligned} ds^2 &= \frac{r}{R}[-2dx^+dx^- + dx_i^2] + \frac{QR^8}{r^2}(dx^+)^2 \\ &\quad + R^2\frac{dr^2}{r^2} + R^2 d\Omega_4^2, \\ G_4 &= C\mathrm{vol}(X^4), \end{aligned} \qquad (36)$$

with Q of dimension energy density in six dimensions. Then after dimensional reduction to IIA, we obtain the 10-dimensional dilaton and string frame metric for D4 branes with null normalizable deformation

$$ds_{st}^2 = \frac{r^{3/2}}{R_4^{3/2}}\left(dx_\parallel^2 + \frac{QR^9}{r^3}(dx^+)^2\right) + R_4^{3/2}\frac{dr^2}{r^{3/2}} + R_4^{3/2} r^{1/2} d\Omega_4^2,$$
$$e^\Phi = g_s \frac{r^{3/4}}{R_4^{3/4}}. \tag{37}$$

In the IIA supergravity equations, the only new piece is $R_{++} \sim -e^{2\Phi} g_{++} F_4^2$, which can be seen to be consistent. Dimensionally reducing the 10-dimensional Einstein metric on the S^4, using (6), and then compactifying the x^+ direction, we obtain

$$ds_E^2 = c_1 \rho^{-4/3}(-c_2 \rho^{-6} dt^2 + dx_i^2 + d\rho^2), \tag{38}$$

with $c_1 \sim R_4^{1/3}(QR^9)^{1/3}$, $c_2 \sim \frac{R_4^9}{QR^9}$. This is of the form (3) with $d = 3$, $z = 4$, $\theta = \frac{1}{3}$.

The 10-dimensional gravity solution breaks down in the IR where perturbative super Yang-Mills theory with null deformation is expected to be a good description. We expect this to be a shock-wave-like state in the gauge theory. In the UV, the description is in terms of null deformations of M5-brane $AdS_7 \times X^4$ solutions, or equivalently null deformations of the dual (2,0) superconformal M5-brane theory. It would thus appear that the dimensional reduction along the 11th circle and the x^+ direction effectively yields a 3 + 1-dimensional nontrivial field theory. It would be interesting to understand this better.

IV. DISCUSSION

We have studied various string/brane configurations and argued that they give rise to effective metrics of the form (3) with Lifshitz scaling and hyperscaling violation. The AdS_5 null normalizable deformation (9) corresponds to $d = 2$, $z = 3$, $\theta = 1$, lying in the family $\theta = d - 1$, which has been argued [28,29] to be a gravitational dual of a theory with hidden Fermi surfaces. Clearly the constructions here are by no means an exhaustive classification; we expect that there exist various others too. We expect that these deformations being lightlike preserve some supersymmetry since the original brane solutions themselves are half-BPS; it would be useful to clarify this.

It is interesting to note that the various z, θ values appearing in the effective metrics (9) and (21), (29) and (32), (35) and (38), all satisfy the null energy conditions (5). This is perhaps not surprising since we are starting with reasonable matter in string and M theory. It is worth noting that the null normalizable deformations have $\theta > 0$, while the null non-normalizable solutions have $\theta \leq 0$; it would be interesting to understand if there is some general correlation here. It is also worth noting that some of the solutions here, e.g. (29) (and others with $d = 1$), have $d - 1 \leq \theta \leq d$, and thus are expected to have violations of the area law for entanglement entropy. We hope to explore these further.

ACKNOWLEDGMENTS

It is a pleasure to thank K. Balasubramanian and S. Trivedi for discussions. This work is partially supported by a Ramanujan Grant, DST, Government of India.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); see also the review article, O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
[2] Some review articles are D. T. Son, Nucl. Phys. B, Proc. Suppl. **195**, 217 (2009); S. A. Hartnoll, Classical Quantum Gravity **26**, 224002 (2009); C. P. Herzog, J. Phys. A **42**, 343001 (2009); J. McGreevy, Adv. High Energy Phys. **2010**, 1 (2010); G. T. Horowitz, arXiv:1002.1722; S. Sachdev, arXiv:1012.0299; S. S. Gubser, arXiv:1012.5312; S. Sachdev, Annu. Rev. Condens. Matter Phys. **3**, 9 (2012).
[3] S. Kachru, X. Liu, and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).
[4] M. Taylor, arXiv:0812.0530.
[5] K. Balasubramanian and K. Narayan, J. High Energy Phys. 08 (2010) 014.
[6] A. Donos and J. P. Gauntlett, J. High Energy Phys. 12 (2010) 002.
[7] R. Gregory, S. L. Parameswaran, G. Tasinato, and I. Zavala, J. High Energy Phys. 12 (2010) 047.
[8] A. Donos, J. P. Gauntlett, N. Kim, and O. Varela, J. High Energy Phys. 12 (2010) 003.
[9] D. Cassani and A. F. Faedo, J. High Energy Phys. 05 (2011) 013.
[10] N. Halmagyi, M. Petrini, and A. Zaffaroni, J. High Energy Phys. 08 (2011) 041.
[11] H. Singh, J. High Energy Phys. 12 (2010) 061; J. High Energy Phys. 04 (2011) 118.
[12] K. Narayan, Phys. Rev. D **84**, 086001 (2011).
[13] W. Chemissany and J. Hartong, Classical Quantum Gravity **28**, 195011 (2011).
[14] W. Li, T. Nishioka, and T. Takayanagi, J. High Energy Phys. 10 (2009) 015.
[15] J. Blaback, U. H. Danielsson, and T. Van Riet, J. High Energy Phys. 02 (2010) 095.

[16] P. Koroteev and M. Libanov, J. High Energy Phys. 02 (2008) 104.
[17] T. Azeyanagi, W. Li, and T. Takayanagi, J. High Energy Phys. 06 (2009) 084.
[18] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong, J. High Energy Phys. 04 (2010) 120.
[19] K. Balasubramanian and J. McGreevy, arXiv:1111.0634.
[20] K. Goldstein, S. Kachru, S. Prakash, and S. P. Trivedi, J. High Energy Phys. 08 (2010) 078; K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi, and A. Westphal, J. High Energy Phys. 10 (2010) 027.
[21] M. Cadoni, G. D'Appollonio, and P. Pani, J. High Energy Phys. 03 (2010) 100.
[22] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer, J. High Energy Phys. 11 (2010) 151.
[23] E. Perlmutter, J. High Energy Phys. 02 (2011) 013.
[24] B. Gouteraux and E. Kiritsis, J. High Energy Phys. 12 (2011) 036.
[25] G. Bertoldi, B. A. Burrington, and A. W. Peet, Phys. Rev. D **82**, 106013 (2010); G. Bertoldi, B. A. Burrington, A. W. Peet, and I. G. Zadeh, Phys. Rev. D **83**, 126006 (2011).
[26] N. Iizuka, N. Kundu, P. Narayan, and S. P. Trivedi, J. High Energy Phys. 01 (2012) 094.
[27] N. Iizuka, S. Kachru, N. Kundu, P. Narayan, N. Sircar, and S. P. Trivedi, arXiv:1201.4861.
[28] N. Ogawa, T. Takayanagi, and T. Ugajin, J. High Energy Phys. 01 (2012) 125.
[29] L. Huijse, S. Sachdev, and B. Swingle, Phys. Rev. B **85**, 035121 (2012).
[30] X. Dong, S. Harrison, S. Kachru, G. Torroba, and H. Wang, arXiv:1201.1905.
[31] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998).
[32] V. Balasubramanian, P. Kraus, and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999); V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, Phys. Rev. D **59**, 104021 (1999).
[33] R. A. Janik and R. B. Peschanski, Phys. Rev. D **73**, 045013 (2006).
[34] D. Grumiller and P. Romatschke, J. High Energy Phys. 08 (2008) 027.
[35] G. T. Horowitz and N. Itzhaki, J. High Energy Phys. 02 (1999) 010; R. Emparan, Phys. Rev. D **64**, 024025 (2001); G. Arcioni, S. de Haro, and M. O'Loughlin, J. High Energy Phys. 07 (2001) 035.
[36] K. Balasubramanian and J. McGreevy, J. High Energy Phys. 01 (2011) 137.
[37] W. D. Goldberger, J. High Energy Phys. 03 (2009) 069; J. L. F. Barbon and C. A. Fuertes, J. High Energy Phys. 09 (2008) 030.
[38] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
[39] R. C. Myers, Phys. Rev. D **60**, 046002 (1999).
[40] R. Emparan, C. V. Johnson, and R. C. Myers, Phys. Rev. D **60**, 104001 (1999).
[41] S. de Haro, S. N. Solodukhin, and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001).
[42] K. Skenderis, Classical Quantum Gravity **19**, 5849 (2002).
[43] J. Gauntlett, S. Kim, O. Varela, and D. Waldram, J. High Energy Phys. 04 (2009) 102.
[44] G. T. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991).
[45] O. Aharony, O. Bergman, D. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091; M. Benna, I. Klebanov, T. Klose, and M. Smedback, J. High Energy Phys. 09 (2008) 072.