PHYSICAL REVIEW D **85,** 106004 (2012)

Notes on Euclidean Wilson loops and Riemann theta functions

Riei Ishizeki,* Martin Kruczenski,† and Sannah Ziama‡

Department of Physics, Purdue University, 525 Northwestern Avenue, W. Lafayette, Indiana 47907-2036, USA

(Received 22 April 2011; published 15 May 2012)

The AdS/CFT correspondence relates Wilson loops in $\mathcal{N} = 4$ super Yang-Mills theory to minimal area surfaces in AdS_5 space. In this paper we consider the case of Euclidean flat Wilson loops, which are related to minimal area surfaces in Euclidean AdS_3 space. Using known mathematical results for such minimal area surfaces, we describe an infinite parameter family of analytic solutions for closed Wilson loops. The solutions are given in terms of Riemann theta functions, and the validity of the equations of motion is proven based on the trisecant identity. The world sheet has the topology of a disk, and the renormalized area is written as a finite, one-dimensional contour integral over the world-sheet boundary. An example is discussed in detail with plots of the corresponding surfaces. Further, for each Wilson loops we explicitly construct a one parameter family of deformations that preserve the area. The parameter is the so-called spectral parameter.

DOI: 10.1103/PhysRevD.85.106004 PACS numbers: 11.25.Tq, 11.15.−q

I. INTRODUCTION

One of the first results of the AdS/CFT correspondence [1] was the computation of Wilson loops and from there the quark antiquark potential as done by Maldacena, Rey and Yee [2]. Although much work has been devoted to the computation of Wilson loops only few explicit examples are known of minimal area surfaces in AdS_5 space. In the case of closed Euclidean Wilson loops (with constant scalar) the most studied one is the circular Wilson loop [3], which is dual to a half-sphere. The only other one we are aware of is the two intersecting arcs (lens shaped) [4]. For infinite Wilson loops, parallel lines [2] and the cusp are known [5]. In the case of multiple contours as, for example, two concentric circles, interesting results were found using integrability [6]. In the language we use here they correspond to elliptic functions, which appear for genus one $g = 1$. In the case of Minkowski signature anti-de Sitter space and, in particular, lightlike lines more is known starting with the lightlike cusp [7] and culminating with a large recent activity [8,9] in relation to scattering amplitudes following [10,11].

In this paper we point out that in the case of flat Euclidean Wilson loops which are dual to minimal area surfaces in Euclidean AdS_3, much can be done by using known results from the mathematical literature [12]. In fact, an infinite parameter family of solution is known in terms of Riemann theta functions. This type of construction using theta functions is described in detail in [13] and was already used in the case of strings moving in $t \times S^3$ by Dorey and Vicedo [14] and in the case of an Euclidean world sheet inside AdS_3 space by Sakai and Satoh in [15]. Here we consider Euclidean Wilson loops inside Euclidean AdS_3 and rederive the original results by perhaps more pedestrian methods based on the trisecant identity for theta functions. In this way theta functions are thought as special functions whose properties fit well with the equations of motion of the string in AdS_3 space much in the same way as trigonometric functions fit the harmonic oscillator equation. Each theta function and therefore each Wilson loop is associated with an auxiliary Riemann surface of given genus g. A relatively simple formula is derived for the area and an example for genus $g = 3$ is worked out in detail. Perhaps our main contribution is to find closed Wilson loops and to derive a formula for the renormalized area that follows the AdS/CFT prescription. The calculations are done at the classical level, it should be interesting to extend them, for example, to one-loop as can be done in the case of the circular Wilson loop [16].

This paper is organized as follows. We start by writing the equations of motion and use the Pohlmeyer reduction procedure to simplify the equations and arrive at the cosh-Gordon equation plus a set of linear equations. In the following section we review the properties of the theta functions and show how they can be used to solve the equations of motion and compute the regularized area. Finally, we construct a particular example of genus three where we show that there are closed Wilson loops that can be described by this method. We plot the corresponding surfaces and compute the areas. Besides, we also describe the mapping of the Wilson loop into a curve embedded inside the Riemann surface. In the last section we give our conclusions.

II. EQUATIONS OF MOTION

In this section we write the equations of motion and simplify them using the Pohlmeyer reduction [17]. In the context of Minkowski space-time this procedure was used by Jevicki and Jin [18] to find new spiky string [19]

*rishizek@purdue.edu
†markru@purdue.edu
‡sziama@purdue.edu

solutions and by Alday and Maldacena [11] to compute certain lightlike Wilson loops. In the case of Euclidean AdS_3 that we are interested in here, we can use embedding coordinates $X_{\mu=0\ldots3}$ parameterizing a space $R^{3,1}$ and subjected to the constraint

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1, \tag{2.1}$$

with an obvious $SO(3,1) \equiv SL(2,\mathbb{C})$ global invariance. The space has an S^2 boundary at infinity. Other useful coordinates are Poincaré coordinates (X, Y, Z) given by

$$X + iY = \frac{X_1 + iX_2}{X_0 - X_3}, \qquad Z = \frac{1}{X_0 - X_3}. \tag{2.2}$$

The boundary is now an R^2 space and located at $Z = 0$. A string is parameterized by world-sheet coordinates $\sigma_a = (\sigma, \tau)$ or equivalently complex coordinates $z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$. The action in conformal gauge is given by

$$S = \frac{1}{2}\int(\partial X_\mu \bar{\partial} X^\mu + \Lambda(X_\mu X^\mu - 1))d\sigma d\tau \tag{2.3}$$

$$= \frac{1}{2}\int \frac{1}{Z^2}(\partial_a X \partial^a X + \partial_a Y \partial^a Y + \partial_a Z \partial^a Z) d\sigma d\tau, \tag{2.4}$$

where Λ is a Lagrange multiplier and the μ indices are raised and lowered with the $R^{3,1}$ metric. An Euclidean classical string is given by functions $X_\mu(z, \bar{z})$ obeying the equations of motion:

$$\partial\bar{\partial} X_\mu = \Lambda X_\mu, \tag{2.5}$$

where Λ, the Lagrange multiplier is given by

$$\Lambda = -\partial X_\mu \bar{\partial} X^\mu. \tag{2.6}$$

These equations should be supplemented by the Virasoro constraints which read

$$\partial X_\mu \partial X^\mu = 0 = \bar{\partial} X_\mu \bar{\partial} X^\mu. \tag{2.7}$$

Later on we will be interested in finding the solutions in Poincaré coordinates (X, Y, Z) but for the moment it is convenient to study the problem in embedding coordinates X_μ. We can rewrite the equations using the matrix

$$\mathbb{X} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} = X_0 + X_i \sigma^i, \tag{2.8}$$

where σ^i denote the Pauli matrices. Notice also that Poincaré coordinates are simply given by

$$Z = \frac{1}{\mathbb{X}_{22}}, \qquad X + iY = \frac{\mathbb{X}_{21}}{\mathbb{X}_{22}}. \tag{2.9}$$

The matrix $\mathbb{X}$ satisfies

$$\mathbb{X}^\dagger = \mathbb{X}, \qquad \det\mathbb{X} = 1, \qquad \partial\bar{\partial}\mathbb{X} = \Lambda\mathbb{X},$$
$$\det(\partial\mathbb{X}) = 0 = \det(\bar{\partial}\mathbb{X}), \tag{2.10}$$

as follows from the definition of $\mathbb{X}$, the constraint (2.1), the equations of motion (2.5) and the Virasoro constraints (2.7). We can solve the constraint $\mathbb{X}^\dagger = \mathbb{X}$ by writing

$$\mathbb{X} = \mathbb{A}\mathbb{A}^\dagger, \qquad \det\mathbb{A} = 1, \qquad \mathbb{A} \in SL(2,\mathbb{C}). \tag{2.11}$$

The equations of motion have a global $SL(2,\mathbb{C}) \equiv SO(3,1)$ symmetry under which

$$\mathbb{X} \to U\mathbb{X}U^\dagger, \qquad \mathbb{A} \to U\mathbb{A}, \qquad U \in SL(2,\mathbb{C}). \tag{2.12}$$

In the new variable there is an $SU(2)$ gauge symmetry

$$\mathbb{A} \to \mathbb{A}\mathcal{U}, \qquad \mathcal{U}(z,\bar{z}) \in SU(2), \tag{2.13}$$

since this leaves $\mathbb{X}$ invariant. We can define the current

$$J = \mathbb{A}^{-1}\partial\mathbb{A}, \qquad \bar{J} = \mathbb{A}^{-1}\bar{\partial}\mathbb{A}, \tag{2.14}$$

which is invariant under the global symmetry and, under the local symmetry transform as

$$J \to \mathcal{U}^\dagger J \mathcal{U} + \mathcal{U}^\dagger \partial\mathcal{U}, \qquad \bar{J} \to \mathcal{U}^\dagger \bar{J} \mathcal{U} + \mathcal{U}^\dagger \bar{\partial}\mathcal{U}. \tag{2.15}$$

From the definition of J, $\bar{J}$, the property that $\det\mathbb{A} = 1$, the equations of motion and the constraints we find:

$$\bar{\partial}J - \partial\bar{J} + [\bar{J}, J] = 0, \tag{2.16}$$

$$\mathrm{Tr}\,J = \mathrm{Tr}\bar{J} = 0, \tag{2.17}$$

$$\partial(\bar{J} + J^\dagger) + \tfrac{1}{2}[J - \bar{J}^\dagger, \bar{J} + J^\dagger] = 0, \tag{2.18}$$

$$\det(\bar{J} + J^\dagger) = 0, \tag{2.19}$$

and the corresponding equations found by Hermitian conjugations of the ones given. In the third equation we used that for traceless 2×2 matrices we have for example:

$$J\bar{J} = \tfrac{1}{2}[J, \bar{J}] + \tfrac{1}{2}\mathrm{Tr}(J\bar{J}), \tag{2.20}$$

and similarly for the other products. The trace part gives the Lagrange multiplier Λ as

$$\Lambda = \tfrac{1}{2}\mathrm{Tr}((J + \bar{J}^\dagger)(J^\dagger + \bar{J})), \tag{2.21}$$

which does not provide an equation but is useful later to determine the world-sheet metric. From the form of the equations it seems convenient to define

$$\mathcal{A} = \tfrac{1}{2}(\bar{J} + J^\dagger), \qquad \mathcal{B} = \tfrac{1}{2}(J - \bar{J}^\dagger). \tag{2.22}$$

The equations read now

$$\mathrm{Tr}\,\mathcal{A} = \mathrm{Tr}\mathcal{B} = 0, \tag{2.23}$$

$$\det\mathcal{A} = 0, \tag{2.24}$$

$$\partial\mathcal{A} + [\mathcal{B}, \mathcal{A}] = 0, \tag{2.25}$$

$$\bar{\partial}\mathcal{B} + \partial\mathcal{B}^\dagger = [\mathcal{B}^\dagger, \mathcal{B}] + [\mathcal{A}^\dagger, \mathcal{A}]. \tag{2.26}$$

The $SU(2)$ gauge symmetry acts on these currents as

$$\mathcal{A} \to \mathcal{U}^\dagger \mathcal{A} \mathcal{U}, \qquad \mathcal{B} \to \mathcal{U}^\dagger \mathcal{B} \mathcal{U} + \mathcal{U}^\dagger \partial \mathcal{U}, \qquad \mathcal{U}(z, \bar{z}) \in SU(2). \tag{2.27}$$

In a sense, $\mathcal{B}$ plays the role of a gauge field. Since $\mathrm{Tr}\mathcal{A} = 0$ we can write in terms of Pauli matrices σ^j:

$$\mathcal{A} = (\mathcal{A}_j^{(1)} + i\mathcal{A}_j^{(2)})\sigma^j, \tag{2.28}$$

with $\mathcal{A}^{1,2}$ two real three-dimensional vectors. The property $\det\mathcal{A} = 0$ implies that they are orthogonal and of the same length. The $SU(2)$ gauge symmetry acts on them as three-dimensional rotation so we can take $\mathcal{A}^{(1)}$ to be along the $\hat{x}$ axis, and $\mathcal{A}^{(2)}$ along the $\hat{y}$ axis. In that way we can choose the gauge such that

$$\mathcal{A} = \tfrac{1}{2} e^{\alpha(z,\bar{z})}(\sigma_1 + i\sigma_2) = e^{\alpha(z,\bar{z})}\sigma_+, \tag{2.29}$$

where $\alpha(z, \bar{z})$ is a real function and $\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Equation $\partial\mathcal{A} + [\mathcal{B}, \mathcal{A}] = 0$ uniquely implies that

$$\mathcal{B} = -\tfrac{1}{2}\partial\alpha\sigma_z + b_2(z, \bar{z})\sigma_+, \tag{2.30}$$

for some function $b_2(z, \bar{z})$. Finally the equation for $\mathcal{B}$ implies that $b_2 = f(z)e^{-\alpha}$ for an arbitrary holomorphic function $f(z)$. It also implies that

$$\partial\bar{\partial}\alpha = e^{2\alpha} + f\bar{f}e^{-2\alpha}. \tag{2.31}$$

So, up to a gauge transformation the most general solution is given by

$$\mathcal{A} = e^{\alpha}\sigma_+, \tag{2.32}$$

$$\mathcal{B} = -\tfrac{1}{2}\partial\alpha\sigma_z + f(z)e^{-\alpha}\sigma_+, \tag{2.33}$$

with α satisfying Eq. (2.31). Finally the function $f(z)$ can be locally eliminated by changing to world-sheet coordinates w such that $dw = \sqrt{f}dz$. If we further redefine $\alpha \to \alpha + \frac{1}{4} \times \ln(f\bar{f})$ and do a gauge transformation with $\mathcal{U} = e^{(i\phi/2)\sigma_z}$ (where $i\phi = \frac{1}{4}f/\bar{f}$) then the result is equivalent to setting locally $f = 1$. If $f(z)$ has zeros, those zeros cannot be eliminated by a coordinate transformation and therefore $f(z)$ cannot be made equal to one globally. In the rest of the paper we restrict ourselves to the case where $f(z)$ is everywhere non vanishing and can therefore, without further loss of generality be assumed that $f(z) = 1$. The case when $f(z)$ has zeros would correspond to new, different solutions and is left for future work.

From the equations of motion for $\mathcal{A}$ and $\mathcal{B}$, namely, Eqs. (2.23), (2.24), (2.25), and (2.26) it can be seen that they are invariant under multiplying $\mathcal{A}$ by a constant of modulus one that we call $\bar{\lambda}(|\lambda| = 1)$. We get

$$\mathcal{A} = \bar{\lambda}e^{\alpha}\sigma_+, \tag{2.34}$$

$$\mathcal{B} = -\tfrac{1}{2}\partial\alpha\sigma_z + e^{-\alpha}\sigma_+, \tag{2.35}$$

$$\partial\bar{\partial}\alpha = 2\cosh(2\alpha), \tag{2.36}$$

where we set $f = 1$ by the reasons indicated before. We emphasize that this can always be done locally but generically not globally. In the generic case $f(z)$ can have zeros which cannot be eliminated by a coordinate redefinition. In this paper we consider the case where $f(z)$ The constant λ can be eliminated by a gauge transformation and a redefinition of α but we keep it for later convenience. It is called the spectral parameter and should not be confused with the coupling constant in the dual gauge theory.

Having computed $\mathcal{A}$ and $\mathcal{B}$ we can use Eq. (2.22) to reconstruct J and $\bar{J}$ obtaining

$$J = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ \lambda e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad \bar{J} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & \bar{\lambda}e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}. \tag{2.37}$$

Finally, we should use Eq. (2.14) to compute $\mathbb{A}$. Summarizing we need first to solve the equation:

$$\partial\bar{\partial}\alpha = 2\cosh 2\alpha, \tag{2.38}$$

then plug α into the definitions for J, $\bar{J}$, namely, Eq. (2.37), and solve for $\mathbb{A}$:

$$\partial\mathbb{A} = \mathbb{A}J, \tag{2.39}$$

$$\bar{\partial}\mathbb{A} = \mathbb{A}\bar{J}. \tag{2.40}$$

Finally, the string solution is determined as $\mathbb{X} = \mathbb{A}\mathbb{A}^\dagger$. The equation for α is nonlinear but the ones for $\mathbb{A}$ are linear since J, $\bar{J}$ are known once α is known. This is the main idea of the Pohlmeyer reduction [17] which we rederive here as it applies to our particular problem. Similar considerations in the context of string theory are well-known, for example, see [11,15,18,20].

Notice that, since $\mathrm{Tr}J = \mathrm{Tr}\bar{J} = 0$ we automatically find that $\det\mathbb{A}$ is constant independent of z, $\bar{z}$. However we need $\det\mathbb{A} = 1$ so we just need to normalize $\mathbb{A}$ dividing by an appropriate constant. Furthermore, it is convenient to write

$$\mathbb{A} = \begin{pmatrix} \psi_1 & \psi_2 \\ \tilde{\psi}_1 & \tilde{\psi}_2 \end{pmatrix}, \tag{2.41}$$

where the vectors $\psi = (\psi_1, \psi_2)$ and $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)$ are linearly independent and satisfy

$$\partial\psi = \psi J, \qquad \bar{\partial}\psi = \psi\bar{J}, \tag{2.42}$$

and the same for $\tilde{\psi}$. They have to be linearly independent so the determinant $(\psi_1\tilde{\psi}_2 - \psi_2\tilde{\psi}_1)$ is non vanishing (but is constant as discussed before). Even with these conditions there is a certain ambiguity in choosing ψ, $\tilde{\psi}$ but those boil down to $SL(2, \mathbb{C}) \equiv SO(3, 1)$ transformations of $\mathbb{X}$.

III. SOLUTIONS

As shown in [12] solutions to Eqs. (2.38), (2.39), and (2.40), can be found using theta functions. In this section we rederive the results of [12] using a perhaps more pedestrian way. The method we use is to consider the theta functions as special functions whose derivatives are such that they are good candidates to solve Eqs. (2.38), (2.39), and (2.40) in much the same way as trigonometric functions are good candidates to solve the harmonic oscillator. The equations are solved by simple substitution and adjustment of the parameters. To start we need to review some properties of the theta functions. Before going into that, we would like to mention that an alternative way to generate new solutions in the case of integrable systems is the dressing method [21]. Such method would allow the construction of new solutions from the ones obtained in this paper. In fact, for the case of Wilson loops these idea was already exploited in [22].

A. Riemann theta functions and their properties

There is vast literature on Riemann theta functions [23]. In this section we review the minimal knowledge necessary to find solutions to the equations. We follow the notation of [24], which also gives a good introduction to Riemann surfaces. Notice also that in the next section we develop an example in detail, which can be read in parallel with this section. Consider a compact Riemann surface of genus g with fundamental cycles a_i, $b_i (i = 1 \dots g)$ and intersections

$$a_i \circ a_j = 0 = b_i \circ b_j, \qquad a_i \circ b_j = \delta_{ij}, \tag{3.1}$$

which means that a_i only intersects b_i. The Riemann surface is taken to be a hyperelliptic one defined by the function:

$$\mu(\lambda) = \sqrt{\lambda \prod_{j=1}^{2g} (\lambda - \lambda_j)}. \tag{3.2}$$

The square root has cuts with branching points at $0, \infty, \lambda_j$ but is well defined in a double cover of the complex plane. This double cover is the Riemann surface we consider. For all values of $\lambda \neq 0, \infty, \lambda_j$ there are two points on the Riemann surface, one in the upper sheet and one in the lower sheet.

Consider now $\omega_{i=1\dots g}$ to be the unique basis of holomorphic Abelian differentials satisfying $\oint_{a_i} \omega_j = \delta_{ij}$, and define the $g \times g$ period matrix

$$\Pi_{ij} = \oint_{b_i} \omega_j. \tag{3.3}$$

It can be proved that Π is a symmetric matrix, and its imaginary part is positive definite allowing the definition of an associated θ function

$$\theta(\zeta) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i ((1/2) n^t \Pi n + n^t \zeta)}. \tag{3.4}$$

The arguments of the θ function are ζ, which is a vector in $\mathbb{C}^g$ and the period matrix Π (which we consider fixed and therefore do not explicitly write as an argument). The sum is done over all $n \in \mathbb{Z}^g$, that is all order g vectors with integer components. All vectors (e.g. n, ζ) are taken to be column vectors (and therefore their transposes n^t, ζ^t are row vectors). Simple but important properties of the theta function are

$$\theta(-\zeta) = \theta(\zeta), \tag{3.5}$$

and the (quasi)-periodicity:

$$\theta(\zeta + \Delta_2 + \Pi \Delta_1) = e^{-2\pi i [\Delta_1^t \zeta + (1/2) \Delta_1^t \Pi \Delta_1]} \theta(\zeta), \tag{3.6}$$

where $\Delta_1, \Delta_2 \in \mathbb{Z}^g$, namely, are vectors with integer components. To shorten some equations, it is also useful to define the θ function with characteristics:

$$\begin{aligned} \hat{\theta}(\zeta) &= \theta\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}(\zeta) \\ &= \exp\left\{2\pi i \left[\frac{1}{8} \Delta_1^t \Pi \Delta_1 + \frac{1}{2} \Delta_1^t \zeta + \frac{1}{4} \Delta_1^t \Delta_2\right]\right\} \\ &\quad \times \theta\left(\zeta + \frac{1}{2} \Delta_2 + \frac{1}{2} \Pi \Delta_1\right), \end{aligned} \tag{3.7}$$

where again $\Delta_1, \Delta_2 \in \mathbb{Z}^g$. We introduced the notation $\hat{\theta}$ for this function because, in the rest of the paper, Δ_1 and Δ_2 are fixed vectors. In particular from now on we are going to consider that $\Delta_1^t \Delta_2$ is an odd integer; in other words, $\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$ is an odd characteristic. In such case, we have

$$\hat{\theta}(-\zeta) = e^{i\pi \Delta_1^t \Delta_2} \hat{\theta}(\zeta) = -\hat{\theta}(\zeta), \tag{3.8}$$

as can be derived from the definition of $\hat{\theta}$ and we used that, in our case, $\Delta_1^t \Delta_2$ is odd. In particular this implies

$$\hat{\theta}(0) = 0 \Rightarrow \theta(\tfrac{1}{2} \Delta_2 + \tfrac{1}{2} \Pi \Delta_1) = 0, \tag{3.9}$$

namely, the vector $a = \frac{1}{2} \Delta_2 + \frac{1}{2} \Pi \Delta_1$ is a zero of the theta function. The (quasi)-periodicity of the theta function implies that

$$\theta\begin{bmatrix} \Delta_1 + 2\varepsilon_1 \\ \Delta_2 + 2\varepsilon_2 \end{bmatrix}(\zeta) = e^{i\pi \Delta_1^t \varepsilon_2} \theta\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}(\zeta) \tag{3.10}$$

for any $\varepsilon_{1,2} \in \mathbb{Z}^g$. Therefore, it only makes sense to consider $\Delta_{1,2}$ modulus two, namely, its components being zero or one.

The most important property of the theta functions that we need in this paper is Fay's trisecant identity:

$$\begin{aligned} &\theta(\zeta)\theta\left(\zeta + \int_{p_2}^{p_1} \omega + \int_{p_3}^{p_4} \omega\right) \\ &\quad = \gamma_{1234} \theta\left(\zeta + \int_{p_2}^{p_1} \omega\right) \theta\left(\zeta + \int_{p_3}^{p_4} \omega\right) \\ &\qquad + \gamma_{1324} \theta\left(\zeta + \int_{p_3}^{p_1} \omega\right) \theta\left(\zeta + \int_{p_2}^{p_4} \omega\right), \end{aligned} \tag{3.11}$$

with

$$\gamma_{ijkl} = \frac{\theta(a + \int_{p_k}^{p_i} \omega) \theta(a + \int_{p_l}^{p_j} \omega)}{\theta(a + \int_{p_l}^{p_i} \omega) \theta(a + \int_{p_k}^{p_j} \omega)}. \tag{3.12}$$

In these formulas p_j are points on the Riemann surface, and a is a nonsingular zero of the Riemann theta function, i.e. the function is zero but not its gradient. In particular, we are going to use $a=\frac{1}{2}\Delta_2+\frac{1}{2}\Pi\Delta_1$, which is a zero as noted before. Also note that the contour integrals $\int_{p_a}^{p_b}\omega_j$ define a vector, which from now on, following standard convention will be abbreviated as

$$\int_{p_a}^{p_b}\omega_j\rightarrow\int_{p_a}^{p_b}. \tag{3.13}$$

The function γ may be viewed as a generalization of the cross-ratio function on $\mathbb{CP}^1$ to functions on Riemann surfaces. Some immediate properties of these function are

$$\begin{aligned}&\gamma_{1233}=\gamma_{1134}=1, \qquad \gamma_{2134}=\gamma_{1234}^{-1},\\ &\gamma_{1214}=0=\gamma_{1232}.\end{aligned} \tag{3.14}$$

One important use of Fay's trisecant formula is that it provides a direct way of obtaining directional derivatives of theta functions or of ratios of them. Taking the derivative with respect to p_1 and then letting $p_2\rightarrow p_1$ we get

$$\begin{aligned}D_{p_1}\ln\left[\frac{\theta(\zeta)}{\theta(\zeta+\int_{p_3}^{p_4})}\right]=&-D_{p_1}\ln\left[\frac{\theta(a+\int_{p_3}^{p_1})}{\theta(a+\int_{p_4}^{p_1})}\right]\\&-\frac{D_{p_1}\theta(a)\theta(a+\int_{p_4}^{p_3})}{\theta(a+\int_{p_4}^{p_1})\theta(a+\int_{p_1}^{p_3})}\\&\times\frac{\theta(\zeta+\int_{p_3}^{p_1})\theta(\zeta+\int_{p_1}^{p_4})}{\theta(\zeta)\theta(\zeta+\int_{p_3}^{p_4})}.\end{aligned} \tag{3.15}$$

Here D_{p_1} indicates a directional derivative defined as (summation over j implied):

$$D_{p_1}F(\zeta)=\omega_j(p_1)\frac{\partial F(\zeta)}{\partial\zeta_j}, \tag{3.16}$$

and should not be confused with a derivative with respect to p_1 that, if appears, we will denote as ∂_{p_1}. Also, the final expression is simplified using the identities (3.14). We can further derive with respect to p_3 and take $p_4\rightarrow p_3$ obtaining

$$\begin{aligned}D_{p_3p_1}\ln\theta(\zeta)=&D_{p_3p_1}\ln\theta\left(a+\int_{p_3}^{p_1}\right)\\&-\frac{D_{p_1}\theta(a)D_{p_3}\theta(a)}{\theta(a+\int_{p_3}^{p_1})\theta(a+\int_{p_1}^{p_3})}\\&\times\frac{\theta(\zeta+\int_{p_3}^{p_1})\theta(\zeta+\int_{p_1}^{p_3})}{\theta^2(\zeta)}.\end{aligned} \tag{3.17}$$

This summarizes the basic properties we need. Much more is known about these functions as can be found in Refs. [23,24].

B. Solution to cosh-Gordon equation

Equation (3.17) shows that the second derivative of the logarithm of a theta function contains the theta function. So solutions of

$$\partial\bar{\partial}\alpha=2\cosh 2\alpha=e^{2\alpha}+e^{-2\alpha} \tag{3.18}$$

should naturally be sought as logs of theta functions. To eliminate the constant term in Eq. (3.17), we subtract two such derivatives and get

$$\begin{aligned}D_{p_1p_3}\ln\frac{\theta(\zeta)}{\theta(\zeta+\int_{p_1}^{p_3})}=&\frac{D_{p_1}\theta(a)D_{p_3}\theta(a)}{\theta(a+\int_{p_3}^{p_1})\theta(a+\int_{p_1}^{p_3})}\\&\times\left\{\frac{\theta(\zeta+2\int_{p_1}^{p_3})\theta(\zeta)}{\theta^2(\zeta+\int_{p_1}^{p_3})}\right.\\&\left.-\frac{\theta(\zeta+\int_{p_1}^{p_3}-2\int_{p_1}^{p_3})\theta(\zeta+\int_{p_1}^{p_3})}{\theta^2(\zeta)}\right\}.\end{aligned} \tag{3.19}$$

To get back the same theta functions we need to exploit their periodicity and therefore require

$$2\int_{p_1}^{p_3}\omega=\Delta_2+\Pi\Delta_1, \tag{3.20}$$

where Δ_2, Δ_1 are integer vectors and Π is the period matrix in Eq. (3.3). This gives

$$\theta\left(\zeta+2\int_{p_1}^{p_3}\right)=e^{-2\pi i\Delta_1^t\zeta-i\pi\Delta_1^t\Pi\Delta_1}\theta(\zeta). \tag{3.21}$$

We obtain

$$\begin{aligned}&D_{p_1p_3}\ln\frac{\theta(\zeta)}{\theta(\zeta+\int_{p_1}^{p_3})}\\&=\frac{D_{p_1}\theta(a)D_{p_3}\theta(a)}{\theta(a+\int_{p_3}^{p_1})\theta(a+\int_{p_1}^{p_3})}e^{-(i\pi/2)\Delta_1^t\Pi\Delta_1}\\&\times\left\{e^{-2\pi i\Delta_1^t\zeta}e^{-(i\pi/2)\Delta_1^t\Pi\Delta_1}\frac{\theta^2(\zeta)}{\theta^2(\zeta+\int_{p_1}^{p_3})}\right.\\&\left.-e^{i\pi\Delta_1^t\Delta_2}e^{2\pi i\Delta_1^t\zeta}e^{(i\pi/2)\Delta_1^t\Pi\Delta_1}\frac{\theta^2(\zeta+\int_{p_1}^{p_3})}{\theta^2(\zeta)}\right\}.\end{aligned} \tag{3.22}$$

We should now choose p_1, p_3 and the path of integration between them such that $\Delta_1^t\Delta_2$ is odd so that $e^{i\pi\Delta_1^t\Delta_2}=-1$. Then we take

$$e^{2\alpha}=-e^{-2\pi i\Delta_1^t\zeta-(i\pi/2)\Delta_1^t\Pi\Delta_1}\frac{\theta^2(\zeta)}{\theta^2(\zeta+\int_{p_1}^{p_3})}=\frac{\theta^2(\zeta)}{\hat{\theta}^2(\zeta)}, \tag{3.23}$$

and

$$\zeta=2\omega(p_1)\bar{z}+2\omega(p_3)z. \tag{3.24}$$

The last choice results in $\partial_z\zeta=2D_{p_3}\zeta$, $\bar{\partial}\zeta=2D_{p_1}\zeta$. The correct normalization for ζ and α follows from the result

$$\frac{D_{p_1}\theta(a)D_{p_3}\theta(a)}{\theta^2(0)} = -\frac{1}{4}e^{-i(\pi/2)\Delta_1^t\Pi\Delta_1}, \tag{3.25}$$

which is explained in the Appendix. In any case, it should be clear at this point that the overall normalization of α can always be adjusted so that Eq. (3.22) becomes the cosh-Gordon equation, $\partial\bar{\partial}\alpha = 2\cosh 2\alpha$. The final and very important point is that the theta functions are generically complex but α should be real. Again, following [12] we impose a reality condition as follows. Suppose there is a $g\times g$ symmetric matrix T such that

$$\bar{\Pi} = -T\Pi T, \qquad \bar{\zeta} = -T\zeta, \qquad T^2 = 1. \tag{3.26}$$

Then, it is easy to prove, using the definition of the theta function that $\theta(\zeta)$, $\hat{\theta}(\zeta)$ are real, whereas, for example, $e^{i\pi\Delta_1^t\zeta}\theta(\zeta + \int_{p_1}^{p_3}\omega) = e^{i\pi\Delta_1^t\zeta}\theta\Big(\zeta + \frac{1}{2}\Delta_2 + \frac{1}{2}\Pi\Delta_1\Big)$ is purely imaginary. This explains the minus sign in (3.23) and proves that α is real. The matrix T is constructed [12] from an involution of the Riemann surface that shuffles the basis of cycles (a_i, b_i). To prove, for example, that $\theta(\zeta)$ is real, we use

$$\bar{\theta}(\zeta) = \sum_{n\in\mathbb{Z}^g} e^{-2\pi i(n^t\bar{\zeta} + (1/2)n^t\bar{\Pi}n)} \tag{3.27}$$

$$= \sum_{n\in\mathbb{Z}^g} e^{-2\pi i(-n^tT\zeta - (1/2)n^tT\Pi Tn)} \tag{3.28}$$

$$= \theta(\zeta), \tag{3.29}$$

where in the last equation we redefine the summation variable $n \to Tn$ and used $T^t = T$, $T^2 = 1$.

C. Solution to equations for ψ, $\tilde{\psi}$

In the previous section we showed in detail how to use the properties of the theta function to solve the cosh-Gordon equation. Now we are going to do the same for the equations determining ψ but in a more sketchy way. Notice that ψ and $\tilde{\psi}$ are two linearly independent solutions of the same equations:

$$\partial\psi_1 = -\tfrac{1}{2}\partial\alpha\psi_1 + \lambda e^{\alpha}\psi_2, \tag{3.30}$$

$$\partial\psi_2 = e^{-\alpha}\psi_1 + \tfrac{1}{2}\partial\alpha\psi_2, \tag{3.31}$$

$$\bar{\partial}\psi_1 = \tfrac{1}{2}\bar{\partial}\alpha\psi_1 - e^{-\alpha}\psi_2, \tag{3.32}$$

$$\bar{\partial}\psi_2 = \frac{1}{\lambda}e^{\alpha}\psi_1 - \frac{1}{2}\bar{\partial}\alpha\psi_2, \tag{3.33}$$

which are the expanded version of Eq. (2.42). As a first step we can define a function $F = e^{\alpha}\frac{\psi_1}{\psi_2}$, which satisfies

$$\partial\ln F = \frac{1}{\lambda}e^{2\alpha}\frac{1}{F} - e^{-2\alpha}F. \tag{3.34}$$

By using the identities for the first derivatives of the theta functions and the value of $e^{2\alpha}$ already given one can readily see that

$$F = -\frac{2D_{p_3}\theta(a)\theta(a + \int_{p_4}^{p_1})}{\theta(a + \int_{p_4}^{p_3})\theta(a + \int_{p_3}^{p_1})}e^{-2\pi i\Delta_1^t\zeta - (i\pi/2)\Delta_1^t\Pi\Delta_1} \times \frac{\theta(\zeta)\theta(\zeta + \int_{p_3}^{p_4})}{\theta(\zeta + \int_{p_1}^{p_3})\theta(\zeta + \int_{p_1}^{p_4})}, \tag{3.35}$$

where we introduced another special point in the Riemann surface that we call p_4. The points $p_{1,3}$ are going to be taken as branching points, in particular, for definiteness we take $p_1 = 0$ and $p_3 = \infty$. On the other hand, p_4 is not a branching point, and we take it to be on the upper sheet with $p_4 = \lambda$, the spectral parameter. Going back to the equations for $\psi_{1,2}$ and using the same techniques, we find that

$$\psi_1 = Ce^{-(1/2)\alpha}\frac{\theta(\zeta + \int_{p_1}^{p_3} + \int_{p_1}^{p_4})}{\theta(\zeta + \int_{p_1}^{p_3})} \times e^{2zD_{p_3}\ln\theta(\int_{p_4}^{p_1}) + 2\bar{z}D_{p_1}\times\ln\theta(a + \int_{p_4}^{p_1}) + 2\pi i z\Delta_1^t\omega(0)},$$

$$\psi_2 = e^{(1/2)\alpha}\frac{\theta(\zeta + \int_{p_1}^{p_4})}{\theta(\zeta)} \times e^{2zD_{p_3}\ln\theta(\int_{p_4}^{p_1}) + 2\bar{z}D_{p_1}\ln\theta(a + \int_{p_4}^{p_1}) + 2\pi i\bar{z}\Delta_1^t\omega(0)}, \tag{3.36}$$

the constant C is determined to be

$$C = \frac{2D_{p_3}\theta(a)\theta(a - \int_{p_1}^{p_4})}{\theta(\int_{p_1}^{p_4})\theta(0)}e^{(i\pi/2)\Delta_1^t\Pi\Delta_1}. \tag{3.37}$$

Again, we emphasize that the technique is to match the equation with the properties of the theta functions and choose the parameters appropriately. Another, linearly independent solution can be obtained by choosing a different point p_4 that we call $\bar{p}_4$. However, it has to be associated to the same value of λ and therefore it can only be the same point but on the other (lower) sheet of the Riemann surface. Namely, both p_4 and $\bar{p}_4$ project on λ. It should be noticed that, when $p_1 = 0$, namely, one of the branching points, this implies

$$\int_{p_1}^{p_4}\omega_j = -\int_{p_1}^{\bar{p}_4}\omega_j, \tag{3.38}$$

because the first integral is done on the upper sheet and the second one on the lower sheet where the function μ changes sign (and therefore ω_j changes sign). We obtain

$$\mathbb{A} = \frac{1}{(\psi_1\tilde{\psi}_2 - \psi_2\tilde{\psi}_1)^{(1/2)}}\begin{pmatrix}\psi_1 & \psi_2\\ \tilde{\psi}_1 & \tilde{\psi}_2\end{pmatrix}. \tag{3.39}$$

The (constant) normalization factor can be computed using the trisecant identity to give

$$(\psi_1\tilde{\psi}_2 - \psi_2\tilde{\psi}_1) = -2D_{p_3}\theta(a)e^{(i\pi/2)\Delta_1^t\Pi\Delta_1} \times \frac{\theta(a+2\int_{p_1}^{p_4})}{\theta^2(\int_{p_1}^{p_4})}e^{2\pi i\Delta_1^t\int_{p_1}^{p_4}} \tag{3.40}$$

$$= 2D_{p_3}\hat{\theta}(0)\frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\theta^2(\int_{p_1}^{p_4})}. \tag{3.41}$$

To finish this section we rewrite the solution using the function $\hat{\theta}$ to obtain

$$\psi_1 = 2\frac{D_{p_3}\hat{\theta}(0)}{\theta(0)}\frac{\hat{\theta}(\int_{p_1}^{p_4})}{\theta(\int_{p_1}^{p_4})}\frac{\hat{\theta}(\zeta+\int_{p_1}^{p_4})}{\hat{\theta}(\zeta)}e^{-(1/2)\alpha}e^{\mu z+\nu\bar{z}} \tag{3.42}$$

$$\psi_2 = \frac{\theta(\zeta+\int_{p_1}^{p_4})}{\theta(\zeta)}e^{(1/2)\alpha}e^{\mu z+\nu\bar{z}}, \tag{3.43}$$

with

$$\mu = -2D_{p_3}\ln\theta\left(\int_{p_1}^{p_4}\right), \qquad \nu = -2D_{p_1}\ln\hat{\theta}\left(\int_{p_1}^{p_4}\right). \tag{3.44}$$

It is straightforward to check directly that these functions satisfy Eqs. (3.30), (3.31), (3.32), and (3.33). The only identities that are needed are

$$-4D_{p_1}\theta(a)D_{p_3}\theta(a)e^{i(\pi/2)\Delta_1^t\Pi\Delta_1} = 4D_{p_1}\hat{\theta}(0)D_{p_3}\hat{\theta}(0) = \theta(0)^2, \tag{3.45}$$

and

$$\lambda = -4e^{-i\pi\Delta_1^t\Pi\Delta_1+2\pi i\Delta_1^t\int_{p_1}^{p_4}}\left[\frac{D_{p_3}\theta(a)\theta(\int_{p_3}^{p_4})}{\theta(a+\int_{p_4}^{p_3})\theta(0)}\right]^2 = -4\left[\frac{D_{p_3}\hat{\theta}(0)\hat{\theta}(\int_{p_1}^{p_4})}{\theta(\int_{p_1}^{p_4})\theta(0)}\right]^2, \tag{3.46}$$

which are explained in the Appendix. The last identity allows us to define (that is to appropriately choose the sign of the square root)

$$\sqrt{-\lambda} \equiv 2\frac{D_{p_3}\hat{\theta}(0)\hat{\theta}(\int_{p_1}^{p_4})}{\theta(\int_{p_1}^{p_4})\theta(0)}. \tag{3.47}$$

Then the final form for $\psi_{1,2}$ is simply

$$\psi_1 = \sqrt{-\lambda}\frac{\hat{\theta}(\zeta+\int_{p_1}^{p_4})}{\hat{\theta}(\zeta)}e^{-(1/2)\alpha}e^{\mu z+\nu\bar{z}} \tag{3.48}$$

$$\psi_2 = \frac{\theta(\zeta+\int_{p_1}^{p_4})}{\theta(\zeta)}e^{(1/2)\alpha}e^{\mu z+\nu\bar{z}}, \tag{3.49}$$

where the sign of the square root is chosen according to the previous equation and μ, ν were defined in Eq. (3.44). We can also compute (remembering that $\int_{p_1}^{p_4} = -\int_{p_1}^{\bar{p}_4}$ because $p_1 = 0$.)

$$\tilde{\psi}_1 = -\sqrt{-\lambda}\frac{\hat{\theta}(\zeta-\int_{p_1}^{p_4})}{\hat{\theta}(\zeta)}e^{-(1/2)\alpha}e^{-\mu z-\nu\bar{z}}, \tag{3.50}$$

$$\tilde{\psi}_2 = \frac{\theta(\zeta-\int_{p_1}^{p_4})}{\theta(\zeta)}e^{(1/2)\alpha}e^{-\mu z-\nu\bar{z}}. \tag{3.51}$$

At this point we can replace $\psi_{1,2}$ and $\tilde{\psi}_{1,2}$ in $\mathbb{A}$ and then in $\mathbb{X}$. This allows us to compute the solution directly in Poincaré coordinates as

$$Z = \left|\frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})}\right|\frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)||e^{\mu z+\nu\bar{z}}|^2}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2}, \tag{3.52}$$

$$X+iY = e^{2\bar{\mu}\bar{z}+2\bar{\nu}z} \times \frac{\theta(\zeta-\int_{p_1}^{p_4})\overline{\theta(\zeta+\int_{p_1}^{p_4})} - \hat{\theta}(\zeta-\int_{p_1}^{p_4})\overline{\hat{\theta}(\zeta+\int_{p_1}^{p_4})}}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2}. \tag{3.53}$$

D. Shape of the Wilson loop

The shape of the Wilson loop is determined by the intersection of the surface with the boundary. The boundary is located at $Z=0$ which, from Eq. (3.52) and for finite z, $\bar{z}$ only happens if

$$Z = 0 \Leftrightarrow \theta(\zeta) = 0 \quad \text{or} \quad \hat{\theta}(\zeta) = 0. \tag{3.54}$$

Later on we are going to deal just with the second case so we determine the shape of the Wilson loop by

$$\hat{\theta}(\zeta) = 0. \tag{3.55}$$

This equation defines a curve in the world sheet, which in turn is mapped to a curve in the (X, Y) plane using the solution to the equations of motion (3.53).

E. Computation of the area

The expectation value of the Wilson loop is determined by the area of the minimal surface we described. In conformal gauge the area is computed as

$$A = 2\int \partial X_\mu\bar{\partial}X^\mu d\sigma d\tau = 2\int \Lambda d\sigma d\tau = 4\int e^{2\alpha}d\sigma d\tau, \tag{3.56}$$

where we used Eqs. (2.6) to write the area in terms of the Lagrange multiplier Λ and then used that

$$\Lambda = 2e^{2\alpha}, \tag{3.57}$$

as follows from Eqs. (2.21) and (2.37) using that $|\lambda| = 1$. We could in principle replace α by its expression in Eq. (3.23) and evaluate the integral numerically but such procedure fails because the area is divergent. The correct procedure is to identify the divergent piece analytically and then extract the finite piece in terms of a finite integral that can be easily evaluated numerically. In order to do so we use Eqs. (3.23) and (3.17) (using $a = \int_{p_1}^{p_3}$) to get

$$e^{2\alpha} = 4\{D_{p_1p_3}\ln\theta(0) - D_{p_1p_3}\ln\hat{\theta}(\zeta)\} \tag{3.58}$$

$$= 4D_{p_1p_3}\ln\theta(0) - \partial\bar{\partial}\ln\hat{\theta}(\zeta). \tag{3.59}$$

The first term in the expression for $e^{2\alpha}$ is a constant and the second one is a total derivative. The first integral is clearly finite, but the second one contains the divergent piece that we need to regulate. In order to do that we observe that for these solutions

$$Z = |\hat{\theta}(\zeta)|h(z, \bar{z}), \tag{3.60}$$

where Z is one of the Poincaré coordinates and

$$h(z,\bar{z}) = \left|\frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})}\right| \frac{|\theta(0)\theta(\zeta)||e^{\mu z+\nu\bar{z}}|^2}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2 + |\theta(\zeta-\int_{p_1}^{p_4})|^2}. \tag{3.61}$$

Furthermore, using Stokes or Gauss theorem we find that for any well-behaved function F:

$$\int d\sigma d\tau \partial\bar{\partial}F = \frac{1}{4}\int d\sigma d\tau \nabla^2 F = \frac{1}{4}\oint \hat{n}\cdot\nabla F d\ell, \tag{3.62}$$

where the contour integral is over the boundary of the Wilson loop (in the world sheet), $\hat{n}$ is an outgoing normal vector and $d\ell$ is the differential of arc length. The area is then

$$A = 16D_{p_1p_3}\ln\theta(0)\int d\sigma d\tau + \oint \hat{n}\cdot\nabla \ln h d\ell - \oint \hat{n}\cdot\nabla\ln Z d\ell. \tag{3.63}$$

The last integral is divergent and we concentrate now on extracting the leading divergence. The correct AdS/CFT prescription is to cut the surface at $Z = \epsilon$ and write the area as (see Fig. 1)

$$A = \frac{L}{\epsilon} + A_f, \tag{3.64}$$

where L should be the length of the Wilson loop and A_f is the finite part which is identified with the expectation value of the Wilson loop through:

$$\langle W\rangle = e^{-(\sqrt{\lambda}/2\pi)A_f}, \tag{3.65}$$

where here λ is the 't Hooft coupling of the gauge theory (not to be confused with the spectral parameter). This prescription is equivalent to subtracting the area $A = \frac{L}{\epsilon}$ of a string ending on the contour of length L and stretching along Z from the boundary to the horizon. To see that the coefficient of the divergence is indeed the length, let us compute

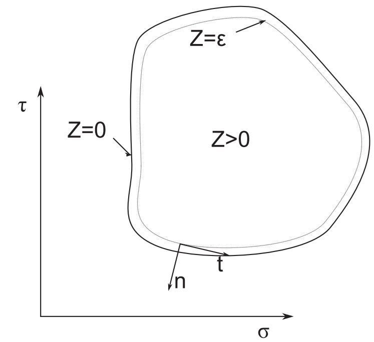

FIG. 1. The boundary is determined by the contour $Z = 0$. However the area is computed by integrating up to a contour $Z = \epsilon \to 0$ and then the leading divergence $\frac{L}{\epsilon}$ is subtracted. Here L is the length of the contour in the boundary (not in this (σ, τ) plane).

$$A_{\text{div.}} = -\oint_{Z=\epsilon}\frac{1}{Z}\hat{n}\cdot\nabla Z d\ell = \frac{1}{\epsilon}\oint_{Z=\epsilon}|\nabla Z| d\ell, \tag{3.66}$$

where we used that the normal is precisely in the opposite direction of ∇Z because the contour is a curve of constant $Z = \epsilon$ and Z increases toward the inside. On the other hand the length in the boundary is given by

$$L = \oint\sqrt{|\hat{t}.\nabla X|^2 + |\hat{t}.\nabla Y|^2}d\ell, \tag{3.67}$$

where $\hat{t}$ is a unit vector tangent to the contour. We can move forward if we write the equation of motion for X as derived form the action (2.4):

$$2\nabla X\cdot\nabla Z = Z\nabla^2 X, \tag{3.68}$$

which, when $Z\to 0$, becomes $\nabla X\cdot\nabla Z = 0$ namely ∇X is perpendicular to the normal and therefore parallel to the tangent $\hat{t}$. The same is true for Y so we find

$$L = \oint\sqrt{|\nabla X|^2 + |\nabla Y|^2}d\ell + \mathcal{O}(\epsilon^2). \tag{3.69}$$

Finally the equation of motion for Z is

$$(\nabla Z)^2 - Z\nabla^2 Z = (\nabla X)^2 + (\nabla Y)^2, \tag{3.70}$$

which for $Z\to 0$ implies that $\sqrt{|\nabla X|^2 + |\nabla Y|^2} = |\nabla Z|$. Therefore the length of the Wilson loop is given by

$$L = \oint|\nabla Z|d\ell - \frac{\epsilon}{2}\oint\frac{\nabla^2 Z}{|\nabla Z|}d\ell, \tag{3.71}$$

and the divergent piece of the area is indeed $A_{\text{div.}} = \frac{L}{\epsilon}$. There is a finite part remaining:

$$A = \frac{L}{\epsilon} + A_f, \tag{3.72}$$

$$A_f = 16D_{p_1p_3}\ln\theta(0)\int d\sigma d\tau + \oint \hat{n}\cdot\nabla\ln h d\ell + \frac{1}{2}\oint\frac{\nabla^2 Z}{|\nabla Z|}d\ell.$$

The integrals are performed on the world sheet parameterized by σ, τ. The first integral is proportional to the area of the world sheet. The last two integrals are done over the world-sheet boundary. The final expression can be simplified by rewriting $Z = |\hat{\theta}(\zeta)|h(z,\bar{z})$ and using that $\hat{\theta}(\zeta)$ vanishes on the boundary where the contour integral is performed. It is then easy to check that $h(z,\bar{z})$ cancels and the final formula for the area is

$$A = \frac{L}{\epsilon} + A_f, \tag{3.73}$$

$$A_f = 16D_{p_1p_3}\ln\theta(0)\int d\sigma d\tau - \frac{1}{2}\oint\frac{\nabla^2\hat{\theta}(\zeta)}{|\nabla\hat{\theta}(\zeta)|}d\ell \tag{3.74}$$

$$= 16D_{p_1p_3}\ln\theta(0)\int d\sigma d\tau + 2\oint\frac{D_{p_1p_3}\hat{\theta}(\zeta)}{|D_{p_1}\hat{\theta}(\zeta)|}d\ell \tag{3.75}$$

$$= 8D_{p_1p_3}\ln\theta(0)\oint(\sigma d\tau - \tau d\sigma) + 2\oint\frac{D_{p_1p_3}\hat{\theta}(\zeta)}{|D_{p_1}\hat{\theta}(\zeta)|}d\ell, \tag{3.76}$$

where in the last step we used the well-known formula for the area of the region encircled by a given curve. The renormalized area is then expressed as a finite one-dimensional contour integral over the boundary of the world sheet. Perhaps it would also be useful to clarify that $|\nabla\hat{\theta}(\zeta)|$ denotes the norm of a real 2-vector whereas $|D_{p_1}\hat{\theta}(\zeta)|$ is the modulus of a complex number. The final expression for the area is quite interesting because it does not depend on the spectral parameter λ. Therefore, the shape of both, the Wilson loop and the dual surface depend on the parameter λ but the area A_f does not. In this way we explicitly find a one parameter family of deformations that preserve the area. It is not obvious at first that the area A_f should be independent of the spectral parameter because, although the definition (3.56) does not contain λ, the regularized area does since, as we said, L depends on λ. It so happens that the finite part A_f does not. The situation is similar to scale transformations that modify L but not A_f. It should be noted that a similar property was observed in [25]. In that case, however, the Wilson loops considered were composed of straight lightlike segments joining at cusps as the one considered in [7]. Nevertheless, at least in principle, an Euclidean Wilson loop can be considered as a limiting case of a lightlike one in the case of an infinite number of cusps. It would be interesting to make this more precise and see if indeed the construction of this paper, in terms of theta functions, can be rederived from the perspective advocated in [25].

IV. EXAMPLE WITH $g = 3$

We are going to consider an example to illustrate the shape of the Wilson loops that are obtained in this way. The main purpose is to show that we find closed Wilson loops whose dual surface is known analytically.

Consider the function

$$\mu = i\sqrt{-i(\lambda+1-i)}\sqrt{-i(\lambda+1+i)}\sqrt{-i(\lambda-\frac{1+i}{2})} \times\sqrt{-i(\lambda-\frac{1-i}{2})}\sqrt{2-\lambda}\sqrt{\lambda}\sqrt{\lambda+\frac{1}{2}}, \tag{4.1}$$

where the square root is taken to have a cut in the negative real axis. So defined, the function μ has cuts in the complex plane as illustrated in Fig. 2 but is smooth in a double cover of the plane which defines a hyperelliptic Riemann surface of genus $g = 3$. The cycles a_i, b_i are taken as in the figure. The Riemann surface has an involution $\lambda \to -\frac{1}{\lambda}$ meaning that knowing the cuts for $|\lambda| > 1$ we can reconstruct the cuts inside the unit circle. This involution is important to construct the matrix T that fixes the correct reality conditions (see [12]). A basis for the holomorphic Abelian differentials is given by

$$\nu_k = \frac{\lambda^{k-1}}{\mu}d\lambda, \qquad k = 1\ldots 3. \tag{4.2}$$

If we compute

$$C_{ij} = \oint_{a_i}\nu_j, \qquad \tilde{C}_{ij} = \oint_{b_i}\nu_j, \tag{4.3}$$

then a normalized basis of holomorphic Abelian differentials is

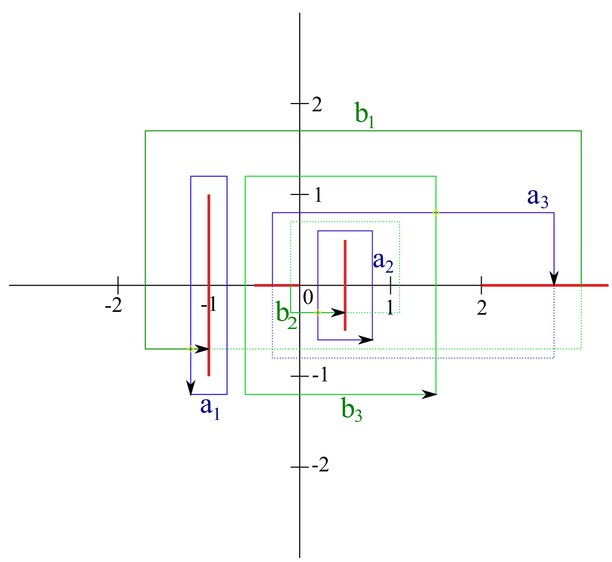

FIG. 2 (color online). Genus three Riemann surface. A basis of fundamental cycles is depicted with solid paths being in the upper sheet and dotted lines in the lower sheet. The circles show their intersection points.

$$\omega_i = \nu_j (C^{-1})_{ji}, \tag{4.4}$$

and the period matrix is

$$\Pi = \tilde{C}C^{-1} = \begin{pmatrix} 0.5 + 0.64972i & 0.14972i & -0.5 \\ 0.14972i & -0.5 + 0.64972i & 0.5 \\ -0.5 & 0.5 & 0.639631i \end{pmatrix}. \tag{4.5}$$

The Jacobi map (with base at 0) is defined as

$$\phi(\lambda)_j = \int_0^\lambda \omega_j = \int_0^\lambda \frac{\lambda^{j-1}}{\mu}(C^{-1})_{ji} d\lambda. \tag{4.6}$$

The function $\theta(\phi(\lambda))$ has three zeros: $\lambda = \infty$, $\frac{1-i}{2}$, $-1+i$. To prove this we can take, on the upper sheet, a path from $\lambda = 0$ to each of the zeros of μ. Coming back along the lower sheet defines a closed path $\mathcal{C}_\lambda$ equivalent to:

$$\begin{aligned}
\lambda = -\frac{1}{2} \to \mathcal{C}_\lambda \equiv b_3 - a_2 \quad & \to \theta\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}(\zeta) \\
\lambda = \frac{1}{2} - \frac{1}{2}i \to \mathcal{C}_\lambda \equiv a_2 + b_2 \quad & \to \theta\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}(\zeta) \\
\lambda = \frac{1}{2} + \frac{1}{2}i \to \mathcal{C}_\lambda \equiv b_2 \quad & \to \theta\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\zeta) \\
\lambda = 2 \to \mathcal{C}_\lambda \equiv a_3 - a_2 \quad & \to \theta\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}(\zeta) \\
\lambda = -1 + i \to \mathcal{C}_\lambda \equiv -a_2 + a_3 + b_1 + b_3 \quad & \to \theta\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}(\zeta) \\
\lambda = -1 - i \to \mathcal{C}_\lambda \equiv -a_1 + a_2 + a_3 + b_1 - b_3 \quad & \to \theta\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}(\zeta) \\
\lambda = \infty \to \mathcal{C}_\lambda \equiv a_1 - a_2 + a_3 + b_3 \quad & \to \theta\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}(\zeta).
\end{aligned} \tag{4.7}$$

The last column gives a theta function with characteristic determined by $\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$ where $\varepsilon_{1,2}$ are given by $\mathcal{C}_\lambda \equiv \sum_{i=1}^3 (\varepsilon_{1i} b_i + \varepsilon_{2i} a_i)$. The point is that this theta function vanishes if and only if $\theta(\phi(\lambda))$ vanishes because $\phi(\lambda) = \frac{1}{2}(\varepsilon_2 + \Pi\varepsilon_1)$ (since the integrals on the upper and lower sheet are equal to half the total integral). If the characteristic is odd (namely $\varepsilon_1^t \varepsilon_2$ is odd) then by symmetry the theta function is zero at the origin and therefore the theta function without characteristic is zero at the corresponding point.

The vector of Riemann constants [24] is computed from the sum of the characteristic of the zeros which is (mod 2)

$$\kappa \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \tag{4.8}$$

and has the property that for any $\lambda_{1,2}$ on the Riemann surface we have

$$\theta\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\phi(\lambda_1) + \phi(\lambda_2)) = 0. \tag{4.9}$$

Moreover, this completely defines the set of zeros of the theta functions we are considering [24].

To write the solution we choose the points $p_1 = 0$ and $p_3 = \infty$ which works well since, as seen above a path between 0 and ∞ defines and odd characteristic that can be taken to be $\Delta_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\Delta_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We therefore define

$$\hat{\theta}(\zeta) = \theta\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}(\zeta). \tag{4.10}$$

We can now write

$$\zeta = 2(\omega(\infty)z + \omega(0)\bar{z}) = \begin{bmatrix} -0.4903\sigma - 0.19069\tau \\ -0.4903\sigma + 0.19069\tau \\ 0.59321\tau \end{bmatrix}, \tag{4.11}$$

where we defined $z = \sigma + i\tau$. The zeros of the function $\hat{\theta}$ in the complex plane z determine the boundary and therefore the shape of the Wilson loops. We plot the contours where $\hat{\theta}$ becomes zero in Fig. 3.

The (quasi)-periodicity of the theta function is evident from the figure but also the existence of closed curves

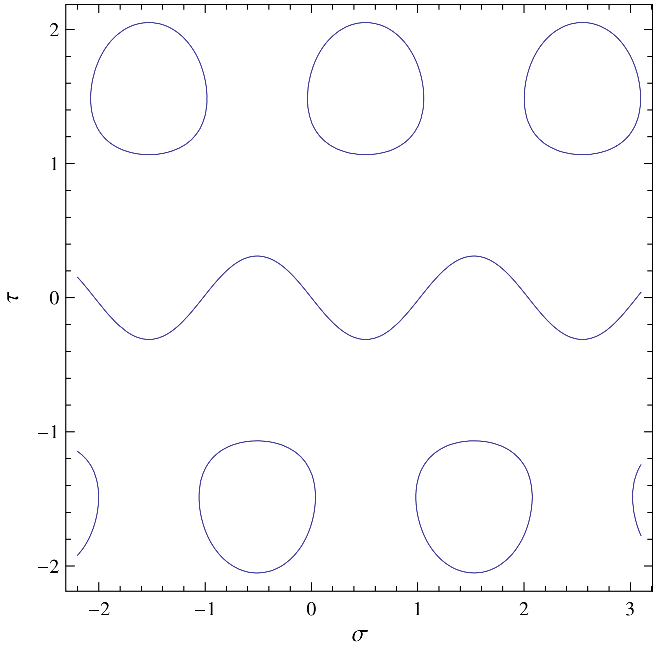

FIG. 3 (color online). Zeros of the function $\hat{\theta}$. We can see the (quasi)-periodicity of the theta function as well as closed contours that map into closed Wilson loops in the boundary.

which in turn will give rise to closed Wilson loops. Choosing the contour displayed in Fig. 4 determines a Wilson loop up to the spectral parameter λ that is still arbitrary. Taking into account that $|\lambda| = 1$ we choose as examples

$$\lambda_1 = i, \qquad \lambda_2 = -\frac{1+i}{\sqrt{2}}. \tag{4.12}$$

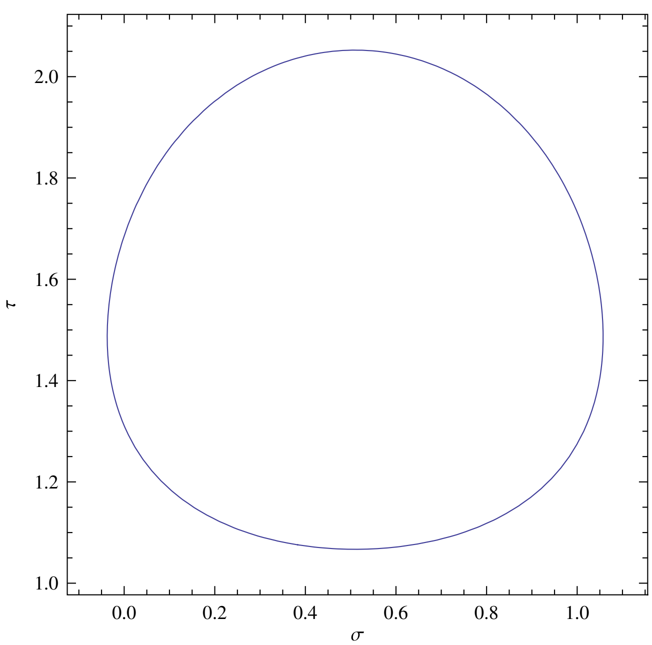

FIG. 4 (color online). Particular contour (taken from Fig. 3) in the z plane chosen to compute the minimal area surface.

Therefore, the point $p_4 = i$ (or $p_4 = -\frac{1+i}{\sqrt{2}}$) in the formulas for the surface (understood to be in the upper sheet). The shape of the Wilson loop can be immediately obtained by mapping the contour into the X, Y plane. The result in displayed in Fig. 5 where we see a nontrivial closed Wilson loop. The minimal surfaces ending on these contours are displayed in Fig. 6. Finally we can compute the length of the Wilson loop and the finite part of the area using Eq. (3.75). The result is

$$L_1 = 13.901, \qquad L_2 = 6.449, \tag{4.13}$$

$$A_f = -6.598 \quad \text{for both.} \tag{4.14}$$

The length $L_{1,2}$ can be changed by a scale transformation but the finite part A_f is scale invariant and independent of λ, the spectral parameter. For comparison, for a circle of radius R the corresponding values are:

$$L^{(\text{circle})} = 2\pi R, \qquad A_f^{(\text{circle})} = -2\pi. \tag{4.15}$$

Although in the end the results are numerical, we emphasize that the shape of the surface is known analytically. Also a relative simple expression was found for the expectation value (after subtracting the infinities) in terms of a one-dimensional finite integral. At the end, the integral was evaluated numerically to get the value for the area. Notice also that the area is smaller than the one for the circle. For a given length L the area is not bounded from below since we can take a contour made out of two parallel lines of length $L/2$ and separated by a distance $\delta \to 0$ in which case the area goes to minus infinity as $A_f \sim -\frac{L}{2\delta}$ [2]. On the other hand, for fixed L, the circle is expected to be an upper bound as shown in [26]. Our result agrees with that bound.

Having described a particular example in detail we want to elaborate further on the properties of these Wilson loops. We know that the zeros of $\hat{\theta}(\zeta)$ determine the boundary of the Wilson loop, but, from Eqs. (4.9) and (4.10), we know that all zeros are given by

$$\zeta = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\Pi\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \phi(\lambda_1) + \phi(\lambda_2), \tag{4.16}$$

for arbitrary points $\lambda_{1,2}$ on the Riemann surface. The vectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ represent the difference (mod 2) between the characteristics of $\hat{\theta}(\zeta) = \theta\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}(\zeta)$ and $\kappa \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. We cannot take random points $\lambda_{1,2}$ on the Riemann surface because we also have

$$\zeta = 2(\omega_\infty z + \omega_0 \bar{z}), \tag{4.17}$$

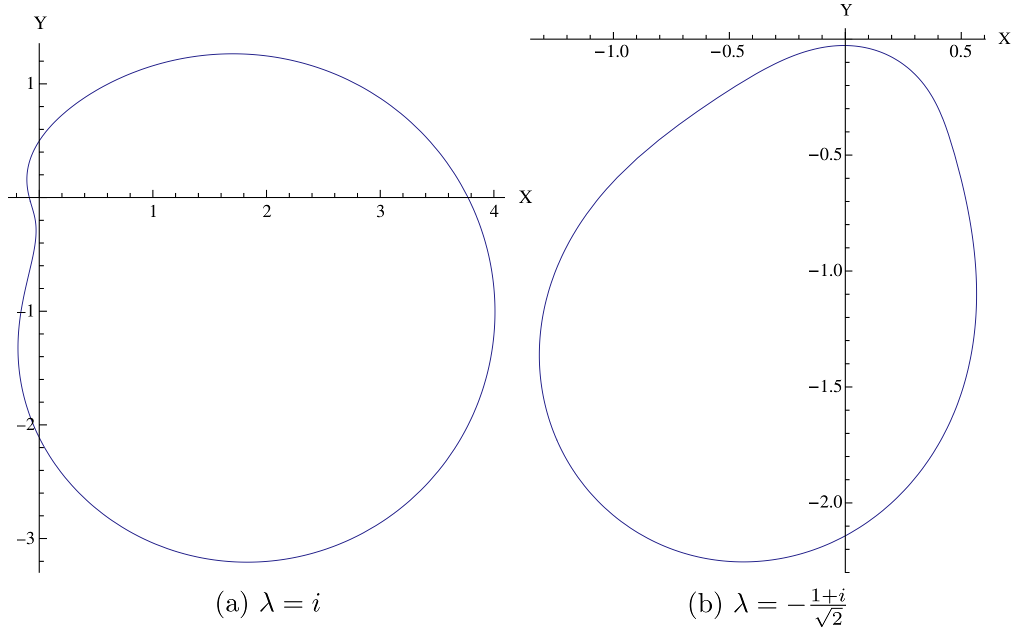

FIG. 5 (color online). Shape of the Wilson loops that we use as an example. It is obtained by mapping the contour in Fig. 4 into the X, Y plane for two values of the spectral parameter λ.

which satisfies

$$\bar{\zeta} = -T\zeta, \qquad T = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.18}$$

With some work it can be seen that we can take

$$\zeta = \frac{1}{2}\begin{bmatrix} 2n_1 \\ 2n_2 \\ 1+2n_3 \end{bmatrix} + \frac{1}{2}\Pi\begin{bmatrix} 1+2m_1 \\ 1+2m_2 \\ 1+2m_3 \end{bmatrix} - \phi(\lambda_1) + \phi\left(-\frac{1}{\bar{\lambda}_1}\right), \tag{4.19}$$

where $n_{1,2,3}$ and $m_{1,2,3}$ are integers which can be absorbed in the definition of the path used to compute the function ϕ. It therefore follows that for genus 3 we can map the Wilson loop into a curve inside the Riemann surface. Namely for each point in the contour displayed in Fig. 4 there is a point λ_1 in the Riemann surface such that

$$2(\omega(\infty)z + \omega(0)\bar{z}) = \frac{1}{2}\begin{bmatrix} 2n_1 \\ 2n_2 \\ 1+2n_3 \end{bmatrix} + \frac{1}{2}\Pi\begin{bmatrix} 1+2m_1 \\ 1+2m_2 \\ 1+2m_3 \end{bmatrix} - \phi(\lambda_1) + \phi\left(-\frac{1}{\bar{\lambda}_1}\right). \tag{4.20}$$

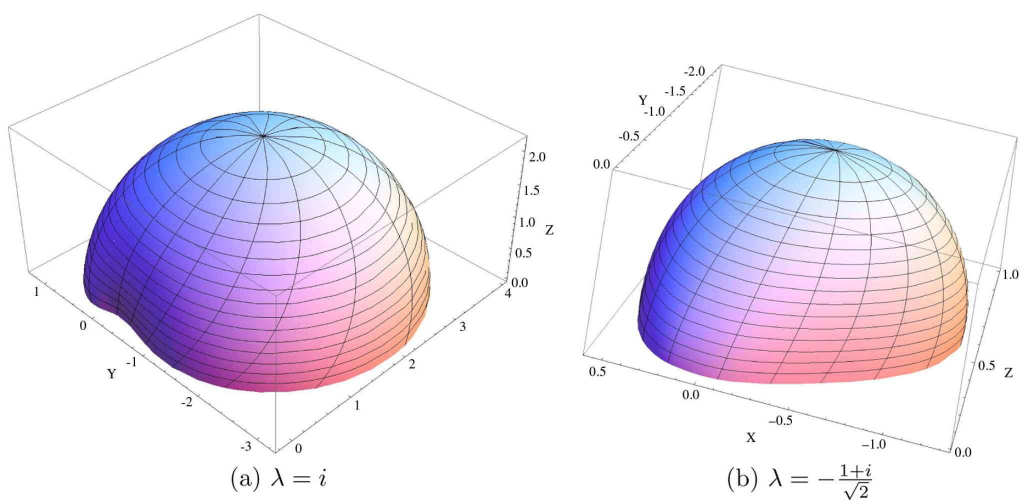

FIG. 6 (color online). Minimal area surfaces ending on the contours illustrated in Fig. 5. We emphasize that the surfaces are known analytically.

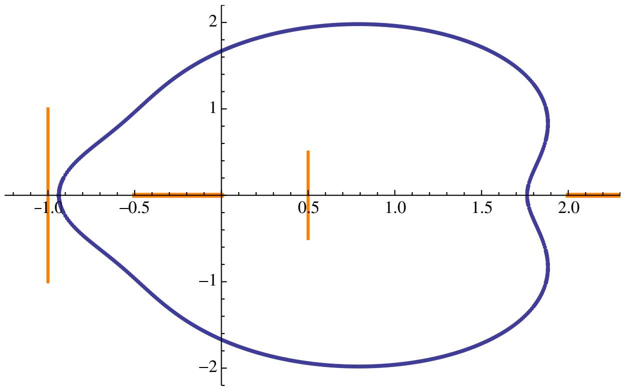

FIG. 7 (color online). For genus three, each Wilson loop computable in terms of theta functions maps into a closed curve inside the corresponding Riemann surface. In this figure we display the closed curve corresponding to the Wilson loop of Fig. 5. We notice it encircles two cuts.

The set of such points describes a curve inside the Riemann surface which is depicted in Fig. 7. The statement is the following: for each point λ_1 in the curve and for a given choice of path used to define the function ϕ there is a set of integers $n_{1,2,3}$ and $m_{1,2,3}$ such that Eq. (4.20) can be solved for z. These values of z lie in the closed curves depicted in Fig. 3 which are the zeros of the function $\hat{\theta}$. Equivalently we can set all integers $n_{1,2,3} = 0$ and $m_{1,2,3} = 0$ but then we have to choose the path used to define ϕ appropriately so that Eq. (4.20) has a solution. Furthermore, with an appropriate choice we can map the curve in Fig. 7 to the curve in Fig. 4 which was the one used in our examples. For higher genus, there should be a set of curves in the Riemann surface for each closed Wilson loop. In this paper we do not explore this issue further but we believe it should be an interesting subject to pursue.

As a final comment, it should be noted that these Wilson loops are not Bogomol'nyi-Prasad-Sommerfeld since, for Euclidean Wilson loops with constant scalar, the only Bogomol'nyi-Prasad-Sommerfeld ones are straight lines [27].

V. CONCLUSIONS

In this paper we discussed minimal area surfaces in AdS_3 space which are dual to Wilson loops in $\mathcal{N} = 4$ SYM. We essentially follow the results of the paper [12] where such solutions were found but provide a different derivation of the solutions and also a formula for the finite part of the area in accordance with the one used in the AdS/CFT correspondence. Finally, we make the observation that closed Wilson loops appear from these solutions in which case the world sheet has the topology of a disk and the area is expressed as a (finite) contour integral over the world-sheet boundary. In this way we construct an infinite parameter family of new examples of Wilson loops whose dual surface is analytically known. It should be noticed that, to our knowledge, only the circular Wilson loop and the lens shaped loop were previously known for closed Euclidean Wilson loop (with constant scalar). Furthermore, the result gives, for each individual Wilson loop a one parameter family of deformation (given by the spectral parameter λ) such that the area remains the same. This is similar to what was observed in the case of the Wilson loops with lightlike cusps [9]. Finally, for genus 3 we pointed out an interesting map between Wilson loops computable in this way and curves inside the Riemann surface. We hope that these new solutions will give rise to a better understanding of Wilson loops in the context of the AdS/CFT correspondence and also in general. It is evident that an important integrable structure lies behind them that should be explored in detail. It would be interesting if these ideas allow us to reconstruct the shape of the Wilson loop from the field theory using some sort of coherent states formalism as in [28].

ACKNOWLEDGMENTS

We are grateful to N. Drukker, G. Michalogiorgakis, A. Tirziu, and P. Ouyang for comments and suggestions. This work was supported in part by the NSF under Grant Nos. PHY-0805948 and PHY-0952630, an AGEP grant, No. 0450373, by the DOE under Grant No. DE-FG02-91ER40681 and by the Alfred P. Sloan Foundation.

APPENDIX

In this appendix we derive a useful identity for the theta functions. Instead of doing a general derivation we show how it works in the example we are dealing with in the main text and then the generalization should be clear. Consider the function

$$h(\lambda) = c\left(\frac{e^{i\pi\Delta_1^t \int_{p_1}^{\lambda}}\theta(a + \int_{p_1}^{\lambda})}{\theta(\int_{p_1}^{\lambda})}\right)^2, \tag{A1}$$

defined on the Riemann surface. Also c is a constant that we choose later. By looking at the results (4.7) one can see that the numerator has zeros at $\lambda = 0$, $\frac{1}{2}\times(1-i)$, $-1+i$ whereas the denominator vanishes for $\lambda = \infty$, $\frac{1}{2}(1-i)$, $-1+i$. It follows that $h(\lambda)$ has a zero at $\lambda = 0$ and a pole at $\lambda = \infty$. To see the behavior near $\lambda = 0$ we use that [remembering that $p_1 = 0$ and Eq. (4.4)]

$$\int_0^{\lambda}\omega_j \simeq \int\frac{d\lambda}{i\sqrt{\lambda}}\lambda C_{1j}^{-1} = -2i\sqrt{\lambda}C_{1j}^{-1}, \qquad \lambda\to 0. \tag{A2}$$

Thus,

$$h(\lambda) \simeq -4c\lambda\left(\frac{D_{p_1}\theta(a)}{\theta(0)}\right)^2. \tag{A3}$$

If we choose

$$c = -\frac{1}{4}\left(\frac{\theta(0)}{D_{p_1}\theta(a)}\right)^2, \tag{A4}$$

then $h(\lambda) \simeq \lambda$, that is, it has a simple zero at $\lambda = 0$. Similarly one can check that it has a simple pole at $\lambda = \infty$. Finally it can be seen to have the right periodicity properties to be well defined on the Riemann surface. The only such function is $h(\lambda) = \lambda$ so we conclude

$$\lambda = -\frac{1}{4}\left(\frac{\theta(0)}{D_{p_1}\theta(a)}\right)^2\left(\frac{e^{i\pi\Delta_1^t \int_{p_1}^{\lambda}}\theta(a + \int_{p_1}^{\lambda})}{\theta(\int_{p_1}^{\lambda})}\right)^2, \tag{A5}$$

as used in the main text. Furthermore, by expanding around $\lambda = \infty$ we find that $h(\lambda) = \lambda$ if we choose the constant c to be

$$c = -4e^{i\pi\Delta_1^t\Pi\Delta_1}\left(\frac{D_{p_3}\theta(a)}{\theta(0)}\right)^2. \tag{A6}$$

Equating the two expressions for c we find

$$\left(\frac{D_{p_3}\theta(a)D_{p_1}\theta(a)}{\theta^2(0)}\right)^2 = \frac{1}{16}e^{-i\pi\Delta_1^t\Pi\Delta_1}, \tag{A7}$$

as we also used in the main text. Although we derived the result for our particular case it is clearly valid in general (see for example [24] for the case of a generic hyperelliptic Riemann surface). Taking the square root of the last equation we find

$$\frac{D_{p_3}\theta(a)D_{p_1}\theta(a)}{\theta^2(0)} = \pm\frac{1}{2}e^{-(i/2)\pi\Delta_1^t\Pi\Delta_1}. \tag{A8}$$

The sign cannot be determined by this reasoning but it is easily found to be minus by a simple numerical computation. The result was used in the main text to find the correct normalization of ζ so that α is a solution to the cosh-Gordon equation.

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[2] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998); S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001).

[3] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, Phys. Rev. D **59**, 105023 (1999); D. J. Gross and H. Ooguri, Phys. Rev. D **58**, 106002 (1998); J. K. Erickson, G. W. Semenoff, and K. Zarembo, Nucl. Phys. **B582**, 155 (2000); N. Drukker and D. J. Gross, J. Math. Phys. (N.Y.) **42**, 2896 (2001); V. Pestun, arXiv:0712.2824.

[4] N. Drukker, S. Giombi, R. Ricci, and D. Trancanelli, J. High Energy Phys. 05 (2008) 017.

[5] N. Drukker, D. J. Gross, and H. Ooguri, Phys. Rev. D **60**, 125006 (1999).

[6] N. Drukker and B. Fiol, J. High Energy Phys. 01 (2006) 056.

[7] M. Kruczenski, J. High Energy Phys. 12 (2002) 024.

[8] See e.g. J. Maldacena and A. Zhiboedov, J. High Energy Phys. 11 (2010) 104; L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena, and E. Sokatchev, J. High Energy Phys. 09 (2011) 123; L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, J. High Energy Phys. 04 (2011) 088.

[9] L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, J. Phys. A **43**, 485401 (2010).

[10] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (2007) 064.

[11] L. F. Alday and J. Maldacena, J. High Energy Phys. 11 (2009) 082.

[12] M. Babich and A. Bobenko, Duke Math. J. **72**, 151 (1993).

[13] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Series in Nonlinear Dynamics (Springer-Verlag, Berlin, Heidelberg, New York, 1994).

[14] N. Dorey and B. Vicedo, J. High Energy Phys. 07 (2006) 014.

[15] K. Sakai and Y. Satoh, J. High Energy Phys. 03 (2010) 077.

[16] M. Kruczenski and A. Tirziu, J. High Energy Phys. 05 (2008) 064; A. Faraggi and L. A. P. Zayas, J. High Energy Phys. 05 (2011) 018.

[17] K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).

[18] A. Jevicki and K. Jin, J. High Energy Phys. 06 (2009) 064.

[19] M. Kruczenski, J. High Energy Phys. 08 (2005) 014.

[20] H. Dorn, N. Drukker, G. Jorjadze, and C. Kalousios, J. High Energy Phys. 04 (2010) 004.

[21] V. E. Zakharov and A. V. Mikhailov, Zh. Eksp. Teor. Fiz. **74**, 1953 (1978) [Sov. Phys. JETP **47**, 1017 (1978)].

[22] M. Spradlin and A. Volovich, J. High Energy Phys. 10 (2006) 012.

[23] D. Mumford, with contributions from C. Musili, M. Nori, E. Previato, and M. Stillman, *Tata Lectures on Theta I (Modern Birkhäuser Classics)*, (Birkhäuser, Boston, 2007); D. Mumford, with contributions from C. Musili, M. Nori, E. Previato, M. Stillman, and H. Umemura, *Tata Lectures on Theta II: Jacobian Theta Functions and Differential Equations (Progress in Mathematics)*, (Birkhäuser, Boston, 2007); John D. Fay, *Lectures Notes in Mathematics* 352 (Springer-Verlag, Berlin Heidelberg, New York, 1973); H. F. Baker, *Abel's Theorem and the Allied Theory, Including the Theory of the Theta Functions* (Cambridge University Press, Cambridge, England, 1897).

[24] H. M. Farkas and I. Kra, *Graduate Texts in Mathematics* (Springer-Verlag, New, Berlin, Heidelberg, 1991), Second ed..

[25] L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, J. Phys. A **43**, 485401 (2010); L. F. Alday and J. Maldacena, J. High Energy Phys. 11 (2007) 068.

[26] S. Alexakis and R. Mazzeo, Commun. Math. Phys. **297**, 621 (2010).

[27] K. Zarembo, Nucl. Phys. **B643**, 157 (2002).

[28] M. Kruczenski, Phys. Rev. Lett. **93**, 161602 (2004).