

Large- N reduction for $\mathcal{N} = 2$ quiver Chern-Simons theories on S^3 and localization in matrix models

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We study reduced matrix models obtained by the dimensional reduction of $\mathcal{N} = 2$ quiver Chern-Simons theories on S^3 to zero dimension and show that if a reduced model is expanded around a particular multiple fuzzy sphere background, it becomes equivalent to the original theory on S^3 in the large- N limit. This is regarded as a novel large- N reduction on a curved space S^3 . We perform the localization method to the reduced model and compute the free energy and the vacuum expectation value of a BPS Wilson loop operator. In the large- N limit, we find an exact agreement between these results and those in the original theory on S^3 .

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I. INTRODUCTION

The localization technique has attracted much attention in recent years as an efficient method of computing a class of physical quantities of our interest. For instance, it enables us to compute exactly BPS Wilson loops or partition functions in four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theories [1] or those in three-dimensional $\mathcal{N} = 2$ Chern-Simons (CS) theories coupled to some matter fields [2–7]. In particular, predictions from their gravity duals, such as the $N^{3/2}$ law of the free energy [8] in the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [9], can be verified explicitly based on the localization method [3] and thus a remarkable progress has been made in testing the gauge/gravity correspondence [10].

In this paper, we use the localization method for another purpose. We apply it to a dimensionally reduced matrix model of a general $\mathcal{N} = 2$ quiver CS theory on S^3 and show that there exists an equivalence in a large- N limit between the reduced model and the original theory on S^3 . This kind of large- N equivalence on S^3 was first discovered in [11] for $\mathcal{N} = 4$ SYM on $R \times S^3$ (see also [12–14] for earlier discussions) and it is regarded as a novel type of the large- N reduction extended to the case of S^3 .

The original large- N reduction initiated by Eguchi and Kawai asserts that a gauge theory on flat space-time in the planar limit is equivalent to a matrix model that is obtained by its dimensional reduction to lower dimensions [15]. This equivalence is significant not only because it realizes the emergence of space-time in matrix models, which is relevant in the context of the matrix model formulation for string theories [16], but also because this equivalence implies that the matrix models provide a nonperturbative formulation of planar gauge theories which is alternative to

the lattice formulation. One may expect that supersymmetric theories can be described in terms of matrix models nonperturbatively based on this equivalence while it is generally difficult in the lattice formulation.¹

It is however well known that this equivalence fails to hold due to the spontaneous $U(1)^D$ symmetry breaking in the original Eguchi-Kawai model [18]. To overcome this difficulty, the quenching and the twisting prescriptions were proposed [18–22]. Although the symmetry breaking can be avoided by introducing such prescriptions, they do not preserve supersymmetry. Because of this, it had been difficult until recently to construct a nonperturbative formulation of supersymmetric gauge theories based on the large- N reduction which keeps supersymmetry manifestly.

The novel large- N reduction was proposed for theories on S^3 [11]. It states that a reduced model, which is obtained by the dimensional reduction of a gauge theory on S^3 to a point, becomes equivalent to the original gauge theory in the large- N limit if the reduced model is expanded around a certain multiple fuzzy sphere background and a continuum limit is taken. In this proposal, the above-mentioned difficulty is avoided thanks to the curvature of S^3 . On S^3 , a gauge theory acquires a mass gap, which is inversely proportional to the radius of S^3 , and hence does not possess a flat direction, which would lead the symmetry breaking and spoil the large- N equivalence. This implies that the prescriptions are not needed in this case, so that a reduced model obtained from a supersymmetric gauge theory still keeps part of the original supersymmetry. One can therefore use the reduced model as a nonperturbative formulation of the supersymmetric theory on S^3 . So far, such formulation has been considered for $\mathcal{N} = 4$ SYM [11], SYM with lower supersymmetry [23], and supersymmetric quiver CS theories [24]. The large- N reduction has also

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been extended to the cases for more general manifolds such as group manifolds [25] and coset spaces [26].

In particular, the large- N reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$ has been studied actively [27–30] since it is relevant to testing the original version of the AdS/CFT correspondence for the type IIB string theory on $\text{AdS}_5 \times S^5$. The reduced model of $\mathcal{N} = 4$ SYM on $R \times S^3$ is given by the plane wave matrix model [31,32]. This model preserves $SU(2|4)$ symmetry, which cannot be realized in the lattice formulation at present, so that it is expected to describe the original theory on S^3 in the continuum limit without any fine-tuning. Based on this formulation, one can analyze numerically the strongly coupled regime of the planar $\mathcal{N} = 4$ SYM, which is mapped to the regime in the string theory where the supergravity or the semiclassical approximation is valid. The methods of the Monte Carlo simulation for matrix models proposed in [33–35] are available for the numerical computation. Thus, it gives a feasible way of testing the AdS/CFT correspondence.²

Although the validity of such nonperturbative formulation of supersymmetric theories has been checked by some perturbative calculations [27–30], it should be checked also for the strong coupling region. In this paper, we consider the large- N reduction for a general $\mathcal{N} = 2$ quiver CS theories on S^3 . Applying the localization method, we compute the partition function and the one-point function of the great circular BPS Wilson loop operator in the reduced model. Then, we prove the large- N equivalence for these quantities to all orders in the perturbation theory. We also find that a saddle point configuration of the reduced matrix model is given as infinitely many copies of that in the original theory up to a cutoff effect, which is negligible in the continuum limit. This fact ensures that the equivalence also holds even in the strongly coupled regime.

So far, a similar test of the large- N reduction has been done for the pure CS theory on S^3 [38,39],³ which is a solvable topological field theory [42]. Since the path integral of the reduced model is easily performed in this case, it is possible to see the agreement of two theories through a direct calculation. Although the theory we consider in this paper contains dynamical degrees of freedom and hence is not so simple, the localization method enables us to verify the large- N equivalence explicitly.

This paper is organized as follows. In Sec. II, we review known results on the computation of the partition function and the BPS Wilson loop operator in a general $\mathcal{N} = 2$ quiver CS theory. In Sec. III, we introduce the reduced model of the theory on S^3 focusing on the supersymmetry transformation. We also perform the path integral of the reduced model by means of the localization. In Sec. IV, we show the large- N equivalence. We first extract a theory

around the multiple fuzzy sphere background which creates S^3 from the result of the localization and see that the theory reduces to a certain eigenvalue integral. By studying this integral, we see the equivalence both in the perturbation theory and in the saddle point equation. Section V is devoted to conclusion. In the Appendices, some details are gathered.

II. LOCALIZATION IN $\mathcal{N} = 2$ QUIVER CS THEORY ON S^3

In this section, we review some known results for the localization in a $\mathcal{N} = 2$ quiver CS theory on S^3 [2–5]. We assume that the gauge group is given by a product of unitary groups, $\otimes_a U(N_a)$, and consider a general matter content. We set the radius of S^3 to be one in this paper and our convention for the theory on S^3 is summarized in Appendix A.

In this theory, there are nilpotent supersymmetries, which we will call Q symbolically in the following. The partition function is invariant under adding Q -exact terms to the action. Hence, the path integral can be localized onto the saddle points of the Q -exact terms. For a gauge multiplet, the role of the Q -exact term is played by the Yang-Mills (YM) action on S^3 and for a matter multiplet by a part of the matter action. The saddle point configuration is given by the flat connection for the gauge field, which is trivial on S^3 , namely, $A_\mu = 0$ up to gauge transformation. Also, all the matter fields are zero at the saddle point. The other bosonic fields in the vector multiplet, σ and D , take nontrivial values at the saddle point. They are given by

$$\sigma = -D = \text{constant}. \quad (2.1)$$

Then, the calculation of the partition function amounts to computing the 1-loop determinant at each saddle point. The result is written as a summation over contributions from all the saddle points, namely, in our case, as an integral over the constant matrix σ for each gauge multiplet. In the following, we will work in the gauge in which σ 's in all the gauge multiplets are diagonalized, so that the integration measure has the Vandermonde determinant as $\int \prod_i d\sigma_i \prod_{i<j} (\sigma_i - \sigma_j)^2$ for each gauge multiplet.

The contribution from a vector multiplet is given by

$$\prod_{i<j} \left(\frac{\sinh(\pi(\sigma_i - \sigma_j))}{\pi(\sigma_i - \sigma_j)} \right)^2, \quad (2.2)$$

up to an overall constant. Note that the denominator cancels the Vandermonde determinant.

Then, let us consider the contribution from a matter multiplet in the bifundamental representation which couples to two different gauge multiplets. The determinant takes the following form,

²See [36,37] for preliminary results of such attempts.

³See also [40,41].

$$\prod_{n>0} \prod_{i,\alpha} \frac{n+1-q+i(\sigma_i-\rho_\alpha)}{n-1-q-i(\sigma_i-\rho_\alpha)}, \quad (2.3)$$

where q is the dimension of the lowest components in the matter multiplet and ρ is the counterpart of σ in the second gauge multiplet. The determinant for an adjoint matter multiplet can be obtained by putting $\rho_i = \sigma_i$ in (2.3). In particular, when $q = \frac{1}{2}$, it is simplified and given by

$$\prod_{i<j} \frac{1}{\cosh(\pi(\sigma_i - \sigma_j))}. \quad (2.4)$$

For example, the ABJM theory with gauge group $U(N_1)_k \times U(N_2)_{-k}$ contains two vector multiplets and four matter chiral multiplets in the bifundamental representation. The partition function is reduced through the above calculation to the so-called ABJM matrix model,

$$\int \prod_i d\sigma_i \prod_\alpha d\rho_\alpha \frac{\prod_{i<j} \sinh^2(\pi(\sigma_i - \sigma_j)) \prod_{\alpha<\beta} \sinh^2(\pi(\rho_\alpha - \rho_\beta))}{\prod_{i,\alpha} \cosh^2(\pi(\sigma_i - \rho_\alpha))} e^{-(2\pi^2/g_s) \sum_i \sigma_i^2 + (2\pi^2/g_s) \sum_\alpha \rho_\alpha^2}, \quad (2.5)$$

where the Gaussian factors are obtained by substituting the saddle point configuration to the original CS actions and the coupling constant is related to the CS level as $g_s = 2\pi i/k$.

A correlation function of Q -closed operators can also be reduced to an eigenvalue integral by the localization method. We consider the one-point function of the BPS Wilson loop,

$$W(\mathcal{C}) = \frac{1}{N} \text{tr} P \exp \left(i \int_0^1 ds (\dot{x}^\mu(s) A_\mu(x) - i |\dot{x}(s)| \sigma(x)) \right), \quad (2.6)$$

where tr stands for the trace in the fundamental representation and N is the rank of the gauge group for A_μ . We consider a great circle on S^3 as the contour \mathcal{C} . It is parametrized as (see Appendix B for our notation for S^3)

$$\{x^\mu(s)\} = (\theta(s), \varphi(s), \psi(s)) = (0, 0, 4\pi s), \quad (2.7)$$

with $s \in [0, 1]$. In this case, the operator is BPS and Q -closed. Evaluating the one-point function around the saddle point, we arrive at

$$W(\mathcal{C}) = \frac{1}{N} \sum_i \langle e^{2\pi\sigma_i} \rangle, \quad (2.8)$$

where $\langle \cdots \rangle$ stands for an average taken with respect to the eigenvalue integral obtained by the localization of the partition function. Note that in the large- N limit, which is our main interest in this paper, a general correlation function of the Wilson loops decomposes to a product of the one-point functions because of the factorization property.

The remaining task of this calculation would be to perform the eigenvalue integral. Although there are several efficient ways of evaluating the integral [3, 7, 43–45], we do not review them here since any explicit solution is not needed in this paper. In the following sections, we consider the reduced model of the quiver CS theory and show that its partition function and the one-point function of a corresponding operator are equivalent to the above eigenvalue integrals of the theory on S^3 in the large- N limit.

III. REDUCED MODEL FOR $\mathcal{N} = 2$ QUIVER CS THEORY ON S^3

In this section, we construct the reduced model of the $\mathcal{N} = 2$ quiver CS theory on S^3 and apply the localization calculation to the model. We first perform the dimensional reduction from S^3 to a point to obtain the reduced model. Then, we perform the localization for each multiplet of the $\mathcal{N} = 2$ supersymmetry. We list the action and the supersymmetry transformations in the original theory on S^3 in Appendix A.

A. Dimensional reduction

Let us first demonstrate the dimensional reduction of the CS term (A1) for a single gauge multiplet. In order to reduce it to a point, we expand the gauge field A in terms of the right-invariant one-form e^a defined in (B3) as

$$A = A_\mu(x) dx^\mu = A_a(x) e^a, \quad (3.1)$$

and drop the coordinate dependence of A_a and of the other fields in the multiplet. Using the Maurer-Cartan Eq. (B5), the derivative of A can be calculated as follows,

$$dA \rightarrow A_a d e^a = \varepsilon^{abc} A_a e^b \wedge e^c, \quad (3.2)$$

where the arrow represents that we have dropped the derivative of A_a . By applying (3.2) to (A1) and performing the integral, which produces only an overall constant factor given by the volume of unit S^3 , we obtain the reduced model of the CS term,

$$S_{\text{CS}}^r = -\frac{1}{g^2} \text{Tr} \left[A_a A^a - \frac{i}{3} \varepsilon^{abc} A_a A_b A_c - \frac{1}{2} \bar{\lambda} \lambda + D\sigma \right]. \quad (3.3)$$

Here, the indices a, b, c are raised and lowered simply by the Kronecker delta and we have introduced a coupling constant g for the reduced model. The value of g will be determined later such that the original continuum theory is reproduced in a continuum limit.

The original theory on S^3 has two kinds of the Killing spinors in the right-invariant frame. One is constant and the other is dependent on the coordinates of S^3 as shown in

(A10). Under the dimensional reduction, the constant Killing spinors survive and they generate the supersymmetry transformations of the reduced model. By dimensionally reducing the supersymmetry transformations in the original theory (A7), we find that (3.3) is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta A_a &= \frac{i}{2}(\bar{\lambda}\gamma_a\epsilon - \bar{\epsilon}\gamma_a\lambda), \\
\delta\sigma &= -\frac{1}{2}(\bar{\lambda}\epsilon - \bar{\epsilon}\lambda), \\
\delta\lambda &= \frac{1}{2}\gamma^{ab}\epsilon F_{ab} - D\epsilon + \gamma^a\epsilon[A_a, \sigma] - \sigma\epsilon, \\
\delta\bar{\lambda} &= \frac{1}{2}\gamma^{ab}\bar{\epsilon}F_{ab} + D\bar{\epsilon} - \gamma^a\bar{\epsilon}[A_a, \sigma] + \sigma\bar{\epsilon}, \\
\delta D &= -\frac{1}{2}[A_a, \bar{\lambda}]\gamma^a\epsilon - \frac{1}{2}\bar{\epsilon}\gamma^a[A_a, \lambda] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] \\
&\quad + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] - \frac{1}{2}\bar{\lambda}\epsilon + \frac{1}{2}\bar{\epsilon}\lambda,
\end{aligned} \tag{3.4}$$

where

$$F_{ab} := 2\varepsilon_{abc}A^c - i[A_a, A_b]. \tag{3.5}$$

The transformations (3.4) consist of two independent parts generated by ϵ and $\bar{\epsilon}$ as $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$. The Killing spinors ϵ and $\bar{\epsilon}$ are constant complex two-component spinors, and decomposed into the upper and the lower components. The transformation generated by each component is nilpotent. To see this, we take two parameters ϵ and ϵ' which have only the upper components and hence satisfy $\epsilon'\epsilon = \epsilon'_1\epsilon_2 + \epsilon'_2\epsilon_1 = 0$. Then, we can see that the supersymmetry is nilpotent, $\delta_{\epsilon'}\delta_\epsilon = 0$. For example, these transformations act on λ as follows,

$$\begin{aligned}
\delta_{\epsilon'}\delta_\epsilon\lambda &= \frac{1}{2}\gamma^{ab}\epsilon(i\varepsilon_{abc}\bar{\lambda}\gamma^c\epsilon' + [\bar{\lambda}\gamma_a\epsilon', A_b]) \\
&\quad - \epsilon\left(-\frac{1}{2}[A_a, \bar{\lambda}\gamma^a\epsilon'] + \frac{i}{2}[\bar{\lambda}\epsilon', \sigma] - \frac{1}{2}\bar{\lambda}\epsilon'\right) \\
&\quad - \frac{1}{2}\gamma^a\epsilon[A_a, \bar{\lambda}\epsilon'] + \frac{i}{2}\gamma^a\epsilon[\bar{\lambda}\gamma_a\epsilon', \sigma] + \frac{1}{2}\epsilon(\bar{\lambda}\epsilon') \\
&= -2\bar{\lambda}(\epsilon\epsilon') + i[\bar{\lambda}(\epsilon\epsilon'), \sigma] - [\gamma^a\bar{\lambda}(\epsilon\epsilon'), A_a] \\
&= 0.
\end{aligned} \tag{3.6}$$

Thus, it is indeed nilpotent. The commutator between δ_ϵ and $\delta_{\bar{\epsilon}}$ becomes a sum of gauge transformation, R -rotation and $SU(2)$ rotation, while the commutator for two barred or two unbarred parameters vanishes (see Appendix C).

We next consider the dimensional reduction of the YM term for a gauge multiplet. By performing the above dimensional reduction to (A2), we obtain

$$\begin{aligned}
S_{\text{YM}}^r &= \text{Tr}\left[\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}(\sigma + D)^2 - \frac{1}{2}[A_a, \sigma]^2\right. \\
&\quad \left. + \frac{1}{2}\bar{\lambda}\gamma^a[A_a, \lambda] - \bar{\lambda}\lambda + \frac{i}{2}\bar{\lambda}[\sigma, \lambda]\right].
\end{aligned} \tag{3.7}$$

This action can be written as a total superderivative:

$$\bar{\epsilon}\epsilon S_{\text{YM}}^r = \delta_{\bar{\epsilon}}\delta_\epsilon\text{Tr}\left[\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\right]. \tag{3.8}$$

We then consider a bifundamental matter multiplet $\{\phi, \psi, F\}$ coupled to two vector multiplets, $\{A_a, \lambda, \sigma, D\}$ and $\{B_a, \eta, \rho, \bar{D}\}$. The matter action on S^3 is given by (A4). Applying the same dimensional reduction to (A4), we obtain

$$\begin{aligned}
S_{\text{matter}}^r &= \text{Tr}\left[\bar{\phi}\nabla(A_a, B_a)^2\phi + \frac{3}{2}\bar{\psi}\psi - \bar{\psi}\gamma^a\nabla(A_a, B_a)\psi\right. \\
&\quad \left. + q(2-q)\bar{\phi}\phi - \frac{2q-1}{2}\bar{\psi}\psi\right. \\
&\quad \left. + i(2q-1)\bar{\phi}\nabla(\sigma, \rho)\phi + i\bar{\psi}\nabla(\sigma, \rho)\psi\right. \\
&\quad \left. + i\bar{\psi}\nabla(\lambda, \eta)\phi - i\bar{\phi}\nabla(\bar{\lambda}, \bar{\eta})\psi + i\bar{\phi}\nabla(D, \bar{D})\phi\right. \\
&\quad \left. + \bar{\phi}\nabla(\sigma, \rho)^2\phi + \bar{F}F\right].
\end{aligned} \tag{3.9}$$

Here, $\nabla(A, B)$ is defined in (A5) and q is the anomalous dimension of the matter multiplet. If the original matter action on S^3 has a superpotential, one can obtain a corresponding potential in the reduced model easily through the same procedure. The action (3.9) is invariant under the following supersymmetry transformations,

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\bar{\phi} &= \epsilon\bar{\psi}, \\
\delta\psi &= \gamma^a\epsilon\nabla(A_a, B_a)\phi + i\epsilon\nabla(\sigma, \rho)\phi - q\phi\epsilon + \bar{\epsilon}F, \\
\delta\bar{\psi} &= \gamma^a\epsilon\nabla(A_a, B_a)\bar{\phi} - i\nabla(\sigma, \rho)\bar{\phi}\bar{\epsilon} - q\bar{\phi}\bar{\epsilon} + \bar{F}\epsilon, \\
\delta F &= (q-2)\psi\epsilon + \epsilon\gamma^a\nabla(A_a, B_a)\psi - i\nabla(\sigma, \rho)\psi\epsilon \\
&\quad - i(\epsilon\nabla(\lambda, \eta))\phi, \\
\delta\bar{F} &= (q-2)\bar{\psi}\bar{\epsilon} + \bar{\epsilon}\gamma^a\nabla(A_a, B_a)\bar{\psi} + i\bar{\epsilon}\nabla(\sigma, \rho)\bar{\psi} \\
&\quad - i(\bar{\epsilon}\nabla(\bar{\lambda}, \bar{\eta}))\bar{\phi}.
\end{aligned} \tag{3.10}$$

One can also check the nilpotency on the matter fields. The matter action can also be written as a total superderivative:

$$\bar{\epsilon}\epsilon S_{\text{matter}}^r = \delta_{\bar{\epsilon}}\delta_\epsilon\text{Tr}[\bar{\psi}\psi - 2i\bar{\phi}\nabla(\sigma, \rho)\phi + 2(q-1)\bar{\phi}\phi]. \tag{3.11}$$

The adjoint matter is given as a special case of the bifundamental matter. Namely, if we identify one gauge multiplet with the other, the action (3.9) and the supersymmetry transformations (3.10) reduce to those for an adjoint matter multiplet.

The reduced model of a general $\mathcal{N} = 2$ quiver CS theory is constructed by combining (3.3) and (3.9) plus an appropriate superpotential term [24]. For example, the ABJM theory contains two copies of supersymmetric CS theory of which gauge groups are $U(N_1)$ and $U(N_2)$ with opposite levels $k, -k$. The matter sector consists of four matter multiplets $\{\phi_I, \psi_I, F_I\}$, which are in the bifundamental representation for $I = 1, 3$ and in the antibifundamental for $I = 2, 4$. The anomalous dimension q is $1/2$.

Vandermonde determinant for each block labeled by s ,

$$\int d\hat{\sigma} \rightarrow \int \prod_{s,i} d\sigma_{si} \prod_s \Delta(\sigma_s)^2, \quad (3.18)$$

$$\Delta(\sigma_s) = \prod_{i>j} (\sigma_{si} - \sigma_{sj}),$$

where σ_{si} 's are eigenvalues of σ_s . In the following, we will work in this gauge.

2. The gauge sector

We expand the matrices in a gauge multiplet around the saddle point given by (3.13) as $A_a = -2L_a + A'_a$, $\sigma = \hat{\sigma} + \sigma'$, and so on, and perform the 1-loop integral with respect to the fluctuations in the gauge multiplet. For this purpose, we need to fix the residual gauge symmetry which leaves the background (3.13) invariant. From the original gauge symmetry written as $A_a \rightarrow UA_aU^\dagger$, $U \in U(K)$, one can read off the transformation law for the fluctuation as

$$A'_a \rightarrow -2[U, L_a]U^\dagger + UA'_aU^\dagger. \quad (3.19)$$

We adopt the standard Faddeev-Popov method and choose the following gauge-fixing condition for A'_a ,

$$[L_a, A'^a] = 0. \quad (3.20)$$

Then, the ghost and the gauge-fixing terms are given by

$$S_{\text{ghost}}^r = -\text{Tr}(ib[L_a, A'^a] + 2\bar{c}[L_a, [L_a, c]]) - \bar{c}[L_a, [A'^a, c]]. \quad (3.21)$$

Note that the zero mode of the ghosts, that is, the mode satisfying $[L_a, c] = 0$ is absent in the above action.

We then perform the 1-loop integral around the saddle point keeping only the quadratic part of the fluctuation. The relevant part in the total action $S_{\text{CS}}^r + S_{\text{YM}}^r + S_{\text{ghost}}^r$ ⁴ for the gauge multiplet is given by

$$\text{Tr} \left\{ (\varepsilon_{abc} A'^c + i[L_a, A'_b] - i[L_b, A'_a])^2 + \frac{1}{2}(\sigma' + D')^2 - \frac{1}{2}[\hat{\sigma}, A'_a]^2 + [L_a, \sigma']^2 - \bar{\lambda}'\gamma^a[L_a, \lambda'] - \bar{\lambda}'\lambda' + \frac{i}{2}\bar{\lambda}'[\hat{\sigma}, \lambda'] - 2\bar{c}'[L_a, [L_a, c']] - ib'[L_a, A'^a] \right\}, \quad (3.22)$$

where the fluctuations are represented by the primed matrices. In (3.22), the zero mode of σ' is not included. Since this mode corresponds to the direction of the moduli of the saddle point, it is treated in the moduli integral in (3.18).

The integration over σ' , D' , c' , \bar{c}' , and b' yields

$$\det'([L_a, [L_a, \cdot]])^{1/2} \delta([L_a, A'^a]), \quad (3.23)$$

⁴We omit t and t' in front of the YM action and the matter action since they are irrelevant for the evaluation of the 1-loop determinant.

up to an overall constant, where \det' means that the zero mode is removed in taking the determinant.

In order to calculate further, let us decompose the matrices to smaller blocks which are defined by the structure of L_a in (3.15). We label each block by a pair (s, t) , where s, t run from $-\Lambda/2$ to $\Lambda/2$, and denote the (s, t) block of A'_a by $A_a^{(s,t)}$. Note that $A_a^{(s,t)}$ is a $(2j_s + 1)M_s \times (2j_t + 1)M_t$ rectangular matrix. Each block component can be expanded in terms of the vector fuzzy spherical harmonics defined in Appendix D as

$$A_a^{(s,t)} = \sum_{\rho=-1}^1 \sum_{\tilde{Q}=|j_s-j_t|}^{j_s+j_t} \sum_{m=-Q}^Q a_{Jm\rho}^{(s,t)} \otimes \hat{Y}_{Jm(j_s j_t)}^\rho, \quad (3.24)$$

where $Q = J + \delta_{\rho,1}$, $\tilde{Q} = J + \delta_{\rho,-1}$, and the sum over \tilde{Q} is taken for $J \geq 0$. $a_{Jm\rho}^{(s,t)}$ is a matrix with size $M_s \times M_t$.

The delta function in (3.23) constrains the $\rho = 0$ component, $a_{Jm0}^{(s,t)}$, in the above expansion to vanish. In fact, using (D10), we find that

$$\delta([L_a, A'^a]) = \prod_{s,t} \prod_{\substack{J \geq |j_s - j_t| \\ J \neq 0}}^{j_s + j_t} \prod_{m=-J}^J \{\sqrt{J(J+1)}\}^{-1} \delta(a_{Jm0}^{(s,t)}). \quad (3.25)$$

We can therefore integrate out $a_{Jm0}^{(s,t)}$ trivially leaving the factor,

$$\prod_{s,t} \prod_{\substack{J \geq |j_s - j_t| \\ J \neq 0}}^{j_s + j_t} \prod_{m=-J}^J \{\sqrt{J(J+1)}\}^{-1} = \{\det'([L_a, [L_a, \cdot]])\}^{-(1/2)}. \quad (3.26)$$

This cancels the other factor in (3.23).

We then perform the integral over A'_a with $\rho = \pm 1$ in the expansion (3.24). By substituting (3.24) to (3.22), the relevant part of the action becomes

$$\text{Tr} \left\{ (\varepsilon_{abc} A'^c + i[L_a, A'_b] - i[L_b, A'_a])^2 - \frac{1}{2}[\hat{\sigma}, A'_a]^2 \right\} = \sum_{s,t} \sum_{i,j} \sum_{\rho=\pm 1} \sum_{J,m} |(a_{Jm\rho}^{(s,t)})_{ij}|^2 \left[2(J+1)^2 + \frac{1}{2}(\sigma_{si} - \sigma_{tj})^2 \right], \quad (3.27)$$

where $i = 1, 2, \dots, M_s$ and $j = 1, 2, \dots, M_t$. We have used the formulas (D11), (D13), and (D15). Then, the integration results in the factor

$$\prod_{s,t} \prod_{i,j} \prod_{\rho=\pm 1} \prod_{J,m} \left[2(J+1)^2 + \frac{1}{2}(\sigma_{si} - \sigma_{tj})^2 \right]^{-1/2}. \quad (3.28)$$

The exponent $-1/2$ comes from the fact that $a_{Jm\rho}^{(s,t)}$ satisfy a kind of the reality condition,

$$a_{Jm\rho}^{(s,t)} = (-1)^{m-(j_s-j_t)+1} a_{J-m\rho}^{(t,s)\dagger}, \quad (3.29)$$

which follows from (D13) and the Hermiticity of A'_a .

To perform the integration over λ' and $\bar{\lambda}'$, we also expand them in terms of the spinor fuzzy spherical harmonics defined in (D8) as

$$\lambda'^{(s,t)\alpha} = \sum_{\kappa=\pm 1} \sum_{\tilde{U}=|j_s-j_t|}^{j_s+j_t} \sum_{m=-U}^U \lambda_{Jm\kappa}^{(s,t)} \otimes \hat{Y}_{Jm(j_s,j_t)\alpha}^\kappa, \quad (3.30)$$

$$\bar{\lambda}'_{\alpha}{}^{(s,t)} = \sum_{\kappa=\pm 1} \sum_{\tilde{U}=|j_s-j_t|}^{j_s+j_t} \sum_{m=-U}^U \bar{\lambda}_{Jm\kappa}^{(s,t)} \otimes (\hat{Y}_{Jm(j_s,j_t)\alpha}^\kappa)^\dagger, \quad (3.31)$$

where α denotes the spinor index, $U = J + \frac{1}{2} \delta_{\kappa,1}$, $\tilde{U} = J + \frac{1}{2} \delta_{\kappa,-1}$, and the summation over \tilde{U} is taken for $J \geq 0$. Plugging these expansions into the action, we obtain

$$\begin{aligned} & \text{Tr} \left\{ -\bar{\lambda}' \gamma^a [L_a, \lambda'] - \bar{\lambda}' \lambda' + \frac{i}{2} \bar{\lambda}' [\hat{\sigma}, \lambda'] \right\} \\ &= \sum_{s,t} \sum_{i,j} \sum_{\kappa=\pm 1} \sum_{J,m} (\bar{\lambda}_{Jm\kappa}^{(s,t)})_{ij} (\lambda_{Jm\kappa}^{(t,s)})_{ji} \left[-\frac{1}{4} - \kappa \left(J + \frac{3}{4} \right) \right. \\ & \quad \left. - \frac{i}{2} (\sigma_{si} - \sigma_{tj}) \right], \end{aligned} \quad (3.32)$$

where we have used the formulas (D12) and (D16). Therefore, the integration with respect to λ' and $\bar{\lambda}'$ gives the factor

$$\prod_{s,t} \prod_{i,j} \prod_{\kappa=\pm 1} \prod_{J,m} \left[\frac{1}{4} + \kappa \left(J + \frac{3}{4} \right) + \frac{i}{2} (\sigma_{si} - \sigma_{tj}) \right]. \quad (3.33)$$

Combining the above results with the Vandermonde determinant for $\hat{\sigma}$, we obtain the 1-loop determinant from the gauge multiplet,

$$\begin{aligned} & \prod_s \Delta(\sigma_s) e^{-\sum_{s,i} ((2j_s+1)/g^2) \sigma_{si}^2} \frac{\prod_{s \neq t} \prod_{i,j} \prod_{J=|j_s-j_t|}^{j_s+j_t} (-J-1+i(\sigma_{si}-\sigma_{tj})/2)^{2J+2}}{\prod_{s < t} \prod_{i,j} \prod_{J=|j_s-j_t|}^{j_s+j_t} ((J+1)^2 + (\sigma_{si}-\sigma_{tj})^2/4)^{2J+3}} \\ & \times \frac{\prod_{s \neq t} \prod_{i,j} \prod_{J=|j_s-j_t|-1/2}^{j_s+j_t-1/2} (J+1/2+i(\sigma_{si}-\sigma_{tj})/2)^{2J+1}}{\prod_{s < t} \prod_{i,j} \prod_{J=|j_s-j_t|-1}^{j_s+j_t-1} ((J+1)^2 + (\sigma_{si}-\sigma_{tj})^2/4)^{2J+1}}. \end{aligned} \quad (3.34)$$

Because of the cancellation between bosons and fermions, this is simplified to

$$e^{-\sum_{s,i} ((2j_s+1)/g^2) \sigma_{si}^2} \prod_s \prod_{i < j} \frac{(\sigma_{si} - \sigma_{sj})^2}{(2j_s+1)^2 + (\sigma_{si} - \sigma_{sj})^2/4} \prod_{s < t} \prod_{i,j} \frac{(j_s - j_t)^2 + (\sigma_{si} - \sigma_{tj})^2/4}{(j_s + j_t + 1)^2 + (\sigma_{si} - \sigma_{tj})^2/4}. \quad (3.35)$$

3. The matter sector

We first consider the matter multiplet in the bifundamental representation (3.9). After integrating out F and \bar{F} trivially, the relevant part of the matter action is given by⁵

$$\begin{aligned} & \text{Tr} [\bar{\phi} \{ 4\nabla(L_a, \tilde{L}_a)^2 + 1 + (\nabla(\hat{\sigma}, \hat{\rho}) - i(1-q))^2 \} \phi \\ & + \bar{\psi} \{ 2\gamma^a \nabla(L_a, \tilde{L}_a) + i\nabla(\hat{\sigma}, \hat{\rho}) + (2-q) \} \psi]. \end{aligned} \quad (3.36)$$

Here, \tilde{L}_a and $\hat{\rho}$ are the saddle point configurations for the second gauge multiplet and are the counterparts of L_a and $\hat{\sigma}$, respectively. They take a similar form to (3.15) and (3.16). We define t , k_t , $\tilde{\Lambda}$, and \tilde{M}_t for \tilde{L}_a and $\hat{\rho}$ as the counterparts of s , j_s , Λ , and M_s in (3.15) and (3.16), respectively.

We decompose the bifundamental (rectangular) matrices to the block components and denote them by $\phi^{(s,t)}$ and $\psi^{(s,t)}$ corresponding to the s th block in L_a and the t th block

in \tilde{L}_a . They are $(2j_s+1)M_s \times (2k_t+1)\tilde{M}_t$ matrices and can be expanded in terms of the fuzzy spherical harmonics as

$$\begin{aligned} \phi^{(s,t)} &= \sum_{J=|j_s-k_t|}^{j_s+k_t} \sum_{m=-J}^J \phi_{Jm}^{(s,t)} \otimes \hat{Y}_{Jm(j_s,k_t)}, \\ \psi_{\alpha}^{(s,t)} &= \sum_{\kappa=\pm 1} \sum_{\tilde{U}=|j_s-k_t|}^{j_s+k_t} \sum_{m=-U}^U \psi_{Jm\kappa}^{(s,t)} \otimes \hat{Y}_{Jm(j_s,k_t)\alpha}^\kappa, \end{aligned} \quad (3.37)$$

where $\phi_{Jm}^{(s,t)}$ and $\psi_{Jm\kappa}^{(s,t)}$ are the $M_s \times \tilde{M}_t$ matrices. In this basis, $\nabla(L_a, \tilde{L}_a)$ can be rewritten as

$$\nabla(L_a, \tilde{L}_a) \phi^{(s,t)} = \sum_{J=|j_s-k_t|}^{j_s+k_t} \sum_{\tilde{m}=-J}^J \phi_{J\tilde{m}}^{(s,t)} \otimes L_a \circ \hat{Y}_{J\tilde{m}(j_s,k_t)}, \quad (3.38)$$

where $L_a \circ$ is defined in (D2). By substituting (3.37), the quadratic part of the matter action becomes

⁵We will omit the primes for the fluctuations of the matters.

$$\begin{aligned}
 & \sum_{s,t} \sum_{i,\alpha} \sum_{J=|j_s-k_t|}^{j_s+k_t} \sum_{m=-J}^J (\bar{\phi}_{Jm}^{(t,s)})_{\alpha i} (\phi_{Jm}^{(s,t)})_{i\alpha} [(2J+1)^2 + (\sigma_{si} - \rho_{t\alpha} - i(1-q))^2] \\
 & + \sum_{s,t} \sum_{i,\alpha} \sum_{\kappa=\pm 1} \sum_{\tilde{U}=|j_s-k_t|}^{j_s+k_t} \sum_{m=-U}^U (\bar{\psi}_{Jm\kappa}^{(t,s)})_{\alpha i} (\psi_{Jm}^{(s,t)})_{i\alpha} \left[2\kappa \left(J + \frac{3}{4} \right) - \left(q - \frac{1}{2} \right) + i(\sigma_{si} - \rho_{t\alpha}) \right], \quad (3.39)
 \end{aligned}$$

where $i = 1, 2, \dots, M_s$ and $\alpha = 1, 2, \dots, \tilde{M}_t$.⁶ Then, we find after a simple calculation that the 1-loop determinant for the bifundamental matter multiplet is given by

$$\prod_{s,t} \prod_{i,\alpha} \prod_{J=|j_s-k_t|}^{j_s+k_t} \frac{2J+2-q+i(\sigma_{si}-\rho_{t\alpha})}{2J+q-i(\sigma_{si}-\rho_{t\alpha})}. \quad (3.40)$$

The 1-loop determinant from the adjoint matter is easily obtained by identifying one gauge multiplet with the other in the above calculation. The result is given by

$$\prod_{s,t} \prod_{i>j} \prod_{J=|j_s-j_t|}^{j_s+j_t} \frac{(2J+2-q)^2 + (\sigma_{si} - \sigma_{tj})^2}{(2J+q)^2 + (\sigma_{si} - \sigma_{tj})^2}. \quad (3.41)$$

In the reduced model of the ABJM theory, $q = 1/2$ and there are two bifundamental and two antibifundamental matters. Hence, the 1-loop determinant from the matter sector is given by

$$\prod_{s,t} \prod_{i,\alpha} \prod_{J=|j_s-k_t|}^{j_s+k_t} \left(\frac{(2J+\frac{3}{2})^2 + (\sigma_{si} - \rho_{t\alpha})^2}{(2J+\frac{1}{2})^2 + (\sigma_{si} - \rho_{t\alpha})^2} \right)^2. \quad (3.42)$$

4. Wilson loop

The Wilson loop operator in the reduced models of theories on S^3 was constructed in [30,40]. It is given as a naive dimensional reduction of the Wilson loop in the theory on S^3 (2.6),

$$\hat{W}(C) = \frac{1}{K} \text{Tr} \left[P \exp \left(i \oint_C ds (\dot{x}^\mu(s) e_\mu^a(x) A_a - i |\dot{x}(s)| \sigma) \right) \right]. \quad (3.43)$$

In the case that the contour is a great circle on S^3 , this operator is BPS in the reduced model, so that we can calculate the correlation function of this operator by the localization technique. In this case, substituting (2.7) simplifies the operator as

$$\hat{W}(C) = \frac{1}{K} \text{Tr} [e^{2\pi i(A_3 - i\sigma)}]. \quad (3.44)$$

Then, applying the localization, we obtain

$$\langle \hat{W}(C) \rangle = \frac{1}{K} \langle \text{Tr} e^{-4\pi i L_3 + 2\pi \hat{\sigma}} \rangle \quad (3.45)$$

⁶It will not cause any confusion to use α both for the spinor index and for the index labeling the diagonal components of ρ_t .

$$= \frac{1}{K} \sum_{s=-\Lambda/2}^{\Lambda/2} (2j_s + 1) \sum_{i=1}^{M_s} \langle e^{2\pi i \sigma_{si}} \rangle, \quad (3.46)$$

where to obtain the second line we have used the fact that each diagonal component of $L_3^{[j_s]}$ takes a value in either integer or half-integer. $\langle \dots \rangle$ stands for an average with respect to the eigenvalue integral for the reduced model.

IV. $\mathcal{N} = 2$ QUIVER CS THEORY ON S^3 FROM REDUCED MODEL

In this section, from the reduced model we realize a quiver CS theory on S^3 with gauge group $\otimes_a U(N_a)$ in the 't Hooft limit in which

$$N_a \rightarrow \infty \quad \text{with} \quad \frac{N_a}{N_b} \quad \text{and} \quad \frac{N_a}{k_a} \quad \text{fixed} \quad (4.1)$$

for any a and b . We assume for simplicity the gauge group to be $U(N_1) \times U(N_2)$ and mainly consider the case of the ABJM theory, which contains two bifundamental multiplets (N_1, \tilde{N}_2) and two antibifundamental multiplets (\tilde{N}_1, N_2) . However, it will be obvious that our argument is applicable to more general $\mathcal{N} = 2$ quiver CS theories. To realize the theory on S^3 , we take the reduced model with gauge group $U(K_1) \times U(K_2)$ where K_i 's ($i = 1, 2$) are much larger than N_i 's such that the Kaluza-Klein (KK) modes on S^3 are embedded in matrices in the reduced model. We denote by g_1 and g_2 the coupling constants for the two CS terms (3.3) in the reduced model.

A. S^3 from matrices

Recall that the partition function in the reduced model is given by a sum of 1-loop contributions around saddle points, each of which is specified by a representation of $SU(2)$, (3.17). In order to obtain the theory on S^3 , we extract the following representations from the sum in (3.17),

$$A_a = -2 \bigoplus_s L_a^{[j_s]} \otimes 1_{N_1}, \quad B_a = -2 \bigoplus_s L_a^{[j_s]} \otimes 1_{N_2}, \quad (4.2)$$

which corresponds to the case of $M_s = N_1$ for A_a and $\tilde{M}_s = N_2$ for B_a in (3.13) and $K_i = N_i \sum_s (2j_s + 1)$. We take j_s as

$$2j_s + 1 = n + s \quad \text{for} \quad -\frac{\Lambda}{2} \leq s \leq \frac{\Lambda}{2}, \quad (4.3)$$

where n is a positive integer. We then take the limits in which

$$\begin{aligned} n \rightarrow \infty, \quad \frac{g_1^2}{n} \rightarrow 0, \quad \frac{g_2^2}{n} \rightarrow 0, \quad \Lambda \rightarrow \infty, \\ n - \Lambda \rightarrow \infty, \quad N_1 \rightarrow \infty, \quad N_2 \rightarrow \infty \end{aligned} \quad (4.4)$$

with the following combinations fixed,

$$t_1 \equiv \frac{N_1 g_1^2}{n}, \quad t_2 \equiv \frac{N_2 g_2^2}{n}, \quad \frac{N_1}{N_2}. \quad (4.5)$$

Here, we explain the reason why the above representation and the limit create S^3 (see [11] for more detail). First, we consider the original theory on S^3 . Since S^3 is viewed as an S^1 -bundle over S^2 , we can perform the KK reduction along the fiber direction. Then, we obtain a theory on S^2 involving infinite KK modes. Reflecting the nontrivial fibration of S^1 , the KK mode with KK momentum \tilde{m} on S^2 can be regarded as a field in a monopole background, where the monopole is sitting at the center of S^2 in R^3 with monopole charge \tilde{m} . As the angular momentum of the field on S^2 in the presence of a monopole is bounded below by its charge $J \geq |\tilde{m}|$, that of the KK mode is also bounded. The same situation can be observed in the mode expansion of a rectangular matrix (3.37) if one identifies the angular momentum J and the monopole charge \tilde{m} on S^2 with J and $j_s - j_t$ in (3.37), respectively. The only difference is the existence of the upper bound of the angular momentum $j_s + j_t$, which can be removed by putting $2j_s + 1 = n + s$ and taking the $n \rightarrow \infty$ limit so that $j_s + j_t \rightarrow \infty$ and $j_s - j_t = \frac{s-t}{2} = \tilde{m}$. Indeed, the rectangular block of the fluctuation is a regularization of a field on S^2 in a monopole background. Thus, expanding the reduced model around the appropriate representation (4.2) such that the full KK modes ($-\infty \leq \tilde{m} \leq \infty$) on S^2 are reproduced, we can obtain the original theory on S^3 . Now, the physical interpretation of n and Λ is clear; n plays a role of UV momentum cutoff on S^2 while Λ plays a role of UV cutoff on S^1 . Therefore, $\{L_a^{[j_s, 1]}\}$ in (4.2) creates S^3 while the multiplicities N_1 and N_2 reproduce the original gauge group $U(N_1) \times U(N_2)$.

In the following calculation of the partition function or the Wilson loop, we first take the $n \rightarrow \infty$ limit shown in (4.4) with n/g_1^2 and n/g_2^2 fixed and later we take the other limits. This is possible since the $n \rightarrow \infty$ limit does not lead to any divergence. In the $n \rightarrow \infty$ limit, we can replace the coefficients of the Gaussian terms in (3.35) with $\frac{n}{g_i^2} = \frac{N_i}{t_i}$. Then, the contribution of the representation (4.2) from the summation in (3.17) is given by

$$\begin{aligned} \int \prod_{s=-(\Lambda/2)}^{(\Lambda/2)} \left(\prod_{i=1}^{N_1} d\sigma_{si} \prod_{\alpha=1}^{N_2} d\rho_{s\alpha} \right) \mathcal{M}_{\text{gauge}} \mathcal{M}_{\text{matter}} \\ \times \exp\left(-\frac{N_1}{t_1} \sum_{s,i} \sigma_{si}^2 + \frac{N_2}{t_2} \sum_{s,\alpha} \rho_{s\alpha}^2\right), \end{aligned} \quad (4.6)$$

where t_1 and t_2 are defined in (4.5) and $\mathcal{M}_{\text{gauge}}$ and $\mathcal{M}_{\text{matter}}$ are 1-loop determinants for the gauge and the matter sector, respectively, in the case of (4.2).

More concretely, $\mathcal{M}_{\text{gauge}}$ is given by

$$\begin{aligned} \mathcal{M}_{\text{gauge}} = \prod_s \prod_{i < j} (\sigma_{si} - \sigma_{sj})^2 \prod_{s < t} \prod_{i,j} \left[1 + \frac{(\sigma_{si} - \sigma_{tj})^2}{(s-t)^2} \right] \\ \times \prod_s \prod_{\alpha < \beta} (\rho_{s\alpha} - \rho_{s\beta})^2 \prod_{s < t} \prod_{\alpha,\beta} \left[1 + \frac{(\rho_{s\alpha} - \rho_{t\beta})^2}{(s-t)^2} \right], \end{aligned} \quad (4.7)$$

where we have dropped the denominator in (3.35) because it becomes independent of σ_{si} or $\rho_{s\alpha}$ in the limit, $n \rightarrow \infty$. In addition, we have dropped the irrelevant constant factor $\prod_{s < t} (s-t)^{2N_1^2 + 2N_2^2}$.

$\mathcal{M}_{\text{matter}}$ depends on the matter content. For a matter multiplet in the bifundamental representation, it is given by

$$\mathcal{M}_{\text{matter}}|_{\text{bifund}} = \prod_{s,t} \prod_{i,\alpha} \prod_{J=|s-t|/2}^{\infty} \frac{2J+2-q+i(\sigma_{si}-\rho_{t\alpha})}{2J+q-i(\sigma_{si}-\rho_{t\alpha})}, \quad (4.8)$$

and for the ABJM theory, it is given by

$$\mathcal{M}_{\text{matter}}|_{\text{ABJM}} = \prod_{s,t} \prod_{i,\alpha} \prod_{J=|s-t|/2}^{\infty} \frac{\left((2J+\frac{3}{2})^2 + (\sigma_{si}-\rho_{t\alpha})^2 \right)^2}{\left((2J+\frac{1}{2})^2 + (\sigma_{si}-\rho_{t\alpha})^2 \right)^2}. \quad (4.9)$$

B. Perturbative proof of large- N equivalence

1. Feynman rule for reduced matrix model

We consider the perturbation theory of (4.6) with respect to the 't Hooft couplings, t_1 and t_2 , in the limit (4.4). Here, $\mathcal{M}_{\text{gauge}}$ and $\mathcal{M}_{\text{matter}}$ are regarded as interactions. We will prove the equivalence between the reduced model (4.6) and the original theory on S^3 by showing one-to-one correspondence of Feynman diagrams between these theories.

To read off the Feynman rules, it is convenient to rewrite (4.6) in a manifestly $U(N_1) \times U(N_2)$ invariant form, which is given by a multimatrix model consisting of matrices σ_s and ρ_s with double-trace interactions.

In (4.7), the factors $\prod_s \prod_{i < j} (\sigma_{si} - \sigma_{sj})^2$ and $\prod_s \prod_{\alpha < \beta} (\rho_{s\alpha} - \rho_{s\beta})^2$ correspond to the Vandermonde determinants for matrices σ_s and ρ_s , and the remaining factor of σ_{si} can be written as

$$\prod_{s < t} \prod_{i, j} \left[1 + \frac{(\sigma_{si} - \sigma_{tj})^2}{(s - t)^2} \right] = \exp \left[\frac{1}{2} \sum_{s \neq t} \sum_{i, j} \ln \left[1 + \frac{(\sigma_{si} - \sigma_{tj})^2}{(s - t)^2} \right] \right] = \exp \left[- \sum_{s \neq t} \sum_{\substack{a, b \in \mathbb{Z}_{\geq 0} \\ a+b \in 2\mathbb{N}}} \frac{K_{ab}}{(s - t)^{a+b}} \text{tr} \sigma_s^a \text{tr} \sigma_t^b \right], \quad (4.10)$$

where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers and $2\mathbb{N}$ is the set of even positive integers. K_{ab} is a numerical factor given by

$$K_{ab} \equiv \frac{(-1)^{(a+b)/2}}{a+b} \binom{a+b}{a} (-1)^a. \quad (4.11)$$

The factor consisting of $\rho_{s\alpha}$ in (4.7) is obtained by just replacing $\sigma \rightarrow \rho$ in (4.10).

For a matter multiplet in the bifundamental representation, if one naively applies the same calculation to the factor (4.8), one obtains

$$\exp \left[- \sum_{s, t} \sum_{j=\frac{|s-t|}{2}}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \left(\frac{i}{2J+2-q} \right)^n - \left(\frac{-i}{2J+q} \right)^n \right\} \right. \\ \left. \times \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} \text{tr} \sigma_s^r \text{tr} \rho_t^{n-r} \right]. \quad (4.12)$$

We find that for $n = 1$, the coefficients of $\text{tr} \sigma_s$ and $\text{tr} \rho_s$, are divergent since they have the form $\sum_J \frac{1}{J}$, and therefore we cannot perform the perturbative expansion. This case should be treated in a nonperturbative framework such as the saddle point approximation, which will be introduced in Sec. IV C. Note that if we restrict σ_s and ρ_s to traceless matrices for each s , $\text{tr} \sigma_s = \text{tr} \rho_s = 0$, we do not have the divergence. In this case, the following argument can be applied and the reduced model properly realizes the perturbative expansion in $\mathcal{N} = 2$ quiver CS theory with a bifundamental matter of $SU(N_1) \times SU(N_2)$ gauge group.

For the reduced model of the ABJM theory, $\mathcal{M}_{\text{matter}}$ can be written as

$$\exp \left[4 \sum_{s, t} \sum_{\substack{a, b \in \mathbb{Z}_{\geq 0} \\ a+b \in 2\mathbb{N}}} \frac{K_{ab}}{2^{a+b}} \left\{ \zeta \left(a + b, \frac{1}{4} + \frac{|s-t|}{2} \right) \right. \right. \\ \left. \left. - \zeta \left(a + b, \frac{3}{4} + \frac{|s-t|}{2} \right) \right\} \text{tr} \sigma_s^a \text{tr} \rho_t^b \right], \quad (4.13)$$

where $\zeta(z, q)$ is the generalized zeta function

$$\zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z}. \quad (4.14)$$

In summary, the reduced model of the ABJM theory is given by

$$\int \left(\prod_{s=-\Lambda/2}^{\Lambda/2} d\sigma_s d\rho_s \right) \exp \left(- \frac{N_1}{t_1} \sum_s \text{tr} \sigma_s^2 + \frac{N_2}{t_2} \sum_s \text{tr} \rho_s^2 \right. \\ \left. + U_{\text{gauge}}^r + U_{\text{matter}}^r \right), \quad (4.15)$$

where U_{gauge}^r and U_{matter}^r are the double-trace interactions:

$$U_{\text{gauge}}^r = - \sum_{s \neq t} \sum_{\substack{a, b \in \mathbb{Z}_{\geq 0} \\ a+b \in 2\mathbb{N}}} \frac{K_{ab}}{(s-t)^{a+b}} (\text{tr} \sigma_s^a \text{tr} \sigma_t^b + \text{tr} \rho_s^a \text{tr} \rho_t^b), \quad (4.16)$$

$$U_{\text{matter}}^r = \sum_{s, t} \sum_{\substack{a, b \in \mathbb{Z}_{\geq 0} \\ a+b \in 2\mathbb{N}}} \frac{4K_{ab}}{2^{a+b}} \left\{ \zeta \left(a + b, \frac{1}{4} + \frac{|s-t|}{2} \right) \right. \\ \left. - \zeta \left(a + b, \frac{3}{4} + \frac{|s-t|}{2} \right) \right\} \text{tr} \sigma_s^a \text{tr} \rho_t^b. \quad (4.17)$$

From this action, we can read off the Feynman rule (see Fig. 1). The propagators are given by

$$\langle \sigma_{sij} \sigma_{tkl} \rangle = \frac{t_1}{2N_1} \delta_{st} \delta_{il} \delta_{jk}, \quad (4.18)$$

$$\langle \rho_{s\alpha\beta} \rho_{t\gamma\delta} \rangle = - \frac{t_2}{2N_2} \delta_{st} \delta_{\alpha\delta} \delta_{\beta\gamma}.$$

The vertex of $\text{tr} \sigma_s^a \text{tr} \sigma_t^b$, or $\text{tr} \rho_s^a \text{tr} \rho_t^b$, gives a factor

$$- \frac{2K_{ab}}{(s-t)^{a+b}}. \quad (4.19)$$

The vertex of $\text{tr} \sigma_s^a \text{tr} \rho_t^b$ gives a factor

$$\frac{4K_{ab}}{2^{a+b}} \left\{ \zeta \left(a + b, \frac{1}{4} + \frac{|s-t|}{2} \right) - \zeta \left(a + b, \frac{3}{4} + \frac{|s-t|}{2} \right) \right\}. \quad (4.20)$$

We write (4.19) and (4.20) collectively as $V_{st}^{(a,b)}$;

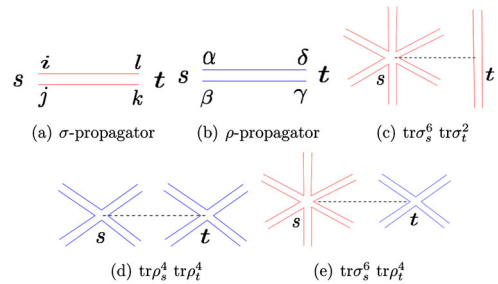


FIG. 1 (color online). The red double line represents σ , and the blue double line represents ρ . The dashed line represents a double-trace interaction. A vertex in a single color, such as (c) or (d), represents an interaction in U_{gauge}^r , and thus $s \neq t$. A vertex in two colors, such as (e), represents an interaction in U_{matter}^r .

$$V_{st}^{(a,b)} = \begin{cases} -2 \frac{K_{ab}}{(s-t)^{a+b}} & \text{for } V_{st}^{(a,b)} \in U_{\text{gauge}}^r \\ \frac{4K_{ab}}{2^{a+b}} \left(\zeta \left(a + b, \frac{1}{4} + \frac{|s-t|}{2} \right) - \zeta \left(a + b, \frac{3}{4} + \frac{|s-t|}{2} \right) \right) & \text{for } V_{st}^{(a,b)} \in U_{\text{matter}}^r \end{cases} \quad (4.21)$$

Here, “ $V_{st}^{(a,b)} \in U_{\text{gauge}}^r$ ” and “ $V_{st}^{(a,b)} \in U_{\text{matter}}^r$ ” mean the vertices coming from the interactions U_{gauge}^r and U_{matter}^r , respectively.

We discuss the calculation of the free energy based on the above Feynman rule. In this calculation, only connected diagrams are relevant as usual. Here, by “connected” we mean that any parts in a diagram are connected by dashed lines or by double lines. Figure 2 shows examples of such diagrams. Since we are interested in the limit (4.4), let us consider what kind of diagrams contributes to the leading order of the $1/N_{1,2}$ expansion. It turns out that the leading contribution is given by the diagrams satisfying the following two conditions:

Condition 1. They are planar with respect to the double lines in the ordinary sense.

Condition 2. They can be separated into two parts by cutting any dashed lines. We call a diagram satisfying the latter condition “tree” diagram since this condition is equivalent to that any dashed lines do not form a loop. We can check the latter condition explicitly for Fig. 2(a) as follows. Since N_1 and N_2 are in the same order in the limit (4.4), we denote the order of them collectively by N . Each propagator gives a factor of N^{-1} , each index loop gives N , and each vertex gives N^0 . While (a) is proportional to $N^{-13} \times N^{15} = N^2$, (b) is proportional to $N^{-12} \times N^{12} = N^0$. Thus, Fig. 2(b) does not contribute in the limit (4.4).

2. Feynman rule for ABJM matrix model

We next construct the Feynman rule for the ABJM matrix model on S^3 , given by (2.5). This can be written as a manifestly $U(N_1) \times U(N_2)$ invariant form as follows,

$$\int d\sigma d\rho \exp \left(-\frac{N_1}{\lambda_1} \text{tr} \sigma^2 + \frac{N_2}{\lambda_2} \text{tr} \rho^2 + U_{\text{gauge}} + U_{\text{matter}} \right), \quad (4.22)$$

where we have defined the 't Hooft couplings λ_i as $\frac{2\pi^2}{g_s} = \frac{N_i}{\lambda_i}$ ($i = 1, 2$), and U_{gauge} and U_{matter} are the double-trace interactions,

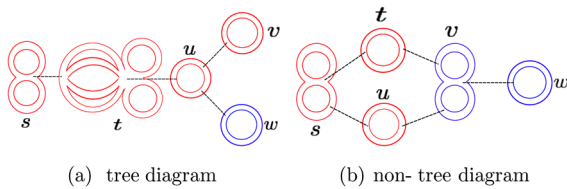


FIG. 2 (color online). Examples of connected diagrams. While the dashed lines in (a) do not form a loop, the dashed lines in (b) do.

$$U_{\text{gauge}} = \sum_{\substack{a,b \\ -a+b \in 2\mathbb{N}}} 2K_{ab} \zeta(a+b) (\text{tr} \sigma^a \text{tr} \sigma^b + \text{tr} \rho^a \text{tr} \rho^b),$$

$$U_{\text{matter}} = \sum_{\substack{a,b \\ a+b \in 2\mathbb{N}}} 4K_{ab} \zeta \left(a + b, \frac{1}{2} \right) \text{tr} \sigma^a \text{tr} \rho^b. \quad (4.23)$$

The vertices in the ABJM matrix model give the following factors,

$$\mathcal{V}^{(a,b)} \equiv \begin{cases} -4K_{ab} \zeta(a+b) & \text{for } \mathcal{V}^{(a,b)} \in U_{\text{gauge}} \\ 4K_{ab} \zeta \left(a + b, \frac{1}{2} \right) & \text{for } \mathcal{V}^{(a,b)} \in U_{\text{matter}} \end{cases}. \quad (4.24)$$

The relevant diagrams in the limit (4.1) are planar and tree as in the case of the reduced model.

Compared with the Feynman rule in the reduced model, the ABJM matrix model does not have the indices s, t, \dots in the Feynman diagrams. We will show that after summing over these indices in the reduced model, each diagram in the reduced model reproduces the corresponding diagram in the ABJM matrix model.

3. Perturbative correspondence of free energy

We compare the free energy of the reduced model with that of the ABJM theory. We will find that, in the limit (4.4), the free energy in the reduced model divided by $\Lambda + 1$ coincides with that in the ABJM theory to all orders in the perturbation theory;

$$\frac{\mathcal{F}_{\text{reduced}}}{\Lambda + 1} = \mathcal{F}_{\text{ABJM}}, \quad (4.25)$$

under the following identification of the coupling constants,

$$t_i = \lambda_i (i = 1, 2). \quad (4.26)$$

We note that under (4.26) the coefficients of the Gaussian terms of (4.15) and (4.22) coincide and so the factors coming from propagators in the reduced model and the ABJM theory trivially agree for the same Feynman diagrams. Therefore, in the following argument, we ignore the factors of propagators and only take care of the factors coming from vertices in these matrix models.

In the reduced model, any tree diagram can be decomposed as

$$\sum_s \sum_t V_{st}^{(a,b)} R_t. \quad (4.27)$$

Here, $V_{st}^{(a,b)}$ is an outermost vertex, that is, a vertex on the tip of a branch in the tree diagram, and R_t represents the rest of the diagram. See Fig. 3. The explicit form of R_t for the case shown in Fig. 3 is given by (E1).

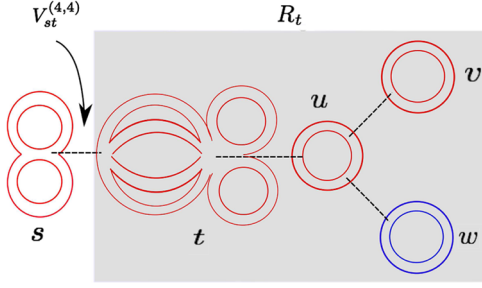


FIG. 3 (color online). This diagram has three outermost vertices, $V_{st}^{(4,4)}$, $V_{uv}^{(2,2)}$, $V_{uw}^{(2,2)}$. Here, we have separated $V_{st}^{(4,4)}$ from the shaded part, R_t .

We will show that in the $\Lambda \rightarrow \infty$ limit, we can replace the sum $\sum_s V_{st}^{(a,b)}$ in (4.27) by the corresponding vertex in the ABJM theory. That is,

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda + 1} \sum_t \sum_s V_{st}^{(a,b)} R_t = \lim_{\Lambda \rightarrow \infty} \frac{\mathcal{V}^{(a,b)}}{\Lambda + 1} \sum_t R_t, \quad (4.28)$$

where $\mathcal{V}^{(a,b)}$ in the right-hand side is given by $\mathcal{V}^{(a,b)} \in U_{\text{gauge}}$ when $V_{st}^{(a,b)} \in U_{\text{gauge}}^r$ and by $\mathcal{V}^{(a,b)} \in U_{\text{matter}}$ when $V_{st}^{(a,b)} \in U_{\text{matter}}^r$. If we establish (4.28), by repeatedly replacing the vertices of the reduced model by those of the ABJM theory, the factor coming from vertices in the reduced model agrees exactly with that in the ABJM matrix model. For an illustration of this procedure, see Fig. 4. Since this equivalence holds for all the tree diagrams, we thus establish the perturbative equivalence of the free energy between the reduced model and the ABJM theory (4.25).

We now prove (4.28). For the vertex $V_{st}^{(a,b)} \in U_{\text{matter}}^r$, the left-hand side in (4.28) is calculated as

$$\begin{aligned} & \lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda + 1} \sum_{t=-(\Lambda/2)}^{(\Lambda/2)} \sum_{s=-(\Lambda/2)}^{(\Lambda/2)} \frac{-2K_{ab}}{(s-t)^{a+b}} R_t \\ &= \lim_{\Lambda \rightarrow \infty} \frac{-2}{\Lambda + 1} \sum_{t=-(\Lambda/2)}^{(\Lambda/2)} K_{ab} R_t \left(2\zeta(a+b) \right. \\ & \quad \left. - \zeta\left(a+b, \frac{\Lambda}{2} + t + 1\right) - \zeta\left(a+b, \frac{\Lambda}{2} - t + 1\right) \right). \quad (4.29) \end{aligned}$$

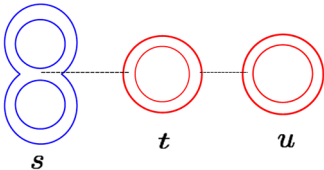


FIG. 4 (color online). The vertices in this diagram give a factor $\frac{1}{\Lambda+1} \sum_{s,t,u} V_{st}^{(2,4)} V_{tu}^{(2,2)}$, where $V_{st}^{(2,4)} \in U_{\text{matter}}^r$ and $V_{tu}^{(2,4)} \in U_{\text{gauge}}^r$. By (4.30), in the $\Lambda \rightarrow \infty$ limit this is equal to $\mathcal{V}^{(2,4)} \mathcal{V}^{(2,2)} \frac{1}{\Lambda+1} \sum_u 1 = \mathcal{V}^{(2,4)} \mathcal{V}^{(2,2)}$, and thus we can recover the factor of the corresponding vertices in the ABJM theory.

The first term above agrees with the right-hand side in (4.28). The remaining terms vanish in the $\Lambda \rightarrow \infty$ limit. To see this, we use the fact that $|R_t|$ has a Λ -independent upper bound denoted by C , which we prove in Appendix E. By using this fact, the absolute value of the remaining terms is bounded from above by

$$\begin{aligned} & \lim_{\Lambda \rightarrow \infty} \frac{2|K_{ab}|C}{\Lambda + 1} \sum_{t=-(\Lambda/2)}^{(\Lambda/2)} \left(\zeta\left(a+b, \frac{\Lambda}{2} + t + 1\right) \right. \\ & \quad \left. + \zeta\left(a+b, \frac{\Lambda}{2} - t + 1\right) \right) \\ &= 4|K_{ab}|C \lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda + 1} \left(\sum_{n=0}^{\Lambda} \frac{1}{(n+1)^{a+b-1}} \right. \\ & \quad \left. + \sum_{m=\Lambda+1}^{\infty} \frac{\Lambda+1}{(m+1)^{a+b}} \right) = 0, \quad (4.30) \end{aligned}$$

where we have used the definition of the generalized zeta function (4.14). The terms in the second line are $\mathcal{O}(\log \Lambda / \Lambda)$ for $a+b=2$, and $\mathcal{O}(1/\Lambda)$ for $a+b>2$.

For the vertex $V_{st}^{(a,b)} \in U_{\text{matter}}^r$, the left-hand side in (4.28) is calculated as

$$\begin{aligned} & \frac{1}{\Lambda + 1} \sum_{t=-(\Lambda/2)}^{(\Lambda/2)} \sum_{s=-(\Lambda/2)}^{(\Lambda/2)} \frac{4K_{ab}}{2^{a+b}} \left(\zeta\left(a+b, \frac{1}{4} + \frac{|s-t|}{2}\right) \right. \\ & \quad \left. - \zeta\left(a+b, \frac{3}{4} + \frac{|s-t|}{2}\right) \right) R_t \\ &= \frac{1}{\Lambda + 1} \sum_{t=-(\Lambda/2)}^{(\Lambda/2)} \frac{4K_{ab}}{2^{a+b}} \left(2^{a+b} \zeta\left(a+b, \frac{1}{2}\right) \right. \\ & \quad \left. - \zeta\left(a+b, \frac{1}{4} + \frac{\Lambda+t}{2}\right) - \zeta\left(a+b, \frac{3}{4} + \frac{\Lambda-t}{2}\right) \right) R_t, \quad (4.31) \end{aligned}$$

where we have used the equality, $\zeta(a+b, \frac{1}{4}) + \zeta(a+b, \frac{3}{4}) = 2^{a+b} \zeta(a+b, \frac{1}{2})$. The first term in (4.31) agrees with the right-hand side in (4.28). The remaining terms vanish in the $\Lambda \rightarrow \infty$ limit, which can be shown in the same way as (4.30). Therefore, the equality (4.28) has been proved.

Here, we briefly comment on the cutoff effect of Λ . For this purpose, we consider the one-point function $\langle \text{tr} \sigma_s^2 \rangle$ for a fixed $s \in \{-\frac{\Lambda}{2}, \dots, \frac{\Lambda}{2}\}$. Figure 5(a) is one of the diagrams which contributes to the one-point function, and leads to the following factor,

$$\sum_{u(\neq s)} \frac{1}{(s-u)^4} = \left(\sum_{n=1}^{(\Lambda/2)-s} + \sum_{n=1}^{s+(\Lambda/2)} \right) \frac{1}{n^4}. \quad (4.32)$$

In the $\Lambda \rightarrow \infty$ limit, while (4.32) goes to $2\zeta(4)$ for all $s \in \{-\frac{\Lambda}{2} + \mathcal{O}(\ln \Lambda), \dots, \frac{\Lambda}{2} - \mathcal{O}(\ln \Lambda)\}$, it deviates from $2\zeta(4)$ for s with $|s \pm \frac{\Lambda}{2}| \sim \mathcal{O}(\Lambda^0)$. Therefore, although (4.32)

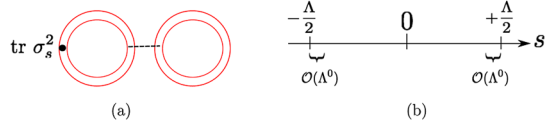


FIG. 5 (color online). (a): A diagram contributing to $\langle \text{tr} \sigma_s^2 \rangle$. (b): Only s near $\pm \frac{\Lambda}{2}$ feels the cutoff effect.

depends on the value of s , the dependence is negligible for almost all s except near the cutoff $\pm \frac{\Lambda}{2}$, and only s within $\mathcal{O}(\Lambda^0)$ from the boundaries feels the cutoff effect [See Fig. 5(b)]. Note that this argument also holds for more general diagrams. Thus, the number of the modes affected by the cutoff is $\mathcal{O}(\Lambda^0)$, which is negligible compared to the total number, $\Lambda + 1$. Since the free energy and one-point functions of Wilson loops are written as an average value over various s , the cutoff effect is negligible in the $\Lambda \rightarrow \infty$ limit.

4. Perturbative correspondence of Wilson loop

We can easily prove the perturbative equivalence between the Wilson loop in the reduced model and that in the ABJM matrix model. While the Wilson loop in the ABJM theory is given by (2.8), the corresponding object in the reduced model is obtained by applying the representation (4.2) to (3.46). Therefore, what we want to show is

$$\frac{1}{N_1} \langle \text{tr} e^{2\pi\sigma} \rangle = \frac{1}{N_1(\Lambda + 1)} \sum_{s=-\Lambda/2}^{\Lambda/2} \langle \text{tr} e^{2\pi\sigma_s} \rangle, \quad (4.33)$$

where the $\langle \dots \rangle$ means the average with respect to the eigenvalue integral of each theory. In perturbation theory, both of the left-hand side and the right-hand side above are calculated by expanding the exponentials. Thus, (4.33) holds if we have, for any positive integer a ,

$$\langle \text{tr} \sigma^a \rangle = \frac{1}{\Lambda + 1} \sum_{s=-\Lambda/2}^{\Lambda/2} \langle \text{tr} \sigma_s^a \rangle. \quad (4.34)$$

We find that this is true from (4.28), and so is (4.33).

C. Large- N equivalence of eigenvalue density

In this section, we investigate the eigenvalue distributions of σ_s and ρ_s in the reduced model for the ABJM matrix model. We show that if s is sufficiently apart from the cutoff $\pm \Lambda/2$ the eigenvalue densities of σ_s and ρ_s coincide with those of σ and ρ in the ABJM matrix model, respectively, and thereby prove the large- N equivalence between the reduced model and the original ABJM theory.

We start with the ABJM matrix model (2.5). From (2.5), the effective action for the eigenvalues are read off as

$$\begin{aligned} S_{\text{eff}} = & \frac{N_1}{\lambda_1} \sum_i \sigma_i^2 - \frac{N_2}{\lambda_2} \sum_\alpha \rho_\alpha^2 - \frac{1}{2} \sum_{i \neq j} \text{lnsinh}^2\{\pi(\sigma_i - \sigma_j)\} \\ & - \frac{1}{2} \sum_{\alpha \neq \beta} \text{lnsinh}^2\{\pi(\rho_\alpha - \rho_\beta)\} \\ & + 2 \sum_{i, \alpha} \text{lncosh}\{\pi(\sigma_i - \rho_\alpha)\}. \end{aligned} \quad (4.35)$$

Varying S_{eff} with respect to σ_i and ρ_α , we obtain the saddle point equation

$$\begin{aligned} 0 = & \frac{N_1}{\pi\lambda_1} \sigma_i - \sum_{j(\neq i)} \text{coth}\{\pi(\sigma_i - \sigma_j)\} \\ & + \sum_\alpha \tanh\{\pi(\sigma_i - \rho_\alpha)\}, \quad (4.36) \\ 0 = & -\frac{N_2}{\pi\lambda_2} \rho_\alpha - \sum_{\beta(\neq \alpha)} \text{coth}\{\pi(\rho_\alpha - \rho_\beta)\} \\ & - \sum_i \tanh\{\pi(\sigma_i - \rho_\alpha)\}. \end{aligned}$$

Let us introduce the eigenvalue densities for σ_i and ρ_α as

$$\rho(x) = \frac{1}{N_1} \sum_{i=1}^{N_1} \delta(x - \sigma_i), \quad \tilde{\rho}(x) = \frac{1}{N_2} \sum_{\alpha=1}^{N_2} \delta(x - \rho_\alpha). \quad (4.37)$$

Using these we can rewrite the saddle point equations as

$$\begin{aligned} 0 = & \frac{1}{\pi\lambda_1} x - \mathcal{f} dy \text{coth}\{\pi(x - y)\} \rho(y) + \frac{N_2}{N_1} \\ & \times \int dy \tanh\{\pi(x - y)\} \tilde{\rho}(y), \quad (4.38) \\ 0 = & -\frac{1}{\pi\lambda_2} x - \mathcal{f} dy \text{coth}\{\pi(x - y)\} \tilde{\rho}(y) + \frac{N_1}{N_2} \\ & \times \int dy \tanh\{\pi(x - y)\} \rho(y), \end{aligned}$$

where \mathcal{f} represents the Cauchy principal integral. In the large- N limit (4.1), the eigenvalues obeying (4.38) form a continuous distribution and so the eigenvalue densities become smooth functions. The explicit solution of (4.38) can be found in [3,43].

Next, let us consider our reduced model (4.6) for the ABJM theory. The effective action for the eigenvalues σ_{si} and ρ_{si} is given by

$$\begin{aligned}
S_{\text{eff}} = & \frac{N_1}{t_1} \sum_{si} \sigma_{si}^2 - \frac{N_2}{t_2} \sum_{s\alpha} \rho_{s\alpha}^2 - \frac{1}{2} \sum_s \sum_{i \neq j} \ln(\sigma_{si} - \sigma_{sj})^2 - \frac{1}{2} \sum_s \sum_{\alpha \neq \beta} \ln(\rho_{s\alpha} - \rho_{s\beta})^2 \\
& - \frac{1}{2} \sum_{s \neq t} \sum_{i,j} \ln \left[1 + \frac{(\sigma_{si} - \sigma_{tj})^2}{(s-t)^2} \right] - \frac{1}{2} \sum_{s \neq t} \sum_{\alpha, \beta} \ln \left[1 + \frac{(\rho_{s\alpha} - \rho_{t\beta})^2}{(s-t)^2} \right] - 2 \sum_{s,t} \sum_{i, \alpha} \sum_{J=|s-t|/2}^{\infty} \left[\ln \left[\left(2J + \frac{3}{2} \right)^2 \right. \right. \\
& \left. \left. + (\sigma_{si} - \rho_{t\alpha})^2 \right] - \ln \left[\left(2J + \frac{1}{2} \right)^2 + (\sigma_{si} - \rho_{t\alpha})^2 \right] \right], \tag{4.39}
\end{aligned}$$

where the summation of s, t is taken over $s, t = -\Lambda/2, \dots, \Lambda/2$. The saddle point equations are

$$\begin{aligned}
0 = & \frac{N_1}{t_1} \sigma_{si} - \sum_{j(\neq i)} \frac{1}{\sigma_{si} - \sigma_{sj}} - \sum_{t(\neq s)} \sum_j \frac{\sigma_{si} - \sigma_{tj}}{(s-t)^2 + (\sigma_{si} - \sigma_{tj})^2} - 2 \sum_{t, \alpha} \sum_{J=|s-t|/2}^{\infty} \left[\frac{\sigma_{si} - \rho_{t\alpha}}{\left(2J + \frac{3}{2} \right)^2 + (\sigma_{si} - \rho_{t\alpha})^2} \right. \\
& \left. - \frac{\sigma_{si} - \rho_{t\alpha}}{\left(2J + \frac{1}{2} \right)^2 + (\sigma_{si} - \rho_{t\alpha})^2} \right], \tag{4.40} \\
0 = & -\frac{N_2}{t_2} \rho_{s\alpha} - \sum_{\beta(\neq \alpha)} \frac{1}{\rho_{s\alpha} - \rho_{s\beta}} - \sum_{t(\neq s)} \sum_{\beta} \frac{\rho_{s\alpha} - \rho_{s\beta}}{(s-t)^2 + (\rho_{s\alpha} - \rho_{t\beta})^2} + 2 \sum_{t, i} \sum_{J=|s-t|/2}^{\infty} \left[\frac{\sigma_{ti} - \rho_{s\alpha}}{\left(2J + \frac{3}{2} \right)^2 + (\sigma_{ti} - \rho_{s\alpha})^2} \right. \\
& \left. - \frac{\sigma_{ti} - \rho_{s\alpha}}{\left(2J + \frac{1}{2} \right)^2 + (\sigma_{ti} - \rho_{s\alpha})^2} \right].
\end{aligned}$$

We introduce the eigenvalue densities of σ_s and ρ_s in the reduced model as

$$\rho^{[s]}(x) = \frac{1}{N_1} \sum_{i=1}^{N_1} \delta(x - \sigma_{si}), \quad \tilde{\rho}^{[s]}(x) = \frac{1}{N_2} \sum_{\alpha=1}^{N_2} \delta(x - \rho_{s\alpha}), \tag{4.41}$$

and rewrite (4.40) as

$$\begin{aligned}
0 = & \frac{1}{t_1} x - \int dy \frac{1}{x-y} \rho^{[s]}(y) - \sum_{t(\neq s)} \int dy \frac{x-y}{(s-t)^2 + (x-y)^2} \rho^{[t]}(y) - \frac{2N_2}{N_1} \sum_t \sum_{J=|s-t|/2}^{\infty} \int dy \left[\frac{x-y}{\left(2J + \frac{3}{2} \right)^2 + (x-y)^2} \right. \\
& \left. - \frac{x-y}{\left(2J + \frac{1}{2} \right)^2 + (x-y)^2} \right] \tilde{\rho}^{[t]}(y), \\
0 = & -\frac{1}{t_2} x - \int dy \frac{1}{x-y} \tilde{\rho}^{[s]}(y) - \sum_{t(\neq s)} \int dy \frac{x-y}{(s-t)^2 + (x-y)^2} \tilde{\rho}^{[t]}(y) - \frac{2N_1}{N_2} \sum_t \sum_{J=|s-t|/2}^{\infty} \int dy \left[\frac{x-y}{\left(2J + \frac{3}{2} \right)^2 + (x-y)^2} \right. \\
& \left. - \frac{x-y}{\left(2J + \frac{1}{2} \right)^2 + (x-y)^2} \right] \rho^{[t]}(y). \tag{4.42}
\end{aligned}$$

We can find a solution to these equations in the $\Lambda \rightarrow \infty$ limit as follows. If one naively takes the $\Lambda \rightarrow \infty$ limit in (4.42), $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x)) = (\rho(x), \tilde{\rho}(x))$ with $\lambda_1 = t_1$ and $\lambda_2 = t_2$ for arbitrary s turns out to be a solution to (4.42), where $(\rho(x), \tilde{\rho}(x))$ is the solution to the saddle point equation (4.38) of the ABJM matrix model. This is because in this case (4.42) reduces to (4.38) (see Appendix F 1). This solution represents infinitely many copies of that of the original ABJM matrix model. Since the free energy and the Wilson loop in the reduced model are given by an average over all s 's as (4.25) and (4.33), they exactly coincide with those in the ABJM matrix model.

The densities $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x))$ with s near the cutoff Λ deviate from $(\rho(x), \tilde{\rho}(x))$. This cutoff effect would spoil the above naive argument if the correlation range between $\rho^{[s]}$'s and $\tilde{\rho}^{[s]}$'s became larger as $\mathcal{O}(\Lambda)$. In this case, the

number of $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x))$ which deviates from $(\rho(x), \tilde{\rho}(x))$ would be $\mathcal{O}(\Lambda)$, namely, the number of the deviating densities and that of the densities coinciding with $(\rho(x), \tilde{\rho}(x))$ would become comparable. Then, the free energy and the Wilson loop in the reduced model given by the average over s 's would not coincide with those in the original model.

However, this is not the case of our reduced model. It turns out that the correlation range is $\mathcal{O}(\Lambda^0)$, and so the above naive argument is indeed valid. In fact, in the saddle point equation for $\rho^{[s]}(x)$ (or $\tilde{\rho}^{[s]}(x)$), the coefficients of $\rho^{[t]}(x)$ and $\tilde{\rho}^{[t]}(x)$ are suppressed if $|t-s|$ is large enough. As shown in Appendix F 2, the contributions from the terms with $|t-s| \geq \ln \Lambda$ can be neglected in the $\Lambda \rightarrow \infty$ limit. Namely, the profile of $\rho^{[s]}(x)$ (or $\tilde{\rho}^{[s]}(x)$) is determined only by $(\rho^{[t]}(x), \tilde{\rho}^{[t]}(x))$'s with $|t-s| \leq \mathcal{O}(\Lambda^0)$,

which means that the correlation range is sufficiently small compared to the system size Λ , so that $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x))$ for $|s| \leq \Lambda/2 - \ln\Lambda$ are not affected by the cutoff. Therefore, in the $\Lambda \rightarrow \infty$ limit $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x)) = (\rho(x), \tilde{\rho}(x))$ still holds except for very narrow region $\Lambda/2 - \ln\Lambda \leq |s| \leq \Lambda/2$. This is consistent with our observation in the perturbation theory mentioned in the last part of Sec. IV B 3. Also, this behavior of the densities is observed in the numerical simulation for the reduced model of the pure CS theory on S^3 [39]. Although $(\rho^{[s]}(x), \tilde{\rho}^{[s]}(x))$ for $\Lambda/2 - \ln\Lambda \leq |s| \leq \Lambda/2$ differs from $(\rho(x), \tilde{\rho}(x))$, their contributions to the physical quantities, such as the free energy and the BPS Wilson loops, are negligible since the physical quantities are calculated as an average taken over all s 's.

Thus, for supersymmetric observables, the large- N equivalence between the reduced model and the ABJM theory is also shown nonperturbatively through the saddle point method of the eigenvalue density. One can also apply the saddle point analysis to general $\mathcal{N} = 2$ quiver CS theories and show the large- N equivalence.

V. CONCLUSION

In this paper, we have studied the large- N reduction for a general $\mathcal{N} = 2$ quiver CS theory on S^3 . We considered the reduced model of the ABJM theory on S^3 as an illustration and explained the calculation of the free energy and the one-point function of the BPS Wilson loop operator in the reduced model. We found that the localization technique reduces the calculation to eigenvalue integrals, as in the original theory on S^3 . To establish the large- N equivalence, we first studied the integrals in the perturbation theory. We constructed the Feynman rule from the eigenvalue integrals, and found that each Feynman diagram in the reduced model coincides with a corresponding diagram in the ABJM matrix model in the continuum limit. Hence, we conclude that these supersymmetric quantities are equivalent in two theories to all orders in the perturbative expansion. Then, we considered the saddle point configuration of the eigenvalues in the reduced model. We found that in the continuum limit the cutoff effect is sufficiently small and that the eigenvalue densities in the reduced model at the saddle point consist of infinitely many copies of those in the original theory. This implies that the expectation values of supersymmetric observables in the reduced model, which are written as the average over all the copies, agree with those in the original theory in the continuum limit. Thus, the large- N equivalence holds also nonperturbatively. Our result gives a strong evidence that the non-perturbative formulation for supersymmetric theories based on the novel large- N reduction works successfully.

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APPENDIX A: $\mathcal{N} = 2$ QUIVER CS THEORY ON S^3

In this Appendix, we summarize our convention for $\mathcal{N} = 2$ quiver CS theory on S^3 [5]. We consider the gauge vector multiplet and the matter chiral multiplets in the adjoint and in the bifundamental representation.

A gauge multiplet contains fermionic (Grassmannian) fields $\{\lambda, \bar{\lambda}\}$ as well as bosonic fields $\{A_\mu, \sigma, D\}$. There are two kinds of supersymmetric action for this multiplet: the CS action and the YM action, which are defined by

$$S_{\text{CS}} = - \int d^3x \text{tr} \left[\varepsilon^{\mu\nu\lambda} \left(A_\mu \partial_\nu A_\lambda - \frac{2i}{3} A_\mu A_\nu A_\lambda \right) + \sqrt{g}(-\bar{\lambda}\lambda + 2D\sigma) \right], \quad (\text{A1})$$

$$S_{\text{YM}} = \int d^3x \sqrt{g} \text{tr} \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} (\sigma + D)^2 + \frac{1}{2} (D_\mu \sigma)^2 + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda - \frac{1}{4} \bar{\lambda} \lambda + \frac{i}{2} \bar{\lambda} [\sigma, \lambda] \right]. \quad (\text{A2})$$

The field strength is defined as usual by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, and the covariant derivative contains the gauge and the spin connections,

$$D_\mu \lambda = \partial_\mu \lambda + \frac{1}{4} \omega_\mu^{bc} \gamma_{bc} \lambda - i[A_\mu, \lambda]. \quad (\text{A3})$$

A matter multiplet in the bifundamental representation consists of bosonic fields $\{\phi, \bar{\phi}, F, \bar{F}\}$ and fermionic fields $\{\psi, \bar{\psi}\}$. We assume that they couple to a gauge multiplet $\{A_\mu, \lambda, \sigma, D\}$ as the fundamental representation and to another gauge multiplet $\{B_\mu, \eta, \rho, \tilde{D}\}$ as the antifundamental. The supersymmetric action for this multiplet is given by⁷

$$S_{\text{matter}} = \int d^3x \sqrt{g} \text{tr} \left[D^\mu \bar{\phi} D_\mu \phi - i \bar{\psi} \gamma^\mu D_\mu \psi + q(2 - q) \times \bar{\phi} \phi - \frac{2q - 1}{2} \bar{\psi} \psi + i(2q - 1) \bar{\phi} \nabla(\sigma, \rho) \phi + i \bar{\psi} \nabla(\sigma, \rho) \psi + i \bar{\psi} \nabla(\lambda, \eta) \phi - i \bar{\phi} \nabla(\bar{\lambda}, \bar{\eta}) \psi + i \bar{\phi} \nabla(D, \tilde{D}) \phi + \bar{\phi} \nabla(\sigma, \rho)^2 \phi + \bar{F} F \right]. \quad (\text{A4})$$

Here, q is the anomalous dimension and $\nabla(A, B)$ is defined as the operator which acts as

⁷In general, the theory may have a superpotential. We ignore it in this paper since it is irrelevant for the localization calculation.

$$\nabla(A, B)\phi := A\phi - \phi B, \quad \nabla(A, B)\bar{\phi} := B\bar{\phi} - \bar{\phi}A \quad (\text{A5})$$

on the bifundamental and on the antibifundamental field, respectively. The covariant derivative acts on the spinors as

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{4}\omega_\mu^{bc}\gamma_{bc}\psi - i\nabla(A_\mu, B_\mu)\psi. \quad (\text{A6})$$

The actions (A1), (A2), and (A4) are invariant under the supersymmetry transformations,

$$\begin{aligned} \delta A_\mu &= \frac{i}{2}(\bar{\lambda}\gamma_\mu\epsilon - \bar{\epsilon}\gamma_\mu\lambda), & \delta\sigma &= -\frac{1}{2}(\bar{\lambda}\epsilon - \bar{\epsilon}\lambda), \\ \delta\lambda &= \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon, \\ \delta\bar{\lambda} &= \frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon}, \\ \delta D &= -\frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon - \frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] \\ &\quad + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] - \frac{i}{6}(\bar{\lambda}\gamma^\mu D_\mu\epsilon + D_\mu\bar{\epsilon}\gamma^\mu\lambda), \end{aligned} \quad (\text{A7})$$

for the gauge multiplet and

$$\begin{aligned} \delta\phi &= \bar{\epsilon}\psi, & \delta\bar{\phi} &= \epsilon\bar{\psi}, \\ \delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\nabla(\sigma, \rho)\phi + \frac{2iq}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F, \\ \delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} - i\nabla(\sigma, \rho)\bar{\phi}\bar{\epsilon} + \frac{2iq}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}\epsilon, \\ \delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\nabla(\sigma, \rho)\psi - i\nabla(\lambda, \eta)\phi) \\ &\quad + \frac{i}{3}(2q-1)D_\mu\epsilon\gamma^\mu\psi, \\ \delta\bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} + i\nabla(\sigma, \rho)\bar{\psi} - i\nabla(\bar{\lambda}, \bar{\eta})\bar{\phi}) \\ &\quad + \frac{i}{3}(2q-1)D_\mu\bar{\epsilon}\gamma^\mu\bar{\psi}, \end{aligned} \quad (\text{A8})$$

for the bifundamental matter multiplet. The Grassmannian parameters ϵ and $\bar{\epsilon}$ satisfy the Killing spinor equation,

$$D_\mu\epsilon = \pm\frac{i}{2}\gamma_\mu\epsilon. \quad (\text{A9})$$

In the right-invariant frame defined in Appendix B, solutions to the Killing spinor equation are given by

$$\epsilon = \epsilon_0 \quad \text{and} \quad \bar{\epsilon} = g\epsilon_0 \quad (\text{A10})$$

for the upper and the lower sign in (A9), respectively, where ϵ_0 is a constant spinor on S^3 and g is a group element of $SU(2)$ defined in (B1).

The supersymmetry transformation δ can be divided into two parts generated by ϵ and $\bar{\epsilon}$ as $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$. While two unbarred or two barred supersymmetries commute, the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ is given by a sum of translation, gauge transformation, Lorentz rotation, dilatation, and R -rotation.

One can also obtain the action and the supersymmetry transformation for an adjoint matter multiplet by identifying one gauge multiplet with the other in (A4) and (A8).

APPENDIX B: OUR CONVENTION FOR S^3

In this Appendix, we summarize our convention for S^3 with a unit radius (see also [11,14]). S^3 is viewed as the $SU(2)$ group manifold. We parametrize an element of $SU(2)$ in terms of the Euler angles as

$$g = e^{-i\varphi\gamma_3/2}e^{-i\theta\gamma_2/2}e^{-i\psi\gamma_3/2}, \quad (\text{B1})$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$, $0 \leq \psi < 4\pi$, and γ_a are the Pauli matrices. The periodicity for these angle variables is given by

$$(\theta, \varphi, \psi) \sim (\theta, \varphi + 2\pi, \psi + 2\pi) \sim (\theta, \varphi, \psi + 4\pi). \quad (\text{B2})$$

The isometry of S^3 corresponds to the left and the right multiplications of $SU(2)$ elements on g . We construct the right-invariant 1-forms under the multiplications,

$$dgg^{-1} = -ie^a\gamma_a. \quad (\text{B3})$$

The explicit form of e^a is given by

$$\begin{aligned} e^1 &= \frac{1}{2}(-\sin\varphi d\theta + \sin\theta\cos\varphi d\psi), \\ e^2 &= \frac{1}{2}(\cos\varphi d\theta + \sin\theta\sin\varphi d\psi), \\ e^3 &= \frac{1}{2}(d\varphi + \cos\theta d\psi). \end{aligned} \quad (\text{B4})$$

It is easy to see that e_a satisfy the Maurer-Cartan equation,

$$de^a - \varepsilon_{abc}e^b \wedge e^c = 0. \quad (\text{B5})$$

We take e^a as the vielbein in this paper. In this frame, the spin connection is simply given by $\omega^{ab} = \varepsilon^{abc}e^c$. The metric is given by

$$ds^2 = e^a e^a = \frac{1}{4}(d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi)^2). \quad (\text{B6})$$

APPENDIX C: COMMUTATOR BETWEEN δ_ϵ AND $\delta_{\bar{\epsilon}}$ IN REDUCED MODEL

The actions of $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ on all the matrices are shown below:

$$\begin{aligned} [\delta_\epsilon, \delta_{\bar{\epsilon}}]A_a &= \Theta_a^b A_b + i[\chi, A_a], & [\delta_\epsilon, \delta_{\bar{\epsilon}}]\sigma &= i[\chi, \sigma], \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\lambda &= \frac{1}{4}\Theta_{ab}\gamma^{ab}\lambda + i[\chi, \lambda] + \alpha\lambda, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\lambda} &= \frac{1}{4}\Theta_{ab}\gamma^{ab}\bar{\lambda} + i[\chi, \bar{\lambda}] - \alpha\bar{\lambda}, \\ [\delta_\epsilon, \delta_{\bar{\epsilon}}]D &= i[\chi, D], \end{aligned} \quad (\text{C1})$$

and

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\phi &= i\nabla(\chi, \tilde{\chi})\phi - q\alpha\phi, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\phi} &= i\nabla(\chi, \tilde{\chi})\bar{\phi} + q\alpha\bar{\phi}, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\psi &= \frac{1}{4}\Theta_{ab}\gamma^{ab}\psi + i\nabla(\chi, \tilde{\chi})\psi + (1-q)\alpha\psi, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\psi} &= \frac{1}{4}\Theta_{ab}\gamma^{ab}\bar{\psi} + i\nabla(\chi, \tilde{\chi})\bar{\psi} - (1-q)\alpha\bar{\psi}, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]F &= i\nabla(\chi, \tilde{\chi})F + (2-q)\alpha F, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{F} &= i\nabla(\chi, \tilde{\chi})\bar{F} - (2-q)\alpha\bar{F},
\end{aligned} \tag{C2}$$

where

$$\begin{aligned}
\Theta^{ab} &:= 2i\varepsilon^{abc}\bar{\epsilon}\gamma_c\epsilon, & \chi &:= -iA_a\bar{\epsilon}\gamma^a\epsilon + \sigma\bar{\epsilon}\epsilon, \\
\tilde{\chi} &:= -iB_a\bar{\epsilon}\gamma^a\epsilon + \rho\bar{\epsilon}\epsilon, & \alpha &:= \bar{\epsilon}\epsilon.
\end{aligned} \tag{C3}$$

The action of $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ on the gauge multiplet $\{B_a, \rho, \eta, \tilde{D}\}$ takes the same form as $\{A_a, \sigma, \lambda, D\}$. We can read off from the above equations that Θ^{ab} are parameters of $SU(2)$ rotation, χ and $\tilde{\chi}$ are gauge transformations for A_a and B_a , respectively, and α is R -rotation.

APPENDIX D: FUZZY SPHERICAL HARMONICS

In this Appendix, we review the fuzzy spherical harmonics which form a basis of rectangular matrices [11, 12].

Let us consider a $(2j_s + 1) \times (2j_t + 1)$ rectangular complex matrix. Such a matrix $M^{(s,t)}$ can be generally expanded as

$$M^{(s,t)} = \sum_{m_s, m_t} M_{m_s, m_t} |j_s, m_s\rangle \langle j_t, m_t|, \tag{D1}$$

by using a basis $\{|jm\rangle | m = -j, -j+1, \dots, j\}$ of the spin j representation space of $SU(2)$ algebra. We define an operation which multiplies the representation matrices of the $SU(2)$ generators from left and right:

$$\begin{aligned}
L_a \circ M^{(s,t)} &= \sum_{m_s, m_t} M_{m_s, m_t} (L_a^{[j_s]} |j_s, m_s\rangle \langle j_t, m_t| - |j_s, m_s\rangle \\
&\quad \times \langle j_t, m_t| L_a^{[j_t]}),
\end{aligned} \tag{D2}$$

where $L_a^{[j]}$ stands for the spin j representation matrix of the generator.

We can construct another basis of the rectangular matrices denoted by $\{\hat{Y}_{Jm(j_s, j_t)}\}$ such that they satisfy

$$\begin{aligned}
(L_a \circ)^2 \hat{Y}_{Jm(j_s, j_t)} &= J(J+1) \hat{Y}_{Jm(j_s, j_t)}, \\
L_\pm \circ \hat{Y}_{Jm(j_s, j_t)} &= \sqrt{(J \mp m)(J \pm m + 1)} \hat{Y}_{J, m \pm 1(j_s, j_t)}, \\
L_3 \circ \hat{Y}_{Jm(j_s, j_t)} &= m \hat{Y}_{Jm(j_s, j_t)}.
\end{aligned} \tag{D3}$$

$\hat{Y}_{Jm(j_s, j_t)}$ are called scalar fuzzy spherical harmonics and defined by

$$\hat{Y}_{Jm(j_s, j_t)} = \sum_{m_s, m_t} (-)^{-j_s + m_t} C_{j_s, m_s, j_t, m_t}^{Jm} |j_s, m_s\rangle \langle j_t, m_t|, \tag{D4}$$

where $C_{j_s, m_s, j_t, m_t}^{Jm}$ are the Clebsch-Gordan coefficients. Their Hermitian conjugates are given by

$$(\hat{Y}_{Jm(j_s, j_t)})^\dagger = (-)^{m-(j_s-j_t)} \hat{Y}_{J-m(j_t, j_s)}, \tag{D5}$$

and they satisfy the orthogonality relation

$$\text{tr}\{(\hat{Y}_{Jm(j_s, j_t)})^\dagger \hat{Y}_{J'm'(j'_s, j'_t)}\} = \delta_{J, J'} \delta_{m, m'}. \tag{D6}$$

Then, we define the vector fuzzy spherical harmonics $\hat{Y}_{Jm(j_s, j_t)a}^\rho$ and the spinor fuzzy spherical harmonics $\hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa$, where $\rho = -1, 0, 1$, $\kappa = -1, 1$. The indices $a = 1, 2, 3$ and $\alpha = 1, 2$ are those for vectors and spinors, respectively.⁸ They are written in terms of the scalar fuzzy spherical harmonics,

$$\hat{Y}_{Jm(j_s, j_t)a}^\rho = i^\rho \sum_{n, p} V_{an} C_{\tilde{Q}p1n}^{Qm} \hat{Y}_{\tilde{Q}p(j_s, j_t)}, \tag{D7}$$

$$\hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa = \sum_p C_{\tilde{U}p\frac{1}{2}\alpha}^{Um} \hat{Y}_{\tilde{U}p(j_s, j_t)}, \tag{D8}$$

where $Q = J + \delta_{\rho, 1}$, $\tilde{Q} = J + \delta_{\rho, -1}$, $U = J + \frac{1}{2}\delta_{\kappa, 1}$, and $\tilde{U} = J + \frac{1}{2}\delta_{\kappa, -1}$. V is a unitary matrix defined by

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \tag{D9}$$

The vector and the spinor harmonics satisfy the following formulas,

$$L_a \circ \hat{Y}_{Jm(j_s, j_t)a}^\rho = \sqrt{J(J+1)} \delta_{\rho, 0} \hat{Y}_{Jm(j_s, j_t)}, \tag{D10}$$

$$i\varepsilon_{abc} L_b \circ \hat{Y}_{Jm(j_s, j_t)c}^\rho + \hat{Y}_{Jm(j_s, j_t)a}^\rho = \rho(J+1) \hat{Y}_{Jm(j_s, j_t)a}^\rho, \tag{D11}$$

$$(\gamma^a)_\alpha^\beta L_a \circ \hat{Y}_{Jm(j_s, j_t)\beta}^\kappa + \frac{3}{4} \hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa = \kappa(J + \frac{3}{4}) \hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa. \tag{D12}$$

Their Hermitian conjugates are given by

$$(\hat{Y}_{Jm(j_s, j_t)a}^\rho)^\dagger = (-)^{m-(j_s-j_t)+1} \hat{Y}_{J-m(j_t, j_s)a}^\rho, \tag{D13}$$

$$(\hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa)^\dagger = (-)^{m-(j_s-j_t)+\kappa\alpha+1} \hat{Y}_{J-m(j_t, j_s)-\alpha}^\kappa. \tag{D14}$$

They also satisfy the orthogonality relations,

$$\text{tr}\{(\hat{Y}_{Jm(j_s, j_t)a}^\rho)^\dagger \hat{Y}_{J'm'(j'_s, j'_t)a}^{\rho'}\} = \delta_{J, J'} \delta_{m, m'} \delta_{\rho, \rho'}, \tag{D15}$$

$$\text{tr}\{(\hat{Y}_{Jm(j_s, j_t)\alpha}^\kappa)^\dagger \hat{Y}_{J'm'(j'_s, j'_t)\alpha}^{\kappa'}\} = \delta_{J, J'} \delta_{m, m'} \delta_{\kappa, \kappa'}. \tag{D16}$$

⁸Here, we mean just a set of three or two matrices by ‘‘vector’’ or ‘‘spinor.’’ This terminology makes sense only when we regard them as the regularized version of the vector and the spinor spherical harmonics on S^2 in the presence of a monopole. See [12] and references therein.

APPENDIX E: FINITENESS OF R_t IN THE LIMIT (4.27)

We prove the finiteness of R_t in the limit of $\Lambda \rightarrow \infty$. We first give the proof for a simple case shown in Fig. 3.

In this case, R_t is given by

$$R_t = \sum_{\substack{u=-(\Lambda/2) \\ (u \neq t)}}^{(\Lambda/2)} \sum_{\substack{v=-(\Lambda/2) \\ (v \neq u)}}^{(\Lambda/2)} \sum_{w=-(\Lambda/2)}^{(\Lambda/2)} V_{tu}^{(8,2)} V_{uv}^{(2,2)} V_{uw}^{(2,2)}, \quad (\text{E1})$$

and using the explicit form (4.21), this becomes

$$\sum_{\substack{u=-(\Lambda/2) \\ (u \neq t)}}^{(\Lambda/2)} \sum_{\substack{v=-(\Lambda/2) \\ (v \neq u)}}^{(\Lambda/2)} \sum_{w=-(\Lambda/2)}^{(\Lambda/2)} \frac{1}{(t-u)^{10}} \frac{1}{(u-v)^4} \times \frac{1}{2^4} \left(\zeta\left(4, \frac{1}{4} + \frac{|u-w|}{2}\right) - \zeta\left(4, \frac{3}{4} + \frac{|u-w|}{2}\right) \right), \quad (\text{E2})$$

where we have omitted the Λ -independent factor, $K_{82}K_{22}K_{22}$, defined in (4.11). To evaluate this in the limit $\Lambda \rightarrow \infty$, we make use of the following inequalities,

$$\begin{aligned} \sum_{\substack{s=-(\Lambda/2) \\ (s \neq t)}}^{(\Lambda/2)} \frac{1}{(s-t)^{2l}} &= 2\zeta(z) - \zeta\left(z, \frac{\Lambda}{2} + s + 1\right) \\ &- \zeta\left(z, \frac{\Lambda}{2} - s + 1\right) < 2\zeta(z), \\ \sum_{s=-(\Lambda/2)}^{(\Lambda/2)} \frac{1}{2^z} \left(\zeta\left(z, \frac{1}{4} + \frac{|s-t|}{2}\right) - \zeta\left(z, \frac{3}{4} + \frac{|s-t|}{2}\right) \right) \\ &= 2^z \zeta\left(z, \frac{1}{2}\right) - \zeta\left(z, \frac{3}{4} + \frac{t + \frac{\Lambda}{2}}{2}\right) \\ &- \zeta\left(z, \frac{3}{4} + \frac{\frac{\Lambda}{2} - t}{2}\right) < 2^z \zeta\left(z, \frac{1}{2}\right). \end{aligned} \quad (\text{E3})$$

By utilizing these, (E2) is bounded from above by

$$2\zeta(10) \times 2\zeta(4) \times \zeta\left(4, \frac{1}{2}\right), \quad (\text{E4})$$

and thus R_t is finite even for $\Lambda \rightarrow \infty$.

As is obvious from this proof, a general R_t can also be bounded from above by a product of ζ functions, and thus we complete the proof of the finiteness of R_t .

APPENDIX F: SADDLE POINT EQUATION IN REDUCED MODEL

1. Naive $\Lambda \rightarrow \infty$ limit

In this Appendix, we show that, if one naively takes the $\Lambda \rightarrow \infty$ limit in (4.42), $\rho^{[s]} = \rho$ and $\tilde{\rho}^{[s]} = \tilde{\rho}$ for all s is a solution of (4.42). To see this, we take the $\Lambda \rightarrow \infty$ limit and substitute the ansatz $\rho^{[s]} = \hat{\rho}$ and $\tilde{\rho}^{[s]} = \hat{\tilde{\rho}}$ for all s into (4.42). [Here, we show only the first equation of (4.42) for simplicity. We can show the second one completely in the same manner.] Then, we obtain

$$\begin{aligned} 0 &= \frac{1}{t_1} x - \sum_{t=-\infty}^{\infty} \int dy \frac{x-y}{(s-t)^2 + (x-y)^2} \hat{\rho}(y) \\ &- \frac{2N_2}{N_1} \sum_{t=-\infty}^{\infty} \sum_{J=\frac{|s-t|}{2}}^{\infty} \int dy \left\{ \frac{x-y}{(2J+\frac{3}{2})^2 + (x-y)^2} \right. \\ &\left. - \frac{x-y}{(2J+\frac{1}{2})^2 + (x-y)^2} \right\} \hat{\tilde{\rho}}(y). \end{aligned} \quad (\text{F1})$$

By using the following formulas,

$$\begin{aligned} \coth x &= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{\pi^2 n^2 + x^2} = \sum_{n=-\infty}^{\infty} \frac{x}{\pi^2 n^2 + x^2}, \\ \tanh x &= \sum_{n=1}^{\infty} \frac{2x}{\pi^2 (n-\frac{1}{2})^2 + x^2} = \sum_{n=-\infty}^{\infty} \frac{x}{\pi^2 (n-\frac{1}{2})^2 + x^2}, \end{aligned} \quad (\text{F2})$$

it is easily seen that the second term in the right-hand side of (F1) is rewritten as

$$\begin{aligned} &\sum_{t=-\infty}^{\infty} \int dy \frac{x-y}{t^2 + (x-y)^2} \hat{\rho}(y) \\ &= \pi \int dy \coth\{\pi(x-y)\} \hat{\rho}(y) \end{aligned} \quad (\text{F3})$$

while the third term is

$$\begin{aligned} &\sum_{t=-\infty}^{\infty} \sum_{J=\frac{|s-t|}{2}}^{\infty} \int dy \left\{ \frac{x-y}{(2J+\frac{3}{2})^2 + (x-y)^2} - \frac{x-y}{(2J+\frac{1}{2})^2 + (x-y)^2} \right\} \hat{\tilde{\rho}}(y) \\ &= \sum_{J=0, \frac{1}{2}, 1, \dots}^{\infty} (2J+1) \int dy \left\{ \frac{x-y}{(2J+\frac{3}{2})^2 + (x-y)^2} - \frac{x-y}{(2J+\frac{1}{2})^2 + (x-y)^2} \right\} \hat{\tilde{\rho}}(y) \\ &= - \sum_{n=0}^{\infty} \int dy \frac{x-y}{(n+\frac{1}{2})^2 + (x-y)^2} \hat{\tilde{\rho}}(y) \\ &= - \frac{\pi}{2} \int dy \tanh\{\pi(x-y)\} \hat{\tilde{\rho}}(y). \end{aligned} \quad (\text{F4})$$

Then, (F1) becomes

$$0 = \frac{1}{t_1} x - \pi \int dy \coth\{\pi(x-y)\} \hat{\rho}(y) + \frac{\pi N_2}{N_1} \int dy \tanh\{\pi(x-y)\} \hat{\tilde{\rho}}(y). \quad (\text{F5})$$

This is nothing but (4.38) under the identification $t_1 = \lambda_1$ and $t_2 = \lambda_2$, and thus $\hat{\rho} = \rho$ and $\hat{\tilde{\rho}} = \tilde{\rho}$ follow.

2. Contributions from $\rho^{[t]}$ and $\tilde{\rho}^{[t]}$ with $|t-s| \geq \ln\Lambda$

Here, we evaluate the contributions from $\rho^{[t]}$ and $\tilde{\rho}^{[t]}$ with $|t-s| \geq \ln\Lambda$ in the first equation of (4.42). We show that they are negligible in the $\Lambda \rightarrow \infty$ limit. The same evaluation can be applied to the second equation of (4.42).

We assume that $\rho^{[t]}$ and $\tilde{\rho}^{[t]}$ have finite supports, and so there exists a region $[a, b]$ which contains all the supports. There also exist two constants c and \tilde{c} satisfying $\rho^{[t]}(x) \leq c$ and $\tilde{\rho}^{[t]}(x) \leq \tilde{c}$, respectively, for arbitrary $x \in \mathbb{R}$ and t .

First, we evaluate such contributions in the second term in (4.42),

$$\begin{aligned} \left(\sum_{t=s+\ln\Lambda}^{\Lambda/2} + \sum_{t=-\Lambda/2}^{s-\ln\Lambda} \right) \int_a^b dy \frac{x-y}{(s-t)^2 + (x-y)^2} \rho^{[t]}(y) &\leq c \left(\sum_{t=s+\ln\Lambda}^{\Lambda/2} + \sum_{t=-\Lambda/2}^{s-\ln\Lambda} \right) \int_a^b dy \frac{x-y}{(s-t)^2 + (x-y)^2} \\ &\simeq c \left(\sum_{n=\ln\Lambda}^{\Lambda/2-s} + \sum_{n=-\Lambda/2-s}^{-\ln\Lambda} \right) \left\{ \int_a^b dy \frac{(x-y)}{n^2} + \mathcal{O}(n^{-4}) \right\} \rightarrow 0 \quad (\Lambda \rightarrow \infty). \end{aligned} \quad (\text{F6})$$

In the same way, those in the third term are evaluated as

$$\begin{aligned} \left(\sum_{t=s+\ln\Lambda}^{\Lambda/2} + \sum_{t=-\Lambda/2}^{s-\ln\Lambda} \right) \sum_{J=\lfloor \frac{|s-t|}{2} \rfloor}^{\infty} \int_a^b dy \left\{ \frac{x-y}{(2J+\frac{3}{2})^2 + (x-y)^2} - \frac{x-y}{(2J+\frac{1}{2})^2 + (x-y)^2} \right\} \tilde{\rho}^{[t]}(y) \\ \leq \tilde{c} \left(\sum_{t=s+\ln\Lambda}^{\Lambda/2} + \sum_{t=-\Lambda/2}^{s-\ln\Lambda} \right) \sum_{J=\lfloor \frac{|s-t|}{2} \rfloor}^{\infty} \int_a^b dy \left\{ \frac{x-y}{(2J+\frac{3}{2})^2 + (x-y)^2} - \frac{x-y}{(2J+\frac{1}{2})^2 + (x-y)^2} \right\} \\ = \tilde{c} \left(\sum_{n=\ln\Lambda}^{\Lambda/2-s} + \sum_{n=-\Lambda/2-s}^{-\ln\Lambda} \right) \sum_{J=\lfloor \frac{|s-t|}{2} \rfloor}^{\infty} \int_a^b dy \left\{ -\frac{x-y}{J^3} + \mathcal{O}(J^{-4}) \right\} \rightarrow 0 \quad (\Lambda \rightarrow \infty). \end{aligned} \quad (\text{F7})$$

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- [1] V. Pestun, [arXiv:0712.2824](#).
[2] A. Kapustin, B. Willett, and I. Yaakov, *J. High Energy Phys.* **03** (2010) 089.
[3] N. Drukker, M. Marino, and P. Putrov, *Commun. Math. Phys.* **306**, 511 (2011).
[4] D.L. Jafferis, [arXiv:1012.3210](#).
[5] N. Hama, K. Hosomichi, and S. Lee, *J. High Energy Phys.* **03** (2011) 127.
[6] N. Drukker, M. Marino, and P. Putrov, *J. High Energy Phys.* **11** (2011) 141.
[7] H. Fuji, S. Hirano, and S. Moriyama, *J. High Energy Phys.* **08** (2011) 001.
[8] I.R. Klebanov and A.A. Tseytlin, *Nucl. Phys.* **B475**, 164 (1996).
[9] O. Aharony, O. Bergman, D.L. Jafferis, and J. Maldacena, *J. High Energy Phys.* **10** (2008) 091.
[10] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998); *Int. J. Theor. Phys.* **38**, 1113 (1999); S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, *Phys. Lett. B* **428**, 105 (1998); E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
[11] T. Ishii, G. Ishiki, S. Shimasaki, and A. Tsuchiya, *Phys. Rev. D* **78**, 106001 (2008).
[12] G. Ishiki, S. Shimasaki, Y. Takayama, and A. Tsuchiya, *J. High Energy Phys.* **11** (2006) 089.
[13] T. Ishii, G. Ishiki, S. Shimasaki, and A. Tsuchiya, *J. High Energy Phys.* **05** (2007) 014.
[14] T. Ishii, G. Ishiki, S. Shimasaki, and A. Tsuchiya, *Phys. Rev. D* **77**, 126015 (2008).
[15] T. Eguchi and H. Kawai, *Phys. Rev. Lett.* **48**, 1063 (1982).
[16] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, *Phys. Rev. D* **55**, 5112 (1997); N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, *Nucl. Phys.* **B498**, 467 (1997); R. Dijkgraaf, E.P. Verlinde, and H.L. Verlinde, *Nucl. Phys.* **B500**, 43 (1997).
[17] F. Sugino, *J. High Energy Phys.* **03** (2004) 067; D.B. Kaplan and M. Unsal, *J. High Energy Phys.* **09** (2005) 042;

- M. Unsal, J. *High Energy Phys.* **04** (2006) 002; J. W. Elliott, J. Giedt, and G. D. Moore, *Phys. Rev. D* **78**, 081701 (2008); S. Catterall, J. *High Energy Phys.* **01** (2009) 040; J. Giedt, *Int. J. Mod. Phys. A* **24**, 4045 (2009); M. Hanada, S. Matsuura, and F. Sugino, *Prog. Theor. Phys.* **126**, 597 (2011); M. Hanada, J. *High Energy Phys.* **11** (2010) 112; I. Kanamori, [arXiv:1202.2101](https://arxiv.org/abs/1202.2101).
- [18] G. Bhanot, U. M. Heller, and H. Neuberger, *Phys. Lett.* **113B**, 47 (1982).
- [19] G. Parisi, *Phys. Lett.* **112B**, 463 (1982).
- [20] D. J. Gross and Y. Kitazawa, *Nucl. Phys.* **B206**, 440 (1982).
- [21] S. R. Das and S. R. Wadia, *Phys. Lett.* **117B**, 228 (1982); **121B**, 456(E) (1983).
- [22] A. Gonzalez-Arroyo and M. Okawa, *Phys. Rev. D* **27**, 2397 (1983).
- [23] M. Hanada, L. Mannelli, and Y. Matsuo, *Phys. Rev. D* **80**, 125001 (2009).
- [24] M. Hanada, L. Mannelli, and Y. Matsuo, J. *High Energy Phys.* **11** (2009) 087.
- [25] H. Kawai, S. Shimasaki, and A. Tsuchiya, *Int. J. Mod. Phys. A* **25**, 3389 (2010).
- [26] H. Kawai, S. Shimasaki, and A. Tsuchiya, *Phys. Rev. D* **81**, 085019 (2010).
- [27] G. Ishiki, S. W. Kim, J. Nishimura, and A. Tsuchiya, *Phys. Rev. Lett.* **102**, 111601 (2009).
- [28] Y. Kitazawa and K. Matsumoto, *Phys. Rev. D* **79**, 065003 (2009).
- [29] G. Ishiki, S. W. Kim, J. Nishimura, and A. Tsuchiya, J. *High Energy Phys.* **09** (2009) 029.
- [30] G. Ishiki, S. Shimasaki, and A. Tsuchiya, J. *High Energy Phys.* **11** (2011) 036.
- [31] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, J. *High Energy Phys.* **04** (2002) 013.
- [32] N. Kim, T. Klose, and J. Plefka, *Nucl. Phys.* **B671**, 359 (2003).
- [33] M. Hanada, J. Nishimura, and S. Takeuchi, *Phys. Rev. Lett.* **99**, 161602 (2007).
- [34] K. N. Anagnostopoulos, M. Hanada, J. Nishimura, and S. Takeuchi, *Phys. Rev. Lett.* **100**, 021601 (2008).
- [35] S. Catterall and T. Wiseman, *Phys. Rev. D* **78**, 041502 (2008).
- [36] M. Honda, G. Ishiki, J. Nishimura, and A. Tsuchiya, *Proc. Sci., LAT2011* (2011) 244 [[arXiv:1112.4274](https://arxiv.org/abs/1112.4274)].
- [37] M. Honda, G. Ishiki, S. W. Kim, J. Nishimura, and A. Tsuchiya, *Proc. Sci., LATTICE2010* (2010) 253 [[arXiv:1011.3904](https://arxiv.org/abs/1011.3904)].
- [38] G. Ishiki, S. Shimasaki, and A. Tsuchiya, *Phys. Rev. D* **80**, 086004 (2009).
- [39] G. Ishiki, S. Shimasaki, and A. Tsuchiya, *Nucl. Phys.* **B834**, 423 (2010).
- [40] T. Ishii, G. Ishiki, K. Ohta, S. Shimasaki, and A. Tsuchiya, *Prog. Theor. Phys.* **119**, 863 (2008).
- [41] G. Ishiki, K. Ohta, S. Shimasaki, and A. Tsuchiya, *Phys. Lett. B* **672**, 289 (2009).
- [42] E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).
- [43] M. Marino, J. *Phys. A* **44**, 463001 (2011).
- [44] M. Marino and P. Putrov, J. *Stat. Mech.* (2012) P03001.
- [45] M. Hanada, M. Honda, Y. Honma, J. Nishimura, S. Shiba, and Y. Yoshida, [arXiv:1202.5300](https://arxiv.org/abs/1202.5300).