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Rolling tachyons for separated brane-antibrane systems

Dan Israël[1,*] and Flavien Kiefer[1,2,†]

[1]*Institut d'Astrophysique de Paris, 98bis Bd Arago, 75014 Paris, France*[‡]

[2]*LPTENS, École Normale Supéieure, 24 rue Lhomond, 75231 Paris cedex 05, France*[§]

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We consider tachyon condensation between a D-brane and an anti–D-brane in superstring theory, when they are separated in their common transverse directions. A simple rolling tachyon solution, which describes the time evolution of the process, is studied from the point of view of boundary conformal field theory. By computing the boundary beta functions of the system, one finds that this theory is conformal and hence corresponds to an exact solution of the string theory equations of motion. By contrast, the time-reversal-symmetric rolling tachyon is not conformal. These results put constraints on the space-time effective actions for the system.

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I. INTRODUCTION

Annihilation of D-branes of opposite Ramond-Ramond charge is one of the fundamental processes of string theory. Tachyon condensation on brane-antibrane systems also has important cosmological applications, either as a tractable model of a time-dependent process in string theory, or concretely in D-brane inflation models [1]. It also appears in holographic models of QCD, to describe chiral symmetry breaking [2].

Whenever the distance between the branes is smaller than the critical value r_c, the ground state in the brane-antibrane open string sectors becomes tachyonic. It was conjectured long ago that the condensation of this complex-valued tachyon leads to the closed string vacuum, corresponding to the minimum of the tachyon potential [3] and partially confirmed by string field theory computations [4].

In the case where the brane and the antibrane are coincident in their common transverse directions, this system has been thoroughly studied using background-independent string field theory [5–7]. In this approach, one considers the two-dimensional world-sheet-conformal field theory on the disk with marginal and relevant boundary perturbations. It allows one to compute the exact off-shell tree-level tachyon potential [8,9].

On-shell configurations corresponding to real-time tachyon condensation on unstable D-branes are also of interest, especially whenever the boundary conformal field theory (BCFT) is known. For unstable D-branes, a first type of solution, known as the *full S-brane* was found by Sen and represents a time-reversal symmetric process [10]. The second type of solution, known as the *half S-brane* [11,12], represents the more realistic case of a tachyon starting, from $t \to -\infty$, at the maximum of its potential. It is straightforward to extend these results to coincident brane-antibrane pairs.

Although the gradient of the tachyon field on the rolling tachyon solutions is very large, it should make sense to consider a space-time effective action that describes slowly varying perturbations thereof. Remarkably, as was shown by Kutasov and Niarchos [13], it is possible to find *unambiguously* the effective action for the tachyon and its first derivative asking only that (i) the rolling tachyon discussed above is a solution to its equations of motion and that (ii) the on-shell Lagrangian on this solution is equal to the disk partition function with the timelike zero-mode unintegrated. Upon a simple field redefinition, it coincides also with the Tachyon-Dirac-Born-Infeld action that was earlier proposed by Garousi [14],[1] and is able to reproduce correctly N-point tachyon amplitudes [17].[2]

Surprisingly, not much of this program has been carried out for the system of a D-brane and an anti–D-brane at finite distance—letting aside the even more interesting and challenging case of brane-antibrane scattering. The brane separation is a modulus at tree level, even though a brane-antibrane potential is generated at one string loop [19]. Hence, we can ask whether tachyon condensation at fixed separation is possible. One may expect different space-time physics compared to the coincident case, especially in the limit where the absolute value of the tachyon mass is small in string units.

With cubic string field theory, an approximation of the tachyon potential as a function of the fixed brane-antibrane separation r was computed a few years ago using level truncation at next-to-leading order in Ref. [20]. In background-independent string field theory, the framework

*israel@iap.fr

†kiefer@iap.fr

‡Unité mixte de Recherche 7095, CNRS—Université Pierre et Marie Curie.

§Unité mixte de Recherche du CNRS et de l'École Normale Supérieure associée à l'Université Pierre et Marie Curie, UMR 8549.

[1]Ramond-Ramond couplings were added to this action in Refs. [15,16].

[2]A different and interesting approach to tachyon effective actions on brane-antibrane pairs was given in Ref. [18].

for studying the T-dual configuration—a brane-antibrane pair compactified along a world-volume direction, with a relative Wilson line—was set in the works [21,22]. There, the world sheet action of the system, including the background space-time gauge fields along with the complex tachyon, was set. Unfortunately, the Abelian gauge field T-dual to r was set to zero in order to simplify the path-integral computation.[3]

Finally, the half S-brane rolling tachyon solution describing condensation at fixed, finite distance is not really understood, let alone the effective action of which it should be a solution. In Ref. [20], this problem was studied using conformal perturbation theory, which is expected to be valid, in space-time terms, for very early times at the onset of tachyon condensation. Surprisingly, it was found that the boundary interaction corresponding to the rolling tachyon ceases to be marginal for a countable set of values of $|r|$ larger than $r_c/\sqrt{2}$.

In this note, we show that, taking into particular account the effect of contact terms that are dictated by world sheet supersymmetry, the rolling tachyon boundary interaction seems to be exactly marginal for all values of $|r|$ below the critical separation. Study of beta functions for the system illuminates the crucial role of the contact term. The latter is able to cancel the powerlike short-distance singularity that arises at second order in perturbation theory for $|r| > 1/2$. At fourth order, it cancels all but one powerlike singularity that is present for $|r| > \sqrt{7}/4$, for which an higher-order contact term is needed. Nevertheless, the potentially dangerous logarithmic singularities, which could occur for certain values of $|r|$, vanish by themselves without the help of the contact term.

We find that the beta functions of the theory are zero to all orders for $|r| < \sqrt{17}/6$, while for larger values of $|r|$, they vanish at least up to order five in perturbation theory. Thus, we expect that the perturbative expansion in the boundary tachyon perturbations does not break conformal invariance on the boundary, for any subcritical separation.

Unexpectedly, we find that the full S-brane rolling tachyon is not a boundary conformal field theory, for any nonzero separation between the branes. In that case, the beta-function for the distance-changing boundary operator does not vanish. It implies that the corresponding space-time tachyon profile is not a solution of the equations of motion. It seems nevertheless that a more general solution than the half S-brane exists, for which the tachyon starts from and comes back to the tachyon vacuum; its physical meaning is not obvious though, since the phase of the complex tachyon cannot stay constant.

From these results, we learn that there should exist a space-time effective action for the system, which is valid for any $0 \leq |r| < r_c$ (to be more precise, the effective action for the tachyon and distance field should admit a solution where the distance is a constant). Effective actions were proposed in the past by Sen [24] and Garousi [25]. However, its domain of validity is not clear. Indeed, it does not allow as a solution a tachyon condensation at fixed distance, even in the regime of small brane separation in string units.

Imposing the existence of the half S-brane solution at fixed distance fixes the effective action up to second order in the tachyon field. In order to get the fully explicit effective action around this rolling tachyon at fixed distance without further hypothesis, we can proceed as Ref. [13] and try to fix all the coefficients of a generic first-order Lagrangian expressed in power series. It fails to give a single answer for two reasons. First, as the full S-brane solution seems to not be allowed, the constraints from the tachyon equations of motion are weaker. Second, we would need to compute the disk partition function, to all orders in the tachyon coupling; for a generic distance, analytical results for the perturbative integrals seem out of reach, from the fourth order.

This work is organized as follows. In Sec. II, we give some background on the brane-antibrane world sheet action on the disk, emphasizing the role of the Fermi multiplets which realize the Chan-Patton degrees of freedom. In Sec. III, we discuss the role of contact terms in canceling the divergences that arise when tachyon perturbations collide. In Sec. IV, we examine the system from the point of view of boundary renormalization group flow and obtain our main results about the marginality of the rolling-tachyon profile. Finally, in the discussion, we give the implications of our results for space-time effective actions. Some lengthy computations are given in the appendices.

II. BRANE-ANTIBRANE WORLD SHEET ACTION

In this section we discuss in detail the boundary world sheet action of the brane/antibrane system, and set our conventions.

A. Superspace action on the disk

As a starting point, one considers the world-sheet action for coincident D1-brane and anti-D1-brane wrapped around a circle in a compactified direction Y, T-dual to the system of interest. We set $\alpha' = 1$ everywhere in the following .

The $\mathcal{N} = (1, 1)$ superspace action on the disk was written in Refs. [21,22,26], including the coupling to background gauge and tachyon fields. In the present context, one considers nontrivial Wilson lines along the circle, T-dual to the brane positions x_1 and x_2 along X, the T-dual of Y. They naturally appear in the form $x^{(\pm)} = x_1 \pm x_2$.

Setting aside the "spectator" dimensions, one considers a pair of $\mathcal{N} = (1, 1)$ superfields on the disk, one timelike ($\mathbb{X}_0$) and the other compactified on a circle ($\mathbb{Y}$), with

[3]Using these results, the space-time effective action with nonzero gauge field profiles was conjectured in Ref. [23] as a plausible covariantization; however, it was not derived from first principles.

e.g. $\mathbb{X}_0 = X_0 + \frac{i}{\sqrt{2}}(\theta\psi_0 + \bar{\theta}\bar{\psi}_0) + \theta\bar{\theta}F_0$. The superspace coordinates are denoted as $\hat{z} = (z, \theta, \bar{\theta})$.

At the boundary of the disk, the Grassmann coordinates satisfy the boundary condition $\theta = \pm\bar{\theta}$. The algebra of the Chan-Patton factors for the brane-antibrane system is conveniently implemented by the canonical quantization of boundary fermions [27] (see below). These boundary fermions are the bottom components of *Fermi superfields* of the boundary $\mathcal{N} = 1$ superspace. For the brane-antibrane system, one needs a complex superfield

$$\mathbf{\Gamma}^{\pm} = \eta^{\pm} + \theta F^{\pm}. \tag{1}$$

with $\mathbf{\Gamma}^- = (\mathbf{\Gamma}^+)^*$.

Then the world-sheet action on the disk,[4] including the tachyon background as well as Wilson lines around the circle, reads:

$$\begin{aligned} S_{\mathrm{BCFT}}(\lambda^+, \lambda^-) = \frac{1}{2\pi}\int_{D^2} & \mathrm{d}^2z\mathrm{d}^2\theta(-D\mathbb{X}^0\bar{D}\mathbb{X}^0 + D\mathbb{Y}\bar{D}\mathbb{Y}) \\ & + i\oint_{S^1}\mathrm{d}u\mathrm{d}\theta\frac{x^{(+)}}{4\pi}D_u\mathbb{Y} \\ & - \oint_{S^1}\mathrm{d}u\mathrm{d}\theta\bigg(\mathbf{\Gamma}^+\bigg(D_u + i\frac{x^{(-)}}{2\pi}D_u\mathbb{Y}\bigg)\mathbf{\Gamma}^- \\ & - \mathbf{\Gamma}^+\mathbb{T}^+ - \mathbf{\Gamma}^-\mathbb{T}^-\bigg), \end{aligned} \tag{2}$$

with the measure $\mathrm{d}^2\theta = \mathrm{d}\theta\mathrm{d}\bar{\theta}$, the superspace holomorphic derivative $D = \partial_\theta + \theta\partial$ and the superspace boundary derivative $D_u = \partial_\theta + \theta\partial_u$, with the boundary coordinate u on S^1.[5]

We consider simple rolling-tachyon profiles of the form

$$\mathbb{T}^{\pm} = \frac{\lambda^{\pm}}{2\pi}e^{\omega\mathbb{X}^0}, \tag{3}$$

with $0 < \omega \leq 1\sqrt{2}$. In order to get a real action, one chooses $(\lambda^+)^* = \lambda^-$. These are actually the tachyons that we are expecting to be solutions of the space-time effective action. It is understood in this expression that the superfield $\mathbb{X}$ is taken on the (super)boundary of the disk.

The space-time gauge field $A^{(-)} = -\frac{x^{(-)}}{4\pi}\mathrm{d}y$ being locally pure gauge, its minimal coupling to the Fermi superfields can be absorbed by a "gauge" transformation.[6] One has to be careful with this transformation if $\mathbb{Y}$-dependent insertions appear in the path integral; a prescription must be chosen (see below):

$$\mathbf{\Gamma}^{\pm} \longrightarrow \mathbf{\Gamma}^{\pm}e^{\pm i(x^{(-)}/2\pi)\mathbb{Y}}. \tag{4}$$

After this field redefinition, the boundary Fermi superfields are free, with the propagator on the real axis:

$$\begin{aligned} \langle\mathbf{\Gamma}^+(\hat{z})\mathbf{\Gamma}^-(\hat{w})\rangle &= \hat{\epsilon}(\hat{z} - \hat{w}) \\ &= \epsilon(z - w) - 2\theta_z\theta_w\delta(z - w), \end{aligned} \tag{5}$$

with the sign function $\epsilon(z) = \Theta(z) - \Theta(-z)$. This implies that $\Delta(\mathbf{\Gamma}^{\pm}) = 0$, i.e. vanishing conformal dimension.

In terms of these new variables, the world sheet action (2) reads

$$\begin{aligned} S_{\mathrm{BCFT}}(\lambda^+, \lambda^-) = \frac{1}{2\pi}\int_{D^2} & \mathrm{d}^2z\mathrm{d}^2\theta(-D\mathbb{X}^0\bar{D}\mathbb{X}^0 + D\mathbb{Y}\bar{D}\mathbb{Y}) \\ & + i\oint_{S^1}\mathrm{d}u\mathrm{d}\theta\frac{x^{(+)}}{4\pi}D_u\mathbb{Y} \\ & - \oint_{S^1}\mathrm{d}u\mathrm{d}\theta(\mathbf{\Gamma}^+D\mathbf{\Gamma}^- - \mathbf{\Gamma}^+\mathbb{T}^+ - \mathbf{\Gamma}^-\mathbb{T}^-), \end{aligned} \tag{6}$$

where the tachyon fields now have the expression

$$\mathbb{T}^{\pm} = \frac{\lambda^{\pm}}{2\pi}e^{\pm i(x^{(-)}/2\pi)\mathbb{Y} + \omega\mathbb{X}^0}. \tag{7}$$

Conformal invariance of the action at leading order then imposes

$$\omega^2 + \left(\frac{x^{(-)}}{2\pi}\right)^2 = \frac{1}{2}. \tag{8}$$

This is the standard mass-shell condition of an open string tachyon with $U(1) \times U(1)$ Wilson lines turned on.

The world-sheet action that describes a system of *separated* brane and antibrane is obtained from the previous one by a T-duality along y. In the bulk, the superfield $\mathbb{Y}$ is traded for the superfield $\mathbb{X}$ that has Dirichlet boundary conditions. Renaming $\mathbb{Y}$ as $\tilde{\mathbb{X}}$, the tachyon interaction of interest reads

$$\mathbb{T}^{\pm} = \frac{\lambda^{\pm}}{2\pi}e^{\pm i(x^{(-)}/2\pi)\tilde{\mathbb{X}} + \omega\mathbb{X}^0}. \tag{9}$$

Action (6) will be our starting point. In the free theory, one has two different boundary conditions on the disk boundary, related to the distinct positions of the branes : $X = x^{(1)}$ or $Y = x^{(2)}$. We introduce the notations

$$x^{(-)} = x^{(1)} - x^{(2)} = 2\pi r \quad x^{(+)} = x^{(1)} + x^{(2)} = 2x_{\mathrm{cm}}, \tag{10}$$

where on the first line, r is such that $\omega^2 + r^2 = 1/2$. On the second line, x_{cm} is simply the center-of-mass coordinate of the system.

B. Action in components and quantization of the Fermi superfields

Starting from the action (6), renaming $\mathbb{Y}$ as $\tilde{\mathbb{X}}$ and integrating over the fermionic coordinates, one gets the action

[4]Our convention is that any amplitude is computed with e^{-S}.

[5]The boundary current superfield $D_u\mathbb{Y}$ is defined to be the boundary superderivative of $\mathbb{Y}$ first taken to the boundary (where $\mathbb{Y}$ has Neumann boundary conditions).

[6]This is a slight abuse of language, as this is *not* a gauge symmetry from the world sheet perspective.

$$S_{\mathrm{BCFT}}(\lambda^+,\lambda^-)=\frac{1}{2\pi}\int_{D^2}\mathrm{d}^2z(-\partial X^0\bar\partial X^0+\partial X\bar\partial X)+i\oint_{S^1}\mathrm{d}u\frac{x^{(+)}}{4\pi}\partial_u\tilde X+\oint_{S^1}\mathrm{d}u\Big(\eta^+\partial_u\eta^- -\frac{\lambda^+}{2\pi}\eta^+\psi^+T^+-\frac{\lambda^-}{2\pi}\eta^-\psi^-T^-\Big)$$
$$-\oint_{S^1}\mathrm{d}u(F^+F^- - F^+T^+ - F^-T^-), \tag{11}$$

with

$$\psi^\pm=\pm ir\sqrt{2}\tilde\psi^x+\omega\sqrt{2}\psi^0 \qquad T^\pm=e^{\pm ir\tilde X+\omega X^0}. \tag{12}$$

Auxiliary fields $F^\pm$ are then integrated to give

$$S_{\mathrm{BCFT}}(\lambda^+,\lambda^-)=\frac{1}{2\pi}\int_{D^2}\mathrm{d}^2z(-\partial X^0\bar\partial X^0+\partial X\bar\partial X)+i\oint_{S^1}\mathrm{d}u\frac{x^{(+)}}{4\pi}\partial_u\tilde X$$
$$+\oint_{S^1}\mathrm{d}u\Big(\eta^+\partial_u\eta^- -\frac{\lambda^+}{2\pi}\eta^+\psi^+T^+-\frac{\lambda^-}{2\pi}\eta^-\psi^-T^- +\varepsilon^{1-4r^2}\frac{\lambda^+\lambda^-}{4\pi^2}T^+T^-\Big). \tag{13}$$

A *contact term* at the end of the second line shows up, with a UV cutoff ε. This term, which does not follow from the equations of motion, contributes nevertheless to correlation functions when $1/2<|r|<1/\sqrt{2}$. Its role will be discussed in Sec. III C.

Finally, as the center-of-mass perturbation completely factorizes and commutes with any operators in Eq. (13), one can set $x^{(+)}=0$ without loss of generality.

Upon quantizing canonically the boundary fermions $\eta^\pm$, one recovers the Chan-Patton algebra corresponding to the brane-antibrane system [22]. It leads to the following identifications:

$$\eta^+\Leftrightarrow\sigma^+=\frac{\sigma^1+i\sigma^2}{2}$$
$$\eta^-\Leftrightarrow\sigma^-=\frac{\sigma^1-i\sigma^2}{2} \tag{14}$$
$$\eta^+\eta^-(z)\Leftrightarrow\frac{[\sigma^+,\sigma^-]}{2}=\frac{\sigma^3}{2},$$

where now the prescription for the path integral is $Z=\mathrm{Tr}\int\mathcal{D}X^i\mathcal{D}\psi^iPe^{-S[X^i,\psi^i]}$, which includes a *path ordering* for the operator insertions and a trace over the *CP* factors. In this context, the tachyon becomes a *boundary-changing operator*; when inserted on the boundary of the disk, it interpolates between the two distinct boundary conditions corresponding to the brane and to the antibrane.

The world-sheet action on the disk takes finally the form

$$S=S_{\mathrm{bulk}}-\oint_{S^1}\mathrm{d}u\Big(\frac{\lambda^+}{2\pi}\sigma^+\otimes\psi^+e^{ir\tilde X+\omega X^0}$$
$$+\frac{\lambda^-}{2\pi}\sigma^-\otimes\psi^-e^{-ir\tilde X+\omega X^0}-\frac{\lambda^+\lambda^-}{4\pi^2}\varepsilon^{1-4r^2}e^{2\omega X^0}\Big). \tag{15}$$

III. PERTURBATIVE INTEGRALS AND CONTACT TERMS

In this section, we discuss in more detail the contact term, quadratic in the tachyon field, which appears in the action (6) after integrating out the auxiliary fields from the Fermi superfields $\mathbf{\Gamma}^\pm$, and quantizing their fermionic components. As was discussed long ago by Green and Seiberg [28] for closed string correlation functions, contact terms, dictated by world-sheet supersymmetry, can cancel unphysical divergences in correlation functions. We shall see below that it indeed cancels the short-distance singularity when two tachyons perturbations collide in the perturbative expansion.

A. Free-field correlators

In order to fix the conventions, we use the following Green functions on the upper half-plane H^+ for a free-field X with Dirichlet boundary conditions, and its T-dual field $\tilde X$:

$$\langle X(z_1)X(z_2)\rangle=-\frac{\eta_{xx}}{2}\ln|z_{12}|^2+\frac{\eta_{xx}}{2}\ln|z_{1\bar 2}|^2$$
$$\langle\tilde X(z_1)\tilde X(z_2)\rangle=-\frac{\eta_{xx}}{2}\ln|z_{12}|^2-\frac{\eta_{xx}}{2}\ln|z_{1\bar 2}|^2 \tag{16}$$
$$\langle X(z_1)\tilde X(z_2)\rangle=-\frac{\eta_{xx}}{2}\ln\frac{z_{12}}{z_{\bar 1\bar 2}}-\frac{\eta_{xx}}{2}\ln\frac{z_{1\bar 2}}{z_{\bar 1 2}},$$

with, e.g., $z_{12}=z_1-z_2$ and $z_{1\bar 2}=z_1-\bar z_2$. Finally, the two-point function for fermions with Dirichlet boundary condition (b.c.) read

$$\langle\psi^x(z_1)\psi^x(z_2)\rangle=\frac{\eta^{xx}}{z_1-z_2} \tag{17a}$$
$$\langle\bar\psi^x(\bar z_1)\bar\psi^x(\bar z_2)\rangle=\frac{\eta^{xx}}{\bar z_1-\bar z_2} \tag{17b}$$
$$\langle\psi^x(z_1)\bar\psi^x(\bar z_2)\rangle=-\frac{\zeta\eta^{xx}}{z_1-\bar z_2}, \tag{17c}$$

where $\zeta=\pm1$ corresponds to the spin structure. It corresponds to the boundary conditions for the supercurrent $G(z)-\zeta\bar G(\bar z)|_{z=\bar z}=0$. For the Virasoro superfield $\mathbb{G}=G+\theta T$, this is naturally associated with the superspace boundary $(z,\theta)=(\bar z,\zeta\bar\theta)$. With Neumann b.c., Eq. (17c) gets a minus sign on the right-hand side.

Finally, the boundary Green function for a superfield $\mathbb{X}$ with Neumann boundary conditions reads

$$\langle \mathbb{X}(\hat{z}_1)\mathbb{X}(\hat{z}_2)\rangle_{\Im z=0,\theta=\zeta\bar{\theta}} = -2\eta_{xx}\ln\hat{z}_{12} = -2\eta_{xx}\ln(z_{12}-\theta_1\theta_2), \quad (18)$$

while it vanishes with Dirichlet b.c.

B. Contact term in the world-sheet action

As was explicited in Sec. II B, upon integrating out the auxiliary fields $F^\pm$ which appear in the Fermi multiplets $\mathbf{\Gamma}^\pm$, one obtains a contact term for the tachyon in the world-sheet action.

The auxiliary field has the two-point function $\langle F^+(u)F^-(v)\rangle = 2\delta(u-v)$. It is regularized at short distances according to

$$\langle F^+(t)F^-(s)\rangle = 2\delta(t-s) \to \delta(|t-s|-\varepsilon). \quad (19)$$

It was shown in Ref. [29] that this point-splitting regularization that we use preserves world-sheet supersymmetry (unless one considers bulk-boundary correlators for which more care is needed).

Then the contact term is given by the following nonlocal interaction on the disk (with $u=e^{it}$, $v=e^{is}$):

$$\begin{aligned}&\frac{1}{2}\int_0^{2\pi}\mathrm{d}t\int_0^{2\pi}\mathrm{d}s\,\delta(|t-s|-\varepsilon)\,{}^\star_\star e^{ir\tilde{X}+\omega X^0}(u)\,{}^\star_\star\,{}^\star_\star e^{-ir\tilde{X}+\omega X^0}(v)\,{}^\star_\star\\ &\quad=\frac{1}{2}\int_0^{2\pi}\mathrm{d}t\int_0^{2\pi}\mathrm{d}s\,\delta(|t-s|-\varepsilon)|u-v|^{2(\omega^2-r^2)}\,{}^\star_\star e^{ir\tilde{X}+\omega X^0}(u)e^{-ir\tilde{X}+\omega X^0}(v)\,{}^\star_\star\\ &\quad=\frac{1}{2}\Big(2\sin\frac{\varepsilon}{2}\Big)^{1-4r^2}\int_0^{2\pi}\mathrm{d}s\Big({}^\star_\star e^{ir\tilde{X}+\omega X^0}(v+\varepsilon)e^{-ir\tilde{X}+\omega X^0}(v)\,{}^\star_\star+{}^\star_\star e^{ir\tilde{X}+\omega X^0}(v)e^{-ir\tilde{X}+\omega X^0}(v+\varepsilon)\,{}^\star_\star\Big)\\ &\quad\times\overset{\varepsilon\to 0}{\simeq}\varepsilon^{1-4r^2}\oint_{S^1}\mathrm{d}u\,{}^\star_\star e^{2\omega X^0}(u)\,{}^\star_\star.\end{aligned} \quad (20)$$

By ${}^\star_\star \star {}^\star_\star$, we denote the boundary normal ordering (see, e.g., Ref [30]).[7] This treatment of the contact term may seem a bit *ad hoc*; however, we will find in the next section that the term (20) appears naturally when one considers the renormalization of the world-sheet action, justifying *a posteriori* this presentation.

We will use in the next section the contact term on the upper half-plane. It is similarly written as

$$\frac{\varepsilon^{1-4r^2}}{2}\int_{-\infty}^{+\infty}dv\Big({}^\star_\star e^{ir\tilde{X}+\omega X^0}(v+\varepsilon)e^{-ir\tilde{X}+\omega X^0}(v)\,{}^\star_\star+{}^\star_\star e^{ir\tilde{X}+\omega X^0}(v)e^{-ir\tilde{X}+\omega X^0}(v+\varepsilon)\,{}^\star_\star\Big)\overset{\varepsilon\to 0}{\simeq}\varepsilon^{1-4r^2}\oint_{\mathbb{R}}\mathrm{d}v\,{}^\star_\star e^{2\omega X^0}(v)\,{}^\star_\star. \quad (21)$$

In order to compute all the counterterms generated from this contact term, one will need to work with its complete nonlocal expression, though the dominant term, here the only divergent one, in its Taylor expansion (in terms of local operators) is sufficient to compute most of them. Indeed, it is found that working directly with the dominant term, a local operator, seems to be equivalent to working with the complete nonlocal contact term. It may be explained by the fact that after Taylor expansion of $T^\pm(x+\epsilon)$ and commutation of the sum and the integral, all other terms in the series of integrated local operators vanish as ϵ goes to zero. One may object that we are forgetting subdominant terms, but, as the UV cutoff is an artifact signaling our lack of ability to manipulate infinite quantities, it is to be understood as being strictly equal to zero from the very beginning. From this point of view, we expect that only the divergent terms in Eq. (21) contribute. Then it should be equivalent to use either the dominant (local) term or the complete (nonlocal) contact term. This statement seems to be confirmed numerically in the fourth-order computations of Sec. IV.

As one can see, in the limit $\varepsilon \to 0$, when one takes the UV cutoff to infinity, the contact term vanishes when $|r| < 1/2$. Therefore, the results of the computations made in Ref. [24], where the contact term was not taken into account, remain unchanged.[8] It can be seen also by working directly with the $\mathcal{N}=1$ boundary superspace amplitudes; the contact terms contributions from the $\mathbf{\Gamma}^\pm$ correlators vanish for $|r| < 1/2$.

However, the contact term diverges when $|r| > 1/2$. This contact term may ensure that the amplitudes do not diverge for $|r| > 1/2$. The divergence associated with the contact term, which arises from the fusion of two-tachyon vertices, corresponds to the unphysical integrated vertex operator

[7] We added a $1/2$ normalization such that to take into account the factor 2 coming from the trace over the *CP* factor, since the contact term is multiplied by the identity matrix.

[8] As a side remark, for the rolling tachyon on a non-BPS D-brane, it was already noticed in Ref. [31] that the contact terms, which were absent in the original computation of the partition function performed in Ref. [32], did not contribute to the final result.

$$\int \mathrm{d}u \int \mathrm{d}\theta\theta \,{}^{\star}_{\star} e^{2\omega \mathbb{X}_0(u)} {}^{\star}_{\star} = \int \mathrm{d}u \,{}^{\star}_{\star} e^{2\omega X_0(u)} {}^{\star}_{\star}, \qquad (22)$$

which is not supersymmetric. Hence, as in Ref. [28], one can understand the contact term as necessary to preserve world-sheet superconformal invariance on the boundary, when $|r| > 1/2$. In other words, divergences corresponding to integrated operators of the form (22) cannot occur for a consistent, hence super-BRST invariant, superstring world-sheet theory. We will discuss below higher-order divergences coming from the fusion of more than two operators, for which the analysis is more involved.

C. Boundary one-point function

In order to illustrate more precisely the role of the contact term, we compute the one-point function on the disk for a tachyon boundary vertex operator. This one-point function does not have to vanish because of the rolling tachyon background and contains potentially a divergence at first order when the inserted tachyon vertex collides with the integrated tachyon coming from the perturbative expansion. We will find that the contact term cancels the two-tachyon divergence for all values of r in the range $1/2 < |r| \leq 1/\sqrt{2}$.

At first order in the couplings $\lambda^{\pm}$, the one-point function for one of the boundary tachyon vertex operators is given by the integrated correlator

$$\begin{aligned}
\mathrm{Tr}\langle \sigma^{\pm} \otimes \psi^{\pm} e^{\pm ir\tilde{X} + \omega X^0}(e^{it_1})\rangle &\sim \frac{\lambda^{\mp}}{2\pi}\,\mathrm{Tr}\sigma^{\pm}\sigma^{\mp} \int_0^{t_1} \mathrm{d}t_2 \langle \psi^{\pm} e^{\pm ir\tilde{X}+\omega X^0}(e^{it_1})\psi^{\mp} e^{\mp ir\tilde{X}+\omega X^0}(e^{it_2})\rangle_0 \\
&\quad + \frac{\lambda^{\mp}}{2\pi}\,\mathrm{Tr}\sigma^{\mp}\sigma^{\pm} \int_{t_1}^{2\pi} \mathrm{d}t_2 \langle \psi^{\mp} e^{\mp ir\tilde{X}+\omega X^0}(e^{it_2})\psi^{\pm} e^{\pm ir\tilde{X}+\omega X^0}(e^{it_1})\rangle_0 \\
&\sim \frac{\lambda^{\mp}}{2\pi}(1-4r^2)\int_{t_1}^{t_1+2\pi} \mathrm{d}t_2 \left| 2\sin\frac{t_1-t_2}{2}\right|^{2\omega^2-2r^2-1} \int_{-\infty}^{+\infty} \mathrm{d}x^0 e^{2\omega x^0}. \qquad (23)
\end{aligned}$$

The integration over t_2 is not defined for $|r| > 1/2$; nevertheless, the result

$$\mathrm{Tr}\,\langle \sigma^{\pm} \otimes \psi^{\pm} e^{\pm ir\tilde{X}+\omega X^0}(e^{it_1})\rangle \sim \frac{\lambda^{\mp}}{2\pi}(1-4r^2)2^{1-4r^2}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-2r^2)}{\Gamma(1-2r^2)} \int_{-\infty}^{+\infty} \mathrm{d}x^0 e^{2\sqrt{(1/2)-r^2}x^0} \qquad (24)$$

is analytic for any $r \in [0, 1/\sqrt{2}]$.

In order to show how the divergence for $|r| > 1/2$ is canceled, we can compute directly this quantity in superspace, using the Fermi multiplets $\mathbf{\Gamma}^{\pm}$. Setting aside for a moment the zero-mode integral over x_0, one considers the superspace integral

$$\begin{aligned}
\int \mathrm{d}\theta_1 \langle \mathbf{\Gamma}^{\pm} e^{\pm ir\tilde{\mathbb{X}}+\omega\mathbb{X}^0}(\hat{z}_1)\rangle &\sim -\frac{\lambda^{\mp}}{2\pi}\int \mathrm{d}\theta_1\mathrm{d}\theta_2 \int \mathrm{d}t_2 \epsilon(\hat{z}_1-\hat{z}_2)\langle e^{\pm ir\tilde{\mathbb{X}}+\omega\mathbb{X}^0}(\hat{z}_1) e^{\mp ir\tilde{\mathbb{X}}+\omega\mathbb{X}^0}(\hat{z}_2)\rangle_0 \\
&\sim -\frac{\lambda^{\mp}}{2\pi} e^{2\omega x^0} \int \mathrm{d}\theta_1\mathrm{d}\theta_2 \int \mathrm{d}t_2 [\epsilon(t_1-t_2) - 2\theta_1\theta_2\delta(t_1-t_2)] \\
&\quad \times \left(\left| 2\sin\frac{t_1-t_2}{2}\right|^{1-4r^2} - \theta_1\theta_2(1-4r^2)\epsilon(t_1-t_2)\left| 2\sin\frac{t_1-t_2}{2}\right|^{-4r^2}\right) \sim \\
&\quad -\frac{\lambda^{\mp}}{2\pi} e^{2\omega x^0} \int \mathrm{d}t_2 \left[(1-4r^2)\left| 2\sin\frac{t_1-t_2}{2}\right|^{-4r^2} + 2\delta(t_1-t_2)\left| 2\sin\frac{t_1-t_2}{2}\right|^{1-4r^2}\right]. \qquad (25)
\end{aligned}$$

Now, we introduce a point-splitting regularization, asking that $|t_1 - t_2| > \varepsilon$. As we wish to keep the contact term in the computation, it is natural to include this point splitting in the Θ and δ distributions that appear in the above integral as

$$\Theta(|t_1-t_2|-\varepsilon) = \Theta(t_1-t_2-\varepsilon) + \Theta(t_2-t_1-\varepsilon) \qquad \delta(|t_1-t_2|-\varepsilon) = \delta(t_1-t_2-\varepsilon) + \delta(t_2-t_1-\varepsilon). \qquad (26)$$

In other words, we "spread" the contact term at the boundary of the interval $|t_1 - t_2| < \varepsilon$. Then, the contribution to the one-point function becomes

$$\begin{aligned}
&-\frac{\lambda^{\mp}}{2\pi}\int \mathrm{d}t_2 \left[(1-4r^2)\Theta(|t_1-t_2|-\varepsilon)\left| 2\sin\frac{t_1-t_2}{2}\right|^{-4r^2} + \delta(|t_1-t_2|-\varepsilon)\left| 2\sin\frac{t_1-t_2}{2}\right|^{1-4r^2}\right] \\
&\quad = -\frac{\lambda^{\mp}}{2\pi}(1-4r^2)\int_{t_1-2\pi+\varepsilon}^{t_1-\varepsilon} \mathrm{d}t_2 \left| 2\sin\frac{t_1-t_2}{2}\right|^{-4r^2} - 2\frac{\lambda^{\mp}}{2\pi}\left| 2\sin\frac{\varepsilon}{2}\right|^{1-4r^2} \\
&\quad \sim -\frac{\lambda^{\mp}}{2\pi}(1-4r^2)2^{1-4r^2}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-2r^2)}{\Gamma(1-2r^2)} + 2\frac{\lambda^{\mp}}{2\pi}\varepsilon^{1-4r^2} - 2\frac{\lambda^{\mp}}{2\pi}\left| 2\sin\frac{\varepsilon}{2}\right|^{1-4r^2}, \qquad (27)
\end{aligned}$$

where two first terms in the last line come from the expansion of the following function:

$$(1-4r^2)2^{2-4r^2}\cos\frac{\varepsilon}{2}F_1\left[\frac{1}{2},\frac{1+4r^2}{2},\frac{3}{2},\cos^2\frac{\varepsilon}{2}\right]. \quad (28)$$

The second term of Eq. (27) is the only divergent one if $4r^2 > 1$. It simplifies to

$$-\frac{\lambda^{\mp}}{2\pi}(1-4r^2)2^{1-4r^2}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-2r^2)}{\Gamma(1-2r^2)}+2\frac{\lambda^{\mp}}{2\pi}\varepsilon^{1-4r^2} - 2\frac{\lambda^{\mp}}{2\pi}\varepsilon^{1-4r^2}. \quad (29)$$

Divergences compensate correctly, so that we eventually have, at first order,

$$\mathrm{Tr}\langle(\sigma^{\pm}\otimes\psi^{\pm}-F^{\pm})e^{\pm ir\tilde{X}+\omega X^0}(z_1)\rangle \sim -\lambda^{\mp}\frac{\Gamma(2-4r^2)}{\Gamma^2(1-2r^2)}\int_{-\infty}^{+\infty}\mathrm{d}x^0 e^{2\sqrt{(1/2)-r^2}x^0}. \quad (30)$$

This quantity is UV-finite but has an IR divergence due to the zero-mode integral. This divergence, which appears when $x^0 \to \infty$, simply signals the breakdown of perturbation theory in $\lambda^{\pm}$. Note that for the homogeneous rolling tachyon on a non-BPS brane, for which the all-orders computation is doable, summing up the whole perturbative expansion gives a finite zero-mode integral.[9]

IV. COMPUTATION OF BETA FUNCTIONS

In this section, we argue that the theory defined in Eq. (6) is exactly conformal, with the rolling-tachyon profile (9), for any value of $|r|$ below $r_c = 1/\sqrt{2}$. This will imply that for the space-time effective action of the brane-antibrane system, there exists a half S-brane rolling-tachyon solution at fixed separation of the equations of motion. This is an important point since the effective action proposed in Ref. [25] did not admit a solution at fixed distance; in fact, in this action, for a nonvanishing tachyon, the distance field has an attractive potential towards the origin.

Our motivation for looking closely at this issue was in part due to the results of Bagchi and Sen [20]. They found that the boundary deformation corresponding to the tachyon (9) was only marginal in the range $0 \le |r| < r_c/\sqrt{2}$. For $r_c/\sqrt{2} \le |r| < r_c$, it was found that for an infinite but countable set of distances, the theory was not conformal. This is puzzling, as we expect that everything goes smoothly up to the critical separation r_c.

At the end of the day, the basic difference between those two approaches is the contact term. However, the latter is not responsible for restoring marginality, since it cannot cancel the logarithmic divergences that could spoil conformal invariance, as we shall see. Rather, the actual computation of the possible conformal symmetry-violating terms in the path integral gives zero thanks to the different contributions that cancel among themselves at a given order. Nevertheless, the contact term is able, as expected, to cancel the powerlike two-tachyon divergences in the perturbative integrals.

The cleanest way to show that the action (6), with the rolling-tachyon perturbation (7), is a BCFT is to compute the boundary beta functions for all the boundary couplings involved. On top of the coupling constants $\lambda^{\pm}$ for the rolling tachyon perturbations, one needs to introduce in the computation a perturbation corresponding to the separation-changing boundary operator $\sigma^3 \otimes i\partial_u X$.[10] The brane-antibrane separation is classically fixed at some value r, but still, in the quantum theory, one has to check that the corresponding beta function vanishes for any r. In other words, that it is not "sourced" by terms in $\lambda^{\pm}$. On top of this, more operators need to be considered in the analysis as $|r|$ increases.

A. Generalities about boundary beta-functions

In order to compute the beta functions for their boundary couplings, we follow mostly the clear presentation of Ref. [33].

One considers a conformal field theory on the upper half-plane $H^+ = \{z, \Im\mathfrak{m} z \ge 0\}$ perturbed by boundary operators that can be marginal or relevant. The action of the theory is defined to be

$$S(\lambda^\mu) = S_{\mathrm{bulk}} + \sum_\mu \ell^{-y_\mu}\lambda^\mu \int \mathrm{d}x \phi_\mu(x) + S_{\mathrm{ct}}, \quad (31)$$

in terms of the renormalized dimensionless couplings $\{\lambda^\mu\}$ and the anomalous dimensions $y_\mu = 1 - h_\mu$. The renormalization scale is denoted by ℓ. The last term S_{ct} stands for boundary counterterms whenever they are necessary. The boundary fields ϕ_μ are normalized as[11]

$$(\phi^*_\mu(\infty)|\phi_\mu(0)) = 1, \quad (32)$$

with ϕ^*_μ the conjugate field to ϕ_μ.[12]

At second order in perturbation theory, one encounters the integral (which lies inside a correlator with arbitrary other insertions)[13]:

[9]If we Wick-rotate the theory to a Euclidean target space, for which perturbation theory is well-defined, the zero-mode integration gives $\delta(2\omega)$, which is zero for any value of $|r| < 1/\sqrt{2}$.

[10]To be exact, we will have to add it in superspace as $\mathbf{\Gamma}^+\mathbf{\Gamma}^- D\mathbb{X}$

[11]The Zamolodchikov correlators are defined as $(\phi_a(\infty)|\phi_b(z_b)) = \lim_{z\to\infty} z^{2h_a}\bar{z}^{2\bar{h}_a}\langle\phi_a(z)\phi_b(z_b)\rangle$.

[12]In the case of theories with several boundary conditions, one has to trace over the Chan-Patton factors, which would be here included inside the fields, e.g., as $\mathrm{Tr}(\phi^*_\mu(\infty)|\phi_\mu(0)) = 1$. Considering deformations by boundary-changing operators, the CP factors induce selection rules.

[13]We will use the convention that operators (with CP factors) are ordered from right to left with increasing boundary parameter; this is the opposite convention than in Ref. [33].

$$\frac{1}{2}\sum_{\mu,\nu}\ell^{h_\mu-h_\nu-2}\int \mathrm{d}x_1 \times \int \mathrm{d}x_2 \phi_\mu(x_1)\phi_\nu(x_2)\Theta(|x_1-x_2|-\varepsilon)\Theta(L-|x_1-x_2|). \tag{33}$$

This integral has been regularized by point splitting with a UV cutoff ε, and with an IR cutoff L. In order to compute the integral, one can use the boundary operator product expansion (OPE):

$$\phi_\mu(x_1)\phi_\nu(x_2)=\sum_\rho \frac{D^\rho_{\mu\nu}}{(x_1-x_2)^{h_\mu+h_\nu-h_\rho}}\phi_\rho(x_2)+\ldots \quad x_1>x_2. \tag{34}$$

In this case, Eq. (33) is rewritten as

$$\sum_{\mu<\nu}\ell^{h_\mu-h_\nu-2}\bigg(\oint_{y+\epsilon}^{y+L}\mathrm{d}x\oint \mathrm{d}y\phi_\mu(x)\phi_\nu(y) + \oint_{y+\epsilon}^{y+L}\mathrm{d}x\oint \mathrm{d}y\phi_\nu(x)\phi_\mu(y)\bigg). \tag{35}$$

1. Minimal substraction scheme

In this scheme, we aim to isolate the divergences that occur in the integral (33) when two perturbations collide. One has to consider separately two cases. The subset of boundary fields $\{\phi_\rho\}$, such that $y_\mu+y_\nu-y_\rho<0$ (which are all relevant) gives a divergent contribution to the action (31) of the form (after removing the IR cutoff):

$$S_d=\frac{1}{2}\sum_{\mu,\nu,\rho}\frac{D^\rho_{\mu\nu}}{y_\mu+y_\nu-y_\rho}\varepsilon^{y_\mu+y_\nu-y_\rho}\ell^{y_\mu+y_\nu}\lambda^\mu\lambda^\nu\int \mathrm{d}x\phi_\rho. \tag{36}$$

In the minimal substraction scheme, this divergence is canceled by a similar counterterm $S_{\mathrm{ct}}=-S_d$.

The subset of boundary fields $\{\phi_\tau\}$, such that $y_\mu+y_\nu-y_\tau=0$ gives logarithmic divergences, or resonances (cutting the integration at the renormalization scale ℓ):

$$S_d=\frac{1}{2}\sum_{\mu,\nu,\tau}D^\tau_{\mu\nu}\ln(\varepsilon/\ell)\ell^{-y^\tau}\lambda^\mu\lambda^\nu\int \mathrm{d}x\phi_\tau. \tag{37}$$

This divergent piece is again canceled by an appropriate counterterm $S_{\mathrm{ct}}=-S_d$. Now, equating the bare couplings to the two corresponding contributions from the renormalized action (31), one gets the beta function at second order:

$$\beta^{\mathrm{MS}}_\rho := \ell\frac{\mathrm{d}\mu_\rho}{\mathrm{d}\ell}=y_\rho\lambda^\rho-\sum_{\mu,\nu|y_\mu+y_\nu=y_\rho}D^\rho_{\mu\nu}\lambda^\mu\lambda^\nu. \tag{38}$$

So, nonlinear contributions at quadratic order occur only in the cases of resonances, if they exist.[14] One can show that, in the minimal substraction scheme, this property holds to all orders in perturbation theory.

Note that there is a sign difference between the above result and what appears in Ref. [33]. This comes from their convention of using e^S instead of e^{-S} as we did. One could obtain the same definition by simply changing the sign of the couplings.

2. Wilsonian scheme

In this scheme, we equate the renormalization scale ℓ with the UV scale ε, viewed as a fundamental high-energy scale. We demand that the renormalized theory does not depend on the UV cutoff scale, i.e. that $\varepsilon\partial_\varepsilon e^{-S_{\mathrm{bdy}}}=0$. Then, the renormalized boundary couplings depend on the UV scale ε (as the regularized perturbative integrals do). At second order, the corresponding beta functions read

$$\beta^{\mathrm{ws}}_\rho := \varepsilon\partial_\varepsilon\mu_\rho=y_\rho\lambda^\rho-\sum_{\mu,\nu}D^\rho_{\mu\nu}\lambda^\mu\lambda^\nu. \tag{39}$$

In contrast with the minimal substraction scheme, Eq. (38), there is no restriction to “resonant” boundary couplings in the sum giving the quadratic term of the beta function (39).[15]

We will see below that both schemes are useful in the study of the rolling-tachyon perturbations when it comes to understanding the role of the contact terms.

B. Beta functions for the brane-antibrane system at second order

Coming back to the brane-antibrane system, we consider the following world-sheet action on the upper half-plane, as a function of the boundary couplings. So, now we take the boundary variable to be $u\in\mathbb{R}$. For convenience, we rescale the coupling according to $\lambda^\pm\to 2\pi\lambda^\pm$:

$$S=S_{\mathrm{bulk}}-\int \mathrm{d}x\bigg(\lambda^+\sigma^+\otimes\psi^+e^{ir\tilde{X}+\omega X^0} + \lambda^-\sigma^-\otimes\psi^-e^{-ir\tilde{X}+\omega X^0}-i\frac{\delta r}{2}\sigma^3\otimes\partial_u\tilde{X}\bigg). \tag{40}$$

We omitted for the moment the contact term, which we will reintroduce later on in the discussion.

1. Distance coupling

Let us start by discussing the beta function for the distance perturbation. According to the general discussion above, one has

[14] Notice that, if the boundary perturbations in Eq. (33) are superficially marginal, the resonances correspond to the appearance of a marginal operator in the boundary OPE.

[15] The linear term, as well as the resonant quadratic terms, which are common to both schemes, can be shown to be “universal,” i. e. independent of the scheme chosen for the computations.

$$\beta_r = (1-h_r)\frac{\delta r}{2} - (D^r_{+-} + D^r_{-+})\lambda^-\lambda^+ - D^r_{r+}\frac{\delta r}{2}\lambda^+ - D^r_{r-}\frac{\delta r}{2}\lambda^- \ldots, \tag{41}$$

where the ellipses here stand for higher-order terms. The first term on the right-hand side vanishes because the conformal dimension of the distance perturbation is one. At the second order, all the structure constants for the three boundary operators under study appear, since, being all of conformal dimension 1, they potentially lead to resonances.

Without much work, we have that $D^r_{\pm\mp} = 0$. The fusion of the tachyon vertex operators $T^\pm$ will never produce the current $\partial_u X$, as the $e^{\omega X_0}$ factors just add up. The structure constants $D^r_{r\pm}$ also have to vanish, since the fusion of $T^\pm$ with the boundary current $i\sigma^3\partial_u\tilde{X}$ comes with the Chan-Patton factor $\sigma^\pm\sigma^3 = \mp\sigma^\pm$; hence, not σ^3. However, this product participates to the beta function of $T^\pm$, as we will see.

At higher orders in perturbation theory, we would find a similar behavior. Namely, the fusion of any number of tachyon vertices cannot produce the distance-changing operator; hence, the beta function for δr does not get tachyon "source terms" (which would be proportional to $(\lambda^+\lambda^-)^n$ at order $2n$). In other words, the distance coupling does not run in the rolling-tachyon background (9).

We can also be less specific and consider, instead of Eq. (9), a more general tachyon profile of the form:

$$\mathbb{T}^\pm = \frac{1}{2\pi}e^{\pm ir\tilde{\mathbb{X}}}(\lambda^\pm e^{\omega\mathbb{X}^0} + \xi^\pm e^{-\omega\mathbb{X}^0}), \tag{42}$$

the hermiticity of the action imposing that $\lambda^- = \bar{\lambda}^+$ and $\xi^- = \bar{\xi}^+$.

The conclusion can be different, as the structure constants $D^r_{\pm\mp}$ do not have to vanish by similar arguments. To this end, we use the 0-picture tachyons OPE:

$$\sigma^+\sigma^- \otimes T^+_{(0)}(z)T^-_{(0)}(w) = \frac{1+\sigma^3}{4}\otimes\left(\ldots + ir\frac{\partial_u\tilde{X}}{z-w} + \ldots\right)$$
$$\sigma^-\sigma^+ \otimes T^-_{(0)}(z)T^+_{(0)}(w) = \frac{1-\sigma^3}{4}\otimes\left(\ldots - ir\frac{\partial_u\tilde{X}}{z-w} + \ldots\right), \tag{43}$$

where we only highlighted the interesting term. It is not difficult then to obtain the second-order beta function for the distance coupling:

$$\beta_r = -\frac{\lambda^+\xi^- + \lambda^-\xi^+}{4\pi^2}r = -\frac{1}{2\pi^2}\Re\mathrm{e}(\lambda^+\bar{\xi}^+)r. \tag{44}$$

The beta function (44) is scheme-independent, as the divergence is logarithmic. If one introduces a real parameter μ, the most general solution of this equation is then

$$\mathbb{T}^\pm = \frac{\lambda^\pm}{2\pi}e^{\pm ir\tilde{\mathbb{X}}}(e^{\omega\mathbb{X}^0} \pm i\mu e^{-\omega\mathbb{X}^0}), \qquad \mu\in\mathbb{R}. \tag{45}$$

Notice that, so far, the marginality of this solution has been checked only at second order. The fourth-order (and higher) beta functions would be nontrivial to compute, and marginality at this order is *a priori* not obvious.

In any case, the physical meaning of the general solution (45) is not clear. If, for instance, one chooses λ^+ real (which is always possible by a shift of $\tilde{X}$), it corresponds to a case where the real part of the tachyon condenses, while the imaginary part evolves in the opposite direction. There is no clear reason why the phase of the tachyon condensate *has* to change by $\pi/2$ during its evolution, as the constraints on the solution (45) suggest. By symmetry arguments, the tachyon potential of the effective action should depend only on its square modulus, i.e. of

$$|T(x^0)|^2 = \frac{\lambda^+\lambda^-}{4\pi^2}(e^{2\omega x^0} + \mu^2 e^{-2\omega x^0}), \tag{46}$$

which, for nonzero μ, goes through a minimal value $|T|^2 = \mu\frac{\lambda^+\lambda^-}{2\pi^2}$ at finite time.

If we consider instead the time-reversal-symmetric tachyon profile (full S-brane) as in the case of the non-BPS brane,

$$\mathbb{T}^\pm = \frac{\lambda^\pm}{2\pi}e^{\pm ir\mathbb{X}}\cosh\omega\mathbb{X}_0, \tag{47}$$

the beta function (44) indicates a renormalization group (RG) running of the distance coupling, unless $r = 0$. This result has far-reaching consequences. Unlike the case of coincident brane-antibrane or of a non-BPS brane, the effective action of the brane-antibrane system at finite distance should be such that, while the half S-brane rolling tachyon is allowed as a solution of its equations of motion, the full S-brane should not be.

2. Tachyon couplings at quadratic order

We now compute the beta functions for the tachyon couplings $\lambda^\pm$ at order $\lambda^+\lambda^-$ for the half–S-brane profile.[16]

The boundary OPEs to consider at quadratic order are the distance-tachyon OPE,

$$-i\sigma^3\otimes\partial_u\tilde{X}(x_1)\sigma^\pm\otimes\psi^\pm e^{\omega X_0 \pm ir\tilde{X}}(x_2) \sim \frac{-2}{x_1-x_2}(\pm\sigma^\pm)\otimes(\pm r)\psi^\pm e^{\omega X_0\pm ir\tilde{X}}(x_2) + \ldots, \tag{48}$$

and the tachyon-tachyon OPE,

[16]Let us remark in passing that, by shifting the zero mode of the timelike field $X_0 \to X_0 + \alpha$, there is a common rescaling of the couplings $\lambda^\pm \to \lambda^\pm e^{\omega\alpha}$. This is a common feature of Liouville-like theories. For this reason, the perturbative expansion in $\lambda^\pm$ does strictly make sense only in the Euclidean theory obtained by $X_0 \to iX_{\mathrm{E}}$.

$$\sigma^+ \otimes {}^\star_\star \psi^+ e^{\omega X_0 + i r \tilde{X}}(x_1) {}^\star_\star \sigma^- \otimes {}^\star_\star \psi^- e^{\omega X_0 - i r \tilde{X}}(x_2) {}^\star_\star \sim -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (1 - 4r^2) \frac{1}{(x_1 - x_2)^{4r^2}} e^{2\omega X_0}(x_1) {}^\star_\star + \ldots, \tag{49}$$

both for $x_1 > x_2$. The ellipses stand for less singular terms.

3. Beta function for $|r| < 1/2$

Whenever $|r| < 1/2$, the OPE (49) does not lead to singularities when integrated. Hence, in the minimal substraction scheme, no corresponding counterterm is needed. This reflects the fact that the contact term is zero in this range.[17] This extends to all orders in perturbation theory.

The case of the OPE (48) is different, as it leads to a logarithmic divergence for any $r \neq 0$. From Eq. (38), the relevant beta functions are of the form[18]

$$\beta_\pm = (1 - h_\pm)\lambda^\pm + (D^\pm_{r\pm} + D^\pm_{\pm r}) \frac{\delta r}{2} \lambda^\pm + \ldots \tag{50}$$

We get at second order that

$$\beta_\pm = (\tfrac{1}{2} - r^2 - \omega^2 - 2r\delta r)\lambda^\pm. \tag{51}$$

This is valid in any scheme, as only universal quantities appear. If one keeps the distance perturbation at zero ($\delta r = 0$), then the rolling-tachyon background is marginal at second order, provided that the on-shell condition $\omega^2 + r^2 = 1/2$, as expected.

Otherwise, the marginality of the perturbation is restored, at this order, if we use instead the on-shell condition

$$\omega^2 + (r + \delta r)^2 = 1/2. \tag{52}$$

This is compatible with the interpretation of the boundary perturbation $\sigma^3 \otimes i\partial_u \tilde{X}$, which changes the relative position of the D-brane and the anti–D-brane. It is T-dual to the relative Wilson line that appears in the action (2).[19] One checks that the normalization of this coupling in Eq. (2) is compatible, through T-duality, with relation (52). This analysis shows that, at least at this order, the rolling-tachyon perturbations $T^\pm$ "adjust themselves" to a change of brane-antibrane separation in order to stay marginal.

4. Beta-functions for $1/2 < |r| < r_c$ and contact term

When $1/2 < |r| < r_c$, the situation is different. The operator $\exp 2\omega X_0$ (which appears also in the contact term) becomes relevant and, hence, should be considered in the discussion. As stated earlier, this operator is unphysical from the superstring theory point of view (at zero superghost number).

The corresponding boundary coupling is denoted by μ_c. The tachyon-tachyon OPE (49) gives a singular perturbative integral at second order:

$$\int \mathrm{d}x_1 \int_{x_1 - L}^{x_1 - \varepsilon} \mathrm{d}x_2 {}^\star_\star \psi^+ e^{\omega X_0 + i r \tilde{X}}(x_1) {}^\star_\star {}^\star_\star \psi^- e^{\omega X_0 - i r \tilde{X}}(x_2) {}^\star_\star \times \overset{r \neq 1/2}{\sim} \varepsilon^{1 - 4r^2} \int \mathrm{d}x_1 {}^\star_\star e^{2\omega X_0}(x_1) {}^\star_\star, \tag{53}$$

after removing the IR cutoff ($L \to \infty$ limit).

In the minimal substraction scheme, the following local counterterm is needed at this order to cancel the divergence:

$$S_{\rm ct} = \lambda^+ \lambda^- \varepsilon^{1 - 4r^2} \int \mathrm{d}x {}^\star_\star e^{2\omega X_0}(x) {}^\star_\star. \tag{54}$$

Naturally, it agrees precisely with the expression of the contact term in the action (15). Since this divergence is powerlike, it does not add any nonlinear term in the minimal scheme beta function $\beta_c^{\rm ms}$ for the coupling μ_c. Hence, the latter can be consistently set to zero in the renormalized theory at this order.

For the distance $|r| = 1/2$, amplitudes are finite without the counterterm, so it is not strictly needed,[20] but it contributes nevertheless finitely to the amplitudes.

In the Wilsonian scheme, the beta function reads, at second order,

$$\beta_c^{\rm ws} = (1 - 4r^2)\mu_c - (1 - 4r^2)\lambda^+ \lambda^-. \tag{55}$$

One sees here an interesting phenomenon. The operator $\exp 2\omega X_0$ is relevant at linear order, but the RG flow gives an IR fixed point for this coupling at quadratic order, for $\mu_c = \lambda^+ \lambda^-$.

Comparing the outcomes of both schemes, one gets the same results but the interpretation is different. In the minimal substraction scheme, the contact term appears as a counterterm, but the corresponding renormalized coupling is consistently set to zero. On the contrary, in the Wilsonian scheme, the RG flow has a fixed point with nonzero renormalized coupling μ_c. Both points of view are "nonsupersymmetric." since in the superspace formulation, this term is present from the beginning and removes the divergence under discussion.

C. Marginality beyond quadratic order

Part of the quadratic order results generalizes immediately to higher orders. Indeed, only the fusion of distance perturbations with, say, T^+ can produce T^+ itself (since the fusion of n tachyons goes as $e^{n\omega X_0}$, as far as the X_0 dependence is concerned). Hence, if we set $\delta r = 0$ from the very beginning, we expect that the beta functions $\beta_\pm$ vanish to all orders in perturbation theory. With the same

[17]In the Wilsonian scheme, the contact term is an *irrelevant* operator in this range.

[18]The sign is opposite here since the sign in front of the tachyon perturbation in Eq. (40) is opposite.

[19]To be more correct, as auxiliary fields from the Fermi superfield couple to this perturbation, some $\pm i\delta r \lambda^\pm \psi^x e^{\pm i r \tilde{X} + \omega X^0}$ correction should be included. We verify that it does not modify the beta function at quadratic order. Moreover, this term shows up naturally if we work directly with the superspace distance perturbation $i\delta r \boldsymbol{\Gamma}^+ \boldsymbol{\Gamma}^- D_u \mathbb{X}$.

[20]But partition function appears to be discontinuous at $|r| = 1/2$ without its contribution.

reasoning, the operator $\exp(2\omega X_0)$ which we had to consider for $|r| > 1/2$ cannot receive higher-order contributions to its beta function.

However, the study of the marginality at higher orders is quite messy when $|r|$ is getting closer to the critical distance, as the fusion of tachyon vertex operators produces more and more relevant boundary operators. For a given value of r, these operators, of the form $e^{2n\omega X_0}$ with $n \in \mathbb{Z}_+$, become (superficially) relevant if $n < (2-4r^2)^{-1/2}$ and are of dimension 1 when they saturate this bound. These resonances occur all for $1/2 \le |r| < 1/\sqrt{2}$; this range was excluded by Bagchi and Sen in their analysis [20] for this precise reason.

A given operator $e^{2n\omega X_0}$ appears first at order $2n$ in the perturbative expansion in the tachyon perturbations; hence, the beta function β_n for its coupling λ_n is of the form

$$\beta_n = (1 - 4n^2\omega^2)\lambda_n + \mathcal{O}((\lambda^+\lambda^-)^n). \tag{56}$$

It is easier then to work in the minimal substraction scheme, where one just has to worry about logarithmic divergences, i.e. resonances. As we emphasized above, if the fusion of (superficially) marginal operators produces a (superficially) marginal operator, it generates a source term in the corresponding minimal scheme beta function. It is nevertheless interesting to consider whether powerlike divergences are also present.

At second order, the potentially marginal operator is nothing but the contact term itself, $e^{2\omega X_0}$, for the distance $|r| = 1/2$. Fortunately, thanks to its fermionic part, the OPE (49) vanishes; hence, there is no logarithmic divergence to cancel.

1. Marginality for $\sqrt{7}/4 < |r| < \sqrt{17}/6$

The next possible resonance occurs when the operator $e^{4\omega X^0}$ becomes of dimension 1, i.e. for $\omega = 1/4$ (equivalently, $|r| = \sqrt{7}/4$). The potential logarithmic divergence would occur at fourth order in perturbation theory. In order to investigate this issue, we compute below all the possible divergent terms that occur at order $(\lambda^+\lambda^-)^2$ from the perturbative integrals, which involve both the tachyon and contact-term vertex operators. In the computations of this subsection, we use the full nonlocal contact term (21), as even the subleading terms contribute *a priori* to the divergences.

The first contribution comes from two contact-term insertions (symbolically contact-contact [CC]). Using the notations $a = 4\omega^2$ and $T^\pm = e^{\pm ir\tilde{X}+\omega X_0}$, it reads

$$\begin{aligned}\mathrm{CC} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\varepsilon^{2a-2}}{4}\int \mathrm{d}x_1 \int_{x_1-L+\varepsilon}^{x_1-2\varepsilon}\mathrm{d}x_2 \\ \times ({}^\star_\star T^+(x_1+\varepsilon)T^-(x_1){}^\star_\star + {}^\star_\star T^-(x_1+\varepsilon)T^+(x_1){}^\star_\star) \\ \times ({}^\star_\star T^+(x_2+\varepsilon)T^-(x_2){}^\star_\star + {}^\star_\star T^-(x_2+\varepsilon)T^+(x_2){}^\star_\star),\end{aligned} \tag{57}$$

the contact term being multiplied by the Chan-Patton identity matrix. The short-distance regularization chosen here prevents any operator to approach another one at less than ε, *before* integration of the auxiliary fields. The most natural IR cutoff prescription is to constrain two ordered operators not to move away from each other by more than ε, also before integration of auxiliary fields. One then gets

$$\begin{aligned}\mathrm{CC} \sim \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Bigg(\frac{1}{2a+1}\left(\frac{L}{\varepsilon}\right)^{2-2a} - \left(\frac{L}{\varepsilon}\right)^{1-2a} \\ - \frac{5-6a-(2a-1)2^{2a+2}{}_2\mathrm{F}_1(1-a,-a-\frac{1}{2};-a+\frac{1}{2};\frac{1}{4})}{4(2a+1)(2a-1)} \\ \times \left(\frac{L}{\varepsilon}\right)^{1-4a}\Bigg) \times L^{4a-1}\int \mathrm{d}x_1\, {}^\star_\star e^{4\omega X_0}{}^\star_\star(x_1).\end{aligned} \tag{58}$$

The second contribution, from two tachyons and a contact term, is more involved, as one has to integrate over two operator positions, leading to various types of singularities. One has to be careful with path ordering of the contact term with the tachyon; we have to distinguish three contributions, symbolically noted CTT, TCT and TTC. One finds that the contributions of CTT and TTC are equal, but TCT is different. We have to sum these three contributions together. Using the notation $C(x) = {}^\star_\star T^+(x+\varepsilon)T^-(x)\times {}^\star_\star + {}^\star_\star T^-(x+\varepsilon)T^+(x){}^\star_\star$, one has

$$\begin{aligned}\mathrm{CTT}+\mathrm{TCT}+\mathrm{TTC} = -\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{\varepsilon^{a-1}}{2}\Bigg(\int \mathrm{d}x_1\int_{x_1-L+\varepsilon}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3\, {}^\star_\star C(x_1){}^\star_\star\, {}^\star_\star\psi^+T^+(x_2){}^\star_\star\, {}^\star_\star\psi^-T^-(x_3){}^\star_\star \\ + \int \mathrm{d}x_1\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L+\varepsilon}^{x_2-\varepsilon}\mathrm{d}x_3\, {}^\star_\star\psi^+T^+(x_1){}^\star_\star\, {}^\star_\star\psi^-T^-(x_3){}^\star_\star \\ \times \int \mathrm{d}x_1\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-2\varepsilon}\mathrm{d}x_3\, {}^\star_\star\psi^+T^+(x_1){}^\star_\star\, {}^\star_\star\psi^-T^-(x_2){}^\star_\star\, {}^\star_\star C(x_3){}^\star_\star\Bigg).\end{aligned} \tag{59}$$

Here, the whole computation is multiplied by the upper part of the identity matrix, since T^+ and T^- are themselves multiplied by σ^+ and σ^-, respectively. One should also take into account the permutated version of Eq. (59), which has ordering T^-T^+ instead of T^+T^-. From symmetry of the OPEs under this permutation, it contributes the same result, but multiplied by the lower part of the identity matrix. Thus, the computation of the divergent terms gives the result (see Appendix A)

$$\mathrm{CTT}+\mathrm{TCT}+\mathrm{TTC}\sim\begin{pmatrix}1&0\\0&1\end{pmatrix}\bigg[-\frac{2}{1+2a}\Big(\frac{L}{\varepsilon}\Big)^{2-2a}+\frac{1}{a}\Big(\frac{L}{\varepsilon}\Big)^{1-2a}+\frac{2(a-1)}{3a}\Big(\frac{L}{\varepsilon}\Big)^{1-a}\bigg(\frac{{}_2\mathrm{F}_1(-a,a+1,a+2,-1)}{a+1}+\frac{{}_2\mathrm{F}_1(-a,a-1,a,-1)}{a-1}+\frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1}\bigg)-V(a)\Big(\frac{L}{\varepsilon}\Big)^{1-4a}\bigg]L^{4a-1}\int\mathrm{d}x_1\,{}^{\star}_{\star}e^{4\omega X_0}\,{}^{\star}_{\star}(x_1). \tag{60}$$

The coefficient $V(a)$ is given by (we did not find a closed form for it)

$$\begin{aligned}V(a)={}&(a-1)\sum_{n=0}^{\infty}\sum_{s=0}^{1}\frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)(3a-s-n)}\bigg(\frac{{}_2\mathrm{F}_1(n-a,1+n-2a;2+n-2a;-1)}{1+n-2a}\\&+\frac{{}_2\mathrm{F}_1(s-a,1+s-2a;2+s-2a;-1)}{1+s-2a}+\frac{{}_2\mathrm{F}_1(n-a,s+n-1-2a;s+n-2a;-1)}{s+n-1-2a}\\&+\frac{{}_2\mathrm{F}_1(s-a,n+s-1-2a;n+s-2a;-1)}{n+s-1-2a}\bigg)+(a-1)\sum_{n,p=0}^{\infty}\frac{\Gamma(a)\Gamma(a-1)}{\Gamma(a-n)\Gamma(1+n)\Gamma(a-1-p)\Gamma(1+p)}\\&\times\frac{{}_2\mathrm{F}_1(1-a,n+p-3a,n+p+1-3a,-1)}{3a-n-p}\times\frac{{}_2\mathrm{F}_1(2+p-a,n+p+1-2a,n+p+2-2a,-1)}{n+p+1-2a}\\&+(a-1)\sum_{n,p=0}^{\infty}\frac{\Gamma(a)\Gamma(a-1)}{\Gamma(a-n)\Gamma(1+n)\Gamma(a-1-p)\Gamma(1+p)}\frac{{}_2\mathrm{F}_1(1-a,n+p-3a,n+p+1-3a,-1)}{3a-n-p}\\&\times\frac{{}_2\mathrm{F}_1(2+p-a,n+p+1-2a,n+p+2-2a,-1}{n+p+1-2a}+(a-1)\sum_{p=0}^{\infty}\sum_{s,t=0}^{1}\frac{\Gamma(a-1)}{\Gamma(a-1-p)\Gamma(1+p)(3a-s-t-p)}\\&\times\frac{{}_2\mathrm{F}_1(2+p-a,s+p+1-2a,s+p+2-2a,-1)}{s+p+1-2a}.\end{aligned}\tag{61}$$

Finally, one has to consider the contribution from four-tachyon insertions in the path integral (TTTT). The method of computation of the multiple integral is explained in Appendix B. After a lengthy computation, one gets[21]

$$\begin{aligned}\mathrm{TTTT}={}&\begin{pmatrix}1&0\\0&1\end{pmatrix}\int\mathrm{d}x_1\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3\int_{x_3-L}^{x_3-\varepsilon}\mathrm{d}x_4\,{}^{\star}_{\star}\psi^+T^+(x_1)\,{}^{\star\star}_{\star\star}\psi^-T^-(x_2)\,{}^{\star\star}_{\star\star}\psi^+T^+(x_3)\,{}^{\star\star}_{\star\star}\psi^-T^-(x_4)\,{}^{\star\star}_{\star\star}\\\sim{}&\begin{pmatrix}1&0\\0&1\end{pmatrix}\bigg\{\frac{1}{2a+1}\Big(\frac{L}{\varepsilon}\Big)^{2-2a}+\frac{a-1}{a}\Big(\frac{L}{\varepsilon}\Big)^{1-2a}-\frac{2(a-1)}{3a}\Big(\frac{L}{\varepsilon}\Big)^{1-a}\\&\times\bigg(\frac{{}_2\mathrm{F}_1(-a,a+1,a+2,-1)}{a+1}+\frac{{}_2\mathrm{F}_1(-a,a-1,a,-1)}{a-1}+\frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1}\bigg)\\&+\frac{(\frac{L}{\varepsilon})^{1-4a}-1}{1-4a}\bigg[(a-1)^2\bigg(\frac{{}_2\mathrm{F}_1(1-2a,a-1,a,-1)}{a-1}+\frac{{}_2\mathrm{F}_1(1-2a,2-3a,3-3a,-1)}{2-3a}\bigg)\\&\times\bigg(\frac{{}_2\mathrm{F}_1(-a,a-1,a,-1)}{a-1}+\frac{{}_2\mathrm{F}_1(-a,1-2a,2-2a,-1)}{1-2a}+\frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1}\\&+\frac{{}_2\mathrm{F}_1(2-a,1-2a,2-2a,-1)}{1-2a}\bigg)+(2(a-1)^2-1)\bigg(\frac{{}_2\mathrm{F}_1(1-a,a,1+a,-1)}{a}\\&+\frac{{}_2\mathrm{F}_1(1-a,1-2a,2-2a,-1)}{1-2a}\bigg)\times\bigg(\frac{{}_2\mathrm{F}_1(1-2a,a,a+1,-1)}{a}+\frac{{}_2\mathrm{F}_1(1-2a,1-3a,2-3a,-1)}{1-3a}\bigg)\bigg]\\&+U(a)\Big(\frac{L}{\varepsilon}\Big)^{1-4a}\bigg\}L^{4a-1}\int\mathrm{d}x_1\,{}^{\star}_{\star}e^{4\omega X_0}\,{}^{\star}_{\star}(x_1),\end{aligned}\tag{62}$$

with $U(a)$ a numerical coefficient which is not singular at $a=1/4$.

[21]The term with ordering $\mathrm{T}^-\mathrm{T}^+\mathrm{T}^-\mathrm{T}^+$ contributes the same result; thus, the total computation is directly multiplied by the identity matrix as in Eq. (60).

As in the previous computation, the coefficient $U(a)$ is known only as a series expansion:

$$\begin{aligned}
U(a) = {} & \frac{(a-1)^2}{4a-1}\left(\frac{{}_2F_1(1-2a,a-1;a;-1)}{a-1}+\frac{{}_2F_1(1-2a,-3a;1-3a;-1)}{3a}\right)\times\left(\frac{{}_2F_1(-a,1-2a;2-2a;-1)}{1-2a}\right.\\
& \left.+\frac{{}_2F_1(-a,a-1;a;-1)}{a-1}+\frac{{}_2F_1(2-a,1-2a;2-2a;-1)}{1-2a}+\frac{{}_2F_1(2-a,a+1;a+2;-1)}{a+1}\right)\\
& +\frac{(2(a-1)^2-1)}{4a-1}\left(\frac{{}_2F_1(1-2a,1-3a;2-3a;-1)}{3a-1}+\frac{{}_2F_1(1-2a,a;a+1;-1)}{a}\right)\\
& \times\left(\frac{{}_2F_1(1-a,1-2a;2-2a;-1)}{1-2a}+\frac{{}_2F_1(1-a,a;a+1;-1)}{a}\right)\\
& +(a-1)^2\sum_{n=0}^{\infty}\frac{\Gamma(a+1)}{\Gamma(n+1)\Gamma(a-n+1)(a+n-1)(3a-n)}\left(\frac{{}_2F_1(n-a,-2a+n+1;-2a+n+2;-1)}{-2a+n+1}\right.\\
& \left.+\frac{{}_2F_1(n-a,-2a+n-1;n-2a;-1)}{-2a+n-1}\right)+(a-1)^2\sum_{n=0}^{\infty}\frac{\Gamma(a-1)}{\Gamma(n+1)\Gamma(a-n-1)(a+n+1)(3a-n-2)}\\
& \times\left(\frac{{}_2F_1(-a+n+2,-2a+n+1;-2a+n+2;-1)}{-2a+n+1}+\frac{{}_2F_1(-a+n+2,-2a+n+3;-2a+n+4;-1)}{-2a+n+3}\right)\\
& +2(2(a-1)^2-1)\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(n+1)\Gamma(a-n)(a+n)(3a-n-1)}\frac{{}_2F_1(-a+n+1,-2a+n+1;-2a+n+2;-1)}{-2a+n+1}.
\end{aligned}\tag{63}$$

The last three sums are rapidly converging; thus, $U(a)$ is known with good accuracy for any value of $a \le 1/4$ (or $\omega \le 1/4$).

Let us now investigate the possible logarithmic divergences, which can only occur from the TTTT integral. Since we have that

$$\frac{(\frac{L}{\varepsilon})^{1-4a}-1}{1-4a}\overset{a\to 1/4}{\longrightarrow}\log\frac{L}{\varepsilon},\tag{64}$$

only the second-to-last term in Eq. (62) could lead to a logarithmic divergence at $\omega = 1/4$. It turns out that, in this limit, the coefficient of this term vanishes exactly. Looking more closely at this computation, one sees that each multiple integral which one gets from the three different fermionic contractions—see Eq. (C5)—has a logarithmic term as expected. However, the sum of them precisely cancels. Hence, the same occurs as at order two; the coefficient in front of the potentially resonant term in the beta function vanishes.[22]

In order to check whether powerlike divergences remain at fourth order, one has to resum the three contributions obtained above. The full contribution at order $((\lambda^+\lambda^-)^2)$ is given by CC + CTT + TCT + TTC + TTTT. Comparing Eqs. (57), (59), and (62), one sees that the coefficients in front of all divergent terms vanish exactly for any value of $\omega \ge 1/4$. Hence, in this range, if one includes the two-tachyon contact term dictated by world-sheet supersymmetry, perturbative expansion is finite.[23]

[22]This is confirmed by a direct evaluation of the TTTT integral at $\omega = 1/4$ (with MATHEMATICA), which gives

$$\begin{aligned}
\mathrm{TTTT} = {} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}4\int dx_1\int_{x_1-L}^{x_1-\varepsilon}dx_2\int_{x_2-L}^{x_2-\varepsilon}dx_3\int_{x_3-L}^{x_3-\varepsilon}dx_4\,{}^\star_\star\psi^+e^{X_0/4+ir\tilde{X}}(x_1)^\star_\star{}^\star_\star\psi^-e^{X_0/4-ir\tilde{X}}(x_2)^\star_\star{}^\star_\star\psi^+e^{X_0/4+ir\tilde{X}}(x_3)^\star_\star{}^\star_\star\psi^-e^{X_0/4-ir\tilde{X}}(x_4)^\star_\star\\
& \sim\left[\frac{2}{3}\left(\frac{L}{\varepsilon}\right)^{3/2}+\left(\frac{7\sqrt{\pi}\Gamma(\frac{5}{4})}{3\Gamma(\frac{3}{4})}-\alpha\right)\left(\frac{L}{\varepsilon}\right)^{3/4}-3\left(\frac{L}{\varepsilon}\right)^{1/2}\right]\int dx_1\,{}^\star_\star e^{X_0}(x_1)^\star_\star,
\end{aligned}$$

with $\alpha \simeq 1.24\ldots$. Logarithmic divergences are again found to vanish.

[23]This is not taking into account possible operator renormalization if there are operator insertions in the path integral.

As stated before, we were not able to compute the coefficient associated to the term of order ϵ^{1-4a}, which becomes divergent for $\omega < 1/4$ in a closed form. Using a numerical evaluation, we find that the sum of the contributions gives a nonzero coefficient for any $\omega < 1/4$. Hence, a powerlike divergence remains in this range. By dimensional counting, this uncanceled divergence corresponds to four-tachyon operators coming close together at the same point. It is not unexpected that this divergence is not canceled by the contact term, as the latter corresponds to a two-tachyon collision. Since this remaining divergence is nonlogarithmic, it does not mean that the boundary theory is not conformal, but rather that it should be renormalized at quartic order. It should be possible to cancel this divergence with a higher-order contact term. It may correspond to additional nonlinear terms in the superspace action (6) (a four-auxiliary field vertex is needed then).

As mentioned in Sec. III B, we also obtained an unexpected result. If we assume that the computations of CTT- and CC-type terms could be equivalently done with the use of the simple dominant term $\varepsilon^{a-1}e^{2\omega X^0}$ in Eq. (21), then we get the following contribution:

$$\begin{aligned}\mathrm{CC}+\mathrm{CTT}+\mathrm{TCT}+\mathrm{TTC} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\bigg\{&-\frac{1}{1+2a}\left(\frac{L}{\varepsilon}\right)^{2-2a}+\frac{1}{a}\left(\frac{L}{\varepsilon}\right)^{1-2a}+\frac{2(a-1)}{3a}\left(\frac{L}{\varepsilon}\right)^{1-a}\\ &\times\bigg(\frac{{}_2\mathrm{F}_1(-a,a+1,a+2,-1)}{a+1}+\frac{{}_2\mathrm{F}_1(-a,a-1,a,-1)}{a-1}+\frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1}\bigg)\\ &-\left(\frac{L}{\varepsilon}\right)^{1-4a}\bigg[2\frac{a-1}{3a}\bigg(\frac{{}_2\mathrm{F}_1(2-a,1-2a;2-2a;-1)}{1-2a}+\frac{{}_2\mathrm{F}_1(-a,1-2a;2-2a;-1)}{1-2a}\\ &-\frac{{}_2\mathrm{F}_1(-a,-1-2a;-2a;-1)}{1+2a}\bigg)+\frac{1}{2a+1}\bigg]\bigg\}L^{4a-1}\int \mathrm{d}x_1\,{}^\star_\star e^{4\omega X_0}\,{}^\star_\star(x_1)\ldots \end{aligned} \tag{65}$$

One recognizes the coefficients of the three first divergences; these are precisely the ones appearing in the sum of Eqs. (57) and (59) Moreover, numerical comparison of the $(\frac{L}{\varepsilon})^{1-4a}$ coefficients gives almost identical results; the tiny difference could reasonably originate from the approximated evaluation of the infinite sums. This seems to show the equivalence of the two computations—Eq. (65) being, of course, significantly easier to perform—and then of the two (local and nonlocal) expressions of the contact term.

C. Marginality to all orders

Computations become intractable for the next resonance, which occurs for $\omega = 1/6$ (or equivalently $|r| = \sqrt{17}/6$), as we have to consider sixth-order perturbation theory, with contributions from both counterterms found so far. However, we assume that the same occurs; the coefficient in front of the logarithmic six-tachyon divergence should vanish as well.

To summarize, we have found that, to all orders in perturbation theory, the theory defined by the boundary action (15) is a boundary conformal field theory when $|r| < \sqrt{17}/6$. In the range $\sqrt{17}/6 < |r| < 1/\sqrt{2}$, the theory is conformal at least up to order 5. We naturally expect that the theory is conformal to all orders in this range as well.

As a side remark, the theory defined by the limit $r \to r_c^-$ seems not well-defined. In this case, all the operators $e^{2n\omega X_0}$ are relevant, and by doing the perturbative expansion in the tachyon couplings, we would need an infinite number of counter-terms. By contrast, the theory defined directly at $r = r_c = 1/\sqrt{2}$ seems fine. The boundary interaction (with $\mathbb{T}^{\pm} \sim e^{\pm i\mathbb{X}/\sqrt{2}}$) is similar to a boundary sine-Gordon theory, with additional CP factors. Other puzzling features of the $r \to r_c^-$ limit will be discussed in the next section.

V. DISCUSSION

We argued in this work that, for all values of the brane-antibrane distance below the critical value r_c, the homogeneous rolling-tachyon solution with a fixed separation is an exact boundary conformal field theory. Thus, a space-time effective action which is valid around this particular solution should have such a tachyon profile as a solution of its equations of motion. An effective action for the brane-antibrane system was proposed by Garousi [25]. In a different parameterization of the tachyon field,[24] it reads

$$\mathcal{L}_{\mathrm{G}}(T,\dot{T},r,0) = -\frac{2}{\cosh\sqrt{\pi}|T|}\times\sqrt{1+4\pi^2r^2|T|^2-|\dot{T}|^2-\pi^2\dot{r}^2}. \tag{66}$$

One checks readily that, with $\dot{r}=0$, $\delta_r\mathcal{L}_{\mathrm{G}} \neq 0$ for any nonzero separation. Hence, this Lagrangian cannot admit solutions with constant brane-antibrane separation. This is not unexpected, since it was obtained by a fermion-number orbifold of the non-Abelian tachyon-DBI action for a pair of coincident non-BPS D-branes. Therefore, it could only be valid for an infinitesimal brane separation. Since $\delta_r\mathcal{L}_{\mathrm{G}}$ is linear in r, it seems to not even be valid in this limit.

[24]This field redefinition was discussed in Ref. [13] for the $r = 0$ case.

In order to find the space-time effective action from first principles, we could proceed as in Ref. [13]. In this approach, one considers a generic space-time Lagrangian of the D0-$\bar{\text{D}}$0 system, depending on the tachyon field τ, its first derivative, the distance field r and its first derivative.[25] Since, as we argued before, rolling-tachyon solutions at constant separation exist. The effective Lagrangian describing nearby field configurations should satisfy the condition

$$\frac{\delta \mathcal{L}(\tau, \dot{\tau}, r, \dot{r})}{\delta r}\Bigg|_{\dot{r}=0,\dot{\tau}=\omega\tau} = 0, \qquad (67)$$

where $\omega^2 = \frac{1}{2} - r^2$, as well as the equation of motion for the tachyon with a profile of the form $\tau = \mu \exp\omega t$.

Solving these equations at quadratic order in the tachyon field, one obtains a unique result, if we ask that for $r = 0$, one should recover the known Lagrangian for the coincident case:

$$\mathcal{L}(\tau, \dot{\tau}, r, 0) = -2 + \sqrt{1-2r^2}\left(\frac{\tau^2}{2} + \frac{\dot{\tau}^2}{1-2r^2}\right) + \ldots \qquad (68)$$

Unlike in the case of the non-BPS brane considered by Kutasov and Niarchos in [13], we did not impose above that the more generic profile $\tau = \zeta e^{\omega t} + \xi e^{-\omega t}$ (with arbitrary coefficients ζ and ξ) is a solution, since it does not correspond to an exactly marginal deformation on the world sheet as long as $r \neq 0$. A straightforward generalization of the effective Lagrangian found by these authors exists for $r \neq 0$ but should not be considered, since, by construction, it allows the time-reversal-symmetric tachyon profile $\tau \sim \cosh\omega t$ as a solution.

Imposing only the half S-brane as a solution leads to an underconstrained (finite) system of equations and not to a single recurrence relation as in Ref. [13]. However, below Eq. (44), we have shown that a solution of this form is marginal at second order, provided $\xi = i\mu\zeta$ with μ real. Besides the necessity to prove its conformal invariance at all orders (by going through an even more tedious analysis as we have done for the half S-brane solution), it would again lead to an underconstrained system. Indeed, this tachyon satisfies the identity $|\dot{\tau}|^2 = \omega^2|\tau|^2$. One can show that, as a consequence, the relation between the coefficients in the Lagrangian does not organize into a single recurrence relation, but rather separates into a system of independent equations which is underconstrained. Thus, it does not lead to a unique effective Lagrangian at higher orders in the tachyon couplings.[26]

The world-sheet theory contains more information about the tachyon effective action, besides imposing that the rolling-tachyon background of interest should be a solution of its equations of motion. Following Refs. [13,17], we expect to get the effective Lagrangian evaluated *on-shell* to be given by the disk partition function, with the timelike zero modes kept unintegrated:

$$\mathcal{L}|_{\tau,\omega\tau,r,0}(x_0) = -Z(r|x_0)|_{\text{disk}}. \qquad (69)$$

With this equation, one can test whether any proposal for the effective Lagrangian of the system is sensible. At second order, one can compare the space-time Lagrangian, given by Eq. (68), with the partition function given in Appendix C:

$$Z(r|x_0) = 2 - \frac{\Gamma(2-4r^2)}{\Gamma^2(1-2r^2)}\lambda^+\lambda^- e^{2\omega x_0} + \mathcal{O}((\lambda^+\lambda^-)^2). \qquad (70)$$

In order to match these two computations, we see that a distance-dependent field redefinition of the tachyon field is necessary:

$$\tau(t) = \left(\frac{1}{\sqrt{1-2r^2}}\frac{\Gamma(2-4r^2)}{\Gamma^2(1-2r^2)}\right)^{1/2}\lambda^+ e^{\omega t}. \qquad (71)$$

As one can see, with this definition, the space-time tachyon vanishes at the critical distance $r = 1/\sqrt{2}$, for any finite value of the world-sheet coupling λ^+. It could be the way string theory deals with the fact that, when $r \to r_c$, the tachyon becomes a light field, and we could wonder how a *local* action along the brane world-volume dimensions—that is *a priori* well-defined, as the tachyon is lighter than all string modes—would make sense, since the separation between the brane and the antibrane is significant in this regime.

The validity of the field redefinition (71) should be tested beyond quadratic order.[27] For this, one would have to first compute analytically the perturbative "screening integrals" at higher order, which does not seem trivial. For the special value of the distance $r = 1/2$, the computation, up to order 8 in the tachyon amplitude, is given in Appendix C:

$$\begin{aligned} Z\left(\frac{1}{2}\,\Big|\,x_0\right) = 2\Bigg(&1 - \frac{\lambda^+\lambda^- e^{\omega x_0}}{2\pi} + \left(1-\frac{\pi^2}{6}\right)\left(\frac{\lambda^+\lambda^- e^{\omega x_0}}{2\pi}\right)^2 \\ &- \left(1 - \frac{128}{3\pi^2}\right)\left(\frac{\lambda^+\lambda^- e^{\omega x_0}}{2\pi}\right)^3 \\ &+ \left(1 + \frac{205}{108} + \frac{3575}{162\pi^2} + \frac{\pi^2}{2} + \frac{\pi^4}{70}\right) \\ &\times\left(\frac{\lambda^+\lambda^- e^{\omega x_0}}{2\pi}\right)^4 + \mathcal{O}\left(\left(\frac{\lambda^+\lambda^- e^{\omega x_0}}{2\pi}\right)^5\right)\Bigg). \end{aligned} \qquad (72)$$

This does not seem to trace back to the Taylor expansion of a known function.

[25]We assume that, by the symmetries of the problem, only even powers of the fields and their derivative appear, i.e one has $\mathcal{L}(|\tau|^2, |\dot{\tau}|^2, r^2, \dot{r}^2)$. Without loss of generality, as the phase of the tachyon for the half S-brane solution under study is constant, we take $\tau(t)$ real.

[26]Note, on the other hand, that Eq. (68) is still valid with Eq. (47) under the replacement $\tau^2 \to |\tau|^2$ and $\dot{\tau}^2 \to |\dot{\tau}|^2$.

[27]One can check already that plugging this redefinition in Garousi's Lagrangian (66) does not lead to a consistent effective Lagrangian.

Since the space-time effective action approach seems to have important limitations for branes-antibranes at finite separation, the boundary string field theory may be more appropriate in order to know the properties of the system. Even though it does not contain information about the dynamics of the system, it allows us to find the exact tachyon potential (as well as the appearance of lower-dimensional branes), and hence can illuminate the fate of the tachyon. These computations seem not to be out of reach. We plan to come back to these issues in the near future.

A heuristic argument gives a good motivation for this study. Following Refs. [34,35], one could describe the result of this condensation by studying the closed string emission from the time-dependent boundary state. It was found in Ref. [34] that, knowing the one-point function on the disk $B(E) = \langle e^{iEX_0}\rangle$, one can compute the density of closed-string states emitted by the decay of a non-BPS brane, which goes as

$$\rho_c \sim \sum_N \frac{1}{E_N} D(N)|B(E_N)|^2, \tag{73}$$

where the asymptotic Hagedorn density of closed-string states at level N has the form $D(N) \sim N^{-\alpha}\exp(4\pi\sqrt{N})$, with $\alpha > 0$, and $E_N \sim 2\sqrt{N}$. The one-point function for an unstable non-BPS D-brane goes as $|B(E)|^2 \sim \exp(-2\pi E)$. Therefore, in this case, the sum is governed by the subleading powerlike corrections to the Hagedorn density and typically diverge, giving the so-called "tachyon dust" of massive closed strings. In the case of nonzero separation, by dimensional analysis, we may expect that $|B(E)| \sim \exp(-\sqrt{2}\pi E/|m_{\text{tach}}(r)|)$. This would lead to a convergent closed string production when $|m_{\text{tach}}| < 1/\sqrt{2}$ (i.e. for $r \neq 0$), signaling that the tachyon does not condense completely at a finite distance.

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APPENDIX A: COMPUTATION OF THE DIVERGENCES IN CTT-TYPE TERMS

We give below one example of computation of the divergence occurring in an integral involving one contact operator insertion. We study here the CTT term. With a bit of care, one can compute them exactly. With the expression of C given in Eq. (21), the CTT term is

$$\begin{aligned} &-\frac{\varepsilon^{a-1}}{2}\int \mathrm{d}x_1 \int_{x_1-L+\varepsilon}^{x_1-\varepsilon}\mathrm{d}x_2 \int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3 \,{}^\star_\star C(x_1){}^\star_\star\,{}^\star_\star T^+(x_2){}^\star_\star\,{}^\star_\star T^-(x_3){}^\star_\star \\ &= (a-1)\frac{\varepsilon^{a-1}}{2}\int \mathrm{d}x_1 \int_{x_1-L+\varepsilon}^{x_1-\varepsilon}\mathrm{d}x_2 \int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3((x_1-x_2+\varepsilon)(x_1-x_3+\varepsilon)^{a-1}(x_1-x_2)^{a-1}(x_1-x_3)(x_2-x_3)^{a-2} \\ &\quad + (x_1-x_2+\varepsilon)^{a-1}(x_1-x_3+\varepsilon)(x_1-x_2)(x_1-x_3)^{a-1}(x_2-x_3)^{a-2}), \end{aligned} \tag{A1}$$

with $a = 4\omega^2$. Note that the IR cutoff is chosen such that two ordered operators do not move away from each other more than L. Then, since $C(x) \sim T^{\pm}(x+\varepsilon)T^{\mp}(x)$, the cutoff for x_2 in relation to x_1 is $L - \varepsilon$. One can get read of the path ordering with the following change of variable :

$$x_2 = -L\delta_1 + x_1 \qquad x_3 = -L\delta_2 + x_2, \tag{A2}$$

such that it gives, introducing $\eta = \varepsilon/L$,

$$\begin{aligned} &(a-1)L^{4a-1}\frac{\eta^{a-1}}{2}\int_\eta^{1-\eta}\mathrm{d}\delta_1\int_\eta^1 \mathrm{d}\delta_2((\delta_1+\eta) \\ &\quad\times(\delta_1+\delta_2+\eta)^{a-1}\delta_1^{a-1}(\delta_1+\delta_2)\delta_2^{a-2} \\ &\quad+(\delta_1+\eta)^{a-1}(\delta_1+\delta_2+\eta)\delta_1(\delta_1+\delta_2)^{a-1}\delta_2^{a-2}) \\ &\quad\times\int \mathrm{d}x_1 e^{4\omega X^0}(x_1). \end{aligned} \tag{A3}$$

The integral over δ_i's can be done with the use of the series representation of $(1+\frac{\eta}{\delta_1+\delta_2})^\alpha$, since $\delta_1+\delta_2 > \eta$, and $(1+\frac{\eta}{\delta_1})^\beta$, since $\delta_1 > \eta$. These are given by

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\frac{\Gamma(1+\alpha)x^n}{\Gamma(1+\alpha-n)\Gamma(1+n)}, \quad \text{with} \quad |x| < 1. \tag{A4}$$

Convergence of the series all along the domain of integration allows us to commute the integral and sum sign,[28] such that one has

$$\begin{aligned} &(a-1)\sum_{s=0}^{1}\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)}\eta^{a-1+s+n} \\ &\quad\times\int_\eta^{1-\eta}\mathrm{d}\delta_1\int_\eta^1\mathrm{d}\delta_2(\delta_1^{a-s}\delta_2^{a-2}(\delta_1+\delta_2)^{a-n} \\ &\quad+\delta_1^{a-n}\delta_2^{a-2}(\delta_1+\delta_2)^{a-s}). \end{aligned} \tag{A5}$$

As one can see, the two integrals to compute are symmetric by permutation of s and n. We then only focus on the first

[28]It is true at least *a fortiori* from the convergence of the integrals and the series of the integrals. Note, besides, that we do not integrate over any pole.

one. There are two ways to proceed now: integrate directly and exactly, since it is possible, or use an indirect method that reintroduces some path ordering. We use the second, and apparently more complicated, method because it is needed to compute TTTT integrals. Indeed, one will see that hypergeometric functions will receive argument z, which has an absolute value less than 1, which is much easier to handle for approximations, since the series representation is known exactly. We separate the first integral of Eq. (A5) into

$$\begin{aligned}
&\int_\eta^{1-\eta} \mathrm{d}\delta_1 \int_\eta^{\delta_1} \mathrm{d}\delta_2 \delta_1^{2a-n-s} \delta_2^{a-2}\left(1+\frac{\delta_2}{\delta_1}\right)^{a-n} + \int_\eta^{1-\eta} \mathrm{d}\delta_1 \int_{\delta_1}^{1} \mathrm{d}\delta_2 \delta_1^{a-s} \delta_2^{2a-2-n}\left(1+\frac{\delta_1}{\delta_2}\right)^{a-n} \\
&\quad= \int_\eta^{1-\eta} \mathrm{d}\delta_1 \delta_1^{2a-n-s}\left[\frac{\delta_2^{a-1}}{a-1}\,{}_2\mathrm{F}_1\left(n-a,a-1,a,-\frac{\delta_2}{\delta_1}\right)\right]_\eta^{\delta_1} \\
&\qquad+ \int_\eta^{1-\eta} \mathrm{d}\delta_1 \delta_1^{a-s}\left[\frac{\delta_2^{2a-1-n}}{2a-1-n}\,{}_2\mathrm{F}_1\left(n-a,1+n-2a,2+n-2a,-\frac{\delta_1}{\delta_2}\right)\right]_{\delta_1}^{1} \\
&\quad= \int_\eta^{1-\eta} \mathrm{d}\delta_1 \delta_1^{3a-1-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-1)2}{a-1} + \frac{{}_2\mathrm{F}_1(n-a,1+n-2a,2+n-2a,-1)}{1+n-2a}\right) \\
&\qquad- \frac{\eta^{a-1}}{a-1}\int_\eta^{1-\eta} \mathrm{d}\delta_1 \delta_{12}^{2a-s-n}\,{}_2\mathrm{F}_1\left(n-a,a-1,a,-\frac{\eta}{\delta_1}\right) \\
&\qquad+ \frac{1}{2a-1-n}\int_\eta^{1-\eta} \mathrm{d}\delta_1 \delta_{12}^{a-s}\,{}_2\mathrm{F}_1(n-a,1+n-2a,2+n-2a,-\delta_1).
\end{aligned} \tag{A6}$$

Let us remark at this stage that the z argument in ${}_2\mathrm{F}_1(a,b,c,z)$ verifies $|z|<1$ in the above integrals. The first one is trivial and gives

$$\begin{aligned}
I_1 &= \frac{(1-\eta)^{3a-s-n}-\eta^{3a-s-n}}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-1)}{a-1} + \frac{{}_2\mathrm{F}_1(n-a,1+n-2a,2+n-2a,-1)}{1+n-2a}\right) \\
&= \frac{1-\eta^{3a-s-n}}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-1)}{a-1} + \frac{{}_2\mathrm{F}_1(n-a,1+n-2a,2+n-2a,-1)}{1+n-2a}\right) + o(\eta).
\end{aligned} \tag{A7}$$

The second one is a bit more involved:

$$\begin{aligned}
I_2 &= -\frac{\eta^{3a-s-n}}{a-1}\left[-\frac{\delta_1^{s+n-1-2a}(a-1)}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-\delta_1)}{a-1} - \frac{{}_2\mathrm{F}_1(n-a,s+n-1-2a,s+n-2a,-\delta_1)}{s+n-1-2a}\right)\right]_{\eta/(1-\eta)}^{1} \\
&= \frac{\eta^{3a-s-n}}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-1)}{a-1} - \frac{{}_2\mathrm{F}_1(n-a,s+n-1-2a,s+n-2a,-1)}{s+n-1-2a}\right) \\
&\quad- \frac{\eta^{a-1}(1-\eta)^{-s-n+1+2a}}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-\frac{\eta}{1-\eta})}{a-1} - \frac{{}_2\mathrm{F}_1(n-a,s+n-1-2a,s+n-2a,-\frac{\eta}{1-\eta})}{s+n-1-2a}\right) \\
&= \frac{\eta^{3a-s-n}}{3a-s-n}\left(\frac{{}_2\mathrm{F}_1(n-a,a-1,a,-1)}{a-1} - \frac{{}_2\mathrm{F}_1(n-a,s+n-1-2a,s+n-2a,-1)}{s+n-1-2a}\right) \\
&\quad+ \frac{\eta^{a-1}}{(a-1)(s+n-1-2a)} + \frac{\eta^a}{a-1}\left(1+\frac{(a-n)(a-1)}{a(s+n-2a)}\right) + o(\eta^{a+1}).
\end{aligned} \tag{A8}$$

On the last line, we used the series representation of ${}_2\mathrm{F}_1$:

$${}_2\mathrm{F}_1(a,b,c,z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!}, \tag{A9}$$

for $|z|<1$. In particular, for $c=b+1$, we have

$${}_2\mathrm{F}_1(-a,b,b+1,-z) = \sum_{k=0}^{\infty}\frac{\Gamma(1+a)b}{\Gamma(1+a-k)\Gamma(1+k)(b+k)}z^k. \tag{A10}$$

Finally, the third one is

$$I_3 = \frac{1}{2a-1-n}\left[-\frac{\delta_1^{a+1-s}(1+n-2a)}{s+n-3a}\left(\frac{{}_2F_1(n-a,1+n-2a,2+n-2a,-\delta_1)}{1+n-2a} - \frac{{}_2F_1(n-a,1+a-s,2+a-s,-\delta_1)}{1+a-s}\right)\right]_\eta^{1-\eta}$$
$$= \frac{(1-\delta_1)^{a+1-s}}{s+n-3a}\left(\frac{{}_2F_1(n-a,1+n-2a,2+n-2a,-1+\eta)}{1+n-2a} - \frac{{}_2F_1(n-a,1+a-s,2+a-s,-1+\eta)}{1+a-s}\right) - o(\eta^{a+1-s})$$
$$= \frac{-1}{3a-s-n}\left(\frac{{}_2F_1(n-a,1+n-2a,2+n-2a,-1)}{1+n-2a} - \frac{{}_2F_1(n-a,1+a-s,2+a-s,-1)}{1+a-s}\right) + o(\eta^{a+1-s}) + o(\eta). \quad \text{(A11)}$$

Collecting these results, one finally get the sum

$$\frac{a-1}{2}\sum_{s=0}^{1}\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)}\eta^{a-1+s+n}\int_\eta^{1-\eta}\mathrm{d}\delta_1\int_\eta^1\mathrm{d}\delta_2(\delta_1^{a-s}\delta_2^{a-2}(\delta_1+\delta_2)^{a-n} + \delta_1^{a-n}\delta_2^{a-2}(\delta_1+\delta_2)^{a-s})$$
$$= \frac{a-1}{2}\sum_{s=0}^{1}\sum_{n=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)}\eta^{a-1+s+n}(I_1+I_2+I_3+(s\leftrightarrow n))$$
$$\sim -\frac{\eta^{2a-2}}{2a+1} + \frac{\eta^{2a-1}}{2a} - \eta^{a-1}\frac{a-1}{3a}\left(\frac{{}_2F_1(-a,a+1,a+2,-1)}{a+1} + \frac{{}_2F_1(-a,a-1,a,-1)}{a-1}\right)$$
$$- \eta^{4a-1}\sum_{n=0}^{\infty}\sum_{s=0}^{1}\frac{\Gamma(a)}{\Gamma(a-n)\Gamma(1+n)(3a-s-n)}\left(\frac{{}_2F_1(n-a,1+n-2a,2+n-2a,-1)}{1+n-2a}\right.$$
$$+ \frac{{}_2F_1(s-a,1+s-2a,2+s-2a,-1)}{1+s-2a} + \frac{{}_2F_1(n-a,s+n-1-2a,s+n-2a,-1)}{s+n-1-2a}$$
$$\left. + \frac{{}_2F_1(s-a,s+n-1-2a,s+n-2a,-1)}{s+n-1-2a}\right). \quad \text{(A12)}$$

A similar computation was done for the TCT and TTC terms, with the correct cutoff prescriptions. Note, however, that CTT = TTC.

APPENDIX B: COMPUTATION OF THE DIVERGENCES IN THE TTTT TERM

The computation of an amplitude with four-tachyon insertions is clearly a lot more involved than the above one, since three integrations have to be done. The straightforward OPE of the four tachyons is doable and gives, from Eq. (C5),

$$\int\mathrm{d}x_1\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3\int_{x_3-L}^{x_3-\varepsilon}\mathrm{d}x_4\,{}^\star_\star\psi^+T^+(x_1){}^\star_\star{}^\star_\star\psi^-T^-(x_2){}^\star_\star{}^\star_\star\psi^+T^+(x_3){}^\star_\star{}^\star_\star\psi^-T^-(x_4){}^\star_\star$$
$$= \int\mathrm{d}x_1 e^{4\omega X^0}\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3\int_{x_3-L}^{x_3-\varepsilon}\mathrm{d}x_4((a-1)^2(x_1-x_2)^{a-2}(x_1-x_3)(x_1-x_4)^{a-1}(x_2-x_3)^{a-1}(x_2-x_4)$$
$$\times(x_3-x_4)^{a-2} - (x_1-x_2)^{a-1}(x_1-x_4)^{a-1}(x_2-x_3)^{a-1}(x_3-x_4)^{a-1} + (a-1)^2(x_1-x_2)^{a-1}(x_1-x_3)$$
$$\times(x_1-x_4)^{a-2}(x_2-x_3)^{a-2}(x_2-x_4)(x_3-x_4)^{a-1}). \quad \text{(B1)}$$

This integrand is too much coupled in its variables and not analytically computable in this form. But one can show using the identity

$$(x_1-x_2)(x_3-x_4) - (x_1-x_3)(x_2-x_4) + (x_1-x_4)(x_2-x_3) = 0 \quad \text{(B2)}$$

that the integrand can be reexpressed as

$$\int\mathrm{d}x_1 e^{4\omega X^0}\int_{x_1-L}^{x_1-\varepsilon}\mathrm{d}x_2\int_{x_2-L}^{x_2-\varepsilon}\mathrm{d}x_3\int_{x_3-L}^{x_3-\varepsilon}\mathrm{d}x_4((a-1)^2(x_1-x_2)^{a-2}(x_2-x_3)^a(x_3-x_4)^{a-2}(x_1-x_4)^a + (2(a-1)^2-1)$$
$$\times(x_1-x_2)^{a-1}(x_1-x_4)^{a-1}(x_2-x_3)^{a-1}(x_3-x_4)^{a-1} + (a-1)^2(x_1-x_2)^a(x_2-x_3)^{a-2}(x_3-x_4)^a(x_1-x_4)^{a-2}). \quad \text{(B3)}$$

If we use the change of variable

$$x_2 = -L\delta_1 + x_1 \qquad x_3 = -L\delta_2 + x_2 \qquad x_4 = -L\delta_3 + x_3, \quad \text{(B4)}$$

the integral becomes

$$\int \mathrm{d}x_1 e^{4\omega X^0} \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^1 \mathrm{d}\delta_2 \int_\eta^1 \mathrm{d}\delta_3((a-1)^2\delta_1^{a-2}\delta_2^a\delta_3^{a-2}(\delta_1+\delta_2+\delta_3)^a + (a-1)^2\delta_1^a\delta_2^{a-2}\delta_3^a(\delta_1+\delta_2+\delta_3)^{a-2} + (2(a-1)^2-1)\delta_1^{a-1}\delta_2^{a-1}\delta_3^{a-1}(\delta_1+\delta_2+\delta_3)^{a-1}). \tag{B5}$$

It is possible to extract the divergences by analytic integration, but we need to be careful since we will need at some point to commute the integrals and sums. For this reason, the z argument in the ${}_2\mathrm{F}_1(a, b, c, z)$ should satisfy $|z| < 1$.

We will not develop the whole computation but give as an example one of the three integrals. Let us study the following one:

$$\int_\eta^1 \mathrm{d}\delta_1 \int_\eta^1 \mathrm{d}\delta_2 \int_\eta^1 \mathrm{d}\delta_3 \delta_1^a \delta_2^{a-2}\delta_3^a(\delta_1+\delta_2+\delta_3)^{a-2}. \tag{B6}$$

Integration of δ_3 imposes to separate the domain of integration in three parts:

$$\begin{aligned}&\delta_1+\delta_2>1 \quad \text{and} \quad \delta_3\in[\eta;1]<\delta_1+\delta_2 \qquad \delta_1+\delta_2<1 \quad \text{and} \quad \delta_3\in[\eta;\delta_1+\delta_2]<\delta_1+\delta_2\\ &\delta_1+\delta_2<1 \quad \text{and} \quad \delta_3\in[\delta_1+\delta_2;1]>\delta_1+\delta_2\,.\end{aligned} \tag{B7}$$

This makes three integrals:

$$\begin{aligned}
I_1 &= \int_\eta^1 \mathrm{d}\delta_1 \int_{1-\delta_1}^1 \mathrm{d}\delta_2 \int_\eta^1 \mathrm{d}\delta_3 \delta_1^a\delta_2^{a-2}\delta_3^a(\delta_1+\delta_2)^{a-2}\left(1+\frac{\delta_3}{\delta_1+\delta_2}\right)^{a-2}\\
I_2 &= \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \int_\eta^{\delta_1+\delta_2} \mathrm{d}\delta_3 \delta_1^a\delta_2^{a-2}\delta_3^a(\delta_1+\delta_2)^{a-2}\left(1+\frac{\delta_3}{\delta_1+\delta_2}\right)^{a-2}\\
I_3 &= \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \int_{\delta_1+\delta_2}^1 \mathrm{d}\delta_3 \delta_1^a\delta_2^{a-2}\delta_3^{2a-2}\left(1+\frac{\delta_1+\delta_2}{\delta_3}\right)^{a-2},
\end{aligned} \tag{B8}$$

which integrate to

$$\begin{aligned}
I_1 &= \int_\eta^1 \mathrm{d}\delta_1 \int_{1-\delta_1}^1 \mathrm{d}\delta_2 \delta_1^a\delta_2^{a-2}(\delta_1+\delta_2)^{a-2}\left[\frac{\delta_3^{a+1}}{a+1}{}_2\mathrm{F}_1\left(2-a, a+1, a+2, -\frac{\delta_3}{\delta_1+\delta_2}\right)\right]_\eta^1\\
I_2 &= \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \delta_1^a\delta_2^{a-2}(\delta_1+\delta_2)^{a-2}\left[\frac{\delta_3^{a+1}}{a+1}{}_2\mathrm{F}_1\left(2-a, a+1, a+2, -\frac{\delta_3}{\delta_1+\delta_2}\right)\right]_\eta^{\delta_1+\delta_2}\\
I_3 &= \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \delta_1^a\delta_2^{a-2}\left[\frac{\delta_3^{2a-1}}{2a-1}{}_2\mathrm{F}_1\left(2-a, 1-2a, 2-2a, -\frac{\delta_1+\delta_2}{\delta_3}\right)\right]_{\delta_1+\delta_2}^1.
\end{aligned} \tag{B9}$$

We will not develop the computations for all the three integrals. Let us focus on the third, which is easier to present. The method is similar for the two other ones.

$$I_3 = \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \delta_1^a\delta_2^{a-2}\left(\frac{1}{2a-1}{}_2\mathrm{F}_1(2-a, 1-2a, 2-2a, -\delta_1-\delta_2) - \frac{(\delta_1+\delta_2)^{2a-1}}{2a-1}{}_2\mathrm{F}_1(2-a, 1-2a, 2-2a, -1)\right). \tag{B10}$$

These are two different integrations to do. We have

$$\begin{aligned}
I_3^1 &= \int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \delta_1^a\delta_2^{a-2}\frac{1}{2a-1}{}_2\mathrm{F}_1(2-a, 1-2a, 2-2a, -\delta_1-\delta_2)\\
I_3^2 &= -\int_\eta^1 \mathrm{d}\delta_1 \int_\eta^{1-\delta_1} \mathrm{d}\delta_2 \frac{(\delta_1+\delta_2)^{2a-1}}{2a-1}{}_2\mathrm{F}_1(2-a, 1-2a, 2-2a, -1).
\end{aligned} \tag{B11}$$

Each of these separates again in three parts:

$$\delta_1\in\left[\eta;\frac{1}{2}\right] \text{ and } \delta_2\in[\eta;\delta_1] \quad \delta_1\in\left[\eta;\frac{1}{2}\right] \text{ and } \delta_2\in[\delta_1;1-\delta_1] \quad \delta_1\in\left[\frac{1}{2};1\right] \text{ and } \delta_2\in[\eta;1-\delta_1]. \tag{B12}$$

There is no known expression for the integration of I_3^1, but it is not much of a problem since we only want to extract divergences. Because $|\delta_1+\delta_2| < 1$, one can express ${}_2\mathrm{F}_1$ as its series expansion given in Eq. (A10). Since the series is convergent everywhere in the integration domain, we can commute the sum and the integral, such that

$$I_3^1 = -\sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(a-1-n)\Gamma(1+n)(1-2a+n)}\bigg(\int_\eta^{1/2} d\delta_1 \int_\eta^{\delta_1} d\delta_2 \delta_1^a \delta_2^{a-2}(\delta_1+\delta_2)^n + \int_\eta^{1/2} d\delta_1 \int_{\delta_1}^{1-\delta_1} d\delta_2 \delta_1^a \delta_2^{a-2}(\delta_1+\delta_2)^n + \int_{1/2}^{1} d\delta_1 \int_\eta^{1-\delta_1} d\delta_2 \delta_1^a \delta_2^{a-2}(\delta_1+\delta_2)^n\bigg). \tag{B13}$$

These integrals are very similar to the ones studied in Appendix A. Following the method presented there, and with a careful power analysis in η, we can obtain

$$\begin{aligned} I_3^1 &= -\sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(a-1-n)\Gamma(1+n)(1-2a+n)} \times \bigg(-\frac{2^{-a-1-n}\eta^{a-1}}{(a+1+n)(a-1)} + o(1) + \frac{(2^{-a-1-n}-1)\eta^{a-1}}{(a+1+n)(a-1)}\bigg) \\ &= -\frac{\eta^{a-1}}{3a(a-1)}\bigg(\frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{2a-1} + \frac{{}_2F_1(2-a,a+1,a+2,-1)}{a+1}\bigg) + o(1). \end{aligned} \tag{B14}$$

The computation of I_3^2 is less difficult. With the method of Appendix A and Eq. (B12), it gives

$$\begin{aligned} I_3^2 = {}&\int_\eta^{1/2} d\delta_1 \delta_1^{4a-2}\bigg(\frac{{}_2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{{}_2F_1(1-2a,2-3a,3-3a,-1)}{2-3a}\bigg)\frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{1-2a} \\ &+ \frac{\eta^{4a-1}}{3a(a-1)(4a-1)}(3a\,{}_2F_1(1-2a,a-1,a,-1) + (a-1)\,{}_2F_1(1-2a,-3a,1-3a,-1)) \\ &\times \frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{1-2a} + \frac{\eta^{a-1}}{3a}\frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{a-1} + o(1). \end{aligned} \tag{B15}$$

We do not integrate explicitly the first term so that the logarithm appears unambiguously at $a = 1/4$. This has to be compared to the second term, which does not become a logarithm, since it is finite at $a = 1/4$. Indeed, for this precise value $a-1 = -3a$, and one gets $\frac{\eta^0}{3a(a-1)}$.

Finally, summing up I_3^1 with I_3^2, one obtains

$$\begin{aligned} I_3 = {}&\int_\eta^{1/2} d\delta_1 \delta_1^{4a-2}\bigg(\frac{{}_2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{{}_2F_1(1-2a,2-3a,3-3a,-1)}{2-3a}\bigg)\frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{1-2a} \\ &+ \frac{\eta^{4a-1}}{3a(a-1)(4a-1)}(3a\,{}_2F_1(1-2a,a-1,a,-1) + (a-1)\,{}_2F_1(1-2a,-3a,1-3a,-1)) \\ &\times \frac{{}_2F_1(2-a,1-2a,2-2a,-1)}{1-2a} - \frac{\eta^{a-1}}{3a}\frac{{}_2F_1(2-a,a+1,a+2,-1)}{a+1} + o(1). \end{aligned} \tag{B16}$$

Similarly, one computes I_1 and I_2, for which we obtain

$$I_1 = o(1) \tag{B17}$$

and

$$\begin{aligned} I_2 = {}&\int_\eta^{1/2} d\delta_1 \delta_1^{4a-2}\bigg(\frac{{}_2F_1(1-2a,a-1,a,-1)}{a-1} + \frac{{}_2F_1(1-2a,2-3a,3-3a,-1)}{2-3a}\bigg)\frac{{}_2F_1(2-a,a+1,a+2,-1)}{a+1} \\ &+ \frac{\eta^{4a-1}}{3a(a-1)(4a-1)}(3a\,{}_2F_1(1-2a,a-1,a,-1) + (a-1)\,{}_2F_1(1-2a,-3a,1-3a,-1))\frac{{}_2F_1(2-a,a+1,a+2,-1)}{a+1} \\ &+ \eta^{4a-1}\sum_{n=0}^{\infty}\frac{\Gamma(a-1)}{\Gamma(n+1)\Gamma(a-n-1)(a+n+1)(3a-n-2)}\bigg(\frac{{}_2F_1(-2a+n+1,-a+n+2;-2a+n+2;-1)}{-2a+n+1} \\ &+ \frac{{}_2F_1(-2a+n+3,-a+n+2;-2a+n+4;-1)}{-2a+n+3}\bigg) - \frac{\eta^{a-1}}{3a}\frac{{}_2F_1(2-a,a+1,a+2,-1)}{a+1} + o(1). \end{aligned} \tag{B18}$$

One expresses then the whole integral (B6) as

$$\int_\eta^1 \mathrm{d}\delta_1 \int_\eta^1 \mathrm{d}\delta_2 \int_\eta^1 \mathrm{d}\delta_3 \delta_1^a \delta_2^{a-2} \delta_3^a (\delta_1+\delta_2+\delta_3)^{a-2}$$
$$\sim \int_\eta^{1/2} \mathrm{d}\delta_1 \delta_1^{4a-2} \bigg(\frac{{}_2\mathrm{F}_1(1-2a,a-1,a,-1)}{a-1} + \frac{{}_2\mathrm{F}_1(1-2a,2-3a,3-3a,-1)}{2-3a} \bigg) \bigg(\frac{{}_2\mathrm{F}_1(2-a,1-2a,2-2a,-1)}{1-2a}$$
$$+ \frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1} \bigg) + \eta^{4a-1} \bigg[\frac{1}{(4a-1)} \bigg(\frac{{}_2\mathrm{F}_1(1-2a,a-1,a,-1)}{a-1} + \frac{{}_2\mathrm{F}_1(1-2a,-3a,1-3a,-1)}{3a} \bigg)$$
$$\times \bigg(\frac{{}_2\mathrm{F}_1(2-a,1-2a,2-2a,-1)}{1-2a} + \frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1} \bigg) + \sum_{n=0}^{\infty} \frac{\Gamma(a-1)}{\Gamma(n+1)\Gamma(a-n-1)(a+n+1)(3a-n-2)}$$
$$\times \bigg(\frac{{}_2\mathrm{F}_1(-2a+n+1,-a+n+2;-2a+n+2;-1)}{-2a+n+1} + \frac{{}_2\mathrm{F}_1(-2a+n+3,-a+n+2;-2a+n+4;-1)}{-2a+n+3} \bigg) \bigg]$$
$$-2 \frac{\eta^{a-1}}{3a} \frac{{}_2\mathrm{F}_1(2-a,a+1,a+2,-1)}{a+1} + o(1). \tag{B19}$$

Similar techniques apply to the two other kinds of integrals.

APPENDIX C: PARTITION FUNCTION TO EIGHTH ORDER

The disk partition function for the system, unintegrated over the timelike zero modes, can be expressed as a series:

$$Z(r|x_0) = \sum_{n=0}^{\infty} (-\lambda^+ \lambda^- e^{2\omega x^0})^n I_n, \tag{C1}$$

with I_n a coefficient which is equal to the sum time-ordered integrals that appear at order n in the perturbative expansion. We can express it in a condensed form as

$$I_n = \int [\mathrm{d}t]_{\underset{>}{2n}} \begin{vmatrix} 1 & 3 & 5 & \dots & 2n-1 \\ 2 & 4 & 6 & \dots & 2n \end{vmatrix}^{-4r^2} \sum_{\text{perm } \mathcal{P}} (-1)^P \begin{vmatrix} a_1 a_2 \\ a_3 a_4 \\ \dots \\ a_{2n-1} a_{2n} \end{vmatrix} (1-4r^2)^{(n/2)-(1/2)\sum_{i=1}^{n}(-1)^{a_{2i-1}-a_{2i}}}, \tag{C2}$$

with the time-ordered measure

$$[\mathrm{d}t]_{\underset{>}{2n}} = \prod_{i=1}^{2n} \frac{\mathrm{d}t_i}{2\pi} \prod_{i=1}^{2n-1} \Theta(t_i - t_{i+1}). \tag{C3}$$

We have also introduced convenient notations for the integrand, defined as

$$\begin{vmatrix} a_1 a_2 \\ a_3 a_4 \\ \dots \\ a_{2n-1} a_{2n} \end{vmatrix} = \prod_{i=1}^{n} \prod_{j=2i+1}^{2n} S(a_{2i-1}, a_j) S(a_{2i}, a_j), \qquad \begin{vmatrix} i_1 i_2 \dots i_p \\ j_1 j_2 \dots j_n \end{vmatrix} = \prod_{\alpha=1}^{p} \prod_{a=1}^{n} S(i_\alpha, j_a), \tag{C4}$$

where $S(i,j) = |2 \sin\frac{t_i - t_j}{2}|$. The sum in Eq. (C2) is done over all permutations within the set $\{1,2,3\dots 2n\}$.[29] Up to $n=2$, the partition function, for given $|r| < 1/\sqrt{2}$, reads

$$Z(r|x_0) = 2 - 2\lambda^+\lambda^- e^{2\omega x^0} \int [\mathrm{d}t]_{\underset{>}{2}} \begin{vmatrix} 1 \\ 2 \end{vmatrix}^{-4r^2} (1-4r^2) + 2(\lambda^+\lambda^- e^{2\omega x^0})^2$$
$$\times \int [\mathrm{d}t]_{\underset{>}{4}} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}^{-4r^2} \bigg((1-4r^2)^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + (1-4r^2)^2 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \bigg) + \dots \tag{C5}$$

The computation at second order in T, for $r \leq 1/2$, gives the result

$$Z(r|x_0) = 2 - \frac{\Gamma(2-4r^2)}{\Gamma^2(1-2r^2)} \lambda^+\lambda^- e^{2\omega x_0} + \mathcal{O}((\lambda^+\lambda^-)^2), \tag{C6}$$

[29]To be precise, we have $P(\{1,2,3\dots 2n\}) = \{a_1, a_2, a_3 \dots a_{2n}\}$.

where we used the Dyson integral [36]:

$$\int_0^{2\pi}\prod_{i=1}^{n}\frac{\mathrm{d}t_i}{2\pi}\prod_{i<j}^{n}|e^{it_i}-e^{it_j}|^{2\alpha}=\frac{\Gamma(1+n\alpha)}{\Gamma^n(1+\alpha)}. \quad \text{(C7)}$$

We notice that the result (C6) is analytic in r for all values below the critical distance $r_c = 1/\sqrt{2}$. The reason for this property should now be familiar to the reader. For $|r| < 1/2$, the contact term vanishes and hence gives no contribution to Eq. (C6). The value $r = 1/2$ is particular. We see that the prefactor of the second-order integral in Eq. (C5) vanishes; at the same time, the contact term gives a finite contribution, ensuring the continuity of the result in Eq. (C6). For any $1/2 < |r| < r_c$, the second-order integral in Eq. (C6) is divergent. As we explained in Sec. III C, the divergence is canceled by the contribution from the contact term, which appears in the world-sheet action (15), where ε is chosen to be the same as the short-distance cutoff in Eq. (C5). The finite part that remains agrees precisely with Eq. (C6). Hence, the presence of the contact term gives (at least at this order) a continuous result all the way to the critical distance.

The $r = 1/2$ case

Finding the complete expression of the disk partition function at any $|r| < 1/\sqrt{2}$ seems to be out of reach, since integrals involve complicated highly coupled multidimensional integrals with path ordering.

For this reason, we want to compute the partition function in the special case where $\omega = 1/2$, which is tractable. We recall that the perturbative expansion of the world-sheet action is given by Eq. (6):

$$\begin{aligned} Z(r,\lambda^+,\lambda^-) &= \langle e^{-\delta S}\rangle \\ &= \langle e^{-(\lambda^+/2\pi)\int \mathrm{d}\hat{t}\boldsymbol{\Gamma}^+T^+(\hat{t})-(\lambda^-/2\pi)\int \mathrm{d}\hat{t}\boldsymbol{\Gamma}^-T^-(\hat{t})}\rangle \\ &= \sum_{n,p=0}^{\infty}(-1)^{n+p}\frac{(\lambda^+)^n}{n!}\frac{(\lambda^-)^p}{p!} \\ &\quad\times\sum_{\mathrm{perm}\pm}\int[\mathrm{d}\hat{t}]_{n+p}\langle\boldsymbol{\Gamma}^\pm(\hat{t}_1)\ldots\boldsymbol{\Gamma}^\pm(\hat{t}_{n+p})\rangle \\ &\quad\times\langle T^\pm(\hat{t}_1)\ldots T^\pm(\hat{t}_{n+p})\rangle, \end{aligned} \quad \text{(C8)}$$

with n and p of the same parity and $T^\pm = e^{(\pm i\tilde{\mathbb{X}}+\omega\mathbb{X}^0)/2}$.

Because of the Fermi multiplets correlators, the only nonvanishing terms are the ones which have as much + as −. The correlators of the Fermi multiplets are easy to compute using Wick theorem and the Green function (5). It leads to one product of supersymmetric sign functions $\prod\hat{\epsilon}(2i,2i+1)$, which decomposes into a sum of $2(n!)^2$ supersymmetric path orderings. One finds that these path orderings are all equivalent under permutations of T^+'s (T^-'s) with T^+'s (T^-'s) and permutations of integration variable. So one chooses one path ordering, symbolically $\hat{t}_1 > \hat{t}_2 > \ldots > \hat{t}_{2n}$ multiplied by a factor $2(n!)^2$. We should then compute

$$\begin{aligned} Z(r,\lambda^+,\lambda^-) &= 2\sum_{n=0}^{\infty}(\lambda^+\lambda^-)^n\int_{>}[\mathrm{d}t]_{2n}\langle T^+(\hat{t}_1)T^-(\hat{t}_2)\ldots T^-(\hat{t}_{2n})\rangle \\ &= 2\sum_{n=0}^{\infty}(\lambda^+\lambda^-)^n e^{inx}\int_{>}[\mathrm{d}\hat{t}]_{2n}\prod_{i<j}^{n}\hat{S}(2i,2j)\hat{S}(2i-1,2j-1) = 2\sum_{n=0}^{\infty}(\lambda^+\lambda^-)^n e^{inx}I_n, \end{aligned} \quad \text{(C9)}$$

with $\hat{S}(i,j) = |2\sin\frac{t_i-t_j}{2}| - \epsilon(i,j)\theta_i\theta_j$.

The computation of the integrals I_n is as follows, using the notation of Eq. (C4). We have first

$$I_1 = -\frac{1}{2\pi}\int[\mathrm{d}t]_1 = -\frac{1}{2\pi}. \quad \text{(C10)}$$

Then I_2, which is still easy, is

$$I_2=\frac{1}{(2\pi)^2}\int\frac{[\mathrm{d}t]_2}{2!}\left|\begin{matrix}1\\2\end{matrix}\right|\left|\begin{matrix}2\\1\end{matrix}\right|-\int\frac{[\mathrm{d}t]_4}{4!}=\frac{1}{(2\pi)^2}-\frac{1}{4!}, \gg \quad \text{(C11)}$$

and

$$\begin{aligned} I_3 &= \frac{1}{2\pi}\int\frac{[\mathrm{d}t]_5}{5!}C_1^5\left|\begin{matrix}1\\2345\end{matrix}\right| \\ &\quad-\frac{1}{(2\pi)^3}\int\frac{[\mathrm{d}t]_3}{3!}\left|\begin{matrix}1\\23\end{matrix}\right|\left|\begin{matrix}2\\13\end{matrix}\right|\left|\begin{matrix}3\\12\end{matrix}\right| \\ &= \frac{2^{12}}{4!(2\pi)^5}-\frac{1}{(2\pi)^3}=\frac{16}{3\pi^5}-\frac{1}{8\pi^3}. \end{aligned} \quad \text{(C12)}$$

I_4 is a bit more complicated to compute, but in terms of integrals, we find

$$\begin{aligned} I_4 &= \int_{>}[\mathrm{d}t]_8\left[\begin{matrix}13\\57\end{matrix}\right]\left[\begin{matrix}24\\68\end{matrix}\right]-\frac{1}{(2\pi)^2}\int\frac{[\mathrm{d}t]_6}{6!}C_2^6\left|\begin{matrix}1\\2\end{matrix}\right|\left|\begin{matrix}2\\1\end{matrix}\right|\left|\begin{matrix}12\\3456\end{matrix}\right|+\frac{1}{(2\pi)^4}\int\frac{[\mathrm{d}t]_4}{4!}\left|\begin{matrix}1\\234\end{matrix}\right|\left|\begin{matrix}2\\134\end{matrix}\right|\left|\begin{matrix}3\\124\end{matrix}\right|\left|\begin{matrix}4\\123\end{matrix}\right| \\ &= \frac{1}{1120}+\frac{143}{144\pi^6}-\frac{55}{192\pi^4}+\frac{13}{480\pi^2}-\left(-\frac{1001}{2592\pi^6}-\frac{175}{432\pi^4}-\frac{1}{240\pi^2}\right)+\frac{1}{16\pi^4} \\ &= \frac{1}{1120}+\frac{3575}{2592\pi^6}+\frac{205}{1728\pi^4}+\frac{1}{32\pi^2}+\frac{1}{16\pi^4}, \end{aligned} \quad \text{(C13)}$$

where we introduced the totally antisymmetric form

$$\begin{bmatrix} ab\ldots \\ cd\ldots \end{bmatrix} = \sum_P (-1)^P \begin{pmatrix} p(a)p(b)\ldots \\ p(c)p(d)\ldots \end{pmatrix} = \begin{pmatrix} ab\ldots \\ cd\ldots \end{pmatrix} - \begin{pmatrix} ac\ldots \\ bd\ldots \end{pmatrix} + \begin{pmatrix} ad\ldots \\ bc\ldots \end{pmatrix} + \ldots, \tag{C14}$$

with the partially antisymmetric form

$$\begin{pmatrix} abc\ldots \\ def\ldots \end{pmatrix} = \epsilon(a,b)\epsilon(a,c)\epsilon(b,c) \times \ldots \times \epsilon(d,e)\epsilon(d,f)\epsilon(e,f) \times \ldots \times \begin{vmatrix} abc\ldots \\ def\ldots \end{vmatrix}. \tag{C15}$$

The bigger n is, the more complicated the corresponding term in the partition function is. This is because more and more contribution of the contact term appears and the path ordering cannot be always removed. For the special value $r = 1/2$, the contact term has indeed a nonzero, but finite, contribution to the final result.

We end up with the following expansion. The terms coming from pure "noncontact" contributions are underlined:

$$\frac{Z(x)}{2} = 1 - \lambda^+\lambda^- \frac{e^{ix}}{2\pi} + \frac{(\lambda^+\lambda^-)^2}{4\pi^2} e^{2ix}\left(1 - \underline{\frac{\pi^2}{6}}\right) - \frac{(\lambda^+\lambda^-)^3}{8\pi^3} e^{3ix}\left(1 - \frac{128}{3\pi^2}\right) + \frac{(\lambda^+\lambda^-)^4}{16\pi^4} e^{4ix}\left(1 + \frac{175}{27} + \frac{1001}{162\pi^2} + \frac{\pi^2}{15} \underline{- \frac{55}{12} + \frac{143}{9\pi^2} + \frac{13\pi^2}{30} + \frac{\pi^4}{70}}\right)\ldots, \tag{C16}$$

where we recognize the trivial expansion

$$1 - \lambda^+\lambda^- \frac{e^{ix}}{2\pi} + (\lambda^+\lambda^-)^2 \frac{e^{2ix}}{4\pi^2} - (\lambda^+\lambda^-)^3 \frac{e^{3ix}}{8\pi^3} + \ldots = \frac{1}{1 + \frac{\lambda^+\lambda^-}{2\pi} e^{ix}}. \tag{C17}$$

In fact, this factorization is exact to all orders; by looking at the integrals I_n, one can see that the maximal contact term is always present and has a standard form, which we recognize as a Vandermonde determinant.

The remaining terms should come from a nontrivial function that multiplies (C17):

$$Z(x) = \frac{2}{1 + \frac{\lambda^+\lambda^-}{2\pi} e^{ix}}\left(1 - \left(\frac{\lambda^+\lambda^-}{2\pi}\right)^2 \frac{\pi^2}{6} e^{2ix} + \left(\frac{128}{3\pi^2} - \frac{\pi^2}{6}\right)\left(\frac{\lambda^+\lambda^-}{2\pi}\right)^3 e^{3ix} + \left(\frac{205}{108} + \frac{10487}{162\pi^2} + \frac{\pi^2}{2} + \frac{\pi^4}{70}\right)\left(\frac{\lambda^+\lambda^-}{2\pi}\right)^4 e^{4ix} + \ldots\right). \tag{C18}$$

This does not seem to come from the Taylor expansion of a simple expression.

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