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Instantons on special holonomy manifolds

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We consider cones over manifolds admitting real Killing spinors and instanton equations on connections on vector bundles over these manifolds. Such cones are manifolds with special (reduced) holonomy. We generalize the scalar ansatz for a connection proposed by Harland and Nölle [D. Harland and C. Nölle, J. High Energy Phys. 03 (2012) 082.] in such a way that instantons are parametrized by constrained matrix-valued functions. Our ansatz reduces instanton equations to matrix model equations which can be further reduced to Newtonian mechanics with particle trajectories obeying first-order gradient flow equations. Generalizations to Kähler-Einstein manifolds and resolved Calabi-Yau cones are briefly discussed. Our construction allows one to associate quiver gauge theories with special holonomy manifolds via dimensional reduction.

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I. INTRODUCTION

Instantons in four dimensions [1] are nonperturbative Bogomolny-Prasad-Sommerfeld (BPS) configurations solving first-order anti-self-duality equations for gauge fields which imply the full Yang-Mills equations. They are important objects in modern field theory [2,3]. Generalization of Yang-Mills instantons to higher dimensions, proposed in [4] and studied in [5–11] (for more literature see references therein), is important both in mathematics [9,10] and string theory [12,13]. Some of their solutions on spaces $\mathbb{R}^n$ with $n > 5$ were obtained in [5,14,15]. Constructions of solutions to the instanton equations on more general curved homogeneous manifolds as well as on cylinders and cones over them were considered in [16,17]. The construction on coset spaces, many of which admit Killing spinors [18], was generalized to cones over manifolds with real Killing spinors [19], not necessarily homogeneous (see also [20] about instantons on Calabi-Yau cones and their resolutions). All these solutions were lifted to solutions of heterotic supergravity equations supplemented by the Bianchi identity [19–23].

Riemannian manifolds (M, g_M) with real Killing spinors[1] often occur in string theory compactifications (see e.g. [19,21–23] and references therein). These manifolds were classified in [18]. Besides the round spheres they are

(i) nearly Kähler 6-manifolds M, SU(3) structure,
(ii) nearly parallel 7-manifolds M, G_2 structure,
(iii) Sasaki-Einstein $(2m+1)$-manifolds M, SU(m) structure, and
(iv) 3-Sasakian $(4m+3)$-manifolds M, Sp(m) structure.

All these manifolds have a connection with a nonvanishing torsion and admit a nonintegrable H structure mentioned above, i.e. a reduction of the structure group SO(n) of the tangent bundle TM to $H \subset$ SO(n). The above manifolds are equipped with canonical 3-form P and 4-form Q defined via the Killing spinors.

Recall that instanton equations on an $(n+1)$-dimensional Riemannian manifold X can be introduced as follows. Suppose there exists a 4-form Q on X. Then there exists an $(n-3)$-form $*Q$, where $*$ is the Hodge operator on X. Let $\mathcal{A}$ be a connection on a bundle over X with the curvature $\mathcal{F}$. Then the generalized anti-self-duality equation on the gauge field $\mathcal{F}$ is [9,10]

$$* \mathcal{F} + *Q \wedge \mathcal{F} = 0. \tag{1.1}$$

For $n+1 > 4$ these equations can be defined on manifolds X with *special holonomy*, i.e. such that the holonomy group G of the Levi-Civita connection on the tangent bundle TX is a subgroup in the group SO$(n+1)$. On such manifolds any solution of Eq. (1.1) satisfies the Yang-Mills equation. The instanton equation (1.1) is also well defined on manifolds X with nonintegrable G structures but then (1.1) implies the Yang-Mills equation with torsion. This torsion term vanishes on manifolds with real Killing spinors [19].

In this paper, we mostly consider $X = \mathcal{C}(M)$, where M is a manifold with real Killing spinors and $\mathcal{C}(M)$ is a cone over M with the metric

$$g_X = dr^2 + r^2 g_M = e^{2\tau}(d\tau^2 + g_M), \quad \text{for } r := e^\tau. \tag{1.2}$$

From (1.2) it follows that the cone $\mathcal{C}(M)$ is conformally equivalent to the cylinder

$$Z = \mathbb{R} \times M \tag{1.3}$$

with the metric

$$g_Z = d\tau^2 + g_M. \tag{1.4}$$

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[1] A Killing spinor on a Riemannian manifold N is a spinor field ϵ which satisfies $\nabla_L \epsilon = \mathrm{i}\lambda L \cdot \epsilon$ for all tangent vectors L, where ∇ is the spinor covariant derivative, $\cdot$ is Clifford multiplication, and λ is a constant. If $\lambda = 0$ then the spinor is called parallel, and N is a manifold with special (reduced) holonomy.

Furthermore, one can show [19] that Eq. (1.1) on the cone $X = \mathcal{C}(M)$ is related to the instanton equation on the cylinder $Z = \mathbb{R} \times M$ as follows:

$$*_X\mathcal{F} + *_X Q_X \wedge \mathcal{F} = e^{(n-3)\tau}(*_Z\mathcal{F} + *_Z Q_Z \wedge \mathcal{F}) = 0, \tag{1.5}$$

where $n + 1 = \dim\mathcal{C}(M) = \dim Z$. In other words, Eq. (1.1) on $\mathcal{C}(M)$ is equivalent to the equation on $\mathbb{R} \times M$ after rescaling (1.2) of the metric. That is why in the following we will consider the instanton equation

$$* \mathcal{F} + *Q_Z \wedge \mathcal{F} = 0 \tag{1.6}$$

on the cylinder Z over M. Here we omit the index Z in the star operator. Note that components of $\mathcal{F}$ on the cone can be obtained from those on the cylinder simply via rescaling (1.2).

In this paper, we generalize the results [19] of Harland and Nölle on investigating instantons on cones over manifolds with Killing spinors. First, in Sec. II, we collect various facts concerning nearly Kähler, nearly parallel G_2, Sasaki-Einstein and 3-Sasakian manifolds M mainly following the description in [19]. We describe metrics on M, canonical connections and various q-forms ($q = 1, 2, \ldots$) as well as their extension to the cylinder $Z = \mathbb{R} \times M$. Then, in Sec. III, we introduce an ansatz for a gauge potential $\mathcal{A}$ which reduces the instanton equation (1.6) on $\mathbb{R} \times M$ to a matrix equation on $\mathbb{R}$. Resolution of natural algebraic constraints on the matrices yields further reduction to a set of first-order equations on functions depending on $\tau \in \mathbb{R}$. These equations are gradient flow equations describing BPS-type trajectories in Newtonian mechanics of particles moving in $\mathbb{R}^N$, where N is the number of functions parametrizing matrices in the ansatz for a gauge potential $\mathcal{A}$. Solutions to these equations give instanton solutions of the Yang-Mills equations on $\mathbb{R} \times M$ and their extension to the cone $\mathcal{C}(M)$. Finally, in Sec. IV, we discuss some generalizations of our construction allowing one to associate quiver gauge theories with such special holonomy manifolds as Kähler-Einstein manifolds and resolved Calabi-Yau cones.

II. MANIFOLDS WITH KILLING SPINORS

A. Nearly Kähler 6-manifolds

Consider the cylinder (1.3) with the metric (1.4), where M is a nearly Kähler 6-manifold. It is defined as a manifold with a 2-form ω and a 3-form P such that

$$d\omega = 3 *_M P, \quad \text{and} \quad dP = 2\omega \wedge \omega =: 4Q. \tag{2.1}$$

For a local orthonormal coframe $\{e^a\}$ on M one can choose

$$\omega = e^{12} + e^{34} + e^{56}, \quad \text{and} \quad P = e^{135} + e^{164} - e^{236} - e^{245}, \tag{2.2}$$

where $a = 1, \ldots, 6$, $e^{a_1 \cdots a_l} := e^1 \wedge \cdots \wedge e^l$, and get

$$*_M P = e^{145} + e^{235} + e^{136} - e^{246}, \quad Q = e^{1234} + e^{1256} + e^{3456}. \tag{2.3}$$

Here $*_M$ denotes the Hodge operator on M. On Z one can introduce the 4-form

$$Q_Z = d\tau \wedge P + Q, \tag{2.4}$$

which is used in the instanton equation (1.6).

The canonical connection $\tilde{\Gamma}$ on M, which is a metric-compatible connection with totally antisymmetric (intrinsic) torsion, has a SU(3) structure group. This connection has components

$$\tilde{\Gamma}^{\,c}_{ab} = \Gamma^c_{ab} + \tfrac{1}{2}P_{cab}, \tag{2.5}$$

where Γ^c_{ab} are components of the Levi-Civita connection and

$$P_{abc} = T^a_{bc} \tag{2.6}$$

are components of the torsion $T^a = \frac{1}{2}T^a_{bc}e^b \wedge e^c$ defined from the Cartan structure equations

$$de^a + \tilde{\Gamma}^a_b \wedge e^b = T^a \tag{2.7}$$

for basis 1-forms e^a.

Note that the structure group of M is SU(3) (or its subgroup) and P induces a G_2 structure on Z since $SU(3) \subset G_2$. Recall that $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$, $\dim\mathfrak{m} = 6$, and one can define the generators of G_2 as 7×7 matrices from $\mathfrak{so}(7)$ with the commutation relations

$$[I_i, I_j] = f^k_{ij}I_k, \qquad [I_i, I_a] = f^b_{ia}I_b, \quad \text{and} \quad [I_a, I_b] = f^i_{ab}I_i + f^c_{ab}I_c, \tag{2.8}$$

where $I_i \in \mathfrak{su}(3)$, $I_a \in \mathfrak{m}$, and f's are structure constants of $\mathfrak{g}_2$. One can choose [19]

$$I^b_{ia} = f^b_{ia}, \qquad I^0_{ia} = -I^a_{i0} = 0, \qquad I^c_{ab} = \tfrac{1}{2}f^c_{ab}, \qquad I^b_{a0} = -I^0_{ab} = \delta^b_a, \tag{2.9}$$

and obtain

$$P_{abc} = -f^c_{ab}. \tag{2.10}$$

Introducing $\mu = (0, a)$, we can denote matrices in (2.9) as $I^\mu_{i\nu}$ and $I^\mu_{a\nu}$. The extension of the canonical connection $\tilde{\Gamma}$ to Z has the same components (2.5) and can be written as

$$\tilde{\Gamma} = \tilde{\Gamma}^i I_i \tag{2.11}$$

with I_i given in (2.9).

B. Nearly parallel G_2-manifolds

Let us consider the cylinder (1.3) over a nearly parallel G_2-manifold M. It is defined as a manifold with a 3-form P (a G_2 structure) preserved by the group $G_2 \subset SO(7)$ such that

$$dP = \gamma *_M P \tag{2.12}$$

for some constant $\gamma \in \mathbb{R}$. For a local orthonormal coframe e^a, $a = 1, \ldots, 7$, on M one can choose

$$P = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} \tag{2.13}$$

and therefore

$$\begin{aligned} *_M P =: Q \\ = e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}. \end{aligned} \tag{2.14}$$

It is easy to see that for the choice (2.13) one obtains $dP = 4Q$, i.e. $\gamma = 4$. The 4-form Q_Z on Z can be chosen similar to (2.4) as

$$Q_Z = d\tau \wedge P + Q. \tag{2.15}$$

This form defines a Spin(7) structure on Z.

One can define generators of the group Spin(7) via the structure constants f_{ij}^k, f_{ia}^b, and f_{ab}^c of the group Spin(7) by using the decomposition $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$, as 8×8 matrices $I_i = (I_{i\nu}^{\mu}) \in \mathfrak{g}_2$ and $I_a = (I_{a\nu}^{\mu}) \in \mathfrak{m}$, $\dim \mathfrak{m} = 7$, $\mu = (0, a)$. The generators I_i, I_a have the same form as in (2.9) but with structure constants f's of Spin(7).

The canonical connection $\tilde{\Gamma}$ on M is not changed after its extension to Z and has the components

$$\tilde{\Gamma} = \tilde{\Gamma}^i I_i \Rightarrow \tilde{\Gamma}_a^i I_{ib}^c = \tilde{\Gamma}_{ab}^c = \Gamma_{ab}^c + \tfrac{1}{3} P_{abc} \tag{2.16}$$

with torsion components

$$T_{bc}^a = \tfrac{2}{3} P_{abc}. \tag{2.17}$$

C. Sasaki-Einstein manifolds

Consider now the cylinder (1.3) with the metric (1.4), where M is a Sasaki-Einstein manifold. It is a $(2m+1)$-dimensional manifold such that the cone $\mathcal{C}(M)$ with the metric (1.2) is a Calabi-Yau $(m+1)$-fold [24]. Such manifolds M have the structure group $SU(m) \subset SO(2m+1)$ and the holonomy group of $\mathcal{C}(M)$ is $SU(m+1)$. Sasaki-Einstein manifolds are endowed with 1-, 2-, 3-, and 4-forms η, ω, P, and Q, which can be defined in an orthonormal basis e^1, e^a, $a = 2, \ldots, 2m+1$, as

$$\begin{aligned} &\eta = e^1, \qquad \omega = e^{23} + e^{45} + \cdots + e^{2m2m+1}, \\ &P = \eta \wedge \omega, \quad \text{and} \quad Q = \tfrac{1}{2}\omega \wedge \omega. \end{aligned} \tag{2.18}$$

One can check that $\eta \lrcorner \omega = 0$ and

$$\begin{aligned} &d\eta = 2\omega, \qquad d *_M \omega = 2m *_M \eta, \\ &dP = 4Q, \quad \text{and} \quad d *_M Q = (2m-2) *_M P. \end{aligned} \tag{2.19}$$

The metric on Z has the form (1.4) with

$$g_M = (e^1)^2 + \exp(2h)\delta_{ab} e^a e^b. \tag{2.20}$$

Note that for the value of h such that

$$\exp(2h) = \frac{2m}{m+1}, \tag{2.21}$$

the torsion of the canonical connection on M (and on Z) becomes antisymmetric [19], but we keep the one-parameter family (2.20) of the Sasakian metric including the case $h = 0$ when the metric is Einstein. Components of the canonical connection $\tilde{\Gamma}$ are

$$\tilde{\Gamma}_{\mu a}^{\ b} = \Gamma_{\mu a}^b + \frac{1}{m} P_{\mu ab}, \qquad -\tilde{\Gamma}_{\mu a}^1 = \tilde{\Gamma}_{\mu 1}^a = \Gamma_{\mu 1}^a + P_{\mu 1 a}, \tag{2.22}$$

where $\mu = (1, a)$ and the torsion of $\tilde{\Gamma}$ is

$$T^1 = P_{1\mu\nu} e^\mu \wedge e^\nu, \quad \text{and} \quad T^a = \frac{m+1}{2m} P_{a\mu\nu} e^\mu \wedge e^\nu. \tag{2.23}$$

As 4-form Q_Z on Z one can take [19]

$$Q_Z = \exp(2h) d\tau \wedge P + \exp(4h) Q, \tag{2.24}$$

where P and Q are given in (2.18).

Let $\hat{\mu} = (0, \mu) = (0, 1, a)$. Then one can define generators of the group $\mathfrak{su}(m+1) = \mathfrak{su}(m) \oplus \mathfrak{m}$ as $(2m+2) \times (2m+2)$ antisymmetric matrices $I_i = (I_{i\hat{\nu}}^{\hat{\mu}}) \in \mathfrak{su}(m)$ and $I_\mu = (I_{\mu\hat{\nu}}^{\hat{\mu}}) \in \mathfrak{m}$ such that nonvanishing components are

$$\begin{aligned} &I_{ia}^b = f_{ia}^b, \qquad I_{1a}^b = -\frac{1}{m} P_{1ab} = (m+1) f_{1a}^b, \\ &-I_{ab}^0 = I_{a0}^b = \delta_a^b, \qquad I_{ab}^1 = -I_{a1}^b = -P_{1ab} = \frac{1}{2} f_{ab}^1, \end{aligned} \tag{2.25}$$

where f_{ia}^b, f_{ab}^1, and f_{1a}^b are parts of the structure constants of $\mathfrak{su}(m+1)$. In terms of these matrices the canonical connection $\tilde{\Gamma}$ on M pulled back to Z can be written as

$$\tilde{\Gamma} = \tilde{\Gamma}^i I_i = e^\mu \tilde{\Gamma}_\mu^i I_i. \tag{2.26}$$

D. 3-Sasakian manifolds

Let us now consider the cylinder (1.3) over a 3-Sasakian manifold M. It is defined as a $(4m+3)$-dimensional manifold such that the cone $\mathcal{C}(M)$ over it is a hyper-Kähler $(4m+4)$-manifold [24], i.e. the holonomy group of $\mathcal{C}(M)$ is $Sp(m+1)$. The structure group of M is $Sp(m)$ and any 3-Sasakian manifold can be endowed with three 1-forms η^α, three 2-forms ω^α, a 3-form P, and a 4-form Q, $\alpha = 1, 2$, and 3 [24]. In a local orthonormal coframe e^α, e^a, $a = 4, \ldots, 4m+3$, these forms can be written as

$$\begin{aligned} &\eta^1 = e^1, \quad \omega^1 = e^{45} + e^{67} + \cdots + e^{4m4m+1} + e^{4m+24m+3}, \\ &\eta^2 = e^2, \quad \omega^2 = e^{46} - e^{57} + \cdots + e^{4m4m+2} - e^{4m+14m+3}, \\ &\eta^3 = e^3, \quad \omega^3 = e^{47} + e^{56} + \cdots + e^{4m4m+3} + e^{4m+14m+2}, \\ &P = \tfrac{1}{3}\eta^{123} + \tfrac{1}{3}\eta^\alpha \wedge \omega^\alpha, \quad \text{and} \quad Q = \tfrac{1}{6}\omega^\alpha \wedge \omega^\alpha. \end{aligned} \tag{2.27}$$

The forms η^α and ω^α satisfy the equations

$$d\eta^\alpha = \varepsilon_{\alpha\beta\gamma}\eta^\beta \wedge \eta^\gamma + 2\omega^\alpha, \quad d\omega^\alpha = 2\varepsilon_{\alpha\beta\gamma}\eta^\beta \wedge \omega^\gamma. \tag{2.28}$$

We introduce indices $\mu = (\alpha, a)$ and $\hat{\mu} = (0, \mu) = (0, \alpha, a)$. Using the splitting

$$\mathfrak{sp}(m+1) = \mathfrak{sp}(m) \oplus \mathfrak{m}, \qquad \dim\mathfrak{m} = 4m+3, \tag{2.29}$$

one can introduce generators $I_i = (I^{\hat{\mu}}_{i\hat{\nu}}) \in \mathfrak{sp}(m)$ and $I_a = (I^{\hat{\mu}}_{a\hat{\nu}}) \in \mathfrak{m}$ of the group Sp$(m+1)$ as matrices from $\mathfrak{so}(4m+4)$. One can take them so that nonvanishing components are [19]

$$I^b_{ia} = f^b_{ia}, \quad I^\gamma_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma} = -3P_{\alpha\beta\gamma} = \tfrac{1}{2}f^\gamma_{\alpha\beta}, \quad I^\beta_{\alpha 0} = \delta^\beta_\alpha, \quad I^\alpha_{ab} = -\omega^\beta_{ab} = -3P_{\alpha ab} = \tfrac{1}{2}f^\alpha_{ab}, \quad I^b_{a0} = \delta^b_a, \tag{2.30}$$

where f's are the structure constants of the group Sp$(m+1)$.

Note that the metric on $Z = \mathbb{R} \times M$ has the form (1.4) with a one-parameter family

$$g_M = \delta_{\alpha\beta}e^\alpha e^\beta + \exp(2h)\delta_{ab}e^a e^b \tag{2.31}$$

of metrics on M. The 4-form Q_Z can be chosen as

$$Q_Z = \tfrac{1}{6}(\exp(4h)\omega^\alpha \wedge \omega^\alpha + \exp(2h)\varepsilon_{\alpha\beta\gamma}\omega^\alpha \wedge \eta^\beta \wedge \eta^\gamma + 2\exp(2h)d\tau \wedge \eta^\alpha \wedge \omega^\alpha + 6 d\tau \wedge \eta^{123}). \tag{2.32}$$

In terms of the matrices (2.30) the canonical connection $\tilde{\Gamma}$ on the cylinder Z over a 3-Sasakian manifold M can be written as

$$\tilde{\Gamma} = \tilde{\Gamma}^i I_i = e^\mu \tilde{\Gamma}^i_\mu I_i. \tag{2.33}$$

It is related with the Levi-Civita connection Γ by formulas

$$-\tilde{\Gamma}^\nu_{\mu\alpha} = \tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + 3P_{\alpha\mu\nu}, \qquad \tilde{\Gamma}^b_{\mu a} = \Gamma^b_{\mu a}, \tag{2.34}$$

and has the torsion

$$T^\alpha = 3P_{\alpha\mu\nu}e^{\mu\nu}, \quad \text{and} \quad T^a = \tfrac{3}{2}P_{a\mu\nu}e^{\mu\nu}, \tag{2.35}$$

which is antisymmetric for the choice $\exp(2h) = 2$ in the metric.

III. INSTANTONS IN HIGHER DIMENSIONS

A. Reduction to matrix equations

Recall that for all cases of manifolds M considered in Sec. II the instanton equation on the cone $\mathcal{C}(M)$ is equivalent to the equation

$$*\mathcal{F} + *Q_Z \wedge \mathcal{F} = 0 \tag{3.1}$$

on the cylinder $Z = \mathbb{R} \times M$ with the metric $g_Z = d\tau^2 + g_M$. The explicit form of the 4-form Q_Z on Z was written down for all cases in Sec. II. Let us denote by G the holonomy group[2] of the Levi-Civita connection on $\mathcal{C}(M)$ and by H the structure group[3] of the canonical connection $\tilde{\Gamma}$ on M (and also on Z). For the Lie algebras $\mathfrak{g} = \mathrm{Lie}G$ and $\mathfrak{h} = \mathrm{Lie}H$ we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{3.2}$$

where $\mathfrak{m}$ is an orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Let $e^0 = d\tau$ and e^μ be an orthonormal basis of T^*Z. Then e^μ form a basis of $T^*M \subset T^*Z$ and their linear span can be identified with the vector space $\mathfrak{m}$.

Consider the generators

$$[\hat{I}_i, \hat{I}_j] = f^k_{ij}\hat{I}_k, \qquad [\hat{I}_i, \hat{I}_\mu] = f^\nu_{i\mu}\hat{I}_\nu, \quad \text{and} \quad [\hat{I}_\mu, \hat{I}_\nu] = f^i_{\mu\nu}\hat{I}_i + f^\sigma_{\mu\nu}\hat{I}_\sigma \tag{3.3}$$

acting on the space V of an irreducible representation of G. These generators satisfy the same commutation relations as the generators I_i, I_a. In Sec. II we wrote down the realization of these generators via the embedding $\mathfrak{g} \subset \mathfrak{so}(n+1)$ with $n + 1 = \dim Z$ as acting (via infinitesimal rotations) on tangent spaces of Z. For this special representation we omit hats and note that $\tilde{\Gamma} = \tilde{\Gamma}^i I_i$ is the canonical connection whose curvature

$$\tilde{R} = d\tilde{\Gamma} + \tilde{\Gamma} \wedge \tilde{\Gamma} = (d\tilde{\Gamma}^i + \tfrac{1}{2}f^i_{jk}\tilde{\Gamma}^j \wedge \tilde{\Gamma}^k)I_i \tag{3.4}$$

satisfies the instanton equation (3.1) (see [19]).

Note that instead of tangent bundle TZ one can consider an arbitrary vector bundle $\mathcal{V} \to Z$ with the structure group G such that fibers are representations V (real, complex, or quaternionic) of the group G. For simplicity, we consider irreducible representations V of the group G. The bundle $\mathcal{V} \to Z$ is associated with the principal bundle $P(Z, G)$. Since H is a closed subgroup of G, it also acts on fibers of $\mathcal{V}$, but in general after restriction to $H \subset G$ the representation V decomposes into a sum of irreducible representations V_{q_r} of H such that $V = \oplus_r V_{q_r}$. We denoted the generators of the group G in the representation V as $\hat{I}_i$, $\hat{I}_\mu$, where $\hat{I}_i \in \mathfrak{h}$ and $\hat{I}_\mu \in \mathfrak{m}$ for the splitting (3.2). Consider a connection

$$\hat{\Gamma} := \tilde{\Gamma}^i \hat{I}_i \tag{3.5}$$

on the bundle $\mathcal{V}$. In general, it is a reducible connection. Here $\tilde{\Gamma}^i$ are components of the canonical connection on the tangent bundle TZ. It is obvious that the curvature $\hat{R} = d\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma}$ of $\hat{\Gamma}$ also satisfies the instanton equation (3.1).

Let us consider matrix-valued functions $X_\mu(\tau) \in \mathrm{End}(V)$ and introduce a connection

[2]This holonomy group G is the group G_2, Spin(7), SU$(m+1)$, and Sp$(m+1)$ for cones over nearly Kähler, nearly parallel G_2, Sasaki-Einstein, and 3-Sasakian manifolds M, respectively.

[3]This structure group is the group SU(3), G_2, SU(m), and Sp(m) for nearly Kähler, nearly parallel G_2, Sasaki-Einstein, and 3-Sasakian manifolds, respectively.

$$\mathcal{A} := \hat{\Gamma} + X_\mu e^\mu \tag{3.6}$$

on the vector bundle $\mathcal{V} \to Z$. Note that for matrices X_μ depending on all coordinates of Z, (3.6) is a general form of a connection on the bundle $\mathcal{V} \to Z$. For X_μ depending only on τ, the instanton equation (3.1) will be reduced to ordinary differential equations on matrices X_μ.

Recall that

$$de^\mu = -\tilde{\Gamma}^\mu_\nu \wedge e^\nu + T^\mu = -\tilde{\Gamma}^i \wedge e^\nu f^\mu_{i\nu} + \tfrac{1}{2} T^\mu_{\sigma\nu} e^\sigma \wedge e^\nu, \tag{3.7}$$

where $f^\mu_{i\nu}$ are structure constants from (3.3). From (3.6) and (3.7) it follows that

$$\begin{aligned}\mathcal{F} &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= d\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma} + \tfrac{1}{2}([X_\mu, X_\nu] + T^\sigma_{\mu\nu} X_\sigma) e^\mu \wedge e^\nu \\ &\quad + \dot{X}_\nu e^0 \wedge e^\nu + \tilde{\Gamma}^i \wedge e^\mu ([\hat{I}_i, X_\mu] - f^\nu_{i\mu} X_\nu),\end{aligned} \tag{3.8}$$

where $\dot{X}_\nu = dX_\nu/d\tau$. Note that $\hat{R} = d\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma}$ satisfies Eq. (3.1) and $\mathcal{F}$ solves the instanton equation (3.1) on Z if the following matrix equations are satisfied:

$$[\hat{I}_i, X_\mu] = f^\nu_{i\mu} X_\nu, \tag{3.9}$$

$$[X_\mu, X_\nu] + T^\sigma_{\mu\nu} X_\sigma = N^\sigma_{\mu\nu} \dot{X}_\sigma + f^i_{\mu\nu} N_i(\tau). \tag{3.10}$$

Here $N^\sigma_{\mu\nu}$ are some constants which we shall specify below for each case, N_i are some $v(\mathfrak{h})$-valued functions defined by Eq. (3.10) after resolving the algebraic constraint equation (3.9) and substituting their solutions X_μ into (3.10). Here v: $\mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$. For X_μ satisfying (3.9) and (3.10), we have

$$\mathcal{F} = d\hat{\Gamma} + \hat{\Gamma} \wedge \hat{\Gamma} + \tfrac{1}{2} N_i f^i_{\mu\nu} e^{\mu\nu} + \dot{X}_\sigma (e^{0\sigma} + \tfrac{1}{2} N^\sigma_{\mu\nu} e^{\mu\nu}), \tag{3.11}$$

where the term with $f^i_{\mu\nu}$ also satisfies (3.1) due to properties of $f^i_{\mu\nu}$ and the last term with $\dot{X}_\sigma$ satisfies (3.1) for choices of $N^\sigma_{\mu\nu}$ specified below for each considered case. Note that the constraint equation (3.9) for some examples of groups G and H was discussed and resolved, e.g., in [25,26] in the context of the equivariant dimensional reductions on coset spaces G/H. For special cases of the ansatz (3.6) instanton solutions were obtained, e.g., in [16,17,19]. A peculiar property of such τ-dependent solutions is that they can be lifted to gauge 5-brane solutions of heterotic supergravity equations as was shown, e.g., in [19,20,23].

B. Reduction for nearly Kähler and nearly parallel G_2 manifolds

Consider a manifold M which is nearly Kähler ($\dim M = n = 6$) or nearly parallel G_2 ($\dim M = n = 7$). For both cases $\mu = a = 1, \ldots, n$ with $n = 6$ or $n = 7$. Note that the 2-forms

$$e^{0a} - \frac{1}{2\rho} P_{abc} e^{bc} \tag{3.12}$$

solve the instanton equation (3.1) on $Z = \mathbb{R} \times M$ for Q_Z and P given in Sec. II. Here $\rho = 2$ for $n = 6$ and $\rho = 3$ for $n = 7$. The generators $\hat{I}_a$ introduced in (3.3) are images of the 2-forms (3.12) under the metric-induced isomorphism $\Lambda^2 Z \cong \mathfrak{so}(7) \supset \mathfrak{g}_2 \supset \mathfrak{m}$ for $n = 6$ and $\Lambda^2 Z \cong \mathfrak{so}(8) \supset \mathfrak{spin}(7) \supset \mathfrak{m}$ for $n = 7$.

For both the nearly Kähler and nearly parallel G_2 cases we have

$$T^a_{bc} = -f^a_{bc}, \quad \text{and} \quad N^a_{bc} = \tfrac{1}{2} f^a_{bc}, \tag{3.13}$$

where N^a_{bc} are defined by comparing the components of $\mathcal{F}$ in (3.11) and the explicit form (3.12) of (parts of) anti-self-dual forms on Z. Thus, we obtain the following matrix equations:

$$[\hat{I}_i, X_a] = f^b_{ia} X_b, \tag{3.14}$$

$$[X_a, X_b] = f^c_{ab}(X_c + \tfrac{1}{2}\dot{X}_c) + f^i_{ab} N_i(\tau). \tag{3.15}$$

Substituting the ansatz $X_a = \phi \hat{I}_a$ with a real function $\phi(\tau)$, we see that (3.14) are satisfied and (3.15) are reduced to the equation

$$\dot{\phi} = 2\phi(\phi - 1) \tag{3.16}$$

obtained in [19] and for N_i we obtain $N_i = \phi^2 \hat{I}_i$. More general equations can be obtained by choosing a more general solution of the constraint equations (3.14). Such solutions for different choices of groups G and H were discussed, e.g., in [25,26]. Constructing solutions to Eqs. (3.9) and (3.10) goes beyond the scope of this short article. This task will be considered elsewhere.

C. Reduction for Sasaki-Einstein manifolds

Consider $Z = \mathbb{R} \times M$ with a Sasaki-Einstein manifold M. In this case $\mu = (1, a)$ with $a = 2, \ldots, 2m+1$ and the 2-forms

$$e^{01} - \frac{1}{m+1} \omega_{ab} e^{ab}, \quad \text{and} \quad \exp(h)(e^{0a} + \omega_{ab} e^{1b}) \tag{3.17}$$

solve the instanton equation (3.1) with Q_Z given in (2.24). From (2.23), (3.8), and (3.17) we obtain

$$\begin{aligned} T^1_{ab} &= -f^1_{ab}, \qquad T^a_{1b} = -f^a_{1b}, \\ N^1_{ab} &= \frac{1}{m+1} f^1_{ab}, \quad \text{and} \quad N^a_{1b} = \frac{m}{m+1} f^a_{1b}. \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.9) and (3.10), we obtain

$$[\hat{I}_i, X_1] = 0, \qquad [\hat{I}_i, X_a] = f^b_{ia} X_b, \tag{3.19}$$

$$[X_1, X_a] = f^b_{1a}\Big(X_b + \frac{m}{m+1}\dot{X}_b\Big), \tag{3.20}$$

$$[X_a, X_b] = f^1_{ab}\left(X_1 + \frac{1}{m+1}\dot{X}_1\right) + f^i_{ab}N_i(\tau). \quad (3.21)$$

If we choose $X_1 = \chi \hat{I}_1$ and $X_a = \psi \hat{I}_a$ then (3.19) is satisfied, for N_i we obtain $N_i = \psi^2 \hat{I}_i$ and (3.20) and (3.21) are reduced to the equations

$$\dot{\chi} = (m+1)(\psi^2 - \chi), \quad \text{and} \quad \dot{\psi} = \frac{m+1}{m}\psi(\chi - 1), \quad (3.22)$$

coinciding with those obtained in [19]. Note that (3.22) is a gradient flow equation

$$\dot{x}^i = g^{ij}\frac{\partial W}{\partial x^j}, \quad \text{for}$$
$$W = (x^1)^2 - 2x^1(x^2)^2 + 2(x^2)^2 - 1, \quad i, j = 1, 2,$$

for the metric on $\mathbb{R}^2$ of the form

$$ds^2 = g_{ij}dx^i dx^j = \frac{2}{m+1}(dx^1)^2 + \frac{4m}{m+1}(dx^2)^2,$$

with $x^1 := \chi$ and $x^2 := \psi$ [19].

D. Reduction for 3-Sasakian manifolds

For $Z = \mathbb{R} \times M$ with a 3-Sasakian manifold M we have $\mu = (\alpha, a)$, $\alpha = 1, 2,$ and 3 and $a = 4, \ldots, 4m+3$. One can check that the image of the generators $\hat{I}_a$ from (3.3) under the map into $\Lambda^2 Z \cong \mathfrak{so}(4m+4) \supset \mathfrak{sp}(m+1) \supset \mathfrak{m}$ is given by the 2-forms

$$e^{0\alpha} - \tfrac{1}{3}\varepsilon_{\alpha\beta\gamma}e^{\beta\gamma}, \quad \text{and} \quad \exp(h)(e^{0a} + \omega^\alpha_{ab}e^{\alpha b}), \quad (3.23)$$

which satisfy Eq. (3.1) with Q_Z given in (2.32). From (2.30), (2.35), (3.8), and (3.23) it follows that

$$T^\alpha_{ab} = -\tfrac{3}{2}f^\alpha_{ab}, \qquad T^b_{a\beta} = f^b_{a\beta}, \qquad T^\alpha_{\beta\gamma} = -f^\alpha_{\beta\gamma}, \quad (3.24)$$

$$N^b_{a\beta} = f^b_{a\beta}, \quad \text{and} \quad N^\alpha_{\beta\gamma} = \tfrac{1}{2}f^\alpha_{\beta\gamma}, \quad (3.25)$$

with other components vanishing. Substituting (3.24) and (3.25) into (3.9) and (3.10), we obtain

$$[\hat{I}_i, X_\alpha] = 0, \qquad [\hat{I}_i, X_a] = f^b_{ia}X_b, \quad (3.26)$$

$$[X_\alpha, X_\beta] = f^\gamma_{\alpha\beta}(X_\gamma + \tfrac{1}{2}\dot{X}_\gamma),$$
$$[X_a, X_\beta] = f^b_{a\beta}(X_b + \dot{X}_b), \quad (3.27)$$

$$[X_a, X_b] = f^\alpha_{ab}X_\alpha + f^i_{ab}N_i(\tau). \quad (3.28)$$

If we choose the ansatz $X_\alpha = \chi \hat{I}_\alpha$ and $X_a = \psi \hat{I}_a$ then (3.26) will be satisfied identically, from (3.28) we obtain $N_i = \psi^2 \hat{I}_i$ and (3.27) and (3.28) reduce to the equations

$$\dot{\chi} = 2\chi(\chi - 1), \qquad \dot{\psi} = \psi(\chi - 1), \quad \text{and} \quad \chi = \psi^2, \quad (3.29)$$

where the last algebraic equation follows from (3.28). These equations coincide with those obtained in [19]. Thus, our ansatz (3.6) which leads to matrix equations (3.9) and (3.10) generalizes the "scalar" ansatz of the paper [19] and allows one to obtain more general instanton solutions. However, obtaining explicit instanton solutions lies beyond the scope of our paper.

IV. GENERALIZATIONS: QUIVER BUNDLES

A. Smaller groups *H*

Recall that we considered nearly Kähler, nearly parallel G_2, Sasaki-Einstein, and 3-Sasakian manifolds M with the structure groups SU(3), G_2, SU(m), and Sp(m), respectively, following Harland and Nölle who considered in their ansatz [19] exactly the above groups with generators in the defining vector representation of the group $SO(n+1) \supset H$, i.e. $V = \mathbb{R}^{n+1}$ with $n = 6, 7, 2m+1,$ and $4m+3$, $m = 1, 2, \ldots$. However, the group H can be smaller than the above-mentioned Lie groups, i.e. often H lies *inside* the group SU(3), G_2, SU(m), and Sp(m), respectively. In this case, the constraint equations (3.9) become weaker and allow more degrees of freedom in matrices X_a. For instance, for the nearly Kähler coset space

$$M = SU(3)/U(1) \times U(1), \quad (4.1)$$

the structure group is $H = U(1) \times U(1)$ that increase the number of functions parametrizing the ansatz (3.6) even for vector representation $V = \mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ with $\mathfrak{g}_2 \supset \mathfrak{su}(3) = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}$. Writing the ansatz (3.6) in terms of $\mathfrak{su}(3)$-valued matrices X_a, one can resolve (3.9) as

$$X_1 = \begin{pmatrix} 0 & 0 & -\phi_1 \\ 0 & 0 & 0 \\ \bar{\phi}_1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & -\bar{\phi}_2 & 0 \\ \phi_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$X_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{\phi}_3 \\ 0 & \phi_3 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & i\phi_1 \\ 0 & 0 & 0 \\ i\bar{\phi}_1 & 0 & 0 \end{pmatrix},$$
$$X_4 = -\begin{pmatrix} 0 & i\bar{\phi}_2 & 0 \\ i\phi_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_6 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\bar{\phi}_3 \\ 0 & i\phi_3 & 0 \end{pmatrix}, \quad (4.2)$$

where ϕ_1, ϕ_2, and ϕ_3 are complex-valued functions of τ and the generators $I_{7,8}$ of the subgroup U(1) × U(1) of SU(3) are chosen in the form

$$I_7 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{and}$$
$$I_8 = \frac{i}{\sqrt{3}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.3)$$

Substituting (4.2) into (3.15), we obtain equations

$$\dot{\phi}_1 = -2\phi_1 + 2\bar{\phi}_2\bar{\phi}_3,\quad \dot{\phi}_2 = -2\phi_2 + 2\bar{\phi}_1\bar{\phi}_3,\quad \dot{\phi}_3 = -2\phi_3 + 2\bar{\phi}_1\bar{\phi}_2, \tag{4.4}$$

and constraints

$$N_7 = \Phi I_7, \qquad N_8 = -\sqrt{3}\Phi I_8, \quad \text{with} \quad \Phi = \phi_1\bar{\phi}_1 = \phi_2\bar{\phi}_2 = \phi_3\bar{\phi}_3, \tag{4.5}$$

for a proper normalization of the structure constants. From (4.5) we see that complex-valued functions ϕ_1, ϕ_2, and ϕ_3 can differ only in their phase parts. For $\phi_1 = \phi_2 = \phi_3 =: \phi$ Eqs. (4.4) reduce to Eq. (3.16) on a real-valued function ϕ.

B. Reducible representations of H and quiver bundles

A similar situation takes place for nearly parallel G_2-manifolds, where as an example one can consider the Aloff-Wallach space SU(3)/U(1) with the structure group $H = \mathrm{U}(1)$ [see the second paper in [17] for discussion of solving Eqs. (3.14)], and also for Sasaki-Einstein and 3-Sasakian manifolds the structure group H can be a closed subgroup of SU(m) and Sp(m), respectively. Even more freedom appears if one considers an irreducible representation V of the holonomy group G of the cone $\mathcal{C}(M)$ which decomposes into a sum of irreducible representations V_{q_r} of the group H,

$$V = \oplus_{r=1}^{\ell} V_{q_r}, \quad \text{with} \quad \sum_{r=1}^{\ell} q_r = q, \tag{4.6}$$

so that

$$\hat{I}_i = \begin{pmatrix} I_i^{q_1} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & I_i^{q_\ell} \end{pmatrix}. \tag{4.7}$$

Here $I_i^{q_r}$ are generators of $q_r \times q_r$ irreducible representations V_{q_r} of H and $V \cong \mathbb{C}^q$ (or $\mathbb{R}^q$ or $\mathbb{H}^q \cong \mathbb{R}^{4q}$). If we assume that H contains a maximal Abelian subgroup of G then the remaining generators $\hat{I}_a$ of G in this representation have the off-diagonal form[4]

$$\hat{I}_a = \begin{pmatrix} 0 & I_a^{q_{12}} & \dots & I_a^{q_{1\ell}} \\ I_a^{q_{21}} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_a^{q_{\ell-1\ell}} \\ I_a^{q_{\ell 1}} & \dots & I_a^{q_{\ell\ell-1}} & 0 \end{pmatrix}, \tag{4.8}$$

where $I_a^{q_{rs}}$ are $q_r \times q_s$ matrices (cf. [11]).

[4]If H does not contain a maximal Abelian subgroup of G or there is a subgroup in G commuting with H then $\hat{I}_a$ in (4.8) will contain diagonal terms $I_a^{q_{rr}}$ with $r = 1, \dots, \ell$.

Thus, one can associate a bounded quiver[5] satisfying a set of relations R to the ansatz (3.6) for a connection on a vector bundle $\mathcal{V} \to \mathcal{C}(M)$ over nearly Kähler, nearly parallel G_2, Sasaki-Einstein, and 3-Sasakian manifolds. In the simplest case of generators (4.7) the matrices X_a solving the constraint equations are obtained from (4.8) by substituting $\phi_{rs} I_a^{q_{rs}}$ instead of $I_a^{q_{rs}}$, where ϕ_{rs} are complex functions of τ. An important fact is that the space $\mathcal{C}(M)$ is not homogeneous and therefore quivers and quiver bundles can appear in dimensional reduction without a G-equivariance condition studied earlier, e.g., in [25,26]. Recall that another way in which quiver gauge theories arise as low-energy effective field theories in string theory is through considering cones and orbifolds with conical singularities and placing D-branes at the orbifold singularities [28]. Our constructions can be lifted as in [23] to heterotic strings and provide a description of NS5-branes and gauge NS5-branes. It would be of interest to further study this brane interpretation and its possible relations with constructions of [28].

C. Kähler-Einstein manifolds and quiver gauge theories

For another example related to quiver gauge theories we consider the manifold

$$\mathcal{Y} = \Sigma \times \mathcal{X}, \tag{4.9}$$

where Σ and $\mathcal{X}$ are 2-dimensional and $2k$-dimensional Kähler-Einstein manifolds with the Kähler form ω on Σ and Ω on $\mathcal{X}$. Let $\hat{\Gamma}$ be the canonical $\mathfrak{u}(k)$-valued Levi-Civita connection on $\mathcal{X}$,

$$\hat{\Gamma} = \Gamma^i \hat{I}_i, \quad \text{with} \quad \hat{I}_i \in \mathfrak{u}(k). \tag{4.10}$$

We consider U(k) as a closed subgroup of the Lie group SU($k+1$). Let $V \cong \mathbb{C}^q$ be an irreducible representation of the group SU($k+1$) decomposed into a sum of irreducible representations $V_{q_r} \cong \mathbb{C}^{q_r}$ of the group U(k) as in (4.7) and $\mathcal{V} \to \mathcal{X}$ is a holomorphic vector bundle over $\mathcal{X}$ associated with the bundle $P(\mathcal{X}, \mathrm{U}(k))$ of Hermitian frames on $\mathcal{X}$. This bundle has the connection (4.10) which is reducible according to (4.7), $\mathcal{V} = \oplus_r \mathcal{V}_{q_r}$.

Consider now ℓ complex vector bundles $E_1, \dots, E_\ell$ over Σ with unitary connections $A^1, \dots, A^\ell$ and ranks $N_1, \dots, N_\ell$. Introduce a complex vector bundle $\mathcal{E} = \oplus_r E_r \otimes \mathcal{V}_{q_r}$ over $\Sigma \times \mathcal{X}$ of rank

$$N = \sum_{r=1}^{\ell} N_r q_r \tag{4.11}$$

[5]A quiver $Q = (Q^0, Q^1)$ is an oriented graph, i.e. a set of vertices Q^0 with a set Q^1 of arrows between the vertices (see, e.g., [27]). A path in Q is a sequence of arrows in Q^1 which compose. A relation of the quiver is a formal finite sum of paths. In our case vertices correspond to vector bundles $\mathcal{V}_{q_r}$ with fibers V_{q_r} and arrows correspond to morphisms $\mathcal{V}_{q_s} \to \mathcal{V}_{q_r}$ of vector bundles.

and assume $c_1(\mathcal{E}) = 0$ without loss of generality, so that the structure group of $\mathcal{E}$ is SU(N). The matrices

$$\tilde{I}_i := \begin{pmatrix} \mathbf{1}_{N_1} \otimes I_i^{q_1} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{1}_{N_\ell} \otimes I_i^{q_\ell} \end{pmatrix} \quad (4.12)$$

are generators of a reducible unitary representation of the group U(k) on the complex vector space $\tilde{V} \cong \mathbb{C}^N$. Introduce a gauge connection

$$A := \begin{pmatrix} A^1 \otimes \mathbf{1}_{q_1} & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A^\ell \otimes \mathbf{1}_{q_\ell} \end{pmatrix} \quad (4.13)$$

on the bundle $E := \oplus_r E_r \otimes \mathbb{C}^{q_r}$ over Σ. It is obvious from (4.13) that $[A, \tilde{I}_i] = 0$.

On the bundle $\mathcal{E} \to \mathcal{Y}$ we introduce a connection

$$\mathcal{A} = A + \Gamma^i \tilde{I}_i + X_a e^a, \quad (4.14)$$

where $X_a \in \mathfrak{su}(N)$ are matrices which depend only on coordinates of Σ and e^a is the basis of 1-forms on $\mathcal{X}$, $a = 1, \dots, 2k$. Note that

$$de^a = -\Gamma^a_b \wedge e^b = -f^a_{ib}\Gamma^i \wedge e^b. \quad (4.15)$$

Using (4.15), for the curvature $\mathcal{F}$ of the connection (4.14) we obtain

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= F + \tilde{R} + (dX_a + [A, X_a]) \wedge e^a + \tfrac{1}{2}[X_a, X_b]e^a \wedge e^b \\ &\quad + ([I_i, X_a] - f^b_{ia} X_b)\Gamma^i \wedge e^a, \end{aligned} \quad (4.16)$$

where

$$\begin{aligned} F &= dA + A \wedge A, \\ \tilde{R} &= d\tilde{\Gamma} + \tilde{\Gamma} \wedge \tilde{\Gamma}, \quad \text{and} \quad \tilde{\Gamma} := \Gamma^i \tilde{I}_i. \end{aligned} \quad (4.17)$$

Suppose that X_a satisfy the constraints

$$[\tilde{I}_i, X_a] = f^b_{ia} X_b \quad (4.18)$$

and impose on $\mathcal{F}$ the Hermitian-Yang-Mills equations [6]

$$\mathcal{F}^{0,2} = 0 \Rightarrow \bar{\partial} X_a + [A^{0,1}, X_a] = 0, \qquad [Y_{\bar{A}}, Y_{\bar{B}}] = 0, \quad (4.19)$$

$$(\omega + \Omega) \lrcorner \mathcal{F} = 0 \Rightarrow \omega^{\alpha\beta}\mathcal{F}_{\alpha\beta} + \lambda \tilde{I}_0 + \Omega^{ab}[X_a, X_b] = 0. \quad (4.20)$$

Here $\bar{\partial} + A^{0,1}$ is the antiholomorphic part of the covariant derivative on Σ,

$$Y_{\bar{1}} := \tfrac{1}{2}(X_1 + iX_{k+1}), \dots, Y_{\bar{k}} := \tfrac{1}{2}(X_k + iX_{2k}),$$

where the constant λ is proportional to the scalar curvature of the Kähler-Einstein manifold $\mathcal{X}$, $\omega_{\alpha\beta}$ and Ω_{ab} are components of the Kähler forms on Σ, and $\mathcal{X}$, $\tilde{I}_0$ is the $\mathfrak{u}(1)$ generator in the decomposition $\mathfrak{u}(k) = \mathfrak{u}(1) \oplus \mathfrak{su}(k) \subset \mathfrak{su}(k+1)$, $\alpha, \beta = 1, 2$ and $A, B, \dots, = 1, \dots, k$. We see that (4.19) and (4.20) are the usual quiver vortex equations on Σ (cf. [11,26]).[6] For $k = 1$ and $\mathcal{X} = \mathbb{C}P^1$, one can obtain [29] the standard vortex equations on a Riemann surface Σ. One can generalize the above construction by taking instead of Σ a Kähler-Einstein manifold of dimension more than 2.

D. Instantons on smooth manifolds

It is of interest to extend the ansatz for a connection $\mathcal{A}$ from cones to their smooth resolutions as proposed in [20] as well as from direct product manifolds, such as $\mathcal{Y}$ in Sec. IV C, to irreducible smooth manifolds with warped product metrics. This is possible.

For illustration we consider noncompact Calabi-Yau $(k+1)$-folds $\mathcal{Z}$ discussed in [20]. They have a metric

$$d\tilde{s}^2 = \frac{dr^2}{f^2(r)} + r^2 f^2(r)\eta^2 + 2r^2 ds^2_{\mathrm{KE}}, \quad (4.21)$$

where

$$f^2 = 1 - \left(\frac{a^2}{r^2}\right)^{k+1}, \quad (4.22)$$

ds^2_{KE} is the standard Kähler metric on a Kähler-Einstein manifold $\mathcal{X}$ which is the base manifold for a projection

$$\pi\colon \mathcal{X}' \to \mathcal{X} \quad (4.23)$$

from Sasaki-Einstein $(2k+1)$-manifold $\mathcal{X}'$ onto $\mathcal{X}$ and η is the 1-form along fibers of the projection (4.23).

Note that

$$d\tilde{s}^2 = r^2 ds^2, \quad (4.24)$$

with

$$ds^2 = \frac{dr^2}{r^2 f^2} + f^2\eta^2 + 2ds^2_{\mathrm{KE}} = \frac{d\tau^2}{f^2} + f^2\eta^2 + 2ds^2_{\mathrm{KE}}, \quad (4.25)$$

i.e. $d\tilde{s}^2$ is conformally equivalent to ds^2. Singularity of the transformation at $r = 0$ is not essential since we are interested in Yang-Mills instantons on the manifold with the metric (4.21) extendable smoothly at $r = 0$. Note also that the instanton equation on $\mathcal{Z}$ is invariant with respect to conformal transformation,

[6] Note that the last equations in (4.19) correspond to quiver relations.

$$\tilde{*}F + \tilde{*}Q_Z \wedge F = \tilde{*}F + (\tilde{\omega} + \tilde{\Omega})^{k-1} \wedge F = r^{2(k-1)}(*F + (\omega + \Omega)^{k-1} \wedge F) = 0, \tag{4.26}$$

since

$$\tilde{\omega} + \tilde{\Omega} = r^2(\omega + \Omega) \tag{4.27}$$

and $\tilde{*} = r^{2(k-1)} *$. Here $\omega = d\tau \wedge \eta$ and Ω is the Kähler form on $\mathcal{X} \hookrightarrow \mathcal{Z}$.

The ansatz for $\mathcal{A}$ on the space $\mathcal{Z}'$ with the metric (4.25) is the same as in (4.14) and leads to the same reduction (4.19) and (4.20) of the instanton equation. Solving these vortex equations, one obtains instantons on $\mathcal{Z}$. One can simplify the task assuming that $\mathcal{A}_\eta$ and X_a depend only on $\tau = \ln r$ and choosing $\mathcal{A}_\tau = 0$. Then (4.19) and (4.20) will be reduced to equations similar to those which were considered in [20].

V. CONCLUSIONS

We have examined in some detail the construction of instantons on cones $\mathcal{C}(M)$ over nearly Kähler and nearly parallel G_2-manifolds M initiated in [16,17] and extended to cones over Sasaki-Einstein and 3-Sasakian manifolds in [19,20]. Having at our disposal a reduced structure group H of a manifold M admitting real Killing spinors and the holonomy group G of the cone $\mathcal{C}(M)$, we introduced a quiver bundle $\mathcal{V}$ over $\mathcal{C}(M)$, determined entirely by the representation theory of the group G and H, and introduced a proper connection $\mathcal{A}$ on this bundle. The ansatz for $\mathcal{A}$ reduces the instanton equations on $\mathcal{C}(M)$ to simpler matrix equations which can be solved in many special cases. It is of interest to construct new instanton solutions on $\mathcal{C}(M)$ by using our generalized ansatz, and to lift them to solutions of heterotic supergravity along the way considered in [19,23].

We have also introduced a quiver bundle $\mathcal{E}$ over a Kähler-Einstein manifold of the form $\Sigma \times \mathcal{X}$ and extended the Levi-Civita connection on the $2k$-dimensional Kähler-Einstein manifold $\mathcal{X}$ to a connection on $\mathcal{E}$ parametrized by $2k$ matrices X_a. We established an equivalence between solutions of Hermitian-Yang-Mills equations on $\Sigma \times \mathcal{X}$ and solutions of some quiver vortex equations on Σ.

Recall that regular Sasaki-Einstein manifolds are U(1) bundles over Kähler-Einstein manifolds and cones over them are Calabi-Yau spaces. Using this correspondence, we have introduced a connection on a quiver bundle $\mathcal{E}$ over smooth resolutions of $(2k+2)$-dimensional Calabi-Yau cones. It is of interest to consider instantons on other special holonomy manifolds.[7] We hope to report on this in the future.

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[7] Some instanton solutions on particular kinds of G_2- and Spin(7)-manifolds were considered in [30].

[1] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).
[2] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Phys. Rep. **66**, 213 (1980).
[3] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1984).
[4] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts, Nucl. Phys. **B214**, 452 (1983).
[5] R. S. Ward, Nucl. Phys. **B236**, 381 (1984).
[6] S. K. Donaldson, Proc. London Math. Soc. **s3-50**, 1 (1985); Duke Math. J. **54**, 231 (1987); K. K. Uhlenbeck and S.-T. Yau, Commun. Pure Appl. Math. **39**, S257 (1986).
[7] M. Mamone Capria and S. M. Salamon, Nonlinearity **1**, 517 (1988); R. Reyes Carrión, Diff. Geom. Appl. **8**, 1 (1998).
[8] L. Baulieu, H. Kanno, and I. M. Singer, Commun. Math. Phys. **194**, 149 (1998); M. Blau and G. Thompson, Phys. Lett. B **415**, 242 (1997); B. S. Acharya, J. M. Figueroa-O'Farrill, B. J. Spence, and M. O'Loughlin, Nucl. Phys. **B514**, 583 (1998).
[9] S. K. Donaldson and R. P. Thomas, *The Geometric Universe* (Oxford University Press, Oxford, 1998); S. K. Donaldson and E. Segal, arXiv:0902.3239.
[10] G. Tian, Ann. Math. **151**, 193 (2000).
[11] A. D. Popov, Lett. Math. Phys. **92**, 253 (2010); A. D. Popov and R. J. Szabo, J. High Energy Phys. 02 (2012) 033.
[12] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987).
[13] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, 2007).
[14] D. B. Fairlie and J. Nuyts, J. Phys. A **17**, 2867 (1984); S. Fubini and H. Nicolai, Phys. Lett. **155B**, 369 (1985); A. D. Popov, Mod. Phys. Lett. A **7**, 2077 (1992); T. A. Ivanova and A. D. Popov, Theor. Math. Phys. **94**, 225 (1993).
[15] E. Corrigan, P. Goddard, and A. Kent, Commun. Math. Phys. **100**, 1 (1985); E. K. Loginov, SIGMA **1**, 002 (2005); J. Broedel, T. A. Ivanova, and O. Lechtenfeld, Mod. Phys. Lett. A **23**, 179 (2008).
[16] T. A. Ivanova, O. Lechtenfeld, A. D. Popov, and T. Rahn, Lett. Math. Phys. **89**, 231 (2009); D. Harland, T. A.

Ivanova, O. Lechtenfeld, and A. D. Popov, Commun. Math. Phys. **300**, 185 (2010); D. Harland and A. D. Popov, J. High Energy Phys. 02 (2012) 107.

[17] I. Bauer, T. A. Ivanova, O. Lechtenfeld, and F. Lubbe, J. High Energy Phys. 10 (2010) 044; A. S. Haupt, T. A. Ivanova, O. Lechtenfeld, and A. D. Popov, Phys. Rev. D **83**, 105028 (2011); K.-P. Gemmer, O. Lechtenfeld, C. Nölle, and A. D. Popov, J. High Energy Phys. 09 (2011) 103.

[18] C. Bär, Commun. Math. Phys. **154**, 509 (1993).

[19] D. Harland and C. Nölle, J. High Energy Phys. 03 (2012) 082.

[20] F. P. Correia, J. High Energy Phys. 12 (2009) 004; 02 (2011) 054.

[21] A. Strominger, Nucl. Phys. **B343**, 167 (1990); J. A. Harvey and A. Strominger, Phys. Rev. Lett. **66**, 549 (1991); T. A. Ivanova, Phys. Lett. B **315**, 277 (1993); M. Gunaydin and H. Nicolai, Phys. Lett. B **351**, 169 (1995).

[22] O. Lechtenfeld, C. Nölle, and A. D. Popov, J. High Energy Phys. 09 (2010) 074; C. Nölle, J. Phys. A **45**, 045402 (2012); A. Chatzistavrakidis, O. Lechtenfeld, and A. D. Popov, J. High Energy Phys. 04 (2012) 114.

[23] K.-P. Gemmer, A. S. Haupt, O. Lechtenfeld, C. Nölle, and A. D. Popov, arXiv:1202.5046.

[24] C. P. Boyer and K. Galicki, arXiv:math/0703231.

[25] O. Garcia-Prada, Commun. Math. Phys. **156**, 527 (1993); L. Alvarez-Consul and O. Garcia-Prada, J. Reine Angew. Math. **2003**, Issue 556, 1 (2003).

[26] A. D. Popov and R. J. Szabo, J. Math. Phys. (N.Y.) **47**, 012306 (2006); O. Lechtenfeld, A. D. Popov, and R. J. Szabo, Prog. Theor. Phys. Suppl. **171**, 258 (2007); J. High Energy Phys. 08 (2008) 093; B. P. Dolan and R. J. Szabo, J. High Energy Phys. 08 (2009) 038.

[27] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras* (Cambridge University Press, Cambridge, 1995).

[28] M. R. Douglas and G. W. Moore, arXiv:hep-th/9603167; C. V. Johnson and R. C. Myers, Phys. Rev. D **55**, 6382 (1997); M. R. Douglas, B. Fiol, and C. Romelsberger, J. High Energy Phys. 09 (2005) 057.

[29] A. D. Popov, Nucl. Phys. **B821**, 452 (2009); Lett. Math. Phys. **84**, 139 (2008).

[30] S. Miyagi, Mod. Phys. Lett. A **14**, 2595 (1999); H. Kanno and Y. Yasui, J. Geom. Phys. **34**, 302 (2000); M. Dunajski and M. Hoegner, arXiv:1109.4537.