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Microscopic twisted-mass Dirac spectrum

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The microscopic spectral density for lattice QCD with two flavors and maximally twisted mass is computed. The results are given for a fixed index of the Dirac operator and include the leading-order a^2 corrections to the chiral Lagrangian due to the discretization errors. The computation is carried out within the framework of Wilson chiral perturbation theory.

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I. INTRODUCTION

Large-scale numerical simulations of twisted-mass lattice QCD [1] are currently investigated in order to access the deep chiral regime of QCD [2]. In the twisted-mass formulation, the standard Hermitian Wilson term is replaced by an anti-Hermitian isospin-violating Wilson term; for a review, see Ref. [3]. Under an axial transformation, this modified Wilson term can be transformed into the standard Wilson term while the mass is transformed into a twisted-mass term. One advantage of this approach is that the fermion determinant of the two-flavor Dirac operator is bounded from below due to the twisted quark mass. This offers automatic control of the problems which the smallest eigenvalues of the Wilson Dirac operator may cause for the numerical stability of standard simulations with Wilson fermions.

The smallest eigenvalues of the Wilson Dirac operator also play a crucial role for the spontaneous breakdown of chiral symmetry [4,5]. Here, we compute analytically the density of these smallest eigenvalues for twisted-mass lattice QCD. This microscopic eigenvalue density is uniquely determined by the symmetries of the lattice theory and hence can be obtained from the low-energy effective theory known as Wilson chiral perturbation theory [6–8]. This effective theory describes the finite volume corrections as well as corrections due to discretization errors caused by the nonzero lattice spacing a. To order a^2, the effects of the lattice spacing are parametrized in terms of three additional low-energy constants. While the values of these constants are specific to the exact implementation on the lattice, it is essential to know their values in order to extract physical observables such as the chiral condensate, Σ, and pion decay constant F_π. The analytical results for the eigenvalue density of the Dirac operator at nonzero twisted mass presented here offer a direct way to test Wilson chiral perturbation theory against lattice data. Moreover, if the test is successful, it provides a direct way to measure the additional low-energy constants as well as the physical ones. Such a test was carried out for the quenched case with a standard (untwisted) mass in Refs. [9,10]. Finally, we discuss constraints on the low-energy constants from QCD inequalities.

This paper is organized as follows: To settle the notation, the next section gives a brief introduction to twisted-mass QCD. In Sec. III, we then turn to the low-energy effective theory known as Wilson chiral perturbation theory. The new results for the microscopic Dirac eigenvalue density at maximally twisted mass are presented in Sec. IV. We discuss the constraints on the additional low-energy constants from the perspective of QCD inequalities the in Sec. V. Finally, we draw conclusions in Sec. VI.

II. BASICS OF TWISTED MASS QCD

Here, we briefly recall the basics of twisted-mass two-flavor QCD in the continuum limit as well as on the lattice; see Ref. [1] for more details. This also introduces the notation used throughout this paper.

A. Twisted mass in the continuum

In the continuum formulation, the twisted-mass fermionic action is given by

$$S = \int d^4x \bar{\psi}(D_\mu \gamma_\mu + m + i z_t \gamma_5 \tau_3)\psi. \qquad (1)$$

Under the axial transformation

$$\psi' = \exp(i\omega\gamma_5\tau_3/2)\psi, \quad \bar{\psi}' = \bar{\psi}\exp(i\omega\gamma_5\tau_3/2), \qquad (2)$$

the mass terms get rotated,

$$m' = m\cos(\omega) - z_t \sin(\omega), \qquad z_t' = m\sin(\omega) + z_t\cos(\omega), \qquad (3)$$

as follows from

$$\exp(i\omega\gamma_5\tau_3) = \cos(\omega) + i\gamma_5\tau_3\sin(\omega). \qquad (4)$$

The continuum covariant derivative term is of course invariant under the axial transformation since $\{\gamma_5, \gamma_\mu\} = 0$. In the continuum, we therefore have

$$\det(D_\mu\gamma_\mu + m + iz_t\gamma_5\tau_3) = \det(D_\mu\gamma_\mu + m' + iz_t'\gamma_5\tau_3). \tag{5}$$

Note that the twisted source, z_t', vanishes completely if we make the rotation with $\tan(\omega) = -z_t/m$.

If we simply want to evaluate the partition function at some nonzero mass (and zero twisted mass), we could start with both m and z_t in the determinant as long as we remember that this corresponds to the value

$$m'(\omega = \arctan(-z_t/m)) = m\sqrt{1 + \left(\frac{z_t}{m}\right)^2} \tag{6}$$

of the quark mass and zero value of the twisted mass.

Maximal twist is obtained at $m = 0$ with $\omega = \pi/2$. For maximal twist,

$$m' = z_t, \qquad z_t' = 0, \tag{7}$$

so that

$$Z(m = 0, z_t; a = 0) = Z(m' = z_t, z_t' = 0; a = 0). \tag{8}$$

B. Twisted mass Wilson fermions on the lattice

With Wilson fermions on the lattice, the discretized covariant derivative

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \frac{ar}{2}\nabla_\mu\nabla_\mu^* \tag{9}$$

is *not* anti-Hermitian and does not anticommute with γ_5. However, D_W is γ_5-Hermitian

$$\gamma_5 D_W \gamma_5 = D_W^\dagger, \tag{10}$$

and the product with γ_5, $D_5(m) \equiv \gamma_5(D_W + m)$ is therefore Hermitian. These properties are unaltered if one adds a clover term to D_W.

The main motivation to introduce the twisted mass becomes obvious when we write the determinant in terms of the eigenvalues, $\lambda_j^5(m)$, of $D_5(m)$:

$$\begin{aligned}\det(D_W + m + iz_t\gamma_5\tau_3) &= \det(D_5(m) + iz_t\tau_3)\\ &= \prod_j(\lambda_j^5(m) + iz_t)(\lambda_j^5(m) - iz_t)\\ &= \prod_j(\lambda_j^5(m)^2 + z_t^2).\end{aligned} \tag{11}$$

The square of the twisted mass sets a lower limit on the terms in the product even when the eigenvalues of $D_5(m)$ are smaller in magnitude than m as happens for $a \neq 0$. The numerical problem with small eigenvalues of D_5 is therefore regulated by the twisted-mass source.

Since the Wilson term breaks the axial-symmetry, the identification (6) for the partition function is no longer valid on the lattice. However, as the Wilson term is a cutoff artifact, one is free to choose the m' as the physical quark mass provided that $z' = 0$. Therefore, if we start with $m = 0$, it is natural to consider the twisted mass z_t as the phyiscal quark mass, cf. Eq. (7).

From Eq. (11), it is then clear that the Dirac spectrum relevant for chiral symmetry breaking at maximal twist is that of $D_5(m = 0)$,

$$\frac{d}{dz_t}\log Z(m = 0, iz_t, -iz_t; a) = \int d\lambda^5 \frac{2z_t}{\lambda^5(m = 0)^2 + z_t^2}\rho_5(\lambda^5(m = 0), z_t; a). \tag{12}$$

Note that, in the twisted-chiral limit, we recover the Banks-Casher [4] relation [11]:

$$\Sigma = \lim_{z_t\to 0}\frac{\pi\rho_5(\lambda^5(m = 0) = 0; z_t; a)}{V}. \tag{13}$$

It is therefore of particular interest to know the analytical form of $\rho_5(\lambda^5(m = 0), z_t; a)$ in the microscopic limit. The microscopic eigenvalue density derived below gives exactly this form.

III. WILSON CHIRAL PERTURBATION THEORY WITH TWISTED MASS

With the twisted source included, the static chiral Lagrangian reads [6–8]

$$\begin{aligned}V\mathcal{L} = {}&\mathrm{Tr}(\hat{m}^\dagger U + \hat{m}U^\dagger) + \mathrm{Tr}(\hat{z}_t^\dagger\tau_3 U - \hat{z}_t\tau_3 U^\dagger)\\ &- \mathrm{Tr}(\hat{a}^\dagger U\hat{a}^\dagger U + \hat{a}U^\dagger\hat{a}U^\dagger),\end{aligned} \tag{14}$$

with the sources

$$\hat{m} = m\Sigma V, \qquad \hat{z}_t = z_t\Sigma V \quad \text{and} \quad \hat{a} = aW_8V. \tag{15}$$

Here, we have set $W_6 = W_7 = 0$ [12].

In the microscopic limit (also known as the ϵ regime) for twisted mass Wilson fermions [15], the zero-momentum modes of the pion fields factorize from the partition function resulting in the z_t dependence,

$$Z_2^\nu(m, z_t; a) = \int_{U(2)}\det{}^\nu(U)e^{V\mathcal{L}}. \tag{16}$$

Here, we have written the expression for a sector with fixed index, ν, of the Dirac operator. The index is defined through

$$\nu \equiv \sum_k \mathrm{sign}\langle k|\gamma_5|k\rangle, \tag{17}$$

where $|k\rangle$ are the eigenstates of D_W. Note that only the real modes of D_W contribute to the index [16]. The index may also be obtained from the flow with m of the eigenvalues of $D_5(m)$ [17].

IV. THE MICROSCOPIC SPECTRUM WITH TWO MAXIMALLY TWISTED FLAVORS

Here, we compute the microscopic spectral density of $D_5(m = 0)$ relevant for two flavors at maximal twisted mass. The computation is carried out for fixed index, ν, of the Wilson Dirac operator.

In order to derive this density, we employ the graded generating functional with index ν. This is given by [13,14,18]

$$Z^{\nu}_{3|1}(Z;a) = \int dUS \det(iU)^{\nu} e^{+(i/2)\mathrm{Trg}(Z[U+U^{-1}])+a^2\mathrm{Trg}(U^2+U^{-2})}, \tag{18}$$

where $Z \equiv \mathrm{diag}(iz_t, -iz_t, z, \tilde{z})$, and the integration is over $Gl(3|1)/U(1)$. The difference from Ref. [19] is that we now have the twisted mass instead of the standard mass. For a discussion of the group manifold, we refer to Ref. [20].

The spectral resolvent is obtained from the graded generating functional by differentiation with respect to the z source and a subsequent quench of the additional flavors by the limit $z \to \tilde{z}$,

$$G^{\nu}_{3|1}(z, z_t; a) = \lim_{\tilde{z}\to z} \frac{d}{dz} Z^{\nu}_{3|1}(iz_t, -iz_t, z, \tilde{z}; a). \tag{19}$$

Finally, the density of eigenvalues, $\rho^{\nu}_5(\lambda^5, z_t; a)$, of D_5 follows from

$$\rho^{\nu}_5(\lambda^5, z_t; a) = \left\langle \sum_k \delta(\lambda^5_k - \lambda^5) \right\rangle_{N_f=2} = \frac{1}{\pi} \mathrm{Im}[G^{\nu}_{3|1}(z = -\lambda^5, z_t; a)]_{\epsilon\to 0}. \tag{20}$$

Our main task is therefore to evaluate the graded generating function. In Ref. [19], it was shown that the generating functional (18) can be rewritten as

$$Z^{\nu}_{3|1}(Z;a) = \frac{e^{-4a^2}}{(16\pi a^2)^2} \int_{-\infty}^{\infty} dsdt \frac{B_{3|1}(S)}{B_{3|1}(Z)} e^{(1/16a^2)\mathrm{Trg}(S^2+Z^2)} \times e^{-t\tilde{z}/8a^2} \det e^{-is_k Z_l/8a^2}{}_{k,l=1,2,3} \times \int dUS \det(iU)^{\nu} e^{+(i/2)\mathrm{Trg}(SU+SU^{-1})}, \tag{21}$$

where the Berezinian is given by

$$B_{3|1}(S) = \frac{(is_3 - is_2)(is_3 - is_1)(is_2 - is_1)}{(t - is_1)(t - is_2)(t - is_3)} \tag{22}$$

and

$$S \equiv \begin{pmatrix} is & 0 \\ 0 & t \end{pmatrix}, \tag{23}$$

with $s = \mathrm{diag}(s_1, s_2, s_3)$.

The integral over U results in the $a = 0$ generating functional which takes the form [21,22]

$$Z^{\nu}_{3|1}(x_1, x_2, x_3, x_4; a=0) = 2\frac{x_4^{\nu}}{x_1^{\nu}x_2^{\nu}x_3^{\nu}} \frac{1}{(x_3^2 - x_2^2)(x_3^2 - x_1^2)(x_2^2 - x_1^2)} \times \det \begin{pmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu+2}(x_1) & x_1^3 I_{\nu+3}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu+2}(x_2) & x_2^3 I_{\nu+3}(x_2) \\ I_\nu(x_3) & x_3 I_{\nu+1}(x_3) & x_3^2 I_{\nu+2}(x_3) & x_3^3 I_{\nu+3}(x_3) \\ (-1)^{\nu} K_\nu(x_4) & x_4(-1)^{\nu+1} K_{\nu+1}(x_4) & x_4^2(-1)^{\nu+2} K_{\nu+2}(x_4) & x_4^3(-1)^{\nu+3} K_{\nu+3}(x_4) \end{pmatrix}. \tag{24}$$

We can thus write

$$Z^{\nu}_{3|1}(Z;a) = \frac{e^{-4a^2}}{(16\pi a^2)^2} \int dsdt \frac{B_{3|1}(S)}{B_{3|1}(Z)} e^{(1/16a^2)\mathrm{Trg}(S^2+Z^2)} \times e^{-t\tilde{z}/8a^2} \det(e^{-is_k Z_l/8a^2})_{k,l=1,2,3} \left(\frac{\prod_k(-is_k)}{-t}\right)^{\nu} Z^{\nu}_{3|1}(\{(s_k^2)^{1/2}\}, (-t^2)^{1/2}; a=0). \tag{25}$$

The next step is to simplify the determinant

$$\det(e^{-is_k Z_j/8\hat{a}^2})_{k,j=1,2,3} = \begin{vmatrix} e^{-is_1 Z_1/8\hat{a}^2} & e^{-is_1 Z_2/8\hat{a}^2} & e^{-is_1 Z_3/8\hat{a}^2} \\ e^{-is_2 Z_1/8\hat{a}^2} & e^{-is_2 Z_2/8\hat{a}^2} & e^{-is_2 Z_3/8\hat{a}^2} \\ e^{-is_3 Z_1/8\hat{a}^2} & e^{-is_3 Z_2/8\hat{a}^2} & e^{-is_3 Z_3/8\hat{a}^2} \end{vmatrix}. \tag{26}$$

Since the other terms in the integrand also combine into an antisymmetric function of the s_k, all terms in the expansions of the determinant as a sum over permutations give the same contributions. In the integrand, we can thus make the replacement

$$\det(e^{-is_k Z_j/8\hat{a}^2})_{k,j=1,2,3} \to 6e^{-is_1 Z_1/8\hat{a}^2 - is_2 Z_2/8\hat{a}^2 - is_3 Z_3/8\hat{a}^2}. \tag{27}$$

The factor $e^{-i(s_1 Z_1 + s_2 Z_2 + s_3 Z_3)/8\hat{a}^2}$ is absorbed into the mixed term in the exponent of $e^{-((1)/(16a^2))\mathrm{Trg}(S-Z)^2}$. The inverse Berezinian of Z becomes

$$\frac{1}{B_{3|1}(\mathcal{Z})} = \frac{(\tilde{z}-z_1)(\tilde{z}-z_2)(\tilde{z}-z)}{(z-z_1)(z-z_2)(z_2-z_1)} = \frac{(\tilde{z}-iz_t)(\tilde{z}+iz_t)(\tilde{z}-z)}{(z-iz_t)(z+iz_t)(-2iz_t)}. \tag{28}$$

This contributes a total factor of $i/2z_t$ to the resolvent G (i.e. after differentiation with respect to z, and the limit $\tilde{z} \to z$ has been taken, so that we necessarily have to differentiate the factor $[\tilde{z}-z]$).

Combining the above expressions, the resolvent for $D_5(m=0)$ takes the form

$$G^\nu_{3|1}(z, m=0, z_t; a) = \frac{i}{\pi^2(16a^2)^2 z_t Z^\nu_{N_f=2}(iz_t, -iz_t; a)} \int ds_1 ds_2 ds_3 dt \frac{(is_2-is_1)(is_3-is_1)(is_3-is_2)}{(t-is_1)(t-is_2)(t-is_3)}$$
$$\times e^{-((1)/(16a^2))[(s_1-z_t)^2+(s_2+z_t)^2+(s_3+iz)^2+(t-z)^2]} \frac{(is_1)^\nu(is_2)^\nu(is_3)^\nu}{(t)^\nu}$$
$$\times Z^\nu_{3|1}((s_1^2)^{1/2}, (s_2^2)^{1/2}, (s_3^2)^{1/2}, (-t^2)^{1/2}; a=0), \tag{29}$$

where the partition function for $a=0$ is given by Eq. (24), and the integration of $s_3 + iz$ is over the real axis. The microscopic eigenvalue density of D_5 in the theory with two flavors at maximally twisted mass follows from Eq. (20). We only need to evaluate the two-flavor maximally-twisted-mass partition function which appears in the normalization of $G_{3|1}$.

The microscopic two flavor maximally-twisted-mass partition function

In order to complete the computation of the microscopic eigenvalue density, we need to evaluate the normalization which is given by the two-flavor maximally-twisted-mass partition function

$$Z^\nu_2(iz_t, -iz_t; a) = \int_{U(2)} dU \det(iU)^\nu e^{+(i/2)\,\mathrm{Tr}(\mathcal{Z}[U+U^{-1}])+a^2\,\mathrm{Tr}(U^2+U^{-2})}, \tag{30}$$

where $\mathcal{Z} \equiv \mathrm{diag}(iz_t, -iz_t)$. Extending the results of Ref. [19] to the twisted-mass case, we find

$$Z^\nu_2(iz_t, -iz_t; a) = \frac{ie^{4a^2}}{z_t\pi(16a^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_1 ds_2 (is_1 - is_2)$$
$$\times e^{-(1/(4a^2))[(s_1-z_t)^2+(s_2+z_t)^2]}$$
$$\times (is_1)^\nu (is_2)^\nu Z^\nu_2(s_1, s_2; a=0), \tag{31}$$

where

$$Z^\nu_2(x_1, x_2; a=0) = \frac{2}{x_1^\nu x_2^\nu (x_2^2 - x_1^2)} \times \det\begin{vmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) \end{vmatrix}. \tag{32}$$

The final step in the calculation is to factorize the four-dimensional integrals in Eq. (29) into the product of two-dimensional integrals. Not only may this factorized form have a deep connection to an underlying integrable hierarchy [23], but it is also highly advantageous for numerical evaluation of the eigenvalue density.

In Appendix A, we show that the spectral resolvent (29) for the microscopic eigenvalue density of D_5 with two flavors at maximally twisted mass can be written as

$$G^\nu_{3|1}(z, z_t; a) = G^\nu_{1|1}(z, z; a) + \frac{Z_2(iz_t, z; a)}{Z^\nu_2(iz_t, -iz_t; a)} \times \frac{z - iz_t}{2iz_t} G^\nu_{1|1}(-iz_t, z; a)$$
$$- \frac{Z^\nu_2(-iz_t, z; a)}{Z^\nu_2(iz_t, -iz_t; a)} \frac{z + iz_t}{2iz_t} G^\nu_{1|1}(iz_t, z; a). \tag{33}$$

Here,

$$G^\nu_{1|1}(z_1, z_2; a) = -\frac{1}{16a^2\pi}\int_{-\infty}^{\infty} dsdt \frac{1}{t + z_2 - is - z_1} \times e^{-(s^2+t^2)/(16a^2)}\left(\frac{is + z_1}{t + z_2}\right)^\nu$$
$$\times Z^\nu_{1|1}(\sqrt{-(is+z_1)^2}, \sqrt{-(t+z_2)^2}, a=0), \tag{34}$$

with

$$Z^\nu_{1|1}(m_1, m_2; a=0) = \left(\frac{m_2}{m_1}\right)^\nu (I_\nu(m_1) m_2 K_{\nu+1}(m_2) + m_1 I_{\nu+1}(m_1) K_\nu(m_2)), \tag{35}$$

and

$$Z^\nu_2(z_1, z_2; a) = \frac{1}{\pi 16a^2}\int_{-\infty}^{\infty} ds_1 ds_2 \frac{1}{(z_2 - z_1)} \times (is_1 + z_1 - is_2 - z_2) e^{-(s_1^2+s_2^2)/(16a^2)} \left(\frac{is_1 + z_1}{is_2 + z_2}\right)^\nu$$
$$\times Z^\nu_2\left(\sqrt{-(is_1+z_1)^2}, \sqrt{-(is_2+z_2)^2}; a=0\right), \tag{36}$$

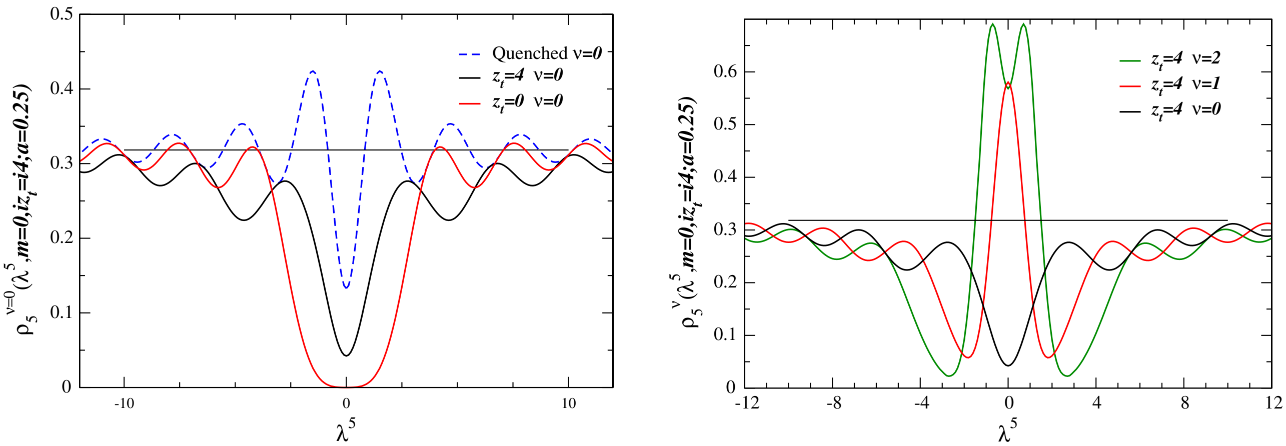

FIG. 1 (color online). The spectrum of $D_5(m = 0)$ for two flavors with maximally twisted mass. *Left:* the sector with zero index of the Dirac operator. As the twisted mass increases, the quenched result (dashed curve) is approached. *Right:* the dependence on the index ν for fixed $z_t = 4$ and $a = 0.25$. As W_8 is decreased, the near zero modes become exact δ functions at $\lambda^5 = 0$. For small lattice spacing, the width of the peak is proportional to $\sqrt{W_8}$. The thin horizontal line in both plots indicates the value $1/\pi$ which is the asymptotic limit of the density for large values of $|\lambda_5|$.

with $Z_2^\nu(x_1, x_2; a = 0)$ given in Eq. (32). Note that the first term on the right-hand side of Eq. (33) gives rise to the quenched density of D_5 at zero untwisted mass, m. A similar factorization of the unquenched density has been observed in the microscopic limit of QCD at nonzero chemical potential [24]. In that case, this structure has been understood in terms of an underlying integrable hierarchy.

With Eq. (33), the spectral density has been expressed in terms of products of double integrals. This form is far easier to evaluate numerically than the four-fold integral given in Eq. (29).

This completes the computation of the microscopic eigenvalue density of $D_5(m = 0)$ for two flavors at maximal twisted mass in sectors with a fixed index of the Wilson Dirac operator. See Fig. 1 for plots of the density. Note, in particular, the behavior of the near zero modes.

V. QCD INEQUALITIES WITH TWISTED QUARK MASS

In this section, we discuss two QCD inequalities. First, a QCD inequality for the microscopic partition function in a sector with fixed ν and, second, a QCD inequality for the pion masses. We will see that both put constraints on the low-energy constants of Wilson chiral perturbation theory.

The twisted-mass $N_f = 2$ QCD partition function is positive definite for all ν. This imposes a positivity requirement of the partition function corresponding the chiral Lagrangian of the Wilson QCD partition function. Because of the identity

$$\begin{aligned} &Z_2^\nu(z_t = 0; W_6, W_7, W_8, a) \\ &\quad = (-1)^\nu Z_2^\nu(z_t = 0; -W_6, -W_7, -W_8, a), \end{aligned} \tag{37}$$

and because for large z_t, the sign of the partition function is independent of the W_k, we necessarily obtain constraints on the W_k. In case $W_6 = W_7 = 0$, we find that $W_8 > 0$. From the small a-expansion of the partition function, we obtain the condition

$$W_8 - W_6 - W_7 > 0, \tag{38}$$

in agreement with the convergence requirements of the graded partition function [14]. Additional constraints can be obtained from mass inequalities for the pion masses which will be discussed in the remainder of this section.

The Dirac operator including the twisted mass,

$$D_W + m + iz\tau_3\gamma_5, \tag{39}$$

has the Hermiticity property

$$\tau_1\gamma_5(D_W + m + iz\tau_3\gamma_5)\gamma_5\tau_1 = (D_W + m + iz\tau_3\gamma_5)^\dagger. \tag{40}$$

Therefore, the inverse Dirac operator

$$S(x, y) = \langle x| \frac{1}{D_W + m + iz\tau_3\gamma_5} |y\rangle \tag{41}$$

satisfies

$$S(x, y)^\dagger = \gamma_5\tau_1 S(y, x)\tau_1\gamma_5. \tag{42}$$

Instead of τ_1, we could of course also have used

$$\cos(\phi)\tau_1 + \sin(\phi)\tau_2 \tag{43}$$

in the Hermiticity relation (40) which leads to the same consequences. All we need is a combination that anticommutes with τ_3 and is unitary. This relation allows us to derive Weingarten-type inequalities [25–27] for the pion masses.

The correlation function of two meson sources $\bar\psi\Gamma\psi(x)$ and $\bar\psi\Gamma\psi(y)$ evaluated for a fixed background gauge field satisfies (Γ is unitary)

$$\begin{aligned}
&\langle\bar{\psi}(x)\Gamma\psi(x)\bar{\psi}(y)\Gamma\psi(y)\rangle\\
&\quad= -\mathrm{Tr}[S(y,x)\Gamma S(x,y)\Gamma] + \mathrm{Tr}[S(x,x)\Gamma]\mathrm{Tr}[S(y,y)\Gamma]\\
&\quad= \mathrm{Tr}[S(y,x)\Gamma i\tau_1\gamma_5 S(y,x)^\dagger i\tau_1\gamma_5\Gamma]\\
&\qquad+ \mathrm{Tr}[S(x,x)\Gamma]\mathrm{Tr}[S(y,y)\Gamma]\\
&\quad\leq \mathrm{Tr}[S(y,x)S(y,x)^\dagger] + \mathrm{Tr}[S(x,x)\Gamma]\mathrm{Tr}[S(y,y)\Gamma]. \qquad (44)
\end{aligned}$$

The bound in the inequality is saturated for $\Gamma = i\gamma_5\tau_1$ (or with $\tau_1 \to \tau_2$ but not with $\tau_1 \to \tau_3$). This inequality has been evaluated for a fixed gauge field background. However, since the fermion determinant is positive for all gauge field configurations, the inequality continues to hold after averaging. If the disconnected diagrams average to zero, we obtain

$$\left\langle\left\langle\bar{\psi}(x)\Gamma\psi(x)\bar{\psi}(y)\Gamma\psi(y)\right\rangle\right\rangle \leq \langle\mathrm{Tr}S(0,x)S(0,x)^\dagger\rangle. \qquad (45)$$

For mesonic channels with mass gap m_Γ, we have

$$\left\langle\left\langle\bar{\psi}(x)\Gamma\psi(x)\bar{\psi}(0)\Gamma\psi(0)\right\rangle\right\rangle \propto \exp(-m_\Gamma|x|) \quad \text{as } x\to\infty. \qquad (46)$$

The inequality for the correlators thus translates into an inequality for the meson masses. From Eq. (45), we then conclude that [28]

$$m_{i\gamma_5\tau_{1,2}} \leq m_\Gamma. \qquad (47)$$

In particular, we have

$$m_{\pi^\pm} \leq m_{\pi^0}. \qquad (48)$$

From leading-order Wilson chiral perturbation theory, one obtains [29,30]

$$(m_\pi^0)^2 - (m_\pi^\pm)^2 = \frac{16a^2(W_8 + 2W_6)}{F_\pi^2}. \qquad (49)$$

If the contribution from disconnected diagrams is not important, we conclude that

$$W_8 + 2W_6 > 0. \qquad (50)$$

The contribution of the disconnected diagrams can be isolated by the introduction of valence quarks. This results in the inequality [31]

$$W_8 > 0, \qquad (51)$$

independent of the value of W_6 and W_7. Lattice simulations for twisted-mass fermions in Ref. [32] show that

$$m_\pi^0 < m_\pi^\pm. \qquad (52)$$

This implies that the contribution of the disconnected diagrams is important for the simulations in Ref. [32]. The possible importance of disconnected diagrams has been studied explicitly in lattice simulations of the respective correlators in Ref. [33]. Using Eq. (49), we thus conclude that for the simulations in Ref. [32],

$$W_8 + 2W_6 < 0. \qquad (53)$$

Combined with the inequality (51) derived in Ref. [31], we obtain the constraint

$$W_6 < 0. \qquad (54)$$

In the quenched case, lattice simulations show that the charged pions are the lightest pseudoscalar Goldstone bosons [34]. This is an agreement with the lore that disconnected diagrams are suppressed in the quenched theory [27].

VI. CONCLUSIONS

We have computed the microscopic spectral density of the massless Hermitian Wilson Dirac operator in the presence of two dynamical flavors at nonzero maximally twisted mass. The characteristic shape of the eigenvalue density in sectors with a fixed index of the Wilson Dirac operator derived in this paper offers a direct way to test Wilson chiral perturbation theory for twisted mass against lattice QCD. If the spectral density obtained on the lattice follows the analytical prediction, the strong dependence of the analytical result on the low-energy constant W_8 offers a direct way to measure the value of W_8. We have reduced the analytical form of the twisted-mass microscopic spectral density to a factorized form which is easily evaluated with standard numerical methods. A similar factorized form of the density for two standard dynamical flavors was recently presented in Ref. [35].

The microscopic results for the spectral density of D_5 at $m = 0$ have been derived for $W_8 > 0$. As was argued in Refs. [13,14], only the theory with $W_8 > 0$ correctly describes lattice QCD with Wilson fermions. In support of this, we have checked that the microscopic partition function for two flavors at maximally twisted mass is a positive definite function in all sectors with fixed index ν of the Wilson Dirac operator.

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APPENDIX A: FACTORIZATION OF $G_{3|1}$

In this appendix, we show that the microscopic eigenvalue density for two flavors of maximally twisted mass can be factorized into two-dimensional integrals.

We start from the resolvent which is given by Eq. (29),

$$G_{3|1}(z, m=0, z_t; a) = \frac{1}{\pi^2(16a^2)^2 Z^{\nu}_{N_f=2}(iz_t, -iz_t; a)} \int ds_1 ds_2 ds_3 dt \frac{i}{z_t} \frac{(is_2 - is_1)(is_3 - is_1)(is_3 - is_2)}{(t - is_1)(t - is_2)(t - is_3)} \times e^{-[(s_1-z_t)^2+(s_2+z_t)^2+(s_3+iz)^2+(t-z)^2]/16a^2} \frac{(is_1 is_2 is_3)^{\nu}}{t^{\nu}} Z^{\nu}_{3|1}((s_1^2)^{1/2}, (s_2^2)^{1/2}, (s_1^2)^{1/2}, (-t^2)^{1/2}; a=0), \tag{A1}$$

and use the notation

$$x_k = (s_k^2)^{1/2}, \qquad k = 1, 2, 3, \qquad x_4 = it. \tag{A2}$$

Our aim is to rewrite this in a factorized form. To this end, we explicitly insert the $a = 0$ partition function given in Eq. (24) and consider the combination

$$\frac{(is_2 - is_1)(is_3 - is_1)(is_3 - is_2)}{(t - is_1)(t - is_2)(t - is_3)} \frac{1}{(x_3^2 - x_2^2)(x_3^2 - x_1^2)(x_2^2 - x_1^2)} \times \det\begin{pmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu+2}(x_1) & x_1^3 I_{\nu+3}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu+2}(x_2) & x_2^3 I_{\nu+3}(x_2) \\ I_\nu(x_3) & x_3 I_{\nu+1}(x_3) & x_3^2 I_{\nu+2}(x_3) & x_3^3 I_{\nu+3}(x_3) \\ (-1)^\nu K_\nu(x_4) & x_4(-1)^{\nu+1} K_{\nu+1}(x_4) & x_4^2(-1)^{\nu+2} K_{\nu+2}(x_4) & x_4^3(-1)^{\nu+3} K_{\nu+3}(x_4) \end{pmatrix}. \tag{A3}$$

Combining the prefactors and using recursion relations for Bessel functions, this can be rewritten as

$$\frac{1}{(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)} \times \det\begin{pmatrix} I_\nu(x_1) & x_1 I_{\nu+1}(x_1) & x_1^2 I_{\nu}(x_1) & x_1^3 I_{\nu+1}(x_1) \\ I_\nu(x_2) & x_2 I_{\nu+1}(x_2) & x_2^2 I_{\nu}(x_2) & x_2^3 I_{\nu+1}(x_2) \\ I_\nu(x_3) & x_3 I_{\nu+1}(x_3) & x_3^2 I_{\nu}(x_3) & x_3^3 I_{\nu+1}(x_3) \\ (-1)^\nu K_\nu(x_4) & x_4(-1)^{\nu+1} K_{\nu+1}(x_4) & x_4^2(-1)^{\nu} K_{\nu}(x_4) & x_4^3(-1)^{\nu+3} K_{\nu+1}(x_4) \end{pmatrix}. \tag{A4}$$

The factorized form is due to the appearance of I_ν/K_ν in the odd columns and $I_{\nu+1}/K_{\nu+1}$ in the even columns. Expanding the determinant results in

$$\begin{aligned} &\frac{1}{(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)}[-(-1)^{\nu+1} I_{\nu+1}(x_3) K_{\nu+1}(x_4) I_\nu(x_1) I_\nu(x_2) x_3 x_4 (x_3^2 - x_4^2)(x_1^2 - x_2^2) \\ &+ (-1)^\nu K_{\nu+1}(x_4) I_{\nu+1}(x_1) I_\nu(x_3) I_\nu(x_2) x_1 x_4 (x_1^2 - x_4^2)(x_2^2 - x_3^2) - (-1)^\nu K_{\nu+1}(x_4) I_{\nu+1}(x_2) I_\nu(x_3) I_\nu(x_1) x_2 x_4 (x_2^2 - x_4^2)(x_1^2 - x_3^2) \\ &+ (-1)^{\nu+1} I_{\nu+1}(x_3) I_{\nu+1}(x_2) K_\nu(x_4) I_\nu(x_1) x_2 x_3 (x_2^2 - x_3^2)(x_1^2 - x_4^2) - (-1)^{\nu+1} I_{\nu+1}(x_3) I_{\nu+1}(x_1) K_\nu(x_4) I_\nu(x_2) x_1 x_3 (x_1^2 - x_3^2)(x_2^2 - x_4^2) \\ &- (-1)^\nu I_{\nu+1}(x_1) I_{\nu+1}(x_2) K_\nu(x_4) I_\nu(x_3) x_1 x_2 (x_1^2 - x_2^2)(x_3^2 - x_4^2)]. \end{aligned} \tag{A5}$$

We then decompose the fractions as

$$\frac{(x_1^2 - x_4^2)(x_2^2 - x_3^2)}{(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)} = \frac{1}{(x_1 + x_2)(x_3 + x_4)} - \frac{1}{(x_1 + x_3)(x_2 + x_4)} \tag{A6}$$

and any cyclic permutations thereof. This results in

$$\frac{1}{(x_1+x_3)(x_2+x_4)}[-(-1)^{\nu+1}I_{\nu+1}(x_3)K_{\nu+1}(x_4)I_\nu(x_1)I_\nu(x_2)x_3x_4-(-1)^\nu K_{\nu+1}(x_4)I_{\nu+1}(x_1)I_\nu(x_3)I_\nu(x_2)x_1x_4$$
$$-(-1)^{\nu+1}I_{\nu+1}(x_3)I_{\nu+1}(x_2)K_\nu(x_4)I_\nu(x_1)x_2x_3-(-1)^\nu I_{\nu+1}(x_1)I_{\nu+1}(x_2)K_\nu(x_4)I_\nu(x_3)x_1x_2]$$
$$+\frac{1}{(x_2+x_3)(x_1+x_4)}[(-1)^{\nu+1}I_{\nu+1}(x_3)K_{\nu+1}(x_4)I_\nu(x_1)I_\nu(x_3)x_4x_4+(-1)^\nu K_{\nu+1}(x_4)I_{\nu+1}(x_2)I_\nu(x_3)I_\nu(x_1)x_2x_4]$$
$$+(-1)^{\nu+1}I_{\nu+1}(x_3)I_{\nu+1}(x_1)K_\nu(x_4)I_\nu(x_2)x_1x_3+(-1)^\nu I_{\nu+1}(x_1)I_{\nu+1}(x_2)K_\nu(x_4)I_\nu(x_3)x_1x_2]$$
$$+\frac{1}{(x_1+x_2)(x_3+x_4)}[(-1)^\nu K_{\nu+1}(x_4)I_{\nu+1}(x_1)I_\nu(x_3)I_\nu(x_2)x_1x_4-(-1)^\nu K_{\nu+1}(x_4)I_{\nu+1}(x_2)I_\nu(x_3)I_\nu(x_1)x_2x_4$$
$$\times(-1)^{\nu+1}I_{\nu+1}(x_3)I_{\nu+1}(x_2)K_\nu(x_4)I_\nu(x_1)x_2x_3-(-1)^{\nu+1}I_{\nu+1}(x_3)I_{\nu+1}(x_1)K_\nu(x_4)I_\nu(x_2)x_1x_3]$$
$$=\frac{(-1)^\nu(x_4K_{\nu+1}(x_4)I_\nu(x_2)+x_2K_\nu(x_4)I_{\nu+1}(x_2)}{x_4+x_2}\frac{x_3I_{\nu+1}(x_3)I_\nu(x_1)-x_1I_{\nu+1}(x_1)+I_\nu(x_3)}{x_1+x_3}$$
$$\times\frac{(-1)^\nu(x_4K_{\nu+1}(x_4)I_\nu(x_1)+x_1K_\nu(x_4)I_{\nu+1}(x_1)}{x_4+x_1}\frac{x_2I_{\nu+1}(x_2)I_\nu(x_3)-x_3I_{\nu+1}(x_2)I_\nu(x_2)}{x_2+x_3}$$
$$\times\frac{(-1)^\nu(x_4K_{\nu+1}(x_4)I_\nu(x_3)+x_3K_\nu(x_4)I_{\nu+1}(x_3)}{x_3+x_4}\frac{x_1I_{\nu+1}(x_1)I_\nu(x_2)-x_2I_{\nu+1}(x_2)+I_\nu(x_1)}{x_1+x_2}. \tag{A7}$$

Using this identity, we can express the resolvent in the factorized form

$$G^\nu_{3|1}(z,m=0,iz_t,-iz_t;a)=G^\nu_{1|1}(z,z;a)+\frac{Z^\nu_2(iz_t,z)(z-iz_t)}{Z^\nu_2(iz_t,-iz_t)2iz_t}G^\nu_{1|1}(-iz_t,z;a)-\frac{Z_2(-iz_t,z)(z+iz_t)}{Z_2(iz_t,-iz_t)2iz_t}G^\nu_{1|1}(iz_t,z;a), \tag{A8}$$

where $Z^\nu_2(z_1,z_2)$ and $G^\nu_{1|1}(z_1,z_2;a)$ are given in Eqs. (36) and (34), respectively. This factorization can also be derived in general terms [36,37].

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