PHYSICAL REVIEW D **85,** 105007 (2012)

Supersymmetric Chern-Simons theory in the presence of a boundary

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(Received 28 December 2011; published 9 May 2012)

In this paper we analyze super-Chern-Simons theory in $\mathcal{N} = 1$ superspace formalism, in the presence of a boundary. We modify the Lagrangian for the Chern-Simons theory in such a way that it is supersymmetric even in the presence of a boundary. Also, even though the Chern-Simons theory is not gauge invariant in the presence of a boundary, if it is suitably coupled to a gauged Wess-Zumino-Witten model, then the resultant theory can be made gauge invariant. Thus, by suitably adding extra boundary degrees of freedom, the gauge and supersymmetry variations of the boundary theory exactly cancel the boundary terms generated by the variations of the bulk Chern-Simons theory. We also discuss how this can be applied to the Aharony-Bergman-Jafferis-Maldacena model in $\mathcal{N} = 1$ superspace, and we then describe the Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetries of the resultant gauge invariant supersymmetric theory.

DOI: 10.1103/PhysRevD.85.105007 PACS numbers: 12.60.Jv, 11.10.Kk, 11.25.Uv, 11.25.Yb

I. INTRODUCTION

The Aharony-Bergman-Jafferis-Maldacena (ABJM) theory is thought to describe the world volume of multiple M2-branes in M theory at low energies [1,2]. It is a three-dimensional Chern-Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ at levels k and $-k$ on the world volume of N M2-branes placed at the fixed point of R^8/Z_k. Although this construction explicitly realizes only $\mathcal{N} = 6$ supersymmetry (SUSY), the supersymmetry is expected to be enhanced to full $\mathcal{N} = 8$ supersymmetry for $k = 1$, 2 [3]. The ABJM theory coincides with the Bagger-Lambert-Gustavsson (BLG) action [4–7], based on the Basu-Harvey equation [8], for the only known example of the Lie 3-algebra.

The BLG model has been analyzed in the $\mathcal{N} = 1$ superfield formalism [9]. First, an octonionic self-dual tensor is used to construction a real superpotential with manifest $SO(7)$ invariance. Then for specially chosen couplings, the component action coincides with the BLG action, and hence the full $SO(8)$ symmetry is restored. After reduction using the novel Higgs mechanism [10], higher-derivative corrections to super-Yang-Mills on D2-branes were analyzed in the $\mathcal{N} = 1$ superspace formalism [11]. Chern-Simons theory with $\mathcal{N} = 1$ supersymmetry has also been studied in relation to axion gauge symmetry which occurs in supergravity theories arising from flux compactifications of superstrings and Scherk-Schwarz generalized dimensional reduction in M theory [12].

The ABJM and BLG actions are formulated for M2-branes without a boundary. However, it is of interest to allow the inclusion of a boundary. Such boundaries correspond to M2-branes ending on other objects in M theory. In [13] appropriate boundary conditions were derived for the ABJM and BLG actions, describing M2-branes ending on M5-branes, M9-branes, or gravitational waves. Boundary conditions in the presence of background flux were derived in [14]. The M5-brane is of particular interest, and certainly one motivation for studying open M2-branes is to learn about the physics of the M5-brane. For example, by considering a system of M2-branes ending on an M5-brane with a constant C field turned on, the BLG model was used to motivate a novel quantum geometry on the M5-brane world volume [15]. Another interesting relation between multiple M2-branes and the M5-brane is the identification of the BLG action (with Nambu-Poisson 3-bracket) as the M5-brane action with a large world volume C field, as reviewed in [16]. While these results involve a model for multiple M2-branes, we note that earlier work using the action for single open M2-branes suggested a form of noncommutative string theory on the M5-brane world volume [17–19]. It would be interesting to understand how these results arising from different approaches are related.

One of our motivations is to make further progress towards a superspace description of the ABJM action with a boundary. Rather than specifying boundary conditions as in [13,14], the idea here is to add additional boundary terms and degrees of freedom to make the action consistent. The prescription is motivated by the symmetries of the bulk action. In particular, we follow the general prescription given in [20] to add boundary terms so that half the bulk supersymmetry is preserved. This procedure has been applied to supersymmetric Abelian Chern-Simons theories in [20] and particularly to various models including Chern-Simons matter theories and the ABJM model in [21]. However, in addition to supersymmetry, it is necessary to consider preservation of gauge symmetry. This issue was considered, with the aim of describing the physics of multiple self-dual strings in [22]. In doing so bosonic Chern-Simons theory on a manifold with a boundary was analyzed. It was found that even though the Chern-Simons theory was not gauge invariant by itself in the presence of a boundary, the sum of it with a

Wess-Zumino-Witten (WZW) model living on the boundary was gauge invariant. Thus, new degrees of freedom were identified on the boundary and these degrees of freedom generated a $U(2N) \times U(2N)$ Kac-Moody current algebra. While it is possible to introduce the fermionic sector and derive a supersymmetric action in component form, it seems somewhat natural to derive a manifestly supersymmetric gauge invariant action, in some sense combining the results of [21,22]. This will be the result of Sec. IV, although for simplicity we limit ourselves to $\mathcal{N} = 1$ superspace and do not address the issue of a background C field as there has been limited progress in extending the ABJM action to include coupling to a general C field [23–25]. Because the issue of preservation of gauge symmetry is specific to the Chern-Simons term, this is considered separately in Sec. III.

While there is a well-known connection between $(2+1)$-dimensional (topological) Chern-Simons theories and $(1+1)$-dimensional conformal field theories (CFTs) [26], the situation is less clear for Chern-Simons matter theories. As shown in [27,28] for pure Chern-Simons theory with suitable boundary conditions, a component of the gauge field, say, A_0, appears linearly in the action and so can be integrated out, imposing the constraint $F_{12} = 0$. This constraint can be solved explicitly (e.g., for a manifold of the form of a disk for each constant time slice) and the result is a $(1+1)$-dimensional WZW model where the bulk gauge potential has been replaced by the boundary gauge degrees of freedom. Now Chern-Simons matter theories are not topological so we should not expect such a connection to $(1+1)$-dimensional CFTs. Of course, in cases such as ABJM theory where the Chern-Simons matter theory is conformal, the boundary theory may still be conformal. However, an important difference to the pure Chern-Simons case is that, due to the gauged scalar kinetic terms, A_0 will no longer appear as a Lagrange multiplier—even the classical equation of motion will couple F_{12} to the scalars rather than simply requiring $F_{12} = 0$. We therefore cannot expect the Chern-Simons action to be replaced by a WZW model in general. However, it is possible to use the principle of gauge invariance in the presence of a boundary to couple the Chern-Simons theory to a boundary theory. The general result is a gauge invariant action coupling the Chern-Simons gauge potential to a boundary WZW model, which reproduces the pure WZW action when starting from a pure Chern-Simons action [22].

Supersymmetric Chern-Simons theories have also been studied as interesting examples of the AdS_4/CFT_3 correspondence [29–33]. Three-dimensional $\mathcal{N} = 1$ superconformal field theories have the property of being supersymmetric without having any holomorphic property. This is a peculiarity of the AdS_4/CFT_3 correspondence with respect to the usual AdS_5/CFT_4. Thus, the results of this paper may be useful in analyzing certain aspects of the AdS_4/CFT_3 correspondence.

We need to fix a gauge before we can quantize any theory which has a gauge symmetry associated with it. This is done by the addition of a gauge-fixing term and a ghost term to the original action. The action thus obtained is invariant under two new symmetries called the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [34,35] and the anti-BRST symmetry [36]. These symmetries are important to show the unitarity of the $\mathcal{S}$ matrix and thus the consistency of the theory at quantum level [37]. The BRST symmetry of the bosonic Chern-Simons theory has been thoroughly investigated [38,39] and the BRST symmetry of the $\mathcal{N} = 1$ Chern-Simons theory has been analyzed in the superspace formalism [40,41]. The BRST and the anti-BRST symmetries of the ABJM theory have also been studied [42]. In this paper we will analyze the BRST and the anti-BRST symmetries of the ABJM theory in the presence of a boundary.

II. PROPERTIES OF SUPERCOVARIANT DERIVATIVES

In this section we shall first review the properties of the supercovariant derivatives for non-Abelian $\mathcal{N} = 1$ gauge fields in three dimensions [43]. Then we shall analyze the effect of having a boundary by generalizing the results of [20] to a non-Abelian case. In order to analyze the properties of the supercovariant derivatives, we first introduce θ_a as two component anticommuting parameters with odd Grassmann parity and let

$$\theta^2 = \tfrac{1}{2}\theta_a C^{ab}\theta_b = \tfrac{1}{2}\theta^a\theta_a. \tag{1}$$

The antisymmetric tensors C^{ab} and C_{ab} can be used to raise and lower spinor indices, and they satisfy $C_{ab}C^{bc} = \delta_a^c$. Now if T_A are Hermitian generators of a Lie algebra $[T_A, T_B] = if_{AB}^C T_C$, in the adjoint representation, then matter fields can be represented by matrix valued complex scalar superfields X and $X^\dagger$ suitably contracted with the generators of this Lie algebra, $X = X^A T_A$, and $X^\dagger = X^{\dagger A} T_A$. Let these superfields transform under infinitesimal gauge transformations as

$$\delta X = i\Lambda X, \qquad \delta X^\dagger = -iX^\dagger\Lambda, \tag{2}$$

where $\Lambda = \Lambda^A T_A$ and the product of these fields is actually a commutator. Now the superderivative, given by

$$D_a = \partial_a + (\gamma^\mu\partial_\mu)_a^b\theta_b, \tag{3}$$

of these superfields does not transform like the original superfields. But we can define a supercovariant derivative for these superfields by requiring it to transform like the original superfields. Thus, we obtain the following expression for the supercovariant derivative of these superfields

$$\nabla_a X = D_a X - i\Gamma_a X, \qquad \nabla_a X^\dagger = D_a X^\dagger + iX^\dagger\Gamma_a, \tag{4}$$

where Γ_a is a matrix valued spinor superfield suitably contracted with generators of a Lie algebra, $\Gamma_a = \Gamma_a^A T_A$.

If this matrix valued spinor superfield is made to transform under gauge transformations as

$$\delta\Gamma_a = \nabla_a \Lambda, \tag{5}$$

then the supercovariant derivative of the scalar superfields X and $X^\dagger$ indeed transforms under gauge transformations like the original fields,

$$\delta\nabla_a X = i\Lambda\nabla_a X, \qquad \delta\nabla_a X^\dagger = -i\nabla_a X^\dagger \Lambda. \tag{6}$$

Now we can derive certain properties of these supercovariant derivatives. The Abelian version of these properties is given in [20]. Now define the components of this superfield Γ_a to be

$$\chi_a = [\Gamma_a]_|, \qquad A = -\tfrac{1}{2}[\nabla^a \Gamma_a]_|,$$
$$A^\mu = -\tfrac{1}{2}[\nabla^a (\gamma^\mu)_a^b \Gamma_b]_|, \qquad E_a = \tfrac{1}{2}[\nabla^b \nabla_a \Gamma_b]_|, \tag{7}$$

where "|" means that the quantity is evaluated at $\theta_a = 0$, and let $\mathcal{D}_\mu$ be the conventional covariant derivative given by

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu. \tag{8}$$

Then it can be shown by direct computation that the supercovariant derivative satisfies

$$\{\nabla_a, \nabla_b\} = -2\nabla_{ab}, \tag{9}$$

where

$$\nabla_{ab} = \partial_{ab} - i\Gamma_{ab}, \qquad \Gamma_{ab} = -\frac{i}{2}[D_{(a}\Gamma_{b)} - i\{\Gamma_a, \Gamma_b\}], \tag{10}$$

and $\partial_{ab} = (\gamma^\mu \partial_\mu)_{ab}$. Now as we are studying $\mathcal{N} = 1$ superfields in three dimensions the indices "a" are two-dimensional and so $[\nabla_a, \nabla_b]$ must be proportional to the antisymmetric tensor C_{ab}. Thus, we find

$$\nabla_a \nabla_b = \tfrac{1}{2}\{\nabla_a, \nabla_b\} + \tfrac{1}{2}[\nabla_a, \nabla_b] = \gamma^\mu_{ab}\mathcal{D}_\mu - C_{ab}\nabla^2. \tag{11}$$

The complete antisymmetrization of three two-dimensional indices vanishes and so we have

$$\nabla_a \nabla_b \nabla_c = \tfrac{1}{2}\nabla_a\{\nabla_b, \nabla_c\} - \tfrac{1}{2}\nabla_b\{\nabla_a, \nabla_c\} + \tfrac{1}{2}\nabla_c\{\nabla_a, \nabla_b\}. \tag{12}$$

Thus, we get

$$\nabla^a \nabla_b \nabla_a = 0, \tag{13}$$

$$\nabla^2 \nabla_a = (\gamma^\mu \nabla)_a \mathcal{D}_\mu. \tag{14}$$

If we put a boundary at fixed x^3, then μ splits into $\mu = (m, 3)$. The induced value of the superderivative D_a and the supercovariant derivative ∇_a on the boundary is denoted by D'_a and ∇'_a, respectively. This boundary superderivative D'_a is obtained by neglecting $\gamma^3\partial_3$ contributions in D_a,

$$D'_a = \partial_a + (\gamma^m \partial_m)_a^b \theta_b. \tag{15}$$

The boundary supercovariant derivative ∇'_a can thus be written as

$$\nabla'_a X' = D'_a X' - i\Gamma'_a X', \qquad \nabla'_a X^{\dagger\prime} = D'_a X^{\dagger\prime} + iX^{\dagger\prime}\Gamma'_a, \tag{16}$$

where X', $X^{\dagger\prime}$, and Γ'_a are the induced values of the bulk fields X, $X^\dagger$, and Γ_a on the boundary. Any boundary field along with the induced value of any quantity, e.g., Λ, on the boundary will be denoted by Λ'. This convention will be followed even for component fields of superfields. The matrix valued spinor superfield Γ'_a transforms under gauge transformations as follows:

$$\delta\Gamma'_a = \nabla'_a \Lambda', \tag{17}$$

where Λ' is the induced value of Λ on the boundary.

Now we define projection operators $P_\pm$ as

$$(P_\pm)_a^b = \tfrac{1}{2}(\delta_a^b \pm (\gamma^3)_a^b). \tag{18}$$

These projection operators can be used to project the supercovariant derivative ∇_a as

$$\nabla_{\pm b} = (P_\pm)_b^a \nabla_a, \tag{19}$$

and $\nabla'_{\pm b}$ as

$$\nabla'_{\pm b} = (P_\pm)_b^a \nabla'_a, \tag{20}$$

where $\nabla'_{\pm a}$ is the induced value of $\nabla_{\pm a}$ on the boundary. These projected values of the supercovariant derivative can now be shown to satisfy

$$\nabla_{+a}\nabla_{+b} = -(P_+\gamma^m)_{ab}\mathcal{D}_m, \tag{21}$$

$$\nabla_{-a}\nabla_{-b} = -(P_-\gamma^m)_{ab}\mathcal{D}_m, \tag{22}$$

$$\nabla_{-a}\nabla_{+b} = -(P_-)_{ab}(\mathcal{D}_3 + \nabla^2), \tag{23}$$

$$\nabla_{+a}\nabla_{-b} = (P_+)_{ab}(\mathcal{D}_3 - \nabla^2). \tag{24}$$

From these relations we can obtain the following algebra for these projected operators:

$$\{\nabla_{+a}, \nabla_{+b}\} = -2(P_+\gamma^m)_{ab}\mathcal{D}_m, \tag{25}$$

$$\{\nabla_{-a}, \nabla_{-b}\} = -2(P_-\gamma^m)_{ab}\mathcal{D}_m, \tag{26}$$

$$\{\nabla_{-a}, \nabla_{+b}\} = -2(P_-)_{ab}\mathcal{D}_3. \tag{27}$$

It will be useful to write Eq. (24) as

$$-\nabla_+\nabla_- = -C^{ab}\nabla_{+a}\nabla_{-b} = -C^{ab}(P_+)_{ab}(\mathcal{D}_3 - \nabla^2)$$
$$= -(P_+)_a^a(\mathcal{D}_3 - \nabla^2) = (\mathcal{D}_3 - \nabla^2). \tag{28}$$

Note that it is also easy to see that the boundary superderivatives satisfy similar relations, and that the supersymmetry splits into left- and right-moving sectors on the boundary since, e.g.,

$$(P_\pm \gamma^m)_{ab} \mathcal{D}_m = (\gamma^\pm)_{ab} \mathcal{D}_\pm, \tag{29}$$

where $\gamma^\pm = \gamma^0 \pm \gamma^1$ and $\mathcal{D}_\pm = \frac{1}{2}(\mathcal{D}_0 \pm \mathcal{D}_1)$.

We have now reviewed properties of supercovariant derivatives and extended results in [20] to non-Abelian theories. In the next section we will use these results to analyze non-Abelian Chern-Simons theory in the presence of a boundary.

III. $\mathcal{N} = 1$ CHERN-SIMONS THEORY

Before we consider a boundary we will review $\mathcal{N} = 1$ non-Abelian Chern-Simons theory on a manifold without a boundary. Now the Lagrangian for $\mathcal{N} = 1$ non-Abelian Chern-Simons theory in superspace formalism can be written (with implicit trace) as [43]

$$\mathcal{L}_{\mathrm{CS},k}(\Gamma) = -\frac{k}{4\pi} \nabla^2 [\Gamma^a \Omega_a]_|, \tag{30}$$

where [43]

$$\Omega_a = \omega_a - \tfrac{1}{6}[\Gamma^b, \Gamma_{ab}], \tag{31}$$

$$\omega_a = \frac{1}{2} D^b D_a \Gamma_b - \frac{i}{2}[\Gamma^b, D_b \Gamma_a] - \frac{1}{6}[\Gamma^b, \{\Gamma_b, \Gamma_a\}], \tag{32}$$

$$\Gamma_{ab} = -\frac{i}{2}[D_{(a}\Gamma_{b)} - i\{\Gamma_a, \Gamma_b\}]. \tag{33}$$

In Eq. (30) a trace over the generators of the Lie algebra is implied. The covariant divergence of ω_a vanishes [11]:

$$\nabla^a \omega_a = 0. \tag{34}$$

The components of the superfield ω_a can now be calculated from Eqs. (7) and (32),

$$[\nabla^a (\gamma^\mu)_a^b \omega_b]_| = \epsilon^{\mu\nu\rho} F_{\nu\rho}, \qquad [\nabla^a \omega_a]_| = 0,$$
$$-[\nabla^b \nabla_a \omega_b]_| = 2(\gamma^\mu \mathcal{D}_\mu)_a^b E_b, \qquad [\omega_a]_| = E_a, \tag{35}$$

where $\epsilon_{\mu\nu\rho}$ is an antisymmetric tensor. So the component form for the Lagrangian for $\mathcal{N} = 1$ non-Abelian Chern-Simons theory can be written as

$$\mathcal{L}_{\mathrm{CS},k} = \frac{k}{4\pi}\Big[\epsilon^{\mu\nu\rho}\Big(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\Big) + E^a E_a + \mathcal{D}_\mu(\chi^a (\gamma^\mu)_a^b E_b)\Big]. \tag{36}$$

Now if the full finite gauge transformation of the superfield Γ_a is written as

$$\Gamma_a \to i u \nabla_a u^{-1}, \tag{37}$$

where

$$u = \exp(i \Lambda^A T_A), \tag{38}$$

then the gauge transformation of the superfield ω_a will be given by

$$\omega_a \to u \omega_a u^{-1}. \tag{39}$$

Under infinitesimal gauge transformations the Lagrangian for the $\mathcal{N} = 1$ non-Abelian Chern-Simons theory transforms as

$$\delta \mathcal{L}_{\mathrm{CS},k}(\Gamma) = -\frac{k}{4\pi} \nabla^2 [(\nabla^a \Lambda) \omega_a]_|. \tag{40}$$

Now using Eq. (34), we get

$$\delta \mathcal{L}_{\mathrm{CS},k}(\Gamma) = -\frac{k}{4\pi} \nabla^2 \nabla^a [\Lambda \omega_a]_| = -\frac{k}{4\pi} (\gamma^\mu \mathcal{D}_\mu \nabla)^a [\Lambda \omega_a]_|. \tag{41}$$

As this is a total derivative, on a manifold without a boundary we have

$$\delta \mathcal{L}_{\mathrm{CS},k} = 0. \tag{42}$$

Thus, the $\mathcal{N} = 1$ non-Abelian Chern-Simons theory is invariant under these gauge transformations on a manifold without a boundary.

After reviewing the gauge invariance of the $\mathcal{N} = 1$ non-Abelian Chern-Simons theory on a manifold without a boundary, we can now discuss the effect of a boundary on it. The effect of a boundary in three dimensions on the SUSY of $\mathcal{N} = 1$ theories, and, in particular, how SUSY can be preserved by adding additional boundary terms, has been recently studied in [20]. The supersymmetric variation of the Lagrangian for $\mathcal{N} = 1$ non-Abelian Chern-Simons theory transforms into a total derivative, so in the absence of a boundary this variation vanishes and the theory is supersymmetric. However, in the presence of a boundary it reduces to a boundary term. This theory can still be made supersymmetric by adding a boundary term whose supersymmetric variation cancels the supersymmetric variation of the original action. The analysis performed for Abelian Chern-Simons theories in [20,21] can be easily generalized to the non-Abelian case for $\mathcal{N} = 1$ SUSY, with the result that the boundary term whose addition will make $\mathcal{N} = 1$ non-Abelian Chern-Simons theory supersymmetric can be written as

$$\mathcal{L}_{b\mathrm{CS},k}(\Gamma) = \frac{k}{4\pi} \mathcal{D}_3 [\Gamma^a \Omega_a]_|. \tag{43}$$

In component form this term can be written as

$$\mathcal{L}_{b\mathrm{CS},k} = \frac{k}{4\pi} \mathcal{D}_3 \Big[\chi^a E_a + \frac{i}{6} \chi^a [(\gamma^\mu A_\mu)_a^b, \chi_b]\Big]. \tag{44}$$

The supersymmetric variation of this boundary term exactly cancels the supersymmetric variation of the bulk Lagrangian, so the sum of the bulk Lagrangian and this boundary term is supersymmetric,

$$\mathcal{L}_{s\mathrm{CS},k}(\Gamma) = \mathcal{L}_{\mathrm{CS},k} + \mathcal{L}_{b\mathrm{CS},k} = \frac{k}{4\pi}(-\nabla^2 + \mathcal{D}_3)[\Gamma^a \Omega_a]_|. \tag{45}$$

It may be noted that only half of the SUSY of the original theory is preserved on the boundary. In this paper we will keep the SUSY corresponding to ∇_- and break the SUSY corresponding to ∇_+ on the boundary.

This supersymmetric Lagrangian with a boundary term is not gauge invariant because following what we did for the $\mathcal{N} = 1$ non-Abelian Chern-Simons theory on a manifold without boundary, the infinitesimal gauge transformation of this Lagrangian is given by

$$\delta \mathcal{L}_{sCS,k}(\Gamma) = \frac{k}{4\pi}(\mathcal{D}_3 - \nabla^2)\nabla^a[\Lambda\omega_a]_|. \tag{46}$$

Now using Eq. (14), this can be written as

$$\delta \mathcal{L}_{sCS,k}(\Gamma) = \frac{k}{4\pi}(\mathcal{D}_3\nabla^a - (\gamma^\mu \mathcal{D}_\mu \nabla)^a)[\Lambda\omega_a]_|. \tag{47}$$

As there is a boundary in the x^3 direction, we get

$$\begin{aligned}\delta \mathcal{L}_{sCS,k}(\Gamma) &= \frac{k}{4\pi}(\mathcal{D}_3\nabla^a - (\gamma^\mu \mathcal{D}_\mu \nabla)^a)[\Lambda\omega_a]_| \\ &\sim \frac{k}{4\pi}(\mathcal{D}_3\nabla^a - (\gamma^3 \mathcal{D}_3 \nabla)^a)[\Lambda\omega_a]_|,\end{aligned} \tag{48}$$

where $\sim$ indicates that we have neglected the total derivative contribution along directions other than x^3, as they will not contribute. Thus, the gauge transformation of this supersymmetric Lagrangian gives a boundary term,

$$\begin{aligned}\delta \mathcal{L}'_{sCS,k}(\Gamma') &= \frac{k}{4\pi}(\delta^a_b - (\gamma^3)^a_b)\nabla'^b[\Lambda'\omega'_a]_| \\ &= \frac{k}{2\pi}(P_-\nabla')^a[\Lambda'\omega'_a]_|.\end{aligned} \tag{49}$$

This boundary term can be written in component form as

$$\delta \mathcal{L}'_{sCS,k} = \frac{k}{2\pi}(\epsilon^{\mu\nu}\lambda' F'_{\mu\nu} + (\lambda^{a\prime}(\gamma^3)^b_a E'_b) + (\lambda^{a\prime} E'_a)), \tag{50}$$

where $\lambda = [\Lambda]_|$, $\lambda_a = [\nabla_a \Lambda]_|$, and the "prime" notations λ', λ'_a, A'_μ, etc., denote the induced values of these fields on the boundary. Because of the presence of this boundary term, the $\mathcal{N} = 1$ non-Abelian Chern-Simons theory is not gauge invariant in the presence of a boundary.

However, it is possible to couple this theory to another boundary theory, such that the total Lagrangian, which is given by the sum of the Lagrangians of both these theories, is gauge invariant. To do so we consider a boundary theory with the following potential term:

$$\mathcal{L}_{pb,k}(v', \Gamma') = \mathcal{L}_{sCS,k}(\Gamma^v) - \mathcal{L}_{sCS,k}(\Gamma), \tag{51}$$

where v' is a boundary scalar superfield, v is an extension of v' into the bulk, and Γ^v denotes the gauge transformation of Γ by v. For v close to the identity, this is a genuine boundary term, while in general we can still consider this to only depend on the boundary in the sense that in the absence of a boundary this term will have no effect since the normalization of the Chern-Simons action is chosen so that the path integral is also invariant under large gauge transformations. See [22] for a more detailed discussion of the bosonic theory. Now the total Lagrangian $\mathcal{L}_{sCS,k}(\Gamma) + \mathcal{L}_{pb,k}(v', \Gamma')$ will clearly be gauge invariant if Γ^v is. This is possible if we require v to transform under gauge transformations as

$$v \to vu^{-1}. \tag{52}$$

To better understand this boundary Lagrangian, we can consider the case where $\Gamma^a = 0$ so that there is no coupling to the bulk fields. In this case the boundary term $\mathcal{L}_{sCS,k}(\Gamma^a = -i(\nabla^a v)v^{-1})$ gives the potential term of the $\mathcal{N} = (1, 0)$ WZW model [44,45]

$$\begin{aligned}\mathcal{L}_{pb,k}(v', \Gamma') = -\frac{k}{2\pi}(P_-\nabla')^a[[(v^{-1}\mathcal{D}_+ v), (v^{-1}\mathcal{D}_3 v)] \\ \times (v^{-1}\nabla_{-a} v)]_|.\end{aligned} \tag{53}$$

We can now add the following supersymmetric gauge invariant kinetic term for the boundary scalar superfield $\hat{v} = v'(\theta_+ = 0)$,

$$\mathcal{L}_{kb,k}(v', \Gamma') = -\frac{|k|}{2\pi}(P_-\nabla')^a[(\hat{v}^{-1}\nabla'_{-a}\hat{v})(\hat{v}^{-1}\mathcal{D}_+\hat{v})]_|, \tag{54}$$

which is a gauging of the kinetic term of the $\mathcal{N} = (1, 0)$ Wess-Zumino-Witten model [44,45]. The other components of v' do not appear in the final action, so there is no need to include their kinetic terms. Note also that we have defined the kinetic term to have the correct sign whether k is positive or negative. The Lagrangian for the boundary theory will now be given by a type of gauged $\mathcal{N} = (1, 0)$ WZW model

$$\mathcal{L}_{b,k}(v', \Gamma') = \mathcal{L}_{kb,k}(v', \Gamma') + \mathcal{L}_{pb,k}(v', \Gamma'), \tag{55}$$

and so the complete gauge and supersymmetry invariant action is given by

$$\mathcal{L}_{sgCS,k}(v', \Gamma) = \mathcal{L}_{sCS,k}(\Gamma) + \mathcal{L}_{b,k}(v', \Gamma'). \tag{56}$$

The component form of $\mathcal{L}_{sCS,k} + \mathcal{L}_{pb,k}$ is obtained by substituting

$$\begin{aligned}&A_\mu \to i\mu(\mathcal{D}_\mu \mu^{-1}), \qquad \chi_a \to \mu\chi_a\mu^{-1} - i\psi_a, \\ &E_a \to \mu E_a \mu^{-1},\end{aligned} \tag{57}$$

where we have defined the components of v to be

$$\mu = v_|, \qquad \psi_a = (D_a v)_| \mu^{-1}, \tag{58}$$

in the original supersymmetric boundary action,

$$\begin{aligned}\mathcal{L}_{sCS,k} = \frac{k}{4\pi}\bigg[&\epsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\Big) + E^a E_a \\ &+ \mathcal{D}_\mu(\chi^a(\gamma^\mu)^b_a E_b) + \mathcal{D}_3(\chi^a E_b) \\ &+ \frac{i}{6}\mathcal{D}_3(\chi^a[(\gamma^\mu A_\mu)^b_a, \chi_b])\bigg].\end{aligned} \tag{59}$$

So we can write

$$\mathcal{L}_{sCS,k} + \mathcal{L}_{pb,k} = \frac{k}{4\pi}\Big[-\epsilon^{\mu\nu\rho}(\mu^{-1}\mathcal{D}_{\mu}\mu)\partial_{\nu}(\mu^{-1}\mathcal{D}_{\rho}\mu) + \frac{2}{3}\epsilon^{\mu\nu\rho}(\mu^{-1}\mathcal{D}_{\mu}\mu)(\mu^{-1}\mathcal{D}_{\nu}\mu)(\mu^{-1}\mathcal{D}_{\rho}\mu) - i\mathcal{D}_{\mu}(\psi^{a}(\gamma^{\mu})_{a}^{b}\mu E_{b}\mu^{-1}) - i\mathcal{D}_{3}(\psi^{a}\mu E_{b}\mu^{-1}) + \mathcal{D}_{3}(\chi^{a}E_{b}) + \frac{i}{6}\mathcal{D}_{3}((\mu\chi^{a}\mu^{-1} - i\psi^{a}) \times [(\mu\gamma^{\mu}\mathcal{D}_{\mu}\mu^{-1})_{a}^{b}, (\mu\chi_{b}\mu^{-1} - i\psi_{b})]) + E^{a}E_{a} + \mathcal{D}_{\mu}(\chi^{a}(\gamma^{\mu})_{a}^{b}E_{b})\Big]. \tag{60}$$

The component form of $\mathcal{L}_{kb,k}$ is the kinetic term for the $\mathcal{N} = (1, 0)$ gauged WZW model,

$$\mathcal{L}_{kb,k} = -\frac{|k|}{2\pi}[-\mu'^{-1}\psi_{-}\mathcal{D}_{+}(\psi_{-}\mu') + (\mu'^{-1}\mathcal{D}_{-}\mu') \times (\mu'^{-1}D_{+}\mu')], \tag{61}$$

where $\psi_{-} = (D_{-}\hat{v})_{|}\mu'^{-1}$ is the single fermionic component of $\hat{v}$.

Thus, $\mathcal{N} = 1$ Chern-Simons theory in the presence of a boundary can be made both gauge and supersymmetry invariant by the addition of a suitable theory on the boundary such that its gauge and supersymmetry variations exactly cancel those of the Chern-Simons theory. Our result generalizes that of [20] which gave the boundary terms to restore supersymmetry but not gauge invariance for Chern-Simons theory, in the case of an Abelian gauge group. It may be remarked that it was already known that the bosonic Chern-Simons theory suitably coupled to a gauged Wess-Zumino-Witten theory on the boundary is gauge invariant [22], and we have now provided a superspace extension of that result, or equivalently a fully gauge invariant extension of the manifestly supersymmetric Chern-Simons with boundary theories considered in [21].

IV. ABJM THEORY

In the previous section we analyzed $\mathcal{N} = 1$ Chern-Simons theory in the presence of a boundary. In this section we shall use the results of the previous section to analyze the ABJM theory in the presence of a boundary. The ABJM theory in the presence of a boundary, in $\mathcal{N} = 1$ superspace formalism, can be formulated as a supersymmetric gauge theory with the gauge group $U(N)_k \times U(N)_{-k}$ and the superfield Lagrangian

$$\mathcal{L}_{\mathrm{ABJM},k} = \mathcal{L}_{\mathrm{CS},k}(\Gamma) + \mathcal{L}_{\mathrm{CS},-k}(\tilde{\Gamma}) + \mathcal{L}_{M,k}, \tag{62}$$

where $\mathcal{L}_{\mathrm{CS},k}$ and $\mathcal{L}_{\mathrm{CS},-k}$ are Chern-Simons theories as discussed in the previous section, and the matter part of the Lagrangian $\mathcal{L}_{M,k}$ is given by

$$\mathcal{L}_{M,k} = \mathcal{L}_{kM} + \mathcal{L}_{pM,k}, \tag{63}$$

where $\mathcal{L}_{pM,k}$ is the potential term given by

$$\mathcal{L}_{pM,k} = -\frac{2\pi}{k}\nabla^2[\epsilon_{IJ}\epsilon^{KL}X^{I}Y_{K}X^{J}Y_{L} + \epsilon^{IJ}\epsilon_{KL}X_{I}^{\dagger}Y^{K\dagger}X^{J\dagger}Y_{L}^{\dagger}]_{|}, \tag{64}$$

and $\mathcal{L}_{kM}$ is the kinetic term given by

$$\mathcal{L}_{kM} = -\tfrac{1}{4}\nabla^2[\nabla^{a}X^{I}\nabla_{a}X_{I}^{\dagger} + \nabla^{a}Y^{I}\nabla_{a}Y_{I}^{\dagger}]_{|}. \tag{65}$$

Here the supercovariant derivatives for the matter fields are given by

$$\begin{aligned}\nabla_{a}X^{I} &= D_{a}X^{I} + i\Gamma_{a}X^{I} - iX^{I}\tilde{\Gamma}_{a},\\ \nabla_{a}Y^{I\dagger} &= D_{a}Y^{I\dagger} + i\Gamma_{a}Y^{I\dagger} - iY^{I\dagger}\tilde{\Gamma}_{a},\\ \nabla_{a}X^{I\dagger} &= D_{a}X^{I\dagger} - iX^{I\dagger}\Gamma_{a} + i\tilde{\Gamma}_{a}X^{I\dagger},\\ \nabla_{a}Y^{I} &= D_{a}Y^{I} - iY^{I}\Gamma_{a} + i\tilde{\Gamma}_{a}Y^{I}.\end{aligned} \tag{66}$$

The full finite gauge transformation under which the ABJM theory, without a boundary, is invariant is given by

$$\begin{aligned}&\Gamma_{a} \to iu\nabla_{a}u^{-1}, \qquad \tilde{\Gamma}_{a} \to i\tilde{u}\nabla_{a}\tilde{u}^{-1}, \qquad X^{I} \to uX^{I}\tilde{u}^{-1},\\ &X^{I\dagger} \to \tilde{u}X^{I\dagger}u^{-1}, \qquad Y^{I} \to \tilde{u}Y^{I}u^{-1}, \qquad Y^{I} \to uY^{I\dagger}\tilde{u}^{-1},\end{aligned} \tag{67}$$

where

$$u = \exp(i\Lambda^{A}T_{A}), \qquad \tilde{u} = \exp(i\tilde{\Lambda}^{A}T_{A}). \tag{68}$$

The infinitesimal gauge transformations of these fields are given by

$$\begin{aligned}&\delta\Gamma_{a} = \nabla_{a}\Lambda, \qquad \delta\tilde{\Gamma}_{a} = \tilde{\nabla}_{a}\tilde{\Lambda},\\ &\delta X^{I} = i(\Lambda X^{I} - X^{I}\tilde{\Lambda}), \qquad \delta X^{I\dagger} = i(\tilde{\Lambda}X^{I\dagger} - X^{I\dagger}\Lambda),\\ &\delta Y^{I} = i(\tilde{\Lambda}Y^{I} - Y^{I}\Lambda), \qquad \delta Y^{I\dagger} = i(\Lambda Y^{I\dagger} - Y^{I\dagger}\tilde{\Lambda}).\end{aligned} \tag{69}$$

We can now discuss the ABJM theory in the presence of a boundary. We can use the analysis in the previous section and a generalization of the work done in [20–22] to analyze the ABJM theory in the presence of a boundary. The supersymmetric variation of the ABJM Lagrangian on a manifold with a boundary is a boundary term. Thus, to retain the SUSY of the theory a suitable boundary piece has to be added in such a way that it cancels the boundary term generated by the supersymmetric variation of the original theory. The sum of this boundary term and the original ABJM Lagrangian can now be written as

$$\mathcal{L}_{s\mathrm{ABJM},k} = \mathcal{L}_{s\mathrm{CS},k}(\Gamma) + \mathcal{L}_{s\mathrm{CS},-k}(\tilde{\Gamma}) + \mathcal{L}_{sM,k}, \tag{70}$$

where $\mathcal{L}_{s\mathrm{CS},k}$ and $\mathcal{L}_{s\mathrm{CS},-k}$ are the Chern-Simons theories on a manifold with a boundary defined in the previous section. The matter part of the Lagrangian $\mathcal{L}_{sM,k}$ is given by

$$\mathcal{L}_{sM,k} = \mathcal{L}_{skM} + \mathcal{L}_{spM,k}, \tag{71}$$

where $\mathcal{L}_{spM,k}$ is the potential term given by

$$\mathcal{L}_{spM,k} = \frac{2\pi}{k}(-\nabla^2 + \mathcal{D}_3)[\epsilon_{IJ}\epsilon^{KL}X^I Y_K X^J Y_L + \epsilon^{IJ}\epsilon_{KL}X_I^\dagger Y^{K\dagger}X^{J\dagger}Y_L^\dagger]_|, \tag{72}$$

and the kinetic term $\mathcal{L}_{skM}$ is now given by

$$\mathcal{L}_{skM} = \tfrac{1}{4}(-\nabla^2 + \mathcal{D}_3)[\nabla^a X^I \nabla_a X_I^\dagger + \nabla^a Y^I \nabla_a Y_I^\dagger]_|. \tag{73}$$

The matter part of the ABJM theory is still invariant under the gauge transformations given by Eq. (69). However, the Chern-Simons part is not invariant under these gauge transformations. Thus, the total Lagrangian for the ABJM theory is not invariant under the gauge transformations given by Eq. (69). However, this is exactly the issue we tackled in the previous section, so we know that we can add a boundary action to modify the ABJM action. The result is the supersymmetric and gauge invariant action:

$$\mathcal{L}_{sg\mathrm{ABJM},k} = \mathcal{L}_{sg\mathrm{CS},k}(v', \Gamma) + \mathcal{L}_{sg\mathrm{CS},-k}(\tilde{v}', \tilde{\Gamma}) + \mathcal{L}_{sM,k}. \tag{74}$$

Furthermore, v' and $\tilde{v}'$ can be extended in to the bulk to produce fields v and $\tilde{v}$ whose finite gauge transformations are given by

$$v \to vu^{-1}, \qquad \tilde{v} \to \tilde{v}\tilde{u}^{-1}. \tag{75}$$

Thus, by introducing new degrees of freedom on the boundary, we have found a superspace description of the boundary ABJM theory which is also gauge invariant. It would be interesting to generalize this to extended superspace[1] so that more supersymmetry was manifest, and to investigate in detail how much supersymmetry is preserved by this theory or similar supersymmetric Chern-Simons theories with matter in the presence of a boundary. Other than some technical complications, it should be possible to extend this analysis to $\mathcal{N} = 2$ superspace, and indeed when the $\mathcal{N} = 2$ Chern-Simons action was derived, its similarity to the $\mathcal{N} = 2$ WZW action was noted [46]. However, an interesting question is whether the full supersymmetry will give further constraints on the boundary action, as we seemingly have the freedom to add any additional supersymmetric gauge invariant boundary terms. One obvious question is whether the boundary theory relates the two $SU(N)$ factors such as through a coupling which preserves the diagonal subgroup. Some such feature may be expected as the $\mathcal{N} = 6$ bulk ABJM action required the specific $SU(N) \times SU(N)$ form of the gauge group, but this is not required by less supersymmetric theories. Going beyond manifest $\mathcal{N} = 2$ supersymmetry is even more difficult, but the ABJM action has been formulated in $\mathcal{N} = 3$ harmonic superspace [2]. Alternatively it may be possible to proceed without an off-shell superspace action using the ectoplasm formalism [47,48], as recently explored for systems with a boundary [49].

[1]The supersymmetric but not gauge invariant case for $\mathcal{N} = 2$ supersymmetry can be found in [21].

V. BRST AND ANTI-BRST SYMMETRIES

In this section we will study the BRST and anti-BRST symmetries of the theory discussed in the previous section. As the sum of the boundary theory and ABJM theory is invariant under gauge transformations, it contains unphysical degrees of freedom. These unphysical degrees of freedom will give rise to constraints in canonical quantization and divergences in the partition function in the path integral quantization. So before we can quantize this theory we will need to eliminate these unphysical degrees of freedom by the addition of a suitable gauge-fixing term and a suitable ghost term to it. The new effective Lagrangian that is obtained by taking the sum of the original classical Lagrangian, the gauge-fixing term, and the ghost term will be invariant under two new sets of transformations called the BRST transformation and the anti-BRST transformation.

In order to write a suitable gauge-fixing term and a suitable ghost for the ABJM theory, we denote the auxiliary superfields by B, $\tilde{B}$. We also denote the ghosts by C, $\tilde{C}$ and the antighosts by $\bar{C}$, $\tilde{\bar{C}}$. It may be noted that whereas the auxiliary fields are regular matrix valued scalar superfields, the ghosts and the antighosts are matrix valued anticommuting superfields. All these superfields are suitably contracted with generators of the Lie algebra in the adjoint representation

$$\begin{aligned} B &= B^A T_A, & \tilde{B} &= \tilde{B}^A T_A, & \tilde{C} &= \tilde{C}^A T_A, \\ C &= C^A T_A, & \bar{C} &= \bar{C}^A T_A. & \tilde{\bar{C}} &= \tilde{\bar{C}}^A T_A. \end{aligned} \tag{76}$$

Now we can write the gauge-fixing term $\mathcal{L}_{\mathrm{gf}}$ and the ghost term $\mathcal{L}_{\mathrm{gh}}$ for the ABJM theory corresponding to the gauge-fixing function [42],

$$D^a \Gamma_a = 0, \qquad D^a \tilde{\Gamma}_a = 0, \tag{77}$$

as follows:

$$\mathcal{L}_{\mathrm{gf}} = -\nabla_+\nabla_-[BD^a\Gamma_a]_| + -\tilde{\nabla}_+\tilde{\nabla}_-[\tilde{B}D^a\tilde{\Gamma}_a]_|, \tag{78}$$

$$\mathcal{L}_{\mathrm{gh}} = -\nabla_+\nabla_-[\bar{C}D^a\nabla_a C]_| + \tilde{\nabla}_+\tilde{\nabla}_-[\tilde{\bar{C}}D^a\tilde{\nabla}_a\tilde{C}]_|. \tag{79}$$

We now define an effective Lagrangian $\mathcal{L}_{\mathrm{eff},k}$ as the sum of the supersymmetric and gauge invariant ABJM Lagrangian, the gauge-fixing term, and the ghost term,

$$\mathcal{L}_{\mathrm{eff},k} = \mathcal{L}_{sg\mathrm{ABJM},k} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}}. \tag{80}$$

The BRST transformations of the matter fields can be written as

$$sX^I = i(CX^I - X^I\tilde{C}), \qquad sX^{I\dagger} = i(\tilde{C}X^{I\dagger} - X^{I\dagger}C),$$
$$sY^I = i(\tilde{C}Y^I - Y^I C), \qquad sY^{I\dagger} = i(CY^{I\dagger} - Y^{I\dagger}\tilde{C}). \tag{81}$$

The BRST transformations of the auxiliary superfields, ghosts, and antighosts can be written as

$$s\Gamma_a = \nabla_a C, \qquad s\tilde{\Gamma}_a = \tilde{\nabla}_a\tilde{C}, \qquad sC = -\tfrac{1}{2}\{C, C\},$$
$$s\tilde{C} = -\tfrac{1}{2}\{\tilde{C}, \tilde{C}\} \qquad s\bar{C} = B, \qquad s\bar{\tilde{C}} = \tilde{B},$$
$$sB = 0, \qquad s\tilde{B} = 0. \tag{82}$$

The BRST transformation of the v and $\tilde{v}$ can be written as

$$sv = -ivC, \qquad s\tilde{v} = -i\tilde{v}\,\tilde{C}. \tag{83}$$

These BRST transformations are nilpotent and thus satisfy $s^2 = 0$. This fact can be used to show that the sum of the gauge-fixing term $\mathcal{L}_{\mathrm{gf}}$ and the ghost term $\mathcal{L}_{\mathrm{gh}}$ is invariant under BRST transformations. It is because the sum of the ghost term and the gauge-fixing term can be written as

$$\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} = -\nabla_+\nabla_- s[\bar{C}D^a\Gamma_a]_| + \tilde{\nabla}_+\tilde{\nabla}_- s[\bar{\tilde{C}}D^a\tilde{\Gamma}_a]_|. \tag{84}$$

Now using the fact that BRST transformations are nilpotent, we get

$$\begin{aligned} s\mathcal{L}_{\mathrm{gf}} + s\mathcal{L}_{\mathrm{gh}} &= -\nabla_+\nabla_- s^2[\bar{C}D^a\Gamma_a]_| \\ &\quad + \tilde{\nabla}_+\tilde{\nabla}_- s^2[\bar{\tilde{C}}D^a\tilde{\Gamma}_a]_| \\ &= 0. \end{aligned} \tag{85}$$

The Lagrangian $\mathcal{L}_{s\mathrm{ABJM},k}$ is not invariant under these BRST transformations as it generates a boundary term which is given by

$$s\mathcal{L}_{s\mathrm{ABJM},k} = \frac{k}{2\pi}(P_-\nabla')^a[C'\omega'_a]_| - \frac{k}{2\pi}(P_-\tilde{\nabla}')^a[\tilde{C}'\tilde{\omega}'_a]_|. \tag{86}$$

Here C' and $\tilde{C}'$ are the induced values of C and $\tilde{C}$ on the boundary. However, this boundary term is exactly canceled by the BRST variation of boundary theory. Thus, the sum of the bulk and the boundary theory is invariant under these BRST transformations, and so we have $s\mathcal{L}_{sg\mathrm{ABJM},k} = 0$. Thus, the effective Lagrangian $\mathcal{L}_{\mathrm{eff},k}$ is invariant under BRST transformations,

$$s\mathcal{L}_{\mathrm{eff},k} = s\mathcal{L}_{sg\mathrm{ABJM},k} + s\mathcal{L}_{\mathrm{gf}} + s\mathcal{L}_{\mathrm{gh}} = 0. \tag{87}$$

We can perform a similar analysis using the anti-BRST transformations. The anti-BRST transformations of the matter fields can be written as

$$\bar{s}X^I = i(\bar{C}X^I - X^I\bar{\tilde{C}}), \qquad \bar{s}X^{I\dagger} = i(\bar{\tilde{C}}X^{I\dagger} - X^{I\dagger}\bar{C}),$$
$$\bar{s}Y^I = i(\bar{\tilde{C}}Y^I - Y^I\bar{C}), \qquad \bar{s}Y^{I\dagger} = i(\bar{C}Y^{I\dagger} - Y^{I\dagger}\bar{\tilde{C}}). \tag{88}$$

The anti-BRST transformations of the auxiliary superfields, ghosts, and antighosts can be written as

$$\bar{s}\Gamma_a = \nabla_a\bar{C}, \qquad \bar{s}\tilde{\Gamma}_a = \tilde{\nabla}_a\bar{\tilde{C}}, \qquad \bar{s}C = -B - \{\bar{C}, C\},$$
$$\bar{s}\,\tilde{C} = -\tilde{B} - \{\bar{\tilde{C}}, \tilde{C}\}, \qquad \bar{s}\,\bar{C} = -\tfrac{1}{2}\{\bar{C}, \bar{C}\},$$
$$\bar{s}\,\bar{\tilde{C}} = -\tfrac{1}{2}\{\bar{\tilde{C}}, \bar{\tilde{C}}\}, \qquad \bar{s}B = \tfrac{1}{2}[B, \bar{C}] \qquad \bar{s}\,\tilde{B} = \tfrac{1}{2}[B, \bar{\tilde{C}}]. \tag{89}$$

The BRST transformation of the v and $\tilde{v}$ fields can be written as

$$\bar{s}v = -iv\bar{C}, \qquad \bar{s}\,\tilde{v} = -i\tilde{v}\,\bar{\tilde{C}}. \tag{90}$$

The anti-BRST transformations also are nilpotent and thus satisfy $\bar{s}^2 = 0$. Furthermore, the sum of the ghost and gauge-fixing terms can also be written as

$$\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} = -\nabla_+\nabla_-\bar{s}[\bar{C}D^a\Gamma_a]_| + \tilde{\nabla}_+\tilde{\nabla}_-\bar{s}[\bar{\tilde{C}}D^a\tilde{\Gamma}_a]_|. \tag{91}$$

Thus, using the fact that anti-BRST transformations are nilpotent, we get

$$\begin{aligned} \bar{s}\mathcal{L}_{\mathrm{gf}} + \bar{s}\mathcal{L}_{\mathrm{gh}} &= -\nabla_+\nabla_-\bar{s}^2[\bar{C}D^a\Gamma_a]_| \\ &\quad + \tilde{\nabla}_+\tilde{\nabla}_-\bar{s}^2[\bar{\tilde{C}}D^a\tilde{\Gamma}_a]_| \\ &= 0. \end{aligned} \tag{92}$$

Here again the Lagrangian $\mathcal{L}_{s\mathrm{ABJM},k}$ is not invariant under these anti-BRST transformations and it generates a boundary term which is given by

$$\bar{s}\mathcal{L}_{s\mathrm{ABJM},k} = \frac{k}{2\pi}(P_-\nabla')^a[\bar{C}'\omega'_a]_| - \frac{k}{2\pi}(P_-\tilde{\nabla}')^a[\bar{\tilde{C}}'\tilde{\omega}'_a]_|. \tag{93}$$

Here $\bar{C}'$ and $\bar{\tilde{C}}'$ are the induced values of $\bar{C}$ and $\bar{\tilde{C}}$ on the boundary. This term is again canceled by the anti-BRST variation of the boundary theory and so we have $\bar{s}\mathcal{L}_{sg\mathrm{ABJM},k} = 0$. Thus, the effective Lagrangian $\mathcal{L}_{\mathrm{eff},k}$ is also invariant under these anti-BRST transformations,

$$\bar{s}\mathcal{L}_{\mathrm{eff},k} = \bar{s}\mathcal{L}_{sg\mathrm{ABJM},k} + \bar{s}\mathcal{L}_{\mathrm{gf}} + \bar{s}\mathcal{L}_{\mathrm{gh}} = 0. \tag{94}$$

So the effective Lagrangian for the supersymmetric and gauge invariant ABJM is invariant under both the BRST and the anti-BRST transformations.

VI. CONCLUSION

In this paper we analyzed the $\mathcal{N} = 1$ Chern-Simons theory in the presence of a boundary. We used the results thus obtained to study the ABJM theory in the presence of a boundary. We first modified the Chern-Simons theory by adding a boundary term to it such that supersymmetry variations of the bulk Chern-Simons theory were canceled by the supersymmetry variations of this boundary term.

The resultant theory was then made gauge invariant by adding new boundary degrees of freedom to it. This new boundary theory was identified as a gauged Wess-Zumino-Witten model. These results were used to obtain a superspace description of the boundary ABJM theory which was also gauge invariant. As the matter part of the ABJM theory is gauge invariant even with a boundary, it was only necessary to include a boundary term to restore SUSY. The Chern-Simons part of the ABJM was modified by both the addition of a term to make it supersymmetric and new boundary degrees of freedom to make it gauge invariant. Thus, we added a suitable theory on the boundary such that its gauge and supersymmetry variations exactly cancel those of the bulk ABJM theory. We also analyzed the BRST and the anti-BRST symmetries of this resultant theory.

Chern-Simons theories are also important in condensed matter physics due to their relevance to the fractional quantum Hall effect [50–53]. The fractional quantum Hall effect is based on the concept of statistical transmutation, i.e., the fact that, in two dimensions, fermions can be described as charged bosons carrying an odd integer number of flux quanta which is achieved by analyzing Chern-Simons fields coupled to the bosons. In this theory electrons in an external magnetic field are described as bosons in a combined external and statistical magnetic field. At special values of the filling fraction the statistical field cancels the external field, in the mean field sense, and the system is described as a gas of bosons feeling no net magnetic field. These bosons condense into a homogeneous ground state. This model describes the quantization of the Hall conductance and the existence of vortex and antivortex excitations. Lately supersymmetric generalization of the fractional quantum Hall effect has also been investigated [54–57]. In particular physical properties of the topological excitations in the supersymmetric quantum Hall liquid have been discussed in a dual supersymmetric Chern-Simons theory [58]. Boundary effects for Chern-Simons theories are also important in condensed matter physics. This is because in quantum Hall systems gapless edge modes exist [59]. These have important consequences for the transport properties of the system [60]. These modes have been studied in the presence of an infinitely steep external confining potential [61,62]. The description of these modes has also been related to the chiral Luttinger liquid description of the edge excitations [63]. Thus, the results of this paper will be useful in analyzing the supersymmetric generalization of gapless edge modes of fractional quantum Hall systems. This can have important consequences for the transport properties of the fractional quantum hall system.

ACKNOWLEDGMENTS

D. J. S. is supported in part by the STFC Consolidated Grant No. ST/J000426/1.

APPENDIX: COMPONENT BRST TRANSFORMATIONS

In this appendix we will first study the gauge transformations of the ABJM theory and the boundary theory in the component form. We will then analyze the BRST and anti-BRST transformations of these theories in the component form. To do so we write ghosts, antighosts, and the auxiliary fields in component form as

$$
\begin{aligned}
&c = [C]_|, \quad \bar{c} = [\bar{C}]_|, \quad b = [B]_|, \quad c_a = [\nabla_a C]_|,\\
&\bar{c}_a = [\nabla_a \bar{C}]_|, \quad b_a = [\nabla_a B]_|, \quad \mathrm{c} = [\nabla^2 C]_|,\\
&\bar{\mathrm{c}} = [\nabla^2 \bar{C}]_|, \quad \mathrm{b} = [\nabla^2 B]_|, \quad \tilde{c} = [\tilde{C}]_|,\\
&\tilde{\bar{c}} = [\tilde{\bar{C}}]_|, \quad \tilde{b} = [\tilde{B}]_|, \quad \tilde{c}_a = [\tilde{\nabla}_a \tilde{C}]_|,\\
&\tilde{\bar{c}}_a = [\tilde{\nabla}_a \tilde{\bar{C}}]_|, \quad \tilde{b}_a = [\tilde{\nabla}_a \tilde{B}]_|, \quad \tilde{\mathrm{c}} = [\tilde{\nabla}^2 \tilde{C}]_|,\\
&\tilde{\bar{\mathrm{c}}} = [\tilde{\nabla}^2 \tilde{\bar{C}}]_|, \quad \tilde{\mathrm{b}} = [\tilde{\nabla}^2 \tilde{B}]_|,
\end{aligned}
\tag{A1}
$$

where the fields c, $\bar{c}$, c, $\bar{\mathrm{c}}$, b_a and $\tilde{c}$, $\tilde{\bar{c}}$, $\tilde{\mathrm{c}}$, $\tilde{\bar{\mathrm{c}}}$, $\tilde{b}_a$ are fermionic fields and the fields c_a, $\bar{c}_a$, b, b and $\tilde{c}_a$, $\tilde{\bar{c}}_a$, b, $\tilde{\mathrm{b}}$ are bosonic fields. The components of the matter fields are given by

$$
\begin{aligned}
&x^I = [X^I]_|, \quad x^I_a = [\nabla_a X^I]_|, \quad \mathrm{x}^I = [\nabla^2 X^I]_|,\\
&y^I = [Y^I]_|, \quad y^I_a = [\nabla_a Y^I]_|, \quad \mathrm{y}^I = [\nabla^2 Y^I]_|,\\
&x^{I\dagger} = [X^{I\dagger}]_|, \quad x^{I\dagger}_a = [\nabla_a X^{I\dagger}]_|, \quad \mathrm{x}^{I\dagger} = [\nabla^2 X^{I\dagger}]_|,\\
&y^{I\dagger} = [Y^{I\dagger}]_|, \quad y^{I\dagger}_a = [\nabla_a Y^{I\dagger}]_|, \quad \mathrm{y}^{I\dagger} = [\nabla^2 Y^{I\dagger}]_|.
\end{aligned}
\tag{A2}
$$

We also write the components of Λ and $\tilde{\Lambda}$ as

$$
\begin{aligned}
&\lambda = [\Lambda]_|, \quad \lambda_a = [\nabla_a \Lambda]_|, \quad \bar{\lambda} = [\nabla^2 \Lambda]_|,\\
&\tilde{\bar{\lambda}} = [\tilde{\nabla}^2 \tilde{\Lambda}]_|, \quad \tilde{\lambda} = [\tilde{\Lambda}]_|, \quad \tilde{\lambda}_a = [\tilde{\nabla}_a \tilde{\Lambda}]_|.
\end{aligned}
\tag{A3}
$$

The component forms of υ and $\tilde{\upsilon}$ are given by

$$
\begin{aligned}
&\mu = [\upsilon]_|, \quad \mu_a = [\nabla_a \upsilon]_|, \quad \nu = [\nabla^2 \upsilon]_|,\\
&\tilde{\nu} = [\tilde{\nabla}^2 \tilde{\upsilon}]_|, \quad \tilde{\mu} = [\tilde{\upsilon}]_|, \quad \tilde{\mu}_a = [\tilde{\nabla}_a \tilde{\upsilon}]_|.
\end{aligned}
\tag{A4}
$$

The component forms of Γ_a and $\tilde{\Gamma}_a$ are given by

$$
\begin{aligned}
&\chi_a = [\Gamma_a]_|, \quad A = -\tfrac{1}{2}[\nabla^a \Gamma_a]_|,\\
&A^\mu = -\tfrac{1}{2}[\nabla^a (\gamma^\mu)^b_a \Gamma_b]_|, \quad E_a = -[\nabla^b \nabla_a \Gamma_b]_|,\\
&\tilde{\chi}_a = [\tilde{\Gamma}_a]_|, \quad \tilde{A} = -\tfrac{1}{2}[\tilde{\nabla}^a \tilde{\Gamma}_a]_|,\\
&\tilde{A}^\mu = -\tfrac{1}{2}[\tilde{\nabla}^a (\gamma^\mu)^b_a \tilde{\Gamma}_b]_|, \quad \tilde{E}_a = -[\tilde{\nabla}^b \tilde{\nabla}_a \tilde{\Gamma}_b]_|.
\end{aligned}
\tag{A5}
$$

Now after writing the components for all superfields we can write the gauge transformations of these component fields. Thus, the component forms of the gauge transformations of matter fields for the ABJM theory are given by

$$
\begin{aligned}
&\delta x^I = i(\lambda x^I - x^I\tilde{\lambda}), \qquad \delta x^{I\dagger} = -i(x^{I\dagger}\lambda - \tilde{\lambda}x^{I\dagger}), \qquad \delta y^I = -i(y^I\lambda - \tilde{\lambda}y^I), \qquad \delta y^{I\dagger} = i(\lambda y^{I\dagger} - y^{I\dagger}\tilde{\lambda}),\\
&\delta x^I_a = i(\lambda_a x^I - x^I\tilde{\lambda}_a) - i(\lambda x^I_a - x^I_a\tilde{\lambda}), \qquad \delta x^{I\dagger}_a = -i(x^{I\dagger}\lambda_a - \tilde{\lambda}_a x^{I\dagger}) + i(x^{I\dagger}_a\lambda - \tilde{\lambda}x^{I\dagger}_a),\\
&\delta y^I_a = -i(y^I\lambda_a - \tilde{\lambda}_a y^I) + i(y^I_a\lambda - \tilde{\lambda}y^I_a), \qquad \delta y^{I\dagger}_a = i(\lambda_a y^{I\dagger} - y^{I\dagger}\tilde{\lambda}_a) - i(\lambda y^{I\dagger}_a - y^{I\dagger}_a\tilde{\lambda}),\\
&\delta \mathrm{x}^I = i(\bar{\lambda}x^I - x^I\tilde{\bar{\lambda}}) + i(\lambda \mathrm{x}^I - \mathrm{x}^I\tilde{\lambda}) - 2i(\lambda^a x^I_a - x^{aI}\tilde{\lambda}_a),\\
&\delta \mathrm{x}^{I\dagger} = -i(x^{I\dagger}\bar{\lambda} - \tilde{\bar{\lambda}}x^{I\dagger}) - i(\mathrm{x}^{I\dagger}\lambda - \tilde{\lambda}\mathrm{x}^{I\dagger}) + 2i(x^{aI\dagger}\lambda_a - \tilde{\lambda}^a x^{I\dagger}_a),\\
&\delta \mathrm{y}^I = -i(y^I\bar{\lambda} - \tilde{\bar{\lambda}}y^I) - i(\mathrm{y}^I\lambda - \tilde{\lambda}\mathrm{y}^I) + 2i(y^{aI}\lambda_a - \tilde{\lambda}^a y^I_a),\\
&\delta \mathrm{y}^{I\dagger} = i(\bar{\lambda}y^{I\dagger} - y^{I\dagger}\tilde{\bar{\lambda}}) + i(\lambda \mathrm{y}^{I\dagger} - \mathrm{y}^{I\dagger}\tilde{\lambda}) - 2i(\lambda^a y^{I\dagger}_a - y^{aI\dagger}\tilde{\lambda}_a). \qquad \text{(A6)}
\end{aligned}
$$

The component forms of the gauge transformation of the gauge fields for the ABJM theory are given by

$$
\begin{aligned}
&\delta\chi_a = \chi_a\lambda + \lambda_a, \qquad \delta A = A\lambda + \bar{\lambda}, \qquad \delta A^\mu = \mathcal{D}_\mu\lambda, \qquad \delta E_a = E_a\lambda, \qquad \delta\tilde{\chi}_a = \tilde{\chi}_a\tilde{\lambda} + \tilde{\lambda}_a,\\
&\delta\tilde{A} = \tilde{A}\,\tilde{\lambda} + \tilde{\bar{\lambda}}, \qquad \delta\tilde{A}^\mu = \tilde{\mathcal{D}}_\mu\tilde{\lambda}, \qquad \delta\tilde{E}_a = \tilde{E}_a\tilde{\lambda}. \qquad \text{(A7)}
\end{aligned}
$$

The component forms of the gauge transformations for v and $\tilde{v}$ are given by

$$
\begin{aligned}
&\delta\mu = -i\mu\lambda, \qquad \delta\nu = -i\nu\lambda - 2i\mu^a\lambda_a - i\mu\bar{\lambda}, \qquad \delta\tilde{\mu} = -i\tilde{\mu}\,\tilde{\lambda}, \qquad \delta\tilde{\nu} = -i\tilde{\nu}\,\tilde{\lambda} - 2i\tilde{\mu}^a\tilde{\lambda}_a - i\tilde{\mu}\,\tilde{\bar{\lambda}},\\
&\delta\mu_a = -i\mu_a\lambda - i\mu\lambda_a, \qquad \delta\tilde{\mu}_a = -i\tilde{\mu}_a\tilde{\lambda} - i\tilde{\mu}\tilde{\lambda}_a. \qquad \text{(A8)}
\end{aligned}
$$

After discussing the component forms of the gauge transformations, we will analyze the component forms of the BRST and the anti-BRST transformations. In component form the BRST transformations of the matter fields in the ABJM theory are given by

$$
\begin{aligned}
&sx^I = i(cx^I - x^I\tilde{c}), \qquad sx^{I\dagger} = -i(x^{I\dagger}c - \tilde{c}x^{I\dagger}), \qquad sy^I = -i(y^Ic - \tilde{c}y^I), \qquad sy^{I\dagger} = i(cy^{I\dagger} - y^{I\dagger}\tilde{c}),\\
&sx^I_a = i(c_a x^I - x^I\tilde{c}_a) - i(cx^I_a - x^I_a\tilde{c}), \qquad sx^{I\dagger}_a = -i(x^{I\dagger}c_a - \tilde{c}_a x^{I\dagger}) + i(x^{I\dagger}_a c - \tilde{c}x^{I\dagger}_a),\\
&sy^I_a = -i(y^Ic_a - \tilde{c}_a y^I) + i(y^I_a c - \tilde{c}y^I_a), \qquad sy^{I\dagger}_a = i(c_a y^{I\dagger} - y^{I\dagger}\tilde{c}_a) - i(cy^{I\dagger}_a - y^{I\dagger}_a\tilde{c}),\\
&s\mathrm{x}^I = i(\mathrm{c}x^I - x^I\tilde{\mathrm{c}}) + i(c\mathrm{x}^I - \mathrm{x}^I\tilde{c}) - 2i(c^a x^I_a - x^{aI}\tilde{c}_a),\\
&s\mathrm{x}^{I\dagger} = -i(x^{I\dagger}\mathrm{c} - \tilde{\mathrm{c}}x^{I\dagger}) - i(\mathrm{x}^{I\dagger}c - \tilde{c}\mathrm{x}^{I\dagger}) + 2i(x^{aI\dagger}c_a - \tilde{c}^a x^{I\dagger}_a),\\
&s\mathrm{y}^I = -i(y^I\mathrm{c} - \tilde{\mathrm{c}}y^I) - i(\mathrm{y}^Ic - \tilde{c}\mathrm{y}^I) + 2i(y^{aI}c_a - \tilde{c}^a y^I_a),\\
&s\mathrm{y}^{I\dagger} = i(\mathrm{c}y^{I\dagger} - y^{I\dagger}\tilde{\mathrm{c}}) + i(c\mathrm{y}^{I\dagger} - \mathrm{y}^{I\dagger}\tilde{c}) - 2i(c^a y^{I\dagger}_a - y^{aI\dagger}\tilde{c}_a). \qquad \text{(A9)}
\end{aligned}
$$

The anti-BRST transformations of the matter fields in the ABJM theory in component form are given by

$$
\begin{aligned}
&\bar{s}x^I = i(\bar{c}x^I - x^I\tilde{\bar{c}}), \qquad \bar{s}x^{I\dagger} = -i(x^{I\dagger}\bar{c} - \tilde{\bar{c}}x^{I\dagger}), \qquad \bar{s}y^I = -i(y^I\bar{c} - \tilde{\bar{c}}y^I), \qquad \bar{s}y^{I\dagger} = i(\bar{c}y^{I\dagger} - y^{I\dagger}\tilde{\bar{c}}),\\
&\bar{s}x^I_a = i(\bar{c}_a x^I - x^I\tilde{\bar{c}}_a) - i(\bar{c}x^I_a - x^I_a\tilde{\bar{c}}), \qquad \bar{s}x^{I\dagger}_a = -i(x^{I\dagger}\bar{c}_a - \tilde{\bar{c}}_a x^{I\dagger}) + i(x^{I\dagger}_a\bar{c} - \tilde{\bar{c}}x^{I\dagger}_a),\\
&\bar{s}y^I_a = -i(y^I\bar{c}_a - \tilde{\bar{c}}_a y^I) + i(y^I_a\bar{c} - \tilde{\bar{c}}y^I_a), \qquad \bar{s}y^{I\dagger}_a = i(\bar{c}_a y^{I\dagger} - y^{I\dagger}\tilde{\bar{c}}_a) - i(\bar{c}y^{I\dagger}_a - y^{I\dagger}_a\tilde{\bar{c}}),\\
&\bar{s}\mathrm{x}^I = i(\bar{\mathrm{c}}x^I - x^I\tilde{\bar{\mathrm{c}}}) + i(\bar{c}\mathrm{x}^I - \mathrm{x}^I\tilde{\bar{c}}) - 2i(\bar{c}^a x^I_a - x^{aI}\tilde{\bar{c}}_a),\\
&\bar{s}\mathrm{x}^{I\dagger} = -i(x^{I\dagger}\bar{\mathrm{c}} - \tilde{\bar{\mathrm{c}}}x^{I\dagger}) - i(\mathrm{x}^{I\dagger}\bar{c} - \tilde{\bar{c}}\mathrm{x}^{I\dagger}) + 2i(x^{aI\dagger}\bar{c}_a - \tilde{\bar{c}}^a x^{I\dagger}_a),\\
&\bar{s}\mathrm{y}^I = -i(y^I\bar{\mathrm{c}} - \tilde{\bar{\mathrm{c}}}y^I) - i(\mathrm{y}^I\bar{c} - \tilde{\bar{c}}\mathrm{y}^I) + 2i(y^{aI}\bar{c}_a - \tilde{\bar{c}}^a y^I_a),\\
&\bar{s}\mathrm{y}^{I\dagger} = i(\bar{\mathrm{c}}y^{I\dagger} - y^{I\dagger}\tilde{\bar{\mathrm{c}}}) + i(\bar{c}\mathrm{y}^{I\dagger} - \mathrm{y}^{I\dagger}\tilde{\bar{c}}) - 2i(\bar{c}^a y^{I\dagger}_a - y^{aI\dagger}\tilde{\bar{c}}_a). \qquad \text{(A10)}
\end{aligned}
$$

In component form the BRST transformations of gauge fields, ghosts, antighosts, and auxiliary fields for the ABJM theory are given by

$$
\begin{aligned}
&s\chi_a = \chi_a c + c_a, \quad sA = Ac + \mathrm{c}, \quad sA^\mu = \mathcal{D}_\mu c, \quad sE_a = E_a c, \quad sc = -\tfrac{1}{2}\{c, c\}, \quad sc_a = [c, c_a],\\
&s\bar{c} = b, \quad s\mathrm{c} = [c^a, c_a] - \{c, \mathrm{c}\}, \quad s\bar{c}_a = b_a, \quad s\bar{\mathrm{c}} = \mathrm{b} \quad s\tilde{\chi}_a = \tilde{\chi}_a \tilde{c} + \tilde{c}_a, \quad s\tilde{A} = \tilde{A}\,\tilde{c} + \tilde{\mathrm{c}},\\
&s\tilde{A}^\mu = \tilde{\mathcal{D}}_\mu \tilde{c}, \quad s\tilde{E}_a = \tilde{E}_a \tilde{c}, \quad s\tilde{c} = -\tfrac{1}{2}\{\tilde{c}, \tilde{c}\}, \quad s\tilde{c}_a = [\tilde{c}, \tilde{c}_a], \quad s\tilde{\bar{c}} = \tilde{b}, \quad s\tilde{\mathrm{c}} = [\tilde{c}^a, \tilde{c}_a] - \{\tilde{c}, \tilde{\mathrm{c}}\},\\
&s\tilde{\bar{c}}_a = \tilde{b}_a, \quad s\tilde{\bar{\mathrm{c}}} = \tilde{\mathrm{b}} \quad sb = 0, \quad sb_a = 0, \quad s\tilde{b} = 0, \quad s\tilde{b}_a = 0, \quad s\tilde{b} = 0 \quad s\tilde{b} = 0.
\end{aligned}
\tag{A11}
$$

The anti-BRST transformations of the gauge fields, ghosts, antighosts, and the auxiliary fields for the ABJM theory in component form are given by

$$
\begin{aligned}
&\bar{s}c = -b - \{\bar{c}, c\}, \quad \bar{s}c_a = -b_a - [\bar{c}_a, c] + [\bar{c}, c_a], \quad \bar{s}\,\bar{c} = -\tfrac{1}{2}\{\bar{c}, \bar{c}\}, \quad \bar{s}\mathrm{c} = -\mathrm{b} - \{\bar{\mathrm{c}}, c\} - \{c, \bar{\mathrm{c}}\} + 2[\bar{c}^a, c_a],\\
&\bar{s}\bar{c}_a = [\bar{c}, \bar{c}_a], \quad \bar{s}\,\bar{\mathrm{c}} = [\bar{c}^a, \bar{c}_a] - \{\bar{c}, \bar{\mathrm{c}}\}, \quad \bar{s}b_a = \tfrac{1}{2}\{b_a, \bar{c}\} + \tfrac{1}{2}[b, \bar{\mathrm{c}}_a], \quad \bar{s}\mathrm{b} = \tfrac{1}{2}[\mathrm{b}, \bar{c}] + \tfrac{1}{2}[b, \bar{\mathrm{c}}] + [b^a, \bar{c}_a],\\
&\bar{s}b = \tfrac{1}{2}[b, \bar{c}], \quad \bar{s}\chi_a = \chi_a \bar{c} + \bar{c}_a, \quad \bar{s}A = A\bar{c} + \bar{\mathrm{c}}, \quad \bar{s}A^\mu = \mathcal{D}_\mu \bar{c}, \quad \bar{s}\,\tilde{c} = -\tilde{b} - \{\tilde{\bar{c}}, \tilde{c}\},\\
&\bar{s}\tilde{c}_a = -\tilde{b}_a - [\tilde{\bar{c}}_a, \tilde{c}] + [\tilde{\bar{c}}, \tilde{c}_a], \quad \bar{s}\,\tilde{\bar{c}} = -\tfrac{1}{2}\{\tilde{\bar{c}}, \tilde{\bar{c}}\}, \quad \bar{s}\,\tilde{\mathrm{c}} = -\tilde{\mathrm{b}} - \{\tilde{\bar{\mathrm{c}}}, \tilde{c}\} - \{\tilde{c}, \tilde{\bar{\mathrm{c}}}\} + 2[\tilde{\bar{c}}^a, \tilde{c}_a], \quad \bar{s}\tilde{\bar{c}}_a = [\tilde{\bar{c}}, \tilde{\bar{c}}_a],\\
&\bar{s}\,\tilde{\bar{\mathrm{c}}} = [\tilde{\bar{c}}^a, \tilde{\bar{c}}_a] - \{\tilde{\bar{c}}, \tilde{\bar{\mathrm{c}}}\}, \quad \bar{s}\tilde{b}_a = \tfrac{1}{2}\{\tilde{b}_a, \tilde{\bar{c}}\} + \tfrac{1}{2}[\tilde{b}, \tilde{\bar{\mathrm{c}}}_a], \quad \bar{s}\,\tilde{\mathrm{b}} = \tfrac{1}{2}[\tilde{\mathrm{b}}, \tilde{\bar{c}}] + \tfrac{1}{2}[\tilde{b}, \tilde{\bar{\mathrm{c}}}] + [\tilde{b}^a, \tilde{\bar{c}}_a], \quad \bar{s}\,\tilde{b} = \tfrac{1}{2}[\tilde{b}, \tilde{\bar{c}}],\\
&\bar{s}\tilde{\chi}_a = \tilde{\chi}_a \tilde{\bar{c}} + \tilde{\bar{c}}_a, \quad \bar{s}\,\tilde{A} = \tilde{A}\,\tilde{\bar{c}} + \tilde{\bar{\mathrm{c}}}, \quad \bar{s}\tilde{A}^\mu = \tilde{\mathcal{D}}_\mu \tilde{\bar{c}}, \quad \bar{s}\tilde{E}_a = \tilde{E}_a \tilde{\bar{c}}, \quad \bar{s}E_a = E_a \bar{c}.
\end{aligned}
\tag{A12}
$$

Furthermore, the BRST transformations of v, $\tilde{v}$ in component form are given by

$$
\begin{aligned}
&s\mu = -i\mu c, \quad s\nu = -i\nu c - 2i\mu^a c_a - i\mu \mathrm{c}, \quad s\tilde{\mu} = -i\tilde{\mu}\,\tilde{c}, \quad s\tilde{\nu} = -i\tilde{\nu}\,\tilde{c} - 2i\tilde{\mu}^a \tilde{c}_a - i\tilde{\mu}\,\tilde{\mathrm{c}},\\
&s\mu_a = -i\mu_a c - i\mu c_a, \quad s\tilde{\mu}_a = -i\tilde{\mu}_a \tilde{c} - i\tilde{\mu}\tilde{c}_a,
\end{aligned}
\tag{A13}
$$

and the anti-BRST transformations of v, $\tilde{v}$ in component form are given by

$$
\begin{aligned}
&\bar{s}\mu = -i\mu\bar{c}, \quad \bar{s}\nu = -i\nu\bar{c} - 2i\mu^a \bar{c}_a - i\mu\bar{\mathrm{c}}, \quad \bar{s}\,\tilde{\mu} = -i\tilde{\mu}\,\tilde{\bar{c}}, \quad \bar{s}\,\tilde{\nu} = -i\tilde{\nu}\,\tilde{\bar{c}} - 2i\tilde{\mu}^a \tilde{\bar{c}}_a - i\tilde{\mu}\,\tilde{\bar{\mathrm{c}}},\\
&\bar{s}\mu_a = -i\mu_a \bar{c} - i\mu\bar{c}_a, \quad \bar{s}\tilde{\mu}_a = -i\tilde{\mu}_a \tilde{\bar{c}} - i\tilde{\mu}\tilde{\bar{c}}_a.
\end{aligned}
\tag{A14}
$$

These are the component forms of the BRST and the anti-BRST transformations of the ABJM theory and the boundary theory it is coupled to.

[1] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
[2] I. L. Buchbinder, E. A. Ivanov, O. Lechtenfeld, N. G. Pletnev, I. B. Samsonov, and B. M. Zupnik, J. High Energy Phys. 03 (2009) 096.
[3] O-Kab Kwon, P. Oh, and J. Sohn, J. High Energy Phys. 08 (2009) 093.
[4] J. Bagger and N. Lambert, Phys. Rev. D **75**, 045020 (2007).
[5] J. Bagger and N. Lambert, Phys. Rev. D **77**, 065008 (2008).
[6] J. Bagger and N. Lambert, J. High Energy Phys. 02 (2008) 105.
[7] A. Gustavsson, Nucl. Phys. **B811**, 66 (2009).
[8] A. Basu and J. A. Harvey, Nucl. Phys. **B713**, 136 (2005).
[9] A. Mauri and A. C. Petkou, Phys. Lett. B **666**, 527 (2008).
[10] S. Mukhi and C. Papageorgakis, J. High Energy Phys. 05 (2008) 085.
[11] S. V. Ketov and S. Kobayashi, Phys. Rev. D **83**, 045003 (2011).
[12] L. Andrianopoli, S. Ferrara, and M. A. Lledo, J. High Energy Phys. 04 (2004) 005.
[13] D. S. Berman, M. J. Perry, E. Sezgin, and D. C. Thompson, J. High Energy Phys. 04 (2010) 025.
[14] C.-S. Chu and G. S. Sehmbi, J. Phys. A **44**, 135 404 (2011).
[15] C.-S. Chu and D. J. Smith, J. High Energy Phys. 04 (2009) 097.
[16] P.-M. Ho, Chin. J. Phys. **48**, 1 (2010).
[17] E. Bergshoeff, D. S. Berman, J. P. van der Schaar, and P. Sundell, Nucl. Phys. **B590**, 173 (2000).
[18] S. Kawamoto and N. Sasakura, J. High Energy Phys. 07 (2000) 014.
[19] D. S. Berman and B. Pioline, Phys. Rev. D **70**, 045007 (2004).
[20] D. V. Belyaev and P. van Nieuwenhuizen, J. High Energy Phys. 04 (2008) 008.
[21] D. S. Berman and D. C. Thompson, Nucl. Phys. **B820**, 503 (2009).
[22] C.-S. Chu and D. J. Smith, J. High Energy Phys. 01 (2010) 001.
[23] N. Lambert and P. Richmond, J. High Energy Phys. 10 (2009) 084.
[24] Y. Kim, O-K. Kwon, H. Nakajima, and D. D. Tolla, J. High Energy Phys. 11 (2010) 069.
[25] J. P. Allen and D. J. Smith, J. High Energy Phys. 08 (2011) 078.

[26] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[27] G. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).

[28] S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. **B326**, 108 (1989).

[29] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **550**, 213 (2002).

[30] J. H. Schwarz, J. High Energy Phys. 11 (2004) 078.

[31] C. Ahn, H. Kim, B. H. Lee, and H. S. Yang, Phys. Rev. D **61**, 066002 (2000).

[32] B. Chen and J. B. Wu, J. High Energy Phys. 09 (2008) 096.

[33] M. Benna, I. Klebanov, T. Klose, and M. Smedback, J. High Energy Phys. 09 (2008) 072.

[34] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) **98**, 287 (1976).

[35] I. V. Tyutin, Lebedev preprint FIAN Report No. 39, 1975.

[36] I. Ojima, Prog. Theor. Phys. **64**, 625 (1980).

[37] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[38] J. Fjelstad and S. Hwang, Phys. Lett. B **466**, 227 (1999).

[39] M. Chaichian, W. F. Chen, and Z. Y. Zhu, Phys. Lett. B **387**, 785 (1996).

[40] L. P. Colatto, M. A. De Andrade, O. M. Del Cima, D. H. T. Franco, J. A. Helayel-Neto, and O. Piguet, J. Phys. G **24**, 1301 (1998).

[41] C. P. Constantinidis, O. Piguet, and W. Spalenza, Eur. Phys. J. C **33**, 443 (2004).

[42] M. Faizal, Phys. Rev. D **84**, 106011 (2011).

[43] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, Front. Phys. **58**, 1 (1983).

[44] C. M. Hull and E. Witten, Phys. Lett. **160B**, 398 (1985).

[45] C. M. Hull and B. Spence, Nucl. Phys. **B345**, 493 (1990).

[46] E. A. Ivanov, Phys. Lett. B **268**, 203 (1991).

[47] S. J. Gates, arXiv:hep-th/9709104.

[48] S. J. Gates, M. T. Grisaru, M. E. Knutt-Wehlau, and W. Siegel, Phys. Lett. B **421**, 203 (1998).

[49] P. S. Howe, T. G. Pugh, K. S. Stelle, and C. Strickland-Constable, J. High Energy Phys. 08 (2011) 081.

[50] D. Orgad and S. Levit, Phys. Rev. B **53**, 7964 (1996).

[51] A. Lopez and E. Fradkin, Phys. Rev. B **69**, 155322 (2004).

[52] B. Skoric and A. M. M. Pruisken, Nucl. Phys. **B559**, 637 (1999).

[53] I. Ichinose and A. Sekiguchi, Nucl. Phys. **B493**, 683 (1997).

[54] K. Hasebe, Phys. Rev. D **72**, 105017 (2005).

[55] K. Hasebe, Phys. Rev. Lett. **94**, 206802 (2005).

[56] K. Hasebe, Phys. Lett. A **372**, 1516 (2008).

[57] E. Ivanov, L. Mezincescu, and P. K. Townsend, J. High Energy Phys. 01 (2006) 143.

[58] K. Hasebe, Phys. Rev. D **74**, 045026 (2006).

[59] X. G. Wen, Adv. Phys. **44**, 405 (1995).

[60] P. L. McEuen, A. Szafer, C. A. Richter, B. W. Alphenaar, J. K. Jain, A. D. Stone, R. G. Wheeler, and R. N. Sacks, Phys. Rev. Lett. **64**, 2062 (1990).

[61] D. Orgad and S. Levit, Phys. Rev. B **53**, 7964 (1996).

[62] T. Morinari and N. Nagaosa, Solid State Commun. **100**, 163 (1996).

[63] F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).